PRELIMINARY EXAMINATION IN ANALYSIS

Part II, Complex Analysis

August 15, 2016

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1. Let f be analytic in a connected open neighborhood of the closed unit disk $|z| \leq 1$. Assume that $|f(z)| = |z+1|$ on the unit circle $|z| = 1$, that $f(1) = 2$, and that f has simple zeros at $\pm i/2$ and no other zeros in the disk $|z| < 1$. Show that these properties determine f uniquely. Calculate $f(0)$.

2. Evaluate the integrals

$$I = \int_C \sqrt{1-z^2}\, dz\,, \qquad J = \int_0^\infty \frac{x\sin(\pi x)}{1-x^2}\, dx\,,$$

for some branch of the square root function (indicate which), where C is the positively oriented circle $|z| = 2$.

3. Is there an entire function f that satisfies $|f(z)| \geq e^{c|z|}$ for some $c > 0$ and all sufficiently large $|z|$? Give an example of such a function or prove that none exists.

4. Let $\mathcal{F}$ be the set of all entire functions f with the following properties. The zeros $a_1, a_2, \ldots$ of f satisfy $\sum_n |a_n|^{-1} \leq 1$. Furthermore, $f(0) = 1$, $|f'(0)| \leq 1$, and $e^{-|z|}|f(z)| \to 0$ as $|z| \to \infty$. Show that every sequence $n \mapsto f_n$ in $\mathcal{F}$ has a subsequence that converges uniformly on compact subsets of $\mathbb{C}$.