PRELIMINARY EXAMINATION:
ANALYSIS — Part II

August 14, 2023

Work all 4 of the following 4 problems.

1. Harmonic and entire functions

(**a**) Consider a real-valued harmonic function u on $\mathbb{R}^2$. Show that there exists a real-valued function v on $\mathbb{R}^2$ such that $u + iv$ is entire on $\mathbb{C} \equiv \mathbb{R}^2$. Show that v is unique up to a constant.

(**b**) Consider f an entire function. Assume than $|f(z)| \geq c > 0$ on $\mathbb{C}$. Show that f is constant.

2. Assume that $\Omega \subset \mathbb{C}$ is a simply connected subdomain, and let $\mathbb{D}$ be the open unit disk of $\mathbb{C}$. Let $f, g : \mathbb{D} \to \Omega$ be two one-to-one and onto holomorphic functions satisfying $f(0) = g(0)$, and $f'(0) > 0$, $g'(0) > 0$. Prove that $f(z) = g(z)$ for all $z \in \mathbb{D}$.

3. Let f be a meromorphic function in $\mathbb{C}$ with finitely many poles, located at $\{z_j\}_{j=1}^{J}$. Prove that

$$\sum_{j=1}^{J} \operatorname{res}(f; z_j) = \operatorname{res}(g; 0)$$

where $g(z) := \frac{1}{z^2} f(\frac{1}{z})$. Here, $\operatorname{res}(f; z_j)$ denote the residue of f at z_j. It is defined by $\operatorname{res}(f; z_j) = a_{-1}$ if $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_j)^n$ is the Laurent series of f at z_j.

4. Show that for any $z \in \mathbb{C}$:

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

[You can use without proof that $\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$.]