PRELIMINARY EXAMINATION: APPLIED MATHEMATICS — Part II

August 19, 2016, 2:40–4:10 p.m.

Work all 3 of the following 3 problems.

1. Let $f \in H^1(\mathbb{R}^d)$ and $0 \leq s \leq 1$. Prove that there is a constant $C > 0$ such that

$$\|f\|_{H^s(\mathbb{R}^d)} \leq C\|f\|^s_{H^1(\mathbb{R}^d)}\, \|f\|^{1-s}_{L^2(\mathbb{R}^d)}.$$

2. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary $\partial\Omega$. For $f(x)$, $g(x)$ and $u_D(x)$ sufficiently smooth, consider the biharmonic boundary value problem (BVP)

$$\begin{aligned} \Delta^2 u &= f && \text{in } \Omega, \\ \Delta u &= g && \text{on } \partial\Omega, \\ u &= u_D && \text{on } \partial\Omega. \end{aligned}$$

(**a**) Reformulate the BVP as a variational problem for $u \in H^2(\Omega) \cap H^1_0(\Omega) + u_D$. Please indicate precisely the spaces in which the functions f, g, and u_D lie.

(**b**) Apply the Lax-Milgram Theorem to show that there is a unique solution to the variational problem.

3. Let $\phi(x) \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

(**a**) Consider the nonlinear initial value problem (IVP)

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + 3u^2\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^2 \partial t} = 0, \quad x \in \mathbb{R}, t > 0, \\ &u(x,0) = \phi(x). \end{aligned}$$

Use the Fourier transform in x to show that the (IVP) can be rewritten in the form

$$\partial_t u = K * (u + u^3), \quad x \in \mathbb{R}, t > 0, \tag{1}$$

$$u(x,0) = \phi(x), \tag{2}$$

for some $K(x) \in L^1(\mathbb{R})$. [Hint: after rewriting the IVP in a convenient way (in particular, you can keep together the terms $u_x + 3u^2 u_x$ by giving this group of terms a temporary name), apply the Fourier transform in x to the IVP, simplify the expression that you obtain and then apply the inverse Fourier transform to formally obtain (1)–(2).]

(**b**) Set up and apply the contraction mapping principle to show that the initial value problem (1)–(2) has a continuous and bounded solution $u = u(x,t)$, at least up to some time $T < \infty$.