Applied Mathematics Preliminary Exam, Part A

August 18, 2023

Work 4 *of the following* 5 *problems.*

1. If f is a map from a Banach space X to a Banach space Y, such that $\psi \circ f$ belongs to the dual of X for every ψ in the dual of Y, show that f is linear and continuous.

2. Let P be a linear operator on a Banach space, satisfying $P^2 = P$. Show that the operator P is continuous if and only if its null space and range are both closed.

3. Show that a compact linear operator maps weakly convergent sequences to strongly convergent sequences.

4. If U is an unitary operator on a Hilbert space, show that $n^{-1}[\mathrm{I}+U+U^2+\ldots+U^{n-1}]$ converges strongly to an orthogonal projection, as $n \to \infty$.

5. For $m = 1, 2, 3, \ldots$, define H_m to be the regular distribution on $\mathbb{R}$ associated with the function $h_m(x) = m^2 \sin(mx)$. Show that $H_m \to 0$ in $\mathcal{D}(\mathbb{R})$ as $m \to \infty$.