Real Analysis Prelim Fall 2021

August 17, 2021

1. Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function, or equivalently,

$$f(x) - f(a) = \int_a^x f', \quad \text{for any } a \leq x \leq b.$$

Show that $f(x)$ is Lipschitz if and only if $f' \in L^\infty([a, b])$.

2. Part **i)** Define the space *Weak* L^p in the unit ball $B_1^n(0)$ in $\mathbb{R}^n$ for $1 < p < \infty$, denoted by $L^{p,w}(B_1^n(0), dx)$.

Part **ii)** Show an example of a function $f(x)$ such that $f \in L^{p,w}(B_1^n(0), dx)$, but not in the classical $L^p(B_1^n(0))$

3. Consider the sequence $\{f_n\}$ in $L^p([0, \pi])$ with $1 \leq p < \infty$, defined by

$$f_n(x) = \cos nx.$$

Show that $\{f_n\}$ converges weakly to zero in $L^p([0, \pi])$, but does not converge strongly to zero in such Banach space.

4. Let Σ be a compact set of functions $f \in L^p([0, 1])$. Let $f^+(x) := \max\{0, f(x)\}$, for all $x \in [0, 1]$, be "non-negative part of $f(x)$".
Show that the subset of Σ, denoted by $\Sigma^+ = \{f^+ \ : \ f \in \Sigma\}$, is also compact.

5. Show that if $f \in L^q(\mathbb{R})$, with $1 \leq q < \infty$, then

$$\int |f(\lambda x) - f(x)|^q dx \ \to \ 0, \qquad \text{whenever } \lambda \to 1.$$