PRELIMINARY EXAMINATION IN ANALYSIS
PART I - REAL ANALYSIS
JANUARY 11, 2016

Please try to solve 4 of the following 5 problems.

(1) For any $r \geq 0$ and any $x \in \mathbb{R}^2$ define $B_r(x) = \{y \in \mathbb{R}^2 : |y - x| \leq r\}$. Let $0 < c < 1$. Let E be a measurable subset of the unit square $Q \subset \mathbb{R}^2$ with the property that for every $x \in Q$ and every $r > 0$ there exists $y \in B_r(x)$ such that $B_{c|x-y|}(y) \subset E$. Prove that $Q \setminus E$ has measure zero.

(2) Show that if $p > 1$ and $f \in L^p([0,\infty), m)$ then the 'mean functional' of f,

$$F(y) := \frac{1}{y}\int_0^y f(t)\, dt = \int_0^1 f(xy)\, dx$$

is also in $L^p([0,\infty), m)$ and moreover

$$\|F\|_p \leq \frac{p}{p-1}\|f\|_p.$$

Hint: consider $f(xy)$ as a function of two variables on $[0,1] \times [0,\infty)$ and use the generalized Minkowski inequality (which states that if $g : X \times Y \to \mathbb{R}$ is any measurable function on the direct product of two sigma-finite measure spaces $(X,\mu), (Y,\nu)$ then

$$\Big\| \|g\|_{L^1(X,\mu)} \Big\|_{L^p(Y,\nu)} \leq \Big\| \|g\|_{L^p(Y,\nu)} \Big\|_{L^1(X,\mu)}).$$

(3) Let (X,d) be a compact metric space. Let $\{\mu_n\}$ be a sequence of positive Borel measures on X that converge in the weak* topology to a finite positive Borel measure μ. Show that for every compact $K \subset X$,

$$\mu(K) \geq \limsup_{n\to\infty} \mu_n(K).$$

(4) Let $1 < p < \infty$. Assume $f \in L^p(\mathbb{R})$ satisfies

$$\sup_{0<|h|<1} \int \left| \frac{f(x+h) - f(x)}{h} \right|^p dx < \infty\,.$$

Show that f has a weak derivative $g \in L^p$, which by definition satisfies $\int \psi g = -\int \psi' f$ for every C^∞ function ψ on $\mathbb{R}$ with compact support.

(5) Assuming $f : [0,1] \to \mathbb{R}$ is absolutely continuous, prove that f is Lipschitz if and only if f' belongs to $L^\infty([0,1])$.