Preliminary Examination: Algebraic topology. August 22, 2018

Instructions: Answer all three questions

Time limit: 90 minutes.

1. Let $X = S^1 \vee S^1$ be the wedge of two circles. For each of the following groups G, decide whether there exists a normal covering space $p : \widetilde{X} \to X$ whose group of covering transformations (also called deck translations) is isomorphic to G. If yes, sketch the covering space. If no, explain why not.

(a): $G = \langle a, b \rangle$, the free group on two generators.

(b): $G = \langle a, b, c \rangle$, the free group on three generators.

(c): $G = S_3 = \langle a, b : a^2 = b^2 = (ab)^3 = 1 \rangle$, the symmetric group on three elements.

2. Prove that there is no continuous map $f : S^2 \to S^1$ whose restriction to the equator $S^1 \subset S^2$ is a homeomorphism.

3.

(a) Let $X = A \cup B$ be the topological subspace of $\mathbb{R}^6$ which is the union of the two subspaces A, B below:

$$A = \{(x_1, x_2, x_3, x_4, 0, 0) : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\} \cong S^3$$
$$B = \{(0, 0, y_3, y_4, y_5, y_6) : y_3^2 + y_4^2 + y_5^2 + y_6^2 = 1\} \cong S^3$$

Calculate the homology groups $H_*(X)$.

(b) Let $Y = V \cup W$ be the subspace of $\mathbb{R}^6$ which is the union of the two coordinate subspaces V, W:

$$V = \{(x_1, x_2, x_3, x_4, 0, 0)\} \cong \mathbb{R}^4$$
$$W = \{(0, 0, y_3, y_4, y_5, y_6)\} \cong \mathbb{R}^4$$

Prove that Y is not homeomorphic to $\mathbb{R}^m$ for any m.