Preliminary Examination in Differential Topology
January 2021

Question 1

Suppose that η is a k-form on an n-manifold X, with $d\eta = 0$; that Y is a closed, oriented k-manifold; and that $f_0 \colon Y \to X$ and $f_1 \colon Y \to X$ are smooth maps. Prove that if f_0 is smoothly homotopic to f_1 then

$$\int_Y f_0^* \eta = \int_Y f_1^* \eta.$$

Carefully state any results that you use.

Question 2

Define $f \colon \mathbb{C}^2 \to \mathbb{C}$ by $f(w,z) = w^2 - z^3$, and let $V = f^{-1}(0)$.

(a) Prove that V intersects the unit sphere $S^3 \subset \mathbb{C}^2$ transversely in a 1-manifold K.
[*Hint:* consider the $\mathbb{R}$-action on $\mathbb{C}^2$ given by $t \cdot (z,w) = (t^2 z, t^3 w)$.]

(b) Prove that the map $\Phi = \arg f = f/|f| \colon \mathbb{C}^2 \setminus V \to S^1 \subset \mathbb{C}$ is a submersion.

(c) Prove that the restricted map $\phi = \arg f \colon S^3 \setminus K \to S^1 \subset \mathbb{C}$ is also a submersion.

Question 3

Let Σ be a compact, oriented 2-manifold, and let $\Gamma \subset \Sigma$ be a compact, oriented 1-dimensional submanifold. Construct a closed 1-form τ_Γ on Σ such that

$$\int_\Sigma \sigma \wedge \tau_\Gamma = \int_\Gamma \sigma|_\Gamma$$

for all closed 1-forms σ on Σ.

Note: You may take for granted that any such Γ, if connected, has an open neighborhood diffeomorphic to $S^1 \times (-1,1)$ by a diffeomorphism carrying Γ to $S^1 \times \{0\}$ (*bonus:* prove this!).