THE UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS

The Preliminary Examination in Probability
Part I

Monday, Jan 12, 2015

Problem 1. Let X and Y be two square-integrable random variables, defined on the same probability space, such that

$$(X(\omega') - X(\omega))(Y(\omega') - Y(\omega)) \geq 0 \text{ for all } \omega, \omega' \in \Omega.$$

Show that $\operatorname{Cov}(X, Y) \geq 0$.

Problem 2. A probability measure μ on (the Borel subsets of) $\mathbb{R}$ is said to be **infinitely divisible**, if, for each $n \in \mathbb{N}$, there exists a probability μ_n on $\mathbb{R}$ such that $\mu = \mu_n * \cdots * \mu_n$ (n-fold convolution).

(1) Show that any $\mu \in \mathbb{R}$ and $\sigma > 0$, the normal distribution, $N(\mu, \sigma^2)$ is infinitely divisible.
(2) Find another example of an infinitely divisible measure on $\mathbb{R}$.
(3) Find an example of a probability measure on $\mathbb{R}$ which is not infinitely divisible. (*Hint:* The sum of two discrete, independent and nonconstant random variables takes at least 4 different values with positive probabilities.)

(*Note:* The set of all probability measures on $\mathbb{R}$ admits the structure of *(commutative) monoid* with respect to the operation of convolution. Infinitely divisible measures are exactly those that admit an "n-th root" for each $n \in \mathbb{N}$.)

Problem 3. Given two independent simple symmetric random walks $\{\tilde{X}_n\}_{n \in \mathbb{N}_0}$ and $\{\tilde{Y}_n\}_{n \in \mathbb{N}_0}$, let $\{X_n\}_{n \in \mathbb{N}_0}$ denote $\{\tilde{X}_n\}_{n \in \mathbb{N}_0}$ stopped when it first hits the level 1, and let $\{Y_n\}_{n \in \mathbb{N}_0}$ be given by

$$Y_0 = 0, \; Y_n = \sum_{k=1}^{n} 2^{-k} (\tilde{Y}_k - \tilde{Y}_{k-1}).$$

Identify the distribution of $\liminf_n (X_n + Y_n)$ and show that the sequence $\{X_n + Y_n\}_{n \in \mathbb{N}_0}$ is *not* uniformly integrable.