Preliminary Examination: Algebraic topology. August 2017

Instructions: Answer all four questions

Time limit: 90 minutes.

1. Let K be a smoothly embedded circle in $\mathbb{R}^3$ (i.e. a knot). Use a Mayer-Vietoris sequence to calculate the homology groups of $X = \mathbb{R}^3 \setminus K$ with $\mathbb{Z}$ coefficients. (The result does not depend on K).

2. Let $X = S^1 \vee S^1$ be the wedge of two circles. Recall that $\pi_1(X) = \langle a, b \rangle$ is free, generated by two loops, one going around each circle.
(a): Draw a picture of a three-fold covering space $p : \widetilde{X} \to X$ which *is not* normal. Write down generators for the corresponding subgroup H of $\pi_1(X)$. What is the size of a minimal generating set for H?
(b): Draw a picture of a three-fold covering space $q : \widetilde{Y} \to X$ which *is* normal. Describe the action of the deck transformation group $G(\widetilde{Y})$ in your picture.

3. Let X be the identification space obtained from a cube $[0,1] \times [0,1] \times [0,1]$ by identifying opposite faces in pairs with a quarter twist in the positive direction. Let Y denote the image of $[0,1] \times [0,1] \times \{0\}$ in X.
(a): Write down presentations for $\pi_1(X, x_0)$ and $\pi_1(Y, x_0)$, where x_0 is one of the vertices. Do not simplify.
(b): Compute the homology groups $H_1(X, \mathbb{Z})$ and $H_1(Y, \mathbb{Z})$.
(c): Does X retract onto Y?

4. Let M_g denote the closed orientable surface of genus g. For what values of n does there exist a connected normal covering space of M_g whose deck transformation group is isomorphic to $\mathbb{Z}^n$?