ALGEBRA PRELIMINARY EXAM: PART I

Problem 1

a) Let p be a prime. Prove that every group of order p^2 is abelian.

b) Determine the number of isomorphism classes of groups of order 45.

Problem 2

a) Let R be an integral domain such that every prime ideal of R is principal. Prove that R is a principal ideal domain.

Hint: Consider the set of ideals of R which are not principal. Prove that if this set is non-empty, then it contains an element I which is maximal under inclusion.

b) Prove that $\mathbb{Z}[\sqrt{-3}]$ is not a principal ideal domain.

Problem 3

Consider the 2-dimensional $\mathbb{R}$ vector space $V = \mathbb{R}^2$, the linear transformation $T_1 : V \to V$ which is the projection onto the y-axis, and the linear transformation T_2 induced by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then view V as an $\mathbb{R}[x]$-module V_i where $x \cdot v = T_i(v)$ for $i = 1, 2$.

Describe the $\mathbb{R}[x]$-module structure of V_i for $i = 1, 2$. Determine whether V_1 and V_2 are isomorphic $\mathbb{R}[x]$-modules.