Algebra Prelim part B

January 7, 2014

Directions: You have 90 minutes. Answer two questions; specify clearly which problems you want graded.

B1. Let R be an integral domain such that every prime ideal of R is principal.

(a) Consider the set of ideals of R which are not principal. Prove that if this set is non-empty, then it contains an element I which is maximal under inclusion.

(b) Prove that R is a principal ideal domain. (Hint: Show that I is principal.)

B2. Suppose that E/F is a Galois field extension and every subextension is normal.

(a) Show that $\mathrm{Gal}(E/F)$ is nilpotent. (Of the various formulations of nilpotence, you may use: a finite group is nilpotent if and only if it is a product of Sylow p-subgroups.)

(b) Give an example to show that it need not be abelian.

B3. Recall that $\mathbf{Q}(t)$ denotes the field of rational functions with coefficients in $\mathbf{Q}$, and similarly with $\mathbf{C}$ in place of Q.

(a) Show that the Galois group of the splitting field K of the polynomial

$$f(x) = x^4 - tx^2 - 1$$

over $\mathbf{Q}(t)$ is the dihedral group of order eight.

(b) Show that K contains $i = \sqrt{-1}$.

(c) Show that the splitting field of f over $\mathbf{C}(t)$ is $\mathbf{Z}/2 \times \mathbf{Z}/2$.