The following three problems are weighted equally. Two complete solutions, or a complete solution and two half-solutions, are required for a passing grade. A correct partial solution is preferrable to a claimed full solution with errors.

Throughout this exam, we say that the random variable $\xi$ is a *coin flip* if

$$P(\xi = 1) = P(\xi = -1) = \frac{1}{2}$$

# Problem 1

Let $\xi_1, \xi_2, \dots$ be independent coin flips. For $\alpha \in (0, 1]$ define

$$X_{n,\alpha} = \frac{1}{n}\sum_{k=1}^{n} k^{\alpha}\xi_k.$$

Determine the set of $\alpha \in (0, 1]$ for which $X_{n,\alpha} \xrightarrow{P} 0$ as $n \to \infty$.

(Hint: to show convergence, compute the variance of $X_{n,\alpha}$. To show non-convergence, apply a suitable Central Limit Theorem to $\sum_{k=\lceil n/2\rceil}^{n} k^{\alpha}\xi_k$, suitably normalized.)

# Problem 2

Let $X$ be a random variable with $X \geq 0$ a.s., and suppose that $E(X) \leq 1$ and $E(X^2) \leq 10$. Given this information, for every $t \geq 0$ find the best possible upper bound for $P(X > t)$. (You should show that your bound holds, and that it cannot be improved.)

# Problem 3

Let $\xi_1, \xi_2, \dots$ be independent coin flips and define

$$S_n = \sum_{i=1}^{n} \xi_i.$$

a) Compute $E(S_{10} \mid \xi_1)$.

b) Compute $E(S_{10}^2 \mid \xi_1)$.

c) Compute $E(\xi_1 \mid S_{10})$.