PRELIMINARY EXAM: DIFFERENTIAL TOPOLOGY

Date: January 9, 2014
Instructions: Do all three problems.
Time Limit: 90 minutes

Problem 1.

(a) Suppose that M is a compact orientable n-manifold (without boundary) and let θ be an $n-1$-form on M. Show that $d\theta$ must vanish at some point.

(b) Give an example of an $n-1$-form θ on a compact orientable n-manifold *with* boundary for which $d\theta$ never vanishes.

Problem 2. Let $Z \subset \mathbb{R}^2$ be the unit circle. Consider the map

$$f : \mathbb{R}^2 \setminus (0,0) \to \mathbb{R}^2$$

given by

$$f(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right) .$$

(a) Prove that f is not transverse to Z.

(b) Display a smooth homotopy

$$F : (\mathbb{R}^2 \setminus (0,0)) \times I \to \mathbb{R}^2$$

such that $F(x,0) = f(x)$ and $g(x) := F(x,1)$ is transverse to Z (and justify this claim).

Problem 3. Let $\tilde{f} : \mathbb{R}^4 \to \mathbb{R}^4$ be the map

$$\tilde{f}(x_0, x_1, x_2, x_3) = (x_0, 2x_1, 3x_2, 4x_3)$$

and let $f : \mathbb{R}P^3 \to \mathbb{R}P^3$ be the map induced by $\tilde{f}$.

(a) Find the fixed points of f.

(b) Compute the local Lefschetz number at each fixed point.

(c) Use this to compute the Euler characteristic of $\mathbb{R}P^3$.