A KREIN–LIKE FORMULA FOR SINGULAR PERTURBATIONS OF SELF–ADJOINT OPERATORS AND APPLICATIONS

by

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Abstract. *Given a self-adjoint operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ and a continuous linear operator $\tau : D(A) \to \mathcal{X}$ with Range$(\tau') \cap \mathcal{H}' = \{0\}$, $\mathcal{X}$ a Banach space, we explicitly construct a family A^τ_Θ of self-adjoint operators such that any A^τ_Θ coincides with the original A on the kernel of τ. Such a family is obtained by giving a Kreĭn-like formula where the role of the deficiency spaces is played by the dual pair $(\mathcal{X}, \mathcal{X}')$. The parameter Θ belongs to the space of symmetric operators from $\mathcal{X}'$ to $\mathcal{X}$. In the case $\mathcal{X} = \mathbb{C}$ one recovers the "$\mathcal{H}_{-2}$–construction" of Kiselev and Simon [KS] and so, to some extent, our results can be considered as an extension of it to the infinite rank case. Various applications to singular perturbations of non necessarily elliptic pseudo-differential operators are given, thus unifying and extending previously known results.*

1. INTRODUCTION

Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator on the Hilbert space $\mathcal{H}$ and suppose that there exists a linear dense set $N \subsetneq D(A)$ which is closed with respect to the graph norm on $D(A)$. If we denote by A_N the restriction of A to N, then A_N is a closed, densely defined, symmetric operator. Since $N \subsetneq D(A)$, A is a non-trivial extension of A_N and so, by the von Neumann theory on self-adjoint extensions of closed symmetric operators (see [N], [DS, §XII.4], [RS, §X.1]), we know that the deficiency indices $n_\pm$, defined as the dimensions of $K_\pm$:=Kernel$(A^*_N \pm i)$, are equal and strictly positive. The family of self-adjoint extensions of A_N is then parametrized by the unitary maps from K_+ onto K_-. When A is strictly positive, a deeper and more explicit construction of the (positive if dim $(K) = +\infty$, K :=Kernel(A^*_N)) self-adjoint extensions of A_N is given by the Birman–Kreĭn–Vishik theory (see [K3], [V], [B], [AS]). In this case the family of (positive) extensions is parametrized by the (positive) quadratic forms on K.

Any self-adjoint extension $\tilde A_N \neq A$ can then be interpreted as a singular per-

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX turbation of A since the two operators differ only on N^c, the set N^c being "thin" since its complement is a linear dense subset of $\mathcal{H}$.

In the case $n_\pm = 1$, Kreĭn obtained, in 1943 (see [K1]), a quite explicit formula relating the resolvents of any two self-adjoint extensions of a given symmetric operator. Such a formula was then extended, by Kreĭn himself in 1946 (see [K2]), to the case $n_\pm = m < +\infty$. In our setting it states the following:

$$\forall\, z \in \rho(A)\cap\rho(\tilde A_N), \qquad (-\tilde A_N + z)^{-1} = (-A+z)^{-1} + \sum_{j,k=1}^m \Gamma(z)^{-1}_{jk}\, \varphi_j(z)\otimes\varphi_k(z^*)\,,$$

where

$$\forall\, z \in \rho(A), \qquad \varphi_k(z) := \varphi_k - (i-z)(-A+z)^{-1}\varphi_k\ ,$$

$\{\varphi_k\}_1^m$ being the set of linear independent solutions of

$$A_N^*\varphi = i\,\varphi\,, \qquad \varphi\in D(A_N^*)\ ,$$

and where the invertible matrix $\Gamma(z)$ satisfies ($\langle\cdot,\cdot\rangle$ denoting the scalar product on $\mathcal{H}$)

$$\forall\, w,z \in \rho(A)\cap\rho(\tilde A_N), \qquad \Gamma(z)_{jk} - \Gamma(w)_{jk} = (z-w)\langle\varphi_j(z^*),\varphi_k(z)\rangle\ .$$

By such a formula, since N is dense, one can then readily define $\tilde A_N$ as

$$D(\tilde A_N) := \{\phi\in\mathcal{H}\ :\ \phi = \phi_z + \sum_{j,k=1}^m \Gamma(z)^{-1}_{jk}\langle\varphi_k(z),\phi_z\rangle\,\varphi_j(z)\,,\ \phi_z\in D(A)\,\}$$

$$(-\tilde A_N + z)\phi := (-A+z)\phi_z\ .$$

Kreĭn's original papers were written in russian, but his results were popularized in some excellent monographs (see e.g. [AG, chap. VII]). Instead, the analogous formula for the case $n_\pm = +\infty$, which was obtained by Saakjan in 1965 (see [S]), is much less known, since the work is not available in english (see however [GMT] and references therein). Due probably to this fact, the Kreĭn formula for $n_\pm = +\infty$ (similar considerations also apply to the Birman–Kreĭn–Vishik theory) was rarely used in concrete applications: we are mainly referring to the much studied case of singular perturbations of the Laplacian supported by null sets (see e.g. [AGHH], [AFHKL], [Br] and references therein). Indeed in situations of this kind other approaches are used: extensions are mainly obtained either as resolvent limits of less singular perturbations or by other constructions often resembling variations of either the Kreĭn formula or the Birman–Kreĭn–Vishik theory. Usually such approaches rely on the elliptic nature of the Laplacian and are not applicable to the study of singular perturbations of hyperbolic operators (this was the original motivation of our work).

Here we show how, when the (not necessarily dense) set N is the kernel of a continuous linear map $\tau : D(A)\to\mathcal{X}$ such that $\mathrm{Range}(\tau')\cap\mathcal{H}' = \{0\}$, $\mathcal{X}$ a Banach space, one can prove, by almost straightforward arguments, a Kreĭn–like formula for self-adjoint extensions of A_N, where the role of $K_\pm$ is played by the dual pair $(\mathcal{X},\mathcal{X}')$ (our construction could be given for $\mathcal{X}$ a locally convex space, but we will not strive here for the maximum of generality).

The formula given here turns out to be relatively simple: for example we do not need to compute A_N^*; in more detail (see theorem 1) one obtains, under a hypothesis which we prove to be satisfied under relatively weak conditions (see proposition 1), and denoting by A^τ_Θ the family of self-adjoint extensions of A_N,

$$(-A^\tau_\Theta+z)^{-1}=(-A+z)^{-1}+G(z)\cdot(\Theta+\Gamma(z))^{-1}\cdot\breve G(z)\ ,$$

where

$$\breve G(z):=\tau\cdot(-A+z)^{-1}\,,\qquad G(z):=C_{\mathcal{H}}^{-1}\cdot\breve G(z^*)'$$

($C_{\mathcal{H}}$ being the canonical isomorphism of $\mathcal{H}$ onto $\mathcal{H}'$) and the conjugate linear operator $\Gamma(z):D\subseteq\mathcal{X}'\to\mathcal{X}$ satisfies the equation

$$\forall\,\ell\in D,\qquad \frac{d}{dz}\,\Gamma(z)\ell=\breve G(z)\cdot G(z)\ell$$

which (see lemma 2) we show to have an explicit (in terms of τ itself) bounded operator solution. Such a solution plays a fundametal role in finding (see lemmata 3 and 4) other nicer (even if unbounded) solutions which we then use in (some of) the examples.

In §3, after showing (example 1) how our construction, in the case $\mathcal{X}=\mathbb{C}$, reproduces the "$\mathcal{H}_{-2}$–construction" given in [KS] and how, in the case A is strictly positive, it gives a variation on the Birman–Kreĭn–Vishik theory which comprises the results in [KKO] (example 2), we use the above Kreĭn–like formula to study singular perturbations of non necessarily elliptic pseudo-differential operators, thus unifying and extending previously known results. More precisely we give the following examples:

- Finitely many point interaction in three dimensions (example 3);
- Infinitely many point interaction in three dimensions (example 4);
- Singular perturbations of the Laplacian in three and four dimensions supported by regular curves (example 5);
- Singular perturbations, supported by null sets with Hausdorff co-dimension less than $2s$, of translation invariant pseudo-differential operators with domain $H^s(\mathbb{R}^n)$ (example 6);
- Singular perturbations of the d'Alembertian in four dimensions supported by time-like straight lines (example 7. In order to limit the lenght of the paper we content ourselves with discussing here only the case of a straight line. A complete study of the case of a generic time-like curve will be the subject of a separate paper. We belive that the detailed study of such a kind of operators will lead to a rigorous framework for the classical and quantum electrodynamics of point particles in the spirit of the results obtained, for the linearized (or dipole) case, in [NP1–3] and [BNP]);

– Singular perturbations, supported by null sets, of translation invariant pseudo-differential operators with domain the Hörmander spaces $H_\varphi(\mathbb{R}^n)$ (example 8).

Definitions and notations

- Given a Banach space $\mathcal{X}$ we denote by $\mathcal{X}'$ its strong dual;
- $L(\mathcal{X},\mathcal{Y})$, resp. $\tilde L(\mathcal{X},\mathcal{Y})$, denotes the space of linear, resp. conjugate linear, operators from the Banach space $\mathcal{X}$ to the Banach space $\mathcal{Y}$.
- $B(\mathcal{X},\mathcal{Y})$, resp. $\tilde B(\mathcal{X},\mathcal{Y})$, denotes the space of bounded, everywhere defined, linear, resp. conjugate linear, operators on the Banach space $\mathcal{X}$ to the Banach space $\mathcal{Y}$. It is a Banach space with the norm $\|A\|_{\mathcal{X},\mathcal{Y}} := \sup\{\|Ax\|_{\mathcal{Y}},\ \|x\|_{\mathcal{X}}=1\}$.
- The closed linear operator operator A' and the conjugate linear closed operator $\tilde A'$ are the adjoints of the densely defined linear operator A and of the densely defined conjugate linear operator $\tilde A$ respectively, i.e.
$$\forall\, x\in D(A)\subseteq\mathcal{X},\quad \forall\tilde\ell\in D(A')\subseteq Y',\qquad (A'\ell)(x)=\ell(Ax),$$
$$\forall\, x\in D(\tilde A)\subseteq\mathcal{X},\quad \forall\tilde\ell\in D(\tilde A')\subseteq Y',\qquad (\tilde A'\ell)(x)=(\,\ell(\tilde Ax)\,)^*,$$
where * denotes complex conjugation.
- $\tilde S(\mathcal{X}',\mathcal{X})$ denotes the space of conjugate linear operators A such that
$$\forall\,\ell_1,\ell_2\in D(A),\qquad \ell_1(A\ell_2)=(\,\ell_2(A\ell_1)\,)^*.$$
- For any $A\in\tilde S(\mathcal{X}',\mathcal{X})$ we define $\gamma(A):=\inf\{\,\ell(A\ell),\ \|\ell\|_{\mathcal{X}'}=1,\ \ell\in D(A)\,\}$.
- $J_{\mathcal{X}}\in B(\mathcal{X},\mathcal{X}'')$ indicates the injective map (an isomorphism when $\mathcal{X}$ is reflexive) defined by $(J_{\mathcal{X}}x)(\ell):=\ell(x)$.
- If $\mathcal{H}$ is a complex Hilbert space with scalar product (conjugate linear w.r.t. the first variable) $\langle\cdot,\cdot\rangle$, then $C_{\mathcal{H}}\in\tilde B(\mathcal{H},\mathcal{H}')$ denotes the isomorphism defined by $(C_{\mathcal{H}}y)(x):=\langle y,x\rangle$. The Hilbert adjoint of the densely defined linear operator A is then given by $A^*=C_{\mathcal{H}}^{-1}\cdot A'\cdot C_{\mathcal{H}}$.
- $\mathcal{F}$ and $*$ denote Fourier transform and convolution respectively.
- $H^s(\mathbb{R}^n)$, $s\in\mathbb{R}$, is the usual scale of Sobolev–Hilbert spaces, i.e. $H^s(\mathbb{R}^n)$ is the space of tempered distributions with a Fourier transform which is square integrable w.r.t. the measure with density $(1+|x|^2)^s$.
- c denotes a generic strictly positive constant which can change from line to line.

2. A KREIN–LIKE FORMULA

Let
$$A:D(A)\to\mathcal{H}$$
be a self-adjoint operator on the complex Hilbert space $\mathcal{H}$. $D(A)$ inherits a Hilbert space structure by introducing the usual scalar product leading to the graph norm

$\|\phi\|_A^2 := \langle\phi,\phi\rangle + \langle A\phi, A\phi\rangle$. Denoting the resolvent set of A by $\rho(A)$ we define, for any $z \in \rho(A)$,

$$R^0(z) := (-A+z)^{-1} : \mathcal{H} \to D(A)\,, \qquad R^0(z) \in B(\mathcal{H}, D(A))\ .$$

We consider now a linear operator

$$\tau : D(A) \to \mathcal{X}\,, \qquad \tau \in B(D(A), \mathcal{X})\ ,$$

where $\mathcal{X}$ is a complex Banach space. By means of A and τ we can define, for any $z \in \rho(A)$, the following operators:

$$\breve{G}(z) := \tau \cdot R^0(z) : \mathcal{H} \to \mathcal{X}\,, \qquad \breve{G}(z) \in B(\mathcal{H}, \mathcal{X})\ ,$$

$$G(z) := C_{\mathcal{H}}^{-1} \cdot \breve{G}(z^*)' : \mathcal{X}' \to \mathcal{H}\,, \qquad G(z) \in \tilde{B}(\mathcal{X}', \mathcal{H})\ .$$

Remark 1. Being $R^0(z)$ surjective, $R^0(z)'$ is injective. If τ has dense range then τ' is injective. Therefore, when τ has dense range, $\breve{G}(z)$ is surjective and $G(z)$ is injective. This implies that the only $\Lambda \in B(\mathcal{X}, \mathcal{X}')$ which solves the operator equation $G(z) \cdot \Lambda \cdot \breve{G}(z) = 0$ is the zero operator.

Lemma 1.

$$(z-w)\,\breve{G}(w) \cdot R^0(z) = \breve{G}(w) - \breve{G}(z)$$

$$(z-w)\,R^0(w) \cdot G(z) = G(w) - G(z)\ .$$

Remark 2. The second relation in the lemma above shows that

$$\forall\, w, z \in \rho(A),\ w \neq z, \qquad \text{Range}(\,G(w) - G(z)\,) \subseteq D(A)\ .$$

Proof of lemma 1. By first resolvent identity one has

$$(z-w)\,R^0(w) \cdot R^0(z) = R^0(w) - R^0(z)\ .$$

Therefore

$$\begin{aligned}(z-w)\,\breve{G}(w) \cdot R^0(z) =&(z-w)\,\tau \cdot R^0(w) \cdot R^0(z)\\ =&\breve{G}(w) - \breve{G}(z)\end{aligned}$$

and, by duality (here $R^0(z)$ is considered as an element of $B(\mathcal{H},\mathcal{H})$),

$$\begin{aligned}&G(w) - G(z) = C_{\mathcal{H}}^{-1} \cdot \Big(\breve{G}(w^*) - \breve{G}(z^*)\Big)'\\ =&C_{\mathcal{H}}^{-1} \cdot \Big((z^*-w^*)\,\breve{G}(z^*) \cdot R^0(w^*)\Big)'\\ =&C_{\mathcal{H}}^{-1} \cdot \Big((z^*-w^*)\,C_{\mathcal{H}} \cdot R^0(w) \cdot C_{\mathcal{H}}^{-1} \cdot \breve{G}(z^*)'\Big)\\ =&(z-w)\,R^0(w) \cdot G(z)\ .\end{aligned}$$

□

We want now to define a new self-adjoint operator which, when restricted to the kernel of τ, coincides with the original A. Since, in the case of a bounded perturbation V, one has, for any $z \in \mathbb{C}\backslash\mathbb{R}$ such that $\|V \cdot R^0(z)\|_{\mathcal{H},\mathcal{H}} < 1$,

$$(-(A+V)+z)^{-1} = R^0(z) + R^0(z) \cdot (\,\mathbb{1} - V \cdot R^0(z)\,)^{-1} \cdot V \cdot R^0(z)\ ,$$

we are lead to write the presumed resolvent as

$$R^\tau(z) = R^0(z) + B(z) \cdot \tau \cdot R^0(z) \equiv R^0(z) + B(z) \cdot \breve{G}(z)\,,$$

where $B(z) \in B(\mathcal{X}, \mathcal{H})$ has to be determined.

Self-adjointness requires $R^\tau(z)^* = R^\tau(z^*)$ or, equivalently,

$$G(z) \cdot B(z^*)' \cdot C_{\mathcal{H}} = B(z) \cdot \breve{G}(z)\ . \tag{1}$$

Therefore if we put $B(z) = G(z) \cdot \Lambda(z)$, $\Lambda(z) \in \tilde{B}(\mathcal{X}, \mathcal{X}')$, then one can check that (1) is implied by (by Remark 1, when τ has dense range, is equivalent to)

$$\Lambda(z)' \cdot J_{\mathcal{X}} = \Lambda(z^*)\ . \tag{2}$$

We now impose the resolvent identity

$$(z-w)\,R^\tau(w)R^\tau(z) = R^\tau(w) - R^\tau(z)\ . \tag{3}$$

Since (we make use of lemma 1)

$$\begin{aligned}
&(z-w)\,R^\tau(w)\cdot R^\tau(z) = (z-w)\,\big(R^0(w)\cdot R^0(z)\\
&+R^0(w)\cdot G(z)\cdot\Lambda(z)\cdot\breve{G}(z) + G(w)\cdot\Lambda(w)\cdot\breve{G}(w)\cdot R^0(z)\\
&+\,G(w)\cdot\Lambda(w)\cdot\breve{G}(w)\cdot G(z)\cdot\Lambda(z)\cdot\breve{G}(z)\big)\\
&=R^0(w) - R^0(z) + G(w)\cdot\Lambda(z)\cdot\breve{G}(z) - G(z)\cdot\Lambda(z)\cdot\breve{G}(z)\\
&+G(w)\cdot\Lambda(w)\cdot\breve{G}(w) - G^0(w)\cdot\Lambda(w)\cdot\breve{G}(z)\\
&+(z-w)\,G(w)\cdot\Lambda(w)\cdot\breve{G}(w)\cdot G(z)\cdot\Lambda(z)\cdot\breve{G}(z)\\
&=R^\tau(w) - R^\tau(z) + G(w)\cdot(\Lambda(z)-\Lambda(w))\cdot\breve{G}(z)\\
&+(z-w)\,G(w)\cdot\Lambda(w)\cdot\breve{G}(w)\cdot G(z)\cdot\Lambda(z)\cdot\breve{G}(z)\ ,
\end{aligned}$$

the relation (3) is implied by (by Remark 1, when τ has dense range, is equivalent to)

$$\Lambda(w) - \Lambda(z) = (z-w)\,\Lambda(w)\cdot\breve{G}(w)\cdot G(z)\cdot\Lambda(z)\ . \tag{4}$$

Suppose now that there exists a (necessarily closed) operator

$$\Gamma(z) : D \subseteq \mathcal{X}' \to \mathcal{X}$$

such that, for some open set $Z \subseteq \rho(A)$ such that $z \in Z$ iff $z^* \in Z$, one has

$$\forall\, z \in Z, \qquad \Gamma(z)^{-1} = \Lambda(z)\ .$$

Then we have that (4) forces $\Gamma(z)$ to satisfy the relation

$$\Gamma(z) - \Gamma(w) = (z - w)\, \breve{G}(w) \cdot G(z)\ . \tag{5}$$

which is equivalent to

$$\forall\, \ell \in D \subseteq \mathcal{X}', \qquad \frac{d}{dz}\,\Gamma(z)\ell = \breve{G}(z) \cdot G(z)\ell\ . \tag{6}$$

Regarding the identity (2), suppose that

$$\forall\, \ell_1, \ell_2 \in D, \qquad \ell_1(\Gamma(z^*)\ell_2) = (\, \ell_2(\Gamma(z)\ell_1)\,)^*\ . \tag{7}$$

This, in the case $\Gamma(z)$ has a bounded inverse given by $\Lambda(z)$ as we are pretending, implies (2) which, if $\Gamma(z)$ is densely defined, is then equivalent (use e.g. [K, thm. 5.30, chap. III]) to

$$\Gamma(z)' = J_{\mathcal{X}} \cdot \Gamma(z^*)\ . \tag{7.1}$$

We will therefore concentrate now on the set of maps

$$\Gamma : \rho(A) \to \tilde{L}(\mathcal{X}', \mathcal{X})$$

which satisfy (6) (equivalently (5)) and (7) (we are implicitly supposing that D, the domain of $\Gamma(z)$, is z–independent).

An explicit representation of the set of such maps is given by the following

Lemma 2. *Given any $z_0 \in \rho(A)$ the map*

$$\hat{\Gamma} : \rho(A) \to \tilde{B}(\mathcal{X}', \mathcal{X}) \qquad \hat{\Gamma}(z) := \tau \cdot \left(\frac{G(z_0) + G(z_0^*)}{2} - G(z) \right) \tag{8}$$

satisfies (6) and (7).

Remark 3. Lemma 2 shows that the set of maps

$$\Gamma : \rho(A) \to \tilde{L}(\mathcal{X}', \mathcal{X})$$

which satisfy (6) and (7) can be parametrized by $\tilde{S}(\mathcal{X}', \mathcal{X})$. Indeed, by (6), any of such maps must differ from $\hat{\Gamma}(z) \in \tilde{B}(\mathcal{X}', \mathcal{X})$ by a z–independent operator in $\tilde{S}(\mathcal{X}', \mathcal{X})$. Therefore any parametrization is of the kind

$$\Gamma_\Theta : \rho(A) \to \tilde{L}(\mathcal{X}', \mathcal{X}) \qquad \Gamma_\Theta(z) = \Theta + \Gamma(z)\,, \quad \Theta \in \tilde{S}(\mathcal{X}', \mathcal{X})\ , \tag{9}$$

where $\Gamma(z)$ is some map which satisfy (6) and (7).

Proof of lemma 2. By lemma 1 one has

$$(z-w)\,\breve{G}(w)\cdot G(z)=\tau(\,G(w)-G(z)\,)=\tau(\,G(z_0)-G(z)\,)-\tau(\,G(z_0)-G(w)\,)$$

and so $\tau\cdot(G(z_0)-G(z))$ solves (5); by linearity also $\hat{\Gamma}(z)$ is a solution.
As regard (7) let us at first note that

$$J_{\mathcal{X}}\cdot\breve{G}(z)=\breve{G}(z)''\cdot J_{\mathcal{H}}$$

and

$$(C'_{\mathcal{H}}\cdot J_{\mathcal{H}}y)\,(x)=(J_{\mathcal{H}}y\,(C_Hx))^*=(C_{\mathcal{H}}x(y))^*=\langle x,y\rangle^*=C_{\mathcal{H}}y(x)\ .$$

Therefore one has

$$\begin{aligned}&\left(\breve{G}(w)\cdot G(z)\right)'=G(z)'\cdot\breve{G}(w)'=\breve{G}(z^*)''\cdot(C'_{\mathcal{H}})^{-1}\cdot\breve{G}(w)'\\=&J_{\mathcal{X}}\cdot\breve{G}(z^*)\cdot J_{\mathcal{H}}^{-1}\cdot(C'_{\mathcal{H}})^{-1}\cdot\breve{G}(w)'=J_{\mathcal{X}}\cdot\breve{G}(z^*)\cdot C_{\mathcal{H}}^{-1}\cdot\breve{G}(w)'\\=&J_{\mathcal{X}}\cdot\breve{G}(z^*)\cdot G(w^*)\end{aligned}$$

which immediately implies that $\hat{\Gamma}(z)$ satisfies (7). □

Lemma 2 does not entirely solve the problem of the search of $\Gamma(z)$ since $\hat{\Gamma}(z)$ can give rise to non-local boundary conditions (see Remark 7 below); moreover $\hat{\Gamma}(z)$ explicitly depends on the choice of a particular $z_0\in\rho(A)$. However the boundedness of $\hat{\Gamma}(z)$ implies a useful criterion for obtaining other maps $\Gamma(z)$ which satisfy (6) and (7):

Lemma 3. *Suppose that*

$$\tilde{\Gamma}(z):D(\tilde{\Gamma})\subseteq\mathcal{X}'\to\mathcal{X}\,,\qquad z\in\rho(A)\,,$$

is a family of conjugate linear, densely defined operators such that

$$\forall\,\ell_1,\ell_2\in D(\tilde{\Gamma})\,,\qquad \ell_2(\tilde{\Gamma}(z^*)\ell_1)=(\,\ell_1(\tilde{\Gamma}(z)\ell_2)\,)^*\tag{10}$$

and

$$\forall\,\ell\in E,\forall\,\ell_1\in D(\tilde{\Gamma})\,,\qquad \frac{d}{dz}\,\ell(\tilde{\Gamma}(z)\ell_1)=\ell(\breve{G}(z)\cdot G(z)\ell_1)\equiv\langle G(z^*)\ell,G(z)\ell_1\rangle\ ,\tag{11}$$

where $E\subseteq\mathcal{X}'$ is either a dense subspace or the dual of some Schauder base in $\mathcal{X}$. Then $\tilde{\Gamma}(z)$ is closable and its closure satisfies (6) and (7).

Proof. By (11) necessarily $\tilde{\Gamma}(z)$ differs from (the restriction to $D(\tilde{\Gamma})$ of) $\hat{\Gamma}(z)$ by a z–independent, densely defined operator $\tilde{\Theta}\in\tilde{S}(\mathcal{X}',\mathcal{X})$. Being densely defined, $\tilde{\Theta}$ has an adjoint and $\tilde{\Theta}\subseteq\Theta'$. Therefore $\tilde{\Theta}$ is closable and so, being $\hat{\Gamma}(z)$ bounded,

$\tilde{\Gamma}(z) = \tilde{\Theta} + \hat{\Gamma}(z)$ is closable. Denoting by Θ the closure of $\tilde{\Theta}$, the closure of $\tilde{\Gamma}(z)$ is given by $\Theta + \hat{\Gamma}(z)$, which satisfies (6) and (7) by lemma 2. $\square$

We can state now our main result:

Theorem 1. *Let $\Gamma_\Theta(z)$ as in (9). Under the hypotheses*

$$Z_\Theta := \left\{ z \in \rho(A) \ : \ \exists\, \Gamma_\Theta(z)^{-1} \in \tilde{B}(\mathcal{X}, \mathcal{X}'),\ \exists\, \Gamma_\Theta(z^*)^{-1} \in \tilde{B}(\mathcal{X}, \mathcal{X}') \right\} \neq \emptyset \quad \text{(h1)}$$

$$\text{Range}\,(\tau') \cap \mathcal{H}' = \{0\}\,, \qquad \text{(h2)}$$

the bounded linear operator

$$R^\tau_\Theta(z) := R^0(z) + G(z) \cdot \Gamma_\Theta(z)^{-1} \cdot \breve{G}(z)\,, \qquad z \in Z_\Theta\,,$$

is a resolvent of the self-adjoint operator A^τ_Θ which coincides with A on the kernel of τ and is defined by

$$D(A^\tau_\Theta) := \left\{ \phi \in \mathcal{H} \ : \ \phi = \phi_z + G(z) \cdot \Gamma_\Theta(z)^{-1} \cdot \tau\, \phi_z, \quad \phi_z \in D(A) \right\}\,,$$

$$(-A^\tau_\Theta + z)\phi := (-A + z)\phi_z\ .$$

Such a definition is z–independent and the decomposition of ϕ entering in the definition of the domain is unique.

Remark 4. Viewing A as a bounded operator on $D(A)$ to $\mathcal{H}$, we can consider the adjoint $(-A + z^*)'$, so that

$$(-A + z^*)' \cdot C_\mathcal{H} : \mathcal{H} \to D(A)'\,, \qquad (-A + z^*)' \cdot C_{\mathcal{H}\,|D(A)} = C_\mathcal{H} \cdot (-A + z)$$

and, by the definition of $G(z)$,

$$(-A + z^*)' \cdot C_\mathcal{H} \cdot G(z) = \tau'\ .$$

Therefore, defining $Q_\phi := \Gamma_\Theta(z)^{-1} \cdot \tau\, \phi_z$, one has

$$C_\mathcal{H} \cdot (-A^\tau_\Theta + z)\phi = (-A + z^*)' \cdot C_\mathcal{H}\phi_z = (-A + z^*)' \cdot C_\mathcal{H}\phi - \tau' Q_\phi\ ,$$

i.e.

$$A^\tau_\Theta \phi = C_\mathcal{H}^{-1} \cdot (A' \cdot C_\mathcal{H}\phi + \tau' Q_\phi)\ .$$

Formally re-writing the last relation as

$$A^\tau_\Theta \phi = A\phi + C_\mathcal{H}^{-1} \cdot \tau' Q_\phi\ ,$$

we can view A^τ_Θ as a perturbation of A, the perturbation being singular since, by (h2), $\tau' Q_\phi \in D(A)' \backslash \mathcal{H}'$.

Remark 5. 1. If $\mathcal{X}$ is reflexive and $\Gamma_\Theta(z)$ is densely defined, then, by (7.1), there follows

$$\Gamma_\Theta(z)^{-1} \in \tilde{B}(\mathcal{X}, \mathcal{X}') \quad \Longrightarrow \quad \Gamma_\Theta(z^*)^{-1} \in \tilde{B}(\mathcal{X}, \mathcal{X}') \ .$$

2. If $Z_\Theta \neq \emptyset$ then Z_Θ is necessarily open. Indeed, by (5),

$$\Gamma_\Theta(z+h) = \Gamma_\Theta(z) + h\, \breve{G}(z) \cdot G(z+h) \ ,$$

and so $\Gamma_\Theta(z+h)^{-1} \in \tilde{B}(\mathcal{X}', \mathcal{X})$ if $z \in Z_\Theta$ and h is sufficiently small.
3. If in the representation (9) there exists $z_0 \in \rho(A)$ such that $\Gamma(z_0) = \Gamma(z_0^*) = 0$ (this is certainly true if $\rho(A) \cap \mathbb{R} \neq \emptyset$ and if one uses representation (8) with $z_0 \in \mathbb{R}$) then obviously Z_Θ is non-empty for any invertible $\Theta \in \tilde{S}(\mathcal{X}', \mathcal{X})$. A more significative criterion leading to (h1) will be given in Proposition 1 below.

Remark 6. 1. By the definition of $G(z)$ one has that (h2) is equivalent to

$$\text{Range}\,(\, G(z)\,) \cap D(A) = \{0\} \ . \tag{h2}$$

2. If Kernel$\,(\tau)$ is dense then (h2) holds true (it is not hard to show that the reverse implication is false). Indeed the density hypothesis implies, if $Q \in \mathcal{X}'$,

$$\forall\, \psi \in \text{Kernel}\,(\tau), \qquad \langle \phi, \psi \rangle = Q(\tau\psi) = \tau' Q(\psi) \quad \Longrightarrow \quad \phi = 0 \ .$$

This, by the definition of $G(z)$, implies

$$R^0(z)\phi = G(z)Q \quad \Longrightarrow \quad \phi = 0 \ ,$$

which gives (h2).

Remark 7. If in the above theorem one uses the representation $\hat{\Gamma}_\Theta(z) := \Theta + \hat{\Gamma}(z)$ given by lemma 2 and Range $\tau = \mathcal{X}$, then one can readily check that the domain of A^τ_Θ is equivalently characterized in term of "generalized boundary conditions": $\phi \in D(A^\tau_\Theta)$ if and only if

$$\exists\, Q_\phi \in D(\Theta) \subseteq \mathcal{X}' \quad \text{such that} \quad \phi - G(z)Q_\phi \in D(A)$$

and

$$\tau\left(\phi - \frac{G(z_0) + G(z_0^*)}{2}\, Q_\phi\right) = \Theta\, Q_\phi \ .$$

If $\mathcal{Y}$ =Range $\tau \subsetneq \mathcal{X}$ then one can not explicitly characterize the regularity of the "charges" Q_ϕ. Indeed in this case one only knows that $Q_\phi \in \Gamma_\Theta(z)^{-1}(\mathcal{Y})$.

Proof of theorem 1. We have already proven that, under our hypotheses, $R^\tau_\Theta(z)$ is a pseudo-resolvent, i.e.

$$(z - w)\, R^\tau_\Theta(w) R^\tau_\Theta(z) = R^\tau_\Theta(w) - R^\tau_\Theta(z) \ . \tag{12}$$

We proceed now as in the proof of [AGHH, thm. II.1.1.1]. By [K, chap. VIII, §1.1] $R^\tau_\Theta(z)$, being a pseudo-resolvent, is the resolvent of a closed operator if and only if it is injective. Since $R^\tau_\Theta(z)\phi=0$ would imply

$$R^0(z)\phi=-G(z)\cdot\Gamma_\Theta(z)^{-1}\cdot\breve G(z)\phi\ ,$$

by (h2) we have $R^0(z)\phi=0$ and so $\phi=0$.

Since, as we have seen before, (7) implies, when $z\in Z_\Theta$,

$$\Gamma_\Theta(z^*)^{-1}=\big(\Gamma_\Theta(z)^{-1}\big)'\cdot J_{\mathcal X}\ ,$$

one has

$$\begin{aligned}&\Big(G(z)\cdot\Gamma_\Theta(z)^{-1}\cdot\breve G(z)\Big)^*\\
=&C_{\mathcal H}^{-1}\cdot\Big(G(z)\cdot\Gamma_\Theta(z)^{-1}\cdot\breve G(z)\Big)'\cdot C_{\mathcal H}\\
=&C_{\mathcal H}^{-1}\cdot\breve G(z)'\cdot\big(\Gamma_\Theta(z)^{-1}\big)'\cdot\breve G(z^*)''\cdot(C'_{\mathcal H})^{-1}\cdot C_{\mathcal H}\\
=&G(z^*)\cdot(\Gamma_\Theta(z)')^{-1}\cdot J_{\mathcal X}\cdot\breve G(z^*)\cdot J_{\mathcal H}^{-1}\cdot(C'_{\mathcal H})^{-1}\cdot C_{\mathcal H}\\
=&G(z^*)\cdot\Gamma_\Theta(z^*)^{-1}\cdot\breve G(z^*)\,,\end{aligned}$$

and so

$$R^\tau_\Theta(z)^*=R^\tau_\Theta(z^*)\ .$$

This gives the denseness of $D(A^\tau_\Theta):=\text{Range}(R^\tau_\Theta(z))$. Indeed $\phi\perp D(A^\tau_\Theta)$, which is equivalent to $\langle R^\tau_\Theta(z^*)\phi,\psi\rangle=0$ for all $\psi\in\mathcal H$, implies $\phi=0$.

Let us now define, on the dense domain $D(A^\tau_\Theta)$, the closed operator

$$A^\tau_\Theta:=R^\tau_\Theta(z)^{-1}-z$$

which, by the resolvent identity (12), is independent of z; it is self-adjoint since

$$((A^\tau_\Theta)^*+z^*)^{-1}=R^\tau_\Theta(z)^*=R^\tau_\Theta(z^*)=(A^\tau_\Theta+z^*)^{-1}\ .$$

To conclude, the uniqueness of the decomposition

$$\phi=\phi_z+G(z)\cdot\Gamma_\Theta(z)^{-1}\cdot\tau\,\phi_z\,,\qquad\phi\in D(A^\tau_\Theta)\,,$$

is an immediate conseguence of (h2). □

The following result states that (h1) holds true under relatively weak hypotheses:

Proposition 1. *Suppose that τ has dense range, and let $\Gamma_\Theta(z)$ as in (9) be closed and densely defined. Then one has*

$$\mathbb C\backslash\mathbb R\cup W^-_\Theta\cup W^+_\Theta\subseteq Z_\Theta\,,$$

where

$$W^\pm_\Theta:=\{\,\lambda\in\mathbb R\cap\rho(A)\ :\ \gamma(\pm\Gamma(\lambda))>-\gamma(\pm\Theta)\,\}\ .$$

Since $Z_\Theta \subseteq \rho(A^\tau_\Theta)$, the above proposition implies a semi-boundedness criterion for the extensions A^τ_Θ:

Corollary 1. *Let $-A$ be bounded from below and suppose that there exist $\lambda_0 \in \rho(A)$ and $\theta_0 \in \mathbb{R}$ such that*

$$\forall\, \lambda \geq \lambda_0 \qquad \gamma(\Gamma(\lambda)) > -\theta_0 \ .$$

Then

$$\inf \sigma(-A^\tau_\Theta) \geq -\lambda_0$$

for any $\Theta \in \tilde{S}(\mathcal{X}', \mathcal{X})$ such that $\gamma(\Theta) \geq \theta_0$.

Proof of Proposition 1. Since $\Theta \in \tilde{S}(\mathcal{X}', \mathcal{X})$ implies $\ell(\Theta\ell) \in \mathbb{R}$ and

$$\begin{aligned}\ell(\Gamma(z)\ell) - (\ell(\Gamma(z)\ell))^* =& \ell(\Gamma(z)\ell) - \Gamma(z)'\ell(\ell) = \ell(\Gamma(z)\ell) - J_{\mathcal{X}} \cdot \Gamma(z^*)\ell(\ell)\\ =& \ell((\Gamma(z) - \Gamma(z^*))\ell) = (z - z^*)\ell(\breve{G}(z^*) \cdot G(z)\ell)\\ =&(z - z^*)\|G(z)\ell\|^2_{\mathcal{H}} \ ,\end{aligned}$$

one has

$$\Gamma_\Theta(z)\ell = 0,\ z \in \mathbb{C}\backslash\mathbb{R} \quad \Rightarrow \quad \mathrm{Im}(z)\|G(z)\ell\|^2_{\mathcal{H}} = 0$$

and

$$\Gamma_\Theta(\lambda)\ell = 0,\ \lambda \in \mathbb{R} \cap \rho(A) \quad \Rightarrow \quad -\ell(\Theta\ell) = \ell(\Gamma(\lambda)\ell) \ .$$

Injectivity of $\Gamma_\Theta(z)$ then follows by injectivity of $G(z)$ (see Remark 1) and the definitions of $W^\pm_\Theta$.
By Hahn–Banach theorem the range of $\Gamma_\Theta(z)$ is dense if and only if its annihilator is equal to $\{0\}$, i.e. if and only if

$$\{\ell_2 \in \mathcal{X}' \ :\ \forall\, \ell_1 \in D(\Gamma_\Theta) \quad \ell_2(\Gamma_\Theta(z)\ell_1) = 0\} = \{0\} \ .$$

Being $D(\Gamma_\Theta)$ dense this is equivalent to

$$\{\ell_2 \in D(\Gamma_\Theta) \ :\ \forall\, \ell_1 \in D(\Gamma_\Theta) \quad \ell_2(\Gamma_\Theta(z)\ell_1) = 0\} = \{0\} \ ,$$

and so relation (7) and injectivity give denseness of the range of $\Gamma_\Theta(z)$. By open mapping theorem, to conclude the proof we need to prove that the range of $\Gamma_\Theta(z)$ is closed. For any $\ell \in D(\Gamma_\Theta)$, $\|\ell\|_{\mathcal{X}'} = 1$, one has

$$\|\Gamma_\Theta(z)\ell\|^2_{\mathcal{X}} \geq |\ell(\Gamma_\Theta(z)\ell)|^2 = (\ell(\Theta\ell) + \mathrm{Re}\,(\ell(\Gamma(z)\ell)))^2 + \mathrm{Im}(z)^2\|G(z)\ell\|^4_{\mathcal{H}} \ .$$

Therefore, when $z \in \mathbb{C}\backslash\mathbb{R} \cup W^-_\Theta \cup W^+_\Theta$, one has

$$\inf\{\, \|\Gamma_\Theta(z)\ell\|_{\mathcal{X}} \ ,\ \|\ell\|_{\mathcal{X}'} = 1\,\} > 0 \ ,$$

and so, since $\Gamma_\Theta(z)$ is closed, it has a closed range by [K, thm. 5.2, chap. IV]. $\square$

Remark 8. By the proposition above, if $\mathcal{X}$ =Range (τ) is finite-dimensional and $\Gamma_\Theta(z)$ is everywhere defined, then (h1) is satisfied with at least $\mathbb{C}\backslash\mathbb{R} \subseteq Z_\Theta$.

Remark 9. By the proposition above, since $\hat{\Gamma}(z)$ is bounded, if one uses the representation $\hat{\Gamma}_\Theta(z)$, with $\Theta \in \tilde{S}(\mathcal{X}', \mathcal{X})$ closed and densely defined, then (h1) is satisfied (with at least $\mathbb{C}\backslash\mathbb{R} \subseteq Z_\Theta$) when τ has dense range.

Remark 10. If $\mathcal{X}$ is a Hilbert space (with scalar product $\langle\cdot,\cdot\rangle$) we can of course use the map $C_\mathcal{X}$ to identify $\mathcal{X}$ with $\mathcal{X}'$ and re-define $G(z)$ as

$$G(z) := C_\mathcal{H}^{-1} \cdot \breve{G}(z^*)' \cdot C_\mathcal{X} : \mathcal{X} \to \mathcal{H} \ .$$

The statements in the above theorem remain then unchanged taking

$$\Gamma_\Theta : \rho(A) \to L(\mathcal{X}, \mathcal{X}) \,, \qquad \Gamma_\Theta(z) = \Theta + \Gamma(z)$$

with Θ such that

$$\forall\, x, y \in D(\Theta), \qquad \langle \Theta x, y\rangle = \langle x, \Theta y\rangle$$

and $\Gamma(z)$ satisfying (6) and

$$\forall\, x, y \in D(\Gamma), \qquad \langle \Gamma(z)x, y\rangle = \langle x, \Gamma(z^*)y\rangle \ .$$

Remark 11. When $\mathcal{X}$ is a Hilbert space, by Theorem 1, since $G(z)$ and $\breve{G}(z)$ are bounded, we have that $R_\Theta^\tau(z) - R^0(z)$ is a trace class operator on $\mathcal{H}$ if and only if $(\Theta + \Gamma(z))^{-1}$ is a trace class operator on $\mathcal{X}$ (see e.g. [K, §1.3, chap. X]). This information can be used (proceeding along the same lines as in [BT]) to infer from $\sigma(A)$ some properties of $\sigma(A_\Theta^\tau)$.

In the Hilbert space case one can give, beside the one appearing in Remark 4, another criterion for obtaining the map Γ_Θ. Indeed one has the following

Lemma 4. *Suppose that there exists a densely defined sesquilinear form $\tilde{\mathcal{E}}(z)$, $z \in \rho(A)$, with z–independent domain $D(\tilde{\mathcal{E}}) \times D(\tilde{\mathcal{E}})$, such that*

$$\forall\, x, y \in D(\tilde{\mathcal{E}}), \qquad \tilde{\mathcal{E}}(z^*)(x, y) = (\, \tilde{\mathcal{E}}(z)(y, x)\,)^* \ , \tag{13}$$

$$\forall\, x, y \in D(\tilde{\mathcal{E}}), \qquad \frac{d}{dz}\, \tilde{\mathcal{E}}(z)(x, y) = \langle G(z^*)x, G(z)y\rangle \ , \tag{14}$$

and such that there exist $z_0 \in \rho(A)$, $M \in \mathbb{R}$ for which $\tilde{\mathcal{E}}(z_0)$ is closable and

$$\forall\, x \in D(\tilde{\mathcal{E}}), \qquad \mathrm{Re}\left(\tilde{\mathcal{E}}(z_0)(x, x)\right) \geq M\,\langle x, x\rangle \ . \tag{15}$$

Then $\tilde{\mathcal{E}}(z)$ is closable for any $z \in \rho(A)$ and, denoting by $\mathcal{E}(z)$ it closure, there exists a densely defined, closed linear operator $\Gamma(z)$ with z–independent domain $D(\Gamma)$, defined by

$$\forall\, x \in D(\mathcal{E}), \forall\, y \in D(\Gamma), \qquad \mathcal{E}(z)(x, y) = \langle x, \Gamma(z)y\rangle \ ,$$

satisfying (6) and the Hilbert space analogue of (7.1), i.e.

$$\Gamma(z)^* = \Gamma(z^*) \ .$$

Proof. By our hypotheses $\tilde{\mathcal{E}}(z)$ necessarily differs from (the restriction to $D(\tilde{\mathcal{E}}) \times D(\tilde{\mathcal{E}})$ of) the bounded sesquilinear form associated to $\hat{\Gamma}(z)$ by a z–independent Hermitean form $\tilde{\mathcal{Q}}$. Therefore

$$\tilde{\mathcal{Q}}(x,y) = \tilde{\mathcal{E}}(z_0)(x,y) - \langle x, \hat{\Gamma}(z_0)y\rangle$$

is a semi-bounded, densely defined, closable Hermitean form. If Θ denotes the unique semi-bounded self-adjoint operator corresponding to the closure of $\tilde{\mathcal{Q}}$ (the existence of Θ is guaranteed by [K, thm. 2.6, chap. VI]), then the operator $\Gamma(z) := \Theta + \hat{\Gamma}(z)$ gives the thesis. $\square$

3. APPLICATIONS

Example 1. *The $\mathcal{H}_{-2}$–construction.*

Let $\mathcal{X} = \mathbb{C}$, $\varphi \in D(A)'\backslash \{0\}$ and put $\tau = \varphi$. Defining

$$\tilde{R}^0(z) := C_{\mathcal{H}}^{-1} \cdot R^0(z^*)' \in B(D(A)', \mathcal{H})$$

one has then

$$\breve{G}(z) : \mathcal{H} \to \mathbb{C}\,, \qquad \breve{G}(z)\phi = \langle \tilde{R}^0(z^*)\varphi, \phi\rangle$$

and

$$G(z) : \mathbb{C} \to \mathcal{H}\,, \qquad G(z)\zeta = \zeta\tilde{R}^0(z)\varphi \ .$$

The hypotheses (h2) is equivalent to the request

$$\varphi \notin \mathcal{H}' \ ,$$

whereas hypotheses (h1) is always satisfied with at least $\mathbb{C}\backslash\mathbb{R} \subseteq Z_\Theta$ since $\mathcal{X}$ is finite dimensional (see Remark 8). Then the self-adjoint operator A^φ_α has resolvent

$$(-A^\varphi_\alpha + z) = (-A+z)^{-1} + \Gamma_\alpha(z)^{-1}\tilde{R}^0(z)\varphi \otimes \tilde{R}^0(z^*)\varphi \ ,$$

where (by lemma 2)

$$\Gamma_\alpha(z) = \alpha + \varphi\left(\frac{\tilde{R}^0(z_0)\varphi + \tilde{R}^0(z_0^*)\varphi}{2} - \tilde{R}^0(z)\varphi\right)\,, \qquad \alpha \in \mathbb{R} \ .$$

This coincides with the "$\mathcal{H}_{-2}$–construction" given in [KS] (there only the case $-A \geq 0$, $z_0 = 1$ was considered). For a similar construction also see [AK] and references therein.

Example 2. *A variation on the Birman–Kreĭn–Vishik theory.*

Let A be a strictly positive self-adjoint operator, so that $0 \in \rho(-A)$, and let $\tau : D(A) \to \mathcal{X}$ satisfy (h2). By Remark 5.3 and theorem 1, for any $\Theta \in \tilde{S}(\mathcal{X}', \mathcal{X})$ which has a bounded inverse, we can define the (strictly positive when Θ is positive, i.e. $\gamma(\Theta) \geq 0$) self-adjoint opertor A^τ_Θ by

$$(A^\tau_\Theta)^{-1} = A^{-1} + G \cdot \Theta^{-1} \cdot \breve{G} \ ,$$

where $G := G(0)$ and $\breve{G} := \breve{G}(0)$. This gives a variation of the Birman–Kreĭn–Vishik approach which comprises the result given in [KKO]. In particular [KKO, example 4.1] can be obtained by taking $\mathcal{H} = L^2(\Omega)$, $A = -\Delta_\Omega + \lambda$, $\lambda > 0$, $\Omega = (0, \pi) \times \mathbb{R}^2$, $D(\Delta_\Omega) = H^2_0(\Omega)$, $\tau : H^2_0(\Omega) \to L^2(0, \pi)$ the evaluation along the segment $\{(x, 0, 0), x \in (0, \pi)\}$, $\Theta = -\Delta_{(0,\pi)}$, $D(\Theta) = H^2_0(0, \pi)$; [KKO, example 4.2] corresponds to $\mathcal{H} = L^2(\mathbb{R}^3)$, $A = -\Delta + \lambda$, $\lambda > 0$, $D(\Delta) = H^2(\mathbb{R}^3)$, whereas τ and Θ are the same as before.

Example 3. *Finitely many point interactions in three dimensions.*

We take $\mathcal{H} = L^2(\mathbb{R}^3)$, $A = \Delta$, $D(A) = H^2(\mathbb{R}^3) \subset C_b(\mathbb{R}^3)$. Considering then a finite set $Y \subset \mathbb{R}^3$, $\#Y = n$, we take as the linear operator τ the linear continuous surjective map

$$\tau_Y : H^2(\mathbb{R}^3) \to \mathbb{C}^n \qquad \tau_Y \phi := \{\phi(y)\}_{y \in Y} \ .$$

Then one has

$$\breve{G}(z) : L^2(\mathbb{R}^3) \to \mathbb{C}^n \ , \qquad \breve{G}(z)\phi = \{\mathcal{G}_z * \phi\,(y)\}_{y \in Y} \ ,$$

where

$$\mathcal{G}_z(x) = \frac{e^{-\sqrt{z}|x|}}{4\pi|x|} \ , \qquad \mathrm{Re}\sqrt{z} > 0 \, , \qquad \mathcal{G}^y_z(x) := \mathcal{G}_z(x - y) \ ,$$

and

$$G(z) : \mathbb{C}^n \to L^2(\mathbb{R}^3) \, , \qquad G(z)\zeta = \sum_{y \in Y} \zeta_y \mathcal{G}^y_z \equiv \mathcal{G}_z * \sum_{y \in Y} \zeta_y \delta_y \ .$$

A straightforward calculation then gives

$$\begin{aligned}
&\left(\breve{G}(z)\cdot G(z)\zeta\right)_y=\sum_{\tilde y\in Y}\zeta_{\tilde y}\langle\mathcal{G}^{\tilde y}_{z^*},\mathcal{G}^y_z\rangle\\
=&\zeta_y\,\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}dk\,\frac{1}{(|k|^2+z)^2}+\sum_{\tilde y\neq y}\zeta_{\tilde y}\,\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}dk\,\frac{e^{-ik\cdot(\tilde y-y)}}{(|k|^2+z)^2}\\
=&\zeta_y\,\frac{1}{2\pi^2}\int_0^\infty dr\,\frac{r^2}{(r^2+z)^2}+\sum_{\tilde y\neq y}\zeta_{\tilde y}\,\frac{1}{2\pi^2|\tilde y-y|}\int_0^\infty dr\,\frac{r\sin(r|\tilde y-y|)}{(r^2+z)^2}\\
=&\zeta_y\,\frac{1}{8\pi\sqrt{z}}+\sum_{\tilde y\neq y}\zeta_{\tilde y}\,\frac{e^{-\sqrt{z}\,|\tilde y-y|}}{8\pi\sqrt{z}}=\frac{d}{dz}\left(\zeta_y\,\frac{\sqrt{z}}{4\pi}-\sum_{\tilde y\neq y}\zeta_{\tilde y}\,\mathcal{G}^{\tilde y y}_z\right)\,,
\end{aligned}$$

where $\mathcal{G}^{\tilde y y}_z:=\mathcal{G}_z(\tilde y-y)$, $\tilde y\neq y$. Defining

$$\tilde{\mathcal{G}}_z:\mathbb{C}^n\to\mathbb{C}^n\qquad(\tilde{\mathcal{G}}_z\zeta)_y:=\sum_{\tilde y\neq y}\zeta_{\tilde y}\,\mathcal{G}^{\tilde y y}_z\,,$$

one can take as $\Gamma_\Theta(z)$ the linear operator

$$\Gamma_\Theta(z)=\Theta+\frac{\sqrt{z}}{4\pi}-\tilde{\mathcal{G}}_z\,,$$

where Θ is any Hermitean $n\times n$ matrix.

Hypotheses (h1) is satisfied with at least $\mathbb{C}\backslash\mathbb{R}\subseteq Z_\Theta$ since $\mathcal{X}$ is finite dimensional (see Remark 8) and hypotheses (h2) is satisfied since $\mathcal{G}^y_z\notin H^2(\mathbb{R}^3)$ for any $y\in Y$. In conclusion on can define the self-adjoint operator Δ^Y_Θ with resolvent given by

$$(-\Delta^Y_\Theta+z)^{-1}=(-\Delta+z)^{-1}+\sum_{y,\tilde y\in Y}\left(\Theta+\frac{\sqrt{z}}{4\pi}-\tilde{\mathcal{G}}_z\right)^{-1}_{y\tilde y}\mathcal{G}^y_z\otimes\mathcal{G}^{\tilde y}_{z^*}\,.$$

This coincides with the operator constructed in [AGHH, §II.1.1].

Example 4. *Infinitely many point interactions in three dimensions.*

We take $\mathcal{H}=L^2(\mathbb{R}^3)$, $A=\Delta$, $D(A)=H^2(\mathbb{R}^3)\subset C_b(\mathbb{R}^3)$. Considering then an infinite and countable set $Y\subset\mathbb{R}^3$ such that

$$\inf_{\substack{y\neq\tilde y\\ y,\tilde y\in Y}}|y-\tilde y|=d>0\,,\tag{16}$$

we take as the linear operator τ the linear map $\tau_Y\phi:=\{\phi(y)\}_{y\in Y}$. The hypothesis (16) ensures its surjectivity and (see [AGHH, page 172])

$$\tau_Y:H^2(\mathbb{R}^3)\to\ell_2(Y)\,,\qquad\tau_Y\in B(H^2(\mathbb{R}^3),\ell_2(Y))$$

Proceeding as in the previous example one has then

$$\breve{G}(z): L^2(\mathbb{R}^3) \to \ell_2(Y)\,, \qquad \breve{G}(z)\phi = \{\mathcal{G}_z * \phi\,(y)\}_{y\in Y}\ ,$$

and

$$G(z): \ell_2(Y) \to L^2(\mathbb{R}^3)$$

is the unique bounded linear operator which, on the dense subspace

$$\ell_0(Y) := \{\zeta \in \ell_2(Y)\ :\ \#\operatorname{supp}(\zeta) < +\infty\}\ ,$$

is defined by

$$G(z)\zeta = \sum_{y\in Y} \zeta_y \mathcal{G}_z^y\ ,$$

i.e.

$$\forall\,\zeta \in \ell_2(Y), \qquad G(z)\zeta = \mathcal{G}_z * \tau_Y'(\zeta)\ ,$$

where $\tau_Y'(\zeta) \in H^{-2}(\mathbb{R}^3)$ is the signed Radon measure defined by $\tau_Y'(\zeta)(\phi) = \langle\zeta^*, \tau_Y\phi\rangle$. Taking $\zeta \in \ell_0(Y)$ one then obtain, proceeding as in example 2,

$$\left(\breve{G}(z)\cdot G(z)\zeta\right)_y = \zeta_y\,\frac{1}{8\pi\sqrt{z}} + \sum_{\tilde y\neq y} \zeta_{\tilde y}\,\frac{e^{-\sqrt{z}\,|\tilde y - y|}}{8\pi\sqrt{z}} = \frac{d}{dz}\left(\zeta_y\,\frac{\sqrt{z}}{4\pi} - \sum_{\tilde y\neq y}\zeta_{\tilde y}\,\mathcal{G}_z^{\tilde y y}\right)\ .$$

Posing

$$\tilde{\mathcal{G}}_z : \ell_0(Y) \to \ell_2(Y)\,, \qquad (\tilde{\mathcal{G}}_z\zeta)_y := \sum_{\tilde y\neq y}\zeta_{\tilde y}\,\mathcal{G}_z^{\tilde y y}\ ,$$

the operator

$$\tilde\Gamma(z) := \frac{\sqrt{z}}{4\pi} - \tilde{\mathcal{G}}_z$$

satisfies (10) and (11). Therefore, by Lemma 3 (with E the canonical basis of $\ell_2(Y)$ and $D(\tilde\Gamma) = \ell_0(Y)$), $\tilde{\mathcal{G}}_z$ is closable and, denoting its closure by the same symbol, one can use the representation

$$\Gamma_\Theta(z) = \Theta + \frac{\sqrt{z}}{4\pi} - \tilde{\mathcal{G}}_z\ ,$$

where Θ is any symmetric operator on $\ell_2(Y)$.

By proposition 1, (h1) is satisfied when Θ is $\Gamma(z)$–bounded (see [K, thm. 1.1. chap. IV]), whereas (h2) is equivalent to $\tau_Y'(\zeta) \notin L^2(\mathbb{R}^3)$ for any $\zeta \neq 0$, which is always true since the support of $\tau_Y'(\zeta)$ is the null set Y. So, by theorem 1, one can define the self-adjoint operator Δ_Θ^Y with resolvent given by

$$(-\Delta_\Theta^Y + z)^{-1} = (-\Delta + z)^{-1} + \sum_{y,\tilde y\in Y}\left(\Theta + \frac{\sqrt{z}}{4\pi} - \tilde{\mathcal{G}}_z\right)^{-1}_{y\tilde y} \mathcal{G}_z^y \otimes \mathcal{G}_{z^*}^{\tilde y}\ .$$

This coincides with the operator constructed (by an approximation method) in [AGHH, §III.1.1].

Example 5. *Singular perturbations of the Laplacian in $L^2(\mathbb{R}^n)$, $n=3$, $n=4$, supported by regular curves.*

We take $\mathcal{H}=L^2(\mathbb{R}^n)$, $A=\Delta$, $D(A)=H^2(\mathbb{R}^n)$, $n=3$ or $n=4$. Consider then a C^2 curve $\gamma: I\subseteq\mathbb{R}\to\mathbb{R}^n$ such that $C:=\gamma(I)$ is a one-dimensional embedded submanifold $C\subset\mathbb{R}^n$ which, when unbounded, is, outside some compact set, globally diffeomorphic to a straight line (these hypotheses on γ will be weakened in the next example). We will suppose C to be parametrized in such a way that $|\dot\gamma|=1$.

We take as linear operator τ the unique linear map

$$\tau_\gamma: H^2(\mathbb{R}^n)\to L^2(I)\,,\qquad \tau_\gamma\in B(H^2(\mathbb{R}^n),L^2(I))$$

such that

$$\forall\phi\in C_0^\infty(\mathbb{R}^n)\,,\qquad \tau_\gamma\phi(s):=\phi(\gamma(s))\;.$$

The existence of such a map is given by combining the results in [BIN, §10] (straight line) with the ones in [BIN, §24] (compact manifold). By [BIN, §25] we have that

$$\text{Range}\,(\tau_\gamma)=H^s(I),\quad \tau_\gamma\in B(H^2(\mathbb{R}^d),H^s(I)),\qquad s=2-\frac{n-1}{2}$$

and so we could take $\mathcal{X}=H^s(I)$. However we prefer to work with $\mathcal{X}=L^2(I)$ even if with this choice τ_γ is not surjective (but has a dense range).

1) The case $n=3$.

One has, proceeding similarly to examples 3 and 4,

$$\breve G(z): L^2(\mathbb{R}^3)\to L^2(I)\,,\qquad \breve G(z)\phi=\tau_\gamma(\mathcal{G}_z*\phi)$$

and

$$G(z): L^2(I)\to L^2(\mathbb{R}^3)\,,\qquad G(z)f=\mathcal{G}_z*\tau'_\gamma(f)\;,$$

where $\tau'_\gamma(\zeta)\in H^{-2}(\mathbb{R}^3)$ is the signed Radon measure defined by $\tau'_\gamma(f)(\phi)=\langle f^*,\tau_\gamma\phi\rangle$. By Fourier transform one has equivalently

$$\mathcal{F}\cdot G(z)f(k)=\frac{1}{(2\pi)^{\frac32}}\,\frac{1}{|k|^2+z}\int_I ds\, f(s)\,e^{-ik\cdot\gamma(s)}\,,\qquad f\in L^1(I)\cap L^2(I)\;,$$

so that, for any $f_1,f_2\in L^1(I)\cap L^2(I)$ one obtains

$$\begin{aligned}&(z-w)\langle f_1,\breve G(w)\cdot G(z)f_2\rangle=(z-w)\,\langle G(w^*)f_1,G(z)f_2\rangle\\
=&\frac{(z-w)}{(2\pi)^3}\int_{I^2}dt\,ds\,f_1^*(t)\,f_2(s)\int_{\mathbb{R}^3}dk\,\frac{e^{-ik\cdot(\gamma(t)-\gamma(s))}}{(|k|^2+w)\,(|k|^2+z)}\\
=&\frac{(z-w)}{2\pi^2}\int_{I^2}dt\,ds\,\frac{f_1^*(t)\,f_2(s)}{|\gamma(t)-\gamma(s)|}\int_0^\infty dr\,\frac{r\sin(r|\gamma(t)-\gamma(s)|)}{(r^2+w)\,(r^2+z)}\\
=&\int_{I^2}dt\,ds\,f_1^*(t)\,f_2(s)\,\frac{e^{-\sqrt{w}\,|\gamma(t)-\gamma(s)|}-e^{-\sqrt{z}\,|\gamma(t)-\gamma(s)|}}{4\pi\,|\gamma(t)-\gamma(s)|}\;.\end{aligned}\tag{17}$$

Suppose now that, in the case I is not compact,

$$\exists\,\lambda_0>0\ :\ \forall\,\lambda\ge\lambda_0,\qquad \sup_{t\in I}\int_I ds\, e^{-\lambda|\gamma(t)-\gamma(s)|}<+\infty\ . \tag{18}$$

By (17) one can then define a linear operator $\breve\Gamma_\epsilon(z)\,:\,L^2_0(I)\to L^2(I)$, $\epsilon>0$, satisfying (5) and (7), by

$$\breve\Gamma_\epsilon(z)f(t):=\int_I ds\, f(s)\left(\frac{\chi_\epsilon(t,s)}{4\pi\,|t-s|}-\frac{e^{-\sqrt{z}\,|\gamma(t)-\gamma(s)|}}{4\pi\,|\gamma(t)-\gamma(s)|}\right)\ ,$$

where $\chi_\epsilon(t,s):=\chi_{[0,\epsilon]}(|t-s|)$ and $L^2_0(I):=\left\{f\in L^2(I)\ :\ f \text{ has compact support}\right\}$. When $f\in C^1_0(I)$ one can then re-write $\breve\Gamma_\epsilon(z)f$ as

$$\begin{aligned}\breve\Gamma_\epsilon(z)f(t)=&\int_I ds\,(\,f(t)-f(s)\,)\,\mathcal{G}_z(\gamma(t)-\gamma(s))\\ &+f(t)\int_I ds\,\frac{\chi_\epsilon(t,s)}{4\pi\,|t-s|}-\frac{e^{-\sqrt{z}\,|\gamma(t)-\gamma(s)|}}{4\pi\,|\gamma(t)-\gamma(s)|}\\ &-\int_I ds\,\chi_\epsilon(t,s)\,\frac{f(t)-f(s)}{4\pi\,|t-s|}\ .\end{aligned}$$

The second term has, as a function of the parameter $\epsilon>0$, a derivative given by $(2\pi\epsilon)^{-1}f(t)$, and the last term is z–independent. Therefore the operator $\tilde\Gamma(z)$: $C^1_0(I)\to L^2(I)$,

$$\begin{aligned}\tilde\Gamma(z)f(t):=&\int_I ds\,(\,f(t)-f(s)\,)\,\mathcal{G}_z(\gamma(t)-\gamma(s))\\ +f(t)&\left(\frac{1}{2\pi}\,\log\left(\epsilon^{-1}\right)+\int_I ds\,\frac{\chi_\epsilon(t,s)}{4\pi\,|t-s|}-\frac{e^{-\sqrt{z}\,|\gamma(t)-\gamma(s)|}}{4\pi\,|\gamma(t)-\gamma(s)|}\right)\end{aligned} \tag{19}$$

is ϵ–independent and satisfies (10) and (11) with $E=L^2(I)$ and $D(\tilde\Gamma)=C^1_0(I)$. Moreover, by Lemma 3, it is closable and its closure $\Gamma(z)$ satisfies (6) and (7).

By Proposition 1, $\Gamma_\Theta(z)$ verifies hypothesis (h1) with Θ any symmetric operator on $L^2(I)$ which is $\Gamma(z)$–bounded (see [K, thm. 1.1. chap. IV]), whereas (h2) is satisfied since $\tau'_\gamma(f)\notin L^2(\mathbb{R}^3)$ for any $f\neq 0$, being the support of $\tau'_\gamma(f)$ given by the null set C.

The corresponding self-adjoint family given by Theorem 1 has resolvents

$$(-\Delta^\gamma_\Theta+z)^{-1}\phi=(-\Delta+z)^{-1}\phi+\mathcal{G}_z*\tau'_\gamma\left((\,\Theta+\Gamma(z)\,)^{-1}\cdot\tau_\gamma(\mathcal{G}_z*\phi)\right)\ .$$

These give singular perturbations of the Laplacian of the same kind obtained (by a quadratic form approach) in [T].

2) The case $n=4$.

Proceeding as in the case $n=3$ one obtains

$$\breve{G}(z):L^2(\mathbb{R}^4)\to L^2(I)\,,\qquad \breve{G}(z)\phi=\tau_\gamma(\mathcal{G}_z*\phi)\ ,$$

$$G(z)f=\mathcal{K}_z*\tau'_\gamma(f),\quad \mathcal{F}\mathcal{K}_z(k):=\frac{1}{|k|^2+z},\quad k\in\mathbb{R}^4$$

and, for any $f_1,f_2\in L^1(I)\cap L^2(I)$,

$$(z-w)\langle f_1,\breve{G}(w)\cdot G(z)f_2\rangle=(z-w)\,\langle G(w^*)f_1,G(z)f_2\rangle$$
$$=\int_{I^2}dt\,ds\,f_1^*(t)\,f_2(s)\ (\,\mathcal{K}_w(\gamma(t)-\gamma(s))-\mathcal{K}_z(\gamma(t)-\gamma(s))\,)\ .$$

Since

$$|\mathcal{K}_z(x)|=\frac{1}{4\pi^2|x|^2}\,(\,1+o\,(|x|)\,)\,,\qquad |x|\ll 1\ ,$$
$$|\mathcal{K}_z(x)|=\frac{1}{2\,(2\pi)^{3/2}|x|^{3/2}}\,e^{-2\mathrm{Re}\sqrt{z}\,|x|}\,(\,1+o\,(1/|x|)\,)\,,\qquad |x|\gg 1\ ,$$

when γ satisfies (18) the linear operator $\breve{\Gamma}_\epsilon(z):L^2_0(I)\to L^2(I)$

$$\breve{\Gamma}_\epsilon(z)f(t)=\int_I ds\,f(s)\,\left(\frac{\chi_\epsilon(t,s)}{4\pi^2\,|t-s|^2}-\mathcal{K}_z(\gamma(t)-\gamma(s))\right)$$

is well defined and satisfies (10) and (11) with $E=L^2(I)$ and $D(\breve{\Gamma})=L^2_0(I)$.

In four dimensions, due to the stronger (w.r.t. $\mathcal{G}_z$) singularity at the origin of $\mathcal{K}_z$, it is no more possible to perform the calculations leading to the analogue of the operator $\tilde{\Gamma}(z)$, and one is forced to use sesquilinear forms and to try then to apply Lemma 4. Defining for brevity

$$k_{(\epsilon)}(t,s):=\frac{\chi_\epsilon(t,s)}{4\pi^2\,|t-s|^2}\,,\qquad k_z(t,s):=\mathcal{K}_z(\gamma(t)-\gamma(s))\ ,$$

one can re-write $\langle f_1,\breve{\Gamma}_\epsilon(z)f_2\rangle$, when $f_1,f_2\in C^1_0(I)$, as

$$\langle f_1,\breve{\Gamma}_\epsilon(z)f_2\rangle=\int_{I^2}dt\,ds\,f_1^*(t)f_2(s)\,(\,k_{(\epsilon)}(t,s)-k_z(t,s)\,)$$
$$=\int_{I^2}dt\,ds\,(\,f_1^*(t)f_2(s)-f_1^*(t)f_2(t)+f_1^*(s)f_2(s)\,)\,(\,k_{(\epsilon)}(t,s)-k_z(t,s)\,)$$
$$=\frac{1}{2}\int_{I^2}dt\,ds\,(\,f_1^*(t)-f_1^*(s)\,)\,(\,f_2(t)-f_2(s)\,)\,(\,k_z(t,s)-k_{(\epsilon)}(t,s)\,)$$
$$+\int_{I^2}dt\,ds\,f_1^*(t)f_2(t)\,(\,k_{(\epsilon)}(t,s)-k_z(t,s)\,)$$
$$=\frac{1}{2}\int_{I^2}dt\,ds\,(\,f_1^*(t)-f_1^*(s)\,)\,(\,f_2(t)-f_2(s)\,)\,k_z(t,s)$$
$$+\int_{I^2}dt\,ds\,f_1^*(t)f_2(t)\,(k_{(\epsilon)}(t,s)-k_z(t,s))$$
$$-\frac{1}{2}\int_{I^2}dt\,ds\,(\,f_1^*(t)-f_1^*(s)\,)\,(\,f_2(t)-f_2(s)\,)\,k_{(\epsilon)}(t,s)\ .$$

Similarly to the three dimensional case the second term has, as a function of the parameter $\epsilon > 0$, a derivative given by $(2\pi^2\epsilon^2)^{-1}\int_I dt\, f_1^*(t)f_2(t)$, and the last term is z–independent. Therefore the sesquilinear form

$$\tilde{\mathcal{E}}(z) : C_0^1(I) \times C_0^1(I) \to \mathbb{C} \ ,$$

$$\begin{aligned}\tilde{\mathcal{E}}(z)(f_1, f_2) := & \frac{1}{2}\int_{I^2} dt\, ds\, (\, f_1^*(t) - f_1^*(s)\,)\,(\, f_2(t) - f_2(s)\,)\, k_z(t,s) \\ & + \int_I dt\, f_1^*(t) f_2(t)\, \left(\frac{1}{2\pi^2\epsilon} + \int_I ds\, (k_{(\epsilon)}(t,s) - k_z(t,s)) \right)\end{aligned} \tag{20}$$

is ϵ–independent and satisfies (13) and (14). It is straightforward to check its closability (see the proof of proposition 2 in [T] if you get stuck), whereas (15) is a conseguence of (18). Being (h2) verified by the same argument as in the case $n = 3$, by Lemma 4, Proposition 1 and Theorem 1, one has a self-adjoint family of self-adjoint operators with resolvents

$$(-\Delta_\Theta^\gamma + z)^{-1}\phi = (-\Delta + z)^{-1}\phi + \mathcal{K}_z * \tau'_\gamma \left((\Theta + \Gamma(z)\,)^{-1} \cdot \tau_\gamma(\mathcal{K}_z * \phi) \right) \ ,$$

where $\Gamma(z)$ is the operator corresponding to the closure of $\tilde{\mathcal{E}}(z)$ and Θ is any symmetric operator on $L^2(I)$ which is $\Gamma(z)$–bounded. This gives singular perturbations of the Laplacian of the same kind obtained in [Sh].

Example 6. *Singular perturbations in $L^2(\mathbb{R}^n)$ given by d–measures.*

A Borel set $F \subset \mathbb{R}^n$ is called a d–set, $d \in (0, n]$, if (see [JW2, Chap. II]) there exists a Borel measure μ in $\mathbb{R}^n$ such that $\text{supp}\,(\mu) = F$ and

$$\exists\, c_1,\, c_2 > 0 \ :\ \forall\, x \in F,\ \forall\, r \in (0,1), \qquad c_1 r^d \le \mu(B_r(x) \cap F) \le c_2 r^d \ , \tag{21}$$

where $B_r(x)$ is the ball of radius r centered at the point x. By [JW2, chap. II, thm. 1], once F is a d–set, μ_F, the d–dimensional Haurdorff measure restricted to F, always satisfies (21) and so F has Hausdorff dimension d in the neighbourhood of any of its points. From the definition there also follows that a finite union of d–sets which intersect on a set of zero d–dimensional Hausdorff measure is a d–set. Examples of d–sets are d–dimensional Lipschitz manifolds (use examples 2.1 and 2.4 in [JW1]) and (when d is not an integer) self-similar fractals of Hausdorff dimension d (see [JW2, chap. II, example 2], [Tr, thm. 4.7]).

Denoting by $j_F : F \to \mathbb{R}^n$ the restriction to the d–set F of the identity map, we take as the linear operator τ the unique continuous map $(0 < n - d < 2s)$

$$\tau_F : H^s(\mathbb{R}^n\,) \to L^2(F)\,, \qquad \tau_F \in B(H^s(\mathbb{R}^n), L^2(F))$$

such that

$$\forall \phi \in C_0^\infty(\mathbb{R}^d)\,, \qquad \tau_F\phi(x) := \phi(j_F(x)) \ .$$

Here $L^2(F)$ denotes the space of (equivalence classes of) functions on F which are square integrable w.r.t. the measure μ. For the existence of such a map τ_F see the proof of [Tr, thm. 18.6]. By [JW, thm. 1, chap. VII] we have that

$$\mathrm{Range}\,(\tau_F) = H^\alpha(F), \quad \tau_F \in B(H^s(\mathbb{R}^d), H^\alpha(F)), \qquad \alpha = s - \frac{n-d}{2}\,,$$

where the Banach space $H^\alpha(F)$ is a Besov–like space which coincides with the usual Sobolev space when F is a regular manifold. In the case $0 < \alpha < 1$, $H^\alpha(F)$ can be defined (see [JW, §1.1, chap. V]) as the set of $f \in L^2(F)$ having finite norm

$$\|f\|^2_{H^\alpha} := \|f\|^2_{L^2} + \int_{|x-y|<1} d\mu(x)\, d\mu(y)\, \frac{|f(x)-f(y)|^2}{|x-y|^{d+2\alpha}}\,.$$

The above situation can be generalized by considering d–measures.

A Borel measure μ on $\mathbb{R}^n$ is said to be a d–measure, $d \in (0, n]$, if

$$\exists\, c > 0\ :\ \forall\, x \in \mathbb{R}^d,\ \forall\, r \in (0,1], \qquad \mu(B_r(x)) \le cr^d\,.$$

Then, by [JW, lemma 1, chap. VIII], when

$$p = \frac{2d}{n-2s_*}, \qquad 0 < s_* \le s, \quad n-d < 2s_* < n\,,$$

and denoting by $L^p(\mu)$ the space of (equivalence classes of) functions with are p–integrable w.r.t. μ, the linear operator

$$\tau_\mu : H^s(\mathbb{R}^n) \to L^p(\mu) \qquad \tau_\mu \in B(H^s(\mathbb{R}^n), L^p(\mu))$$

$$f(\tau_\mu\phi) := \int_{\mathbb{R}^n} d\mu(x)\, f(x)\phi(x)\,, \qquad f \in L^q(\mu),\ \frac{1}{p} + \frac{1}{q} = 1\,,$$

is well defined. Since μ_F, when F is a d–set, is a d–measure, the previous results tell us that in this case we can take $n - d = 2s_* < 2s$ (so that $p = 2$) and τ_{μ_F} concides with τ_F. An interesting example of a d-measure is the one given by the occupation time of Brownian motion: given $\gamma \in C(\mathbb{R}_+, \mathbb{R}^n)$, $n \ge 3$, let us define the Radon measure

$$\mu_\gamma(A) := \int_0^\infty dt\, \chi_A(\gamma(t))\,.$$

Then, by estimates on Brownian motion occupation times and by a Borel–Cantelli argument (see [CT]), one has that, for arbitrarily small positive ϵ and almost surely with respect to Wiener measure,

$$\mu_\gamma(B_r(x)) \le cr^{2-\epsilon}\,;$$

moreover the Hausdorff dimension of the support of μ_γ is equal to two.

Let us now consider the self-adjoint pseudo-differential operator ($s \ge 0$)

$$\psi(D) : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\,, \qquad \psi(D)\phi := \mathcal{F}^{-1}(\psi\,\mathcal{F}\phi)\,,$$

where ψ is a real-valued Borel function such that

$$\frac{1}{c}\,(1+|x|^2)^{s/2}\le 1+|\psi(x)|\le c\,(1+|x|^2)^{s/2}\ .$$

One has

$$\breve G(z):L^2(\mathbb{R}^n)\to L^p(\mu)\,,\qquad \breve G(z)\phi:=\tau_\mu(\mathcal{K}^\psi_z*\phi)\ ,$$

where

$$\mathcal{K}^\psi_z:=\mathcal{F}^{-1}\frac{1}{-\psi+z}\,,\qquad \mathcal{K}^\psi_z*\phi:=(2\pi)^{-n/2}\,\mathcal{F}^{-1}\left(\frac{\mathcal{F}\phi}{-\psi+z}\right)$$

and

$$G(z):L^q(\mu)\to L^2(\mathbb{R}^n)\,,\qquad G(z)f:=\mathcal{K}^\psi_z*\tau'_\mu(f)\ ,$$

where $\tau'_\mu(f)\in H^{-s}(\mathbb{R}^n)$ is the signed measure defined by $\tau'_\mu(f)(\phi)=\langle f^*,\tau_\mu\phi\rangle\equiv f(\tau_\mu\phi)$ and

$$\mathcal{K}^\psi_z*\tau'_\mu(f):=(2\pi)^{-n/2}\,\mathcal{F}^{-1}\left(\frac{\mathcal{F}\tau'_\mu(f)}{-\psi+z}\right)\ .$$

When one uses the representation $\hat\Gamma_\Theta(z)$, by lemma 2 one has, if

$$\mathcal{K}^\psi_z*\nu(x)=\int_{\mathbb{R}^n}d\nu(y)\,\mathcal{K}^\psi_z(x-y)\,,\qquad \nu\in H^{-s}(\mathbb{R}^n)\ ,$$

and

$$\tilde{\mathcal{K}}^\psi_{z_0}:=\frac{1}{2}\,(\mathcal{K}^\psi_{z_0}+\mathcal{K}^\psi_{z_0^*})\equiv\mathrm{Re}(\mathcal{K}^\psi_{z_0})\ ,$$

$$\hat\Gamma(z):L^q(\mu)\to L^p(\mu)\,,\qquad \frac{1}{p}+\frac{1}{q}=1\,,$$

$$f_1(\hat\Gamma(z)f_2)=\int_{\mathbb{R}^{2n}}d\mu(x)\,d\mu(y)\,f_1(x)f_2(y)\,\left(\tilde{\mathcal{K}}^\psi_{z_0}(x-y)-\mathcal{K}^\psi_z(x-y)\right)\ .$$

By its definition and by Hahn–Banach theorem we have that τ_μ has dense range when

$$\left\{f\in L^q(\mu)\ :\ \forall\,\phi\in H^s(\mathbb{R}^n)\quad\int_{\mathbb{R}^n}d\mu(x)\,f(x)\phi(x)=0\right\}=\{0\}\ .$$

Therefore τ_μ has dense range when the Bessel s_*–capacity of $\mathrm{supp}(\mu)$, $s_*\le s$, is not zero, and this is true (by Frostman lemma, see e.g. [MK, thm. 7.1]) when

$$n-d(\mu)<2s_*\le n,\qquad 0<s_*\le s\ ,$$

where $d(\mu)$ denotes the Hausdorff dimension of $\mathrm{supp}(\mu)$. By Remark 9, when τ_μ has dense range, $\hat\Gamma_\Theta$ satisfies (h1) for any $\Theta\in\tilde S(L^q,L^p)$ which is closed and densely defined; (h2) is equivalent to $\tau'_\mu(f)\notin L^2(\mathbb{R}^n)$, $f\ne 0$, which is surely satisfied when the support of μ has zero Lebesgue measure. Therefore, under these conditions, the trace τ_μ gives rise to the family of self-adjoint extensions $\psi(D)^\mu_\Theta$ with resolvent

$$(-\psi(D)^\mu_\Theta + z)^{-1}\phi = (-\psi(D) + z)^{-1}\phi + \mathcal{K}^\psi_z * \tau'_\mu(\,\hat\Gamma_\Theta(z)^{-1} \cdot \tau_\mu(\mathcal{K}^\psi_z * \phi)\,)\ .$$

Let us note that, since $p \in [2, \infty)$ in (19), when μ is a finite measure we can view τ_μ as a map into the Hilbert space $L^2(\mu)$. In this case we can then try to apply lemma 4 in order to find other maps $\Gamma(z)$ which satisfy (6) and (7). Supposing that $\psi(x) = \psi(-x)$, so that $\mathcal{K}^\psi_z(x-y) = \mathcal{K}^\psi_z(y-x)$, and that

$$\int_{\mathbb{R}^{2n}} d\mu(x)\, d\mu(y)\, |x-y|^2\, |\mathcal{K}^\psi_z(x-y)|\ < +\infty$$

we have, for any $f_1, f_2 \in C^1_0(\mathbb{R}^n)$, and proceeding similarly to example 5 (case $n = 4$),

$$\begin{aligned}
\langle f_1, \hat\Gamma(z) f_2\rangle &= \int_{\mathbb{R}^{2n}} d\mu(x)\, d\mu(y)\, f_1^*(x) f_2(y)\, \left(\tilde{\mathcal{K}}^\psi_{z_0}(x-y) - \mathcal{K}^\psi_z(x-y)\right)\\
&= \frac{1}{2}\int_{\mathbb{R}^{2n}} d\mu(x)\, d\mu(y)\, (\,f_1^*(x) - f_1^*(y)\,)\,(\,f_2(x) - f_2(y)\,)\, \mathcal{K}^\psi_z(x-y)\\
&\quad + \int_{\mathbb{R}^{2n}} d\mu(x)\, d\mu(y)\, f_1^*(x) f_2(x)\, \left(\tilde{\mathcal{K}}^\psi_{z_0}(x-y) - \mathcal{K}^\psi_z(x-y)\right)\\
&\quad - \frac{1}{2}\int_{\mathbb{R}^{2n}} d\mu(x)\, d\mu(y)\, (\,f_1^*(x) - f_1^*(y)\,)\,(\,f_2(x) - f_2(y)\,)\, \tilde{\mathcal{K}}^\psi_{z_0}(x-y)\ .
\end{aligned}$$

Therefore, being the last term z–independent, the sequilinear form

$$\tilde{\mathcal{E}}(z) : C^1_0(\mathbb{R}^n) \times C^1_0(\mathbb{R}^n) \to \mathbb{C}$$

$$\begin{aligned}
\tilde{\mathcal{E}}(z)(f_1, f_2) &= \frac{1}{2}\int_{\mathbb{R}^{2n}} d\mu(x)\, d\mu(y)\, (\,f_1^*(x) - f_1^*(y)\,)\,(\,f_2(x) - f_2(y)\,)\, \mathcal{K}^\psi_z(x-y)\\
&\quad + \int_{\mathbb{R}^{2n}} d\mu(x)\, d\mu(y)\, f_1^*(x) f_2(x)\, \left(\tilde{\mathcal{K}}^\psi_{z_0}(x-y) - \mathcal{K}^\psi_z(x-y)\right)
\end{aligned}$$

satisfies (13) and (14). In the case $\tilde{\mathcal{K}}^\psi_{z_0} \ge 0$ one has

$$\mathrm{Re}(\tilde{\mathcal{E}}(z_0)(f, f)) \ge \mathrm{Re}(\langle f, \hat\Gamma(z_0) f\rangle)$$

and so, being $\hat\Gamma(z_0)$ bounded, (15) is satisfied. Moreover $\tilde{\mathcal{E}}(z)$ is closable since it differs from the bounded sesquilinear form corresponding to $\hat\Gamma(z)$ by the positive Hermitean form

$$\frac{1}{2}\int_{\mathbb{R}^{2n}} d\mu(x)\, d\mu(y)\, (\,f_1^*(x) - f_1^*(y)\,)\,(\,f_2(x) - f_2(y)\,)\, \tilde{\mathcal{K}}^\psi_{z_0}(x-y)$$

which is readily checked to be closable. So, by Lemma 4, Proposition 1 and Theorem 1, one obtains a family of self-adjoint extensions which, in the particular case

$\psi(D) = \Delta$, is the same obtained, by an approximation method, in [AFHKL] (also see [C]) and generalizes, although with a different $\Gamma(z)$, the situation discussed in example 5. In this regard suppose that the 1–set C is the range of a Lipschitz path $\gamma : I \subseteq \mathbb{R} \to C \subset \mathbb{R}^n$, $n = 3$ or $n = 4$, $|\dot\gamma| = 1$ a.e., (so that τ_{μ_C} has dense range in $L^2(C) \simeq L^2(I)$). Under the hypotheses (18) one can again consider, when $n = 3$ the operator $\tilde\Gamma(z)$ appearing in (19) and, when $n = 4$ the sesquilinear form $\tilde{\mathcal{E}}(z)$ appearing in (20), the only difference being that now the domain of definition of such objects is $C^1_0(I\backslash I_*)$, with

$$I_* := \{t \in I \,:\, \gamma \text{ is not differentiable at } t\} \cup \{t \in I \,:\, \exists\, s \neq t \quad \text{s.t.} \quad \gamma(t) = \gamma(s)\}$$

(of course, in order $C^1_0(I\backslash I_*)$ to be still a dense set, one has to suppose that the closure of I_* is a null set). However in the case $n = 4$ the problem of the semi-boundedness of $\tilde{\mathcal{E}}(z)$ arises: indeed one can show (see [Sh]) that $\tilde{\mathcal{E}}(z)$ is unbounded from below in the case γ has angle points. This phenomenon is similar to the one related to unboundedness from below of Schrödinger operators describing $n(> 2)$ point interacting particles (see [MF], [DFT] and references therein).

Example 7. *Singular perturbations of the d'Alembertian in $L^2(\mathbb{R}^4)$ supported by time-like straight lines.*

We take $\mathcal{H} = L^2(\mathbb{R}^4)$,

$$A = \Box := -\Delta_{(1)} \otimes \mathbb{I} + \mathbb{I} \otimes \Delta_{(3)}\,,$$

$\Delta_{(d)}$ being the Laplacian in d dimensions, and ($h \in \mathbb{R}$, $k \in \mathbb{R}^3$ denoting the variables dual to $t \in \mathbb{R}$, $x \in \mathbb{R}^3$)

$$D(\Box) = \left\{\Phi \in L^2(\mathbb{R}^4) \;:\; (h^2 - |k|^2)\mathcal{F}\Phi(h,k) \in L^2(\mathbb{R}^4)\right\}\,.$$

Let $l(s) = y + ws$, $y, w \in \mathbb{R}^4$, be a time-like straight line, i.e.

$$w = (\gamma_v, \gamma_v\, v), \quad v \in \mathbb{R}^3, \quad |v| < 1, \quad \gamma_v := \left(1 - |v|^2\right)^{-1/2}\,.$$

Consider now the unique surjective linear operator

$$\tau_0 : D(\Box) \to H^{-\frac12}(\mathbb{R})\,, \qquad \tau_0 \in B(D(\Box), H^{-\frac12}(\mathbb{R}))$$

such that

$$\forall \Phi \in C^\infty_0(\mathbb{R}^4), \qquad \tau_0\Phi(s) := \phi(s, 0)\,.$$

The existence of such a τ_0 is ensured by [H, thm. 2.2.8] (see the next example). Let then $\Pi_{y,v}$ be the unitary operator which compose any function in $L^2(\mathbb{R}^4)$ with the Lorentz boost corresponding to v and then with the translation by y, so that $\Pi_{y,v} \in B(D(\Box), D(\Box))$. Defining

$$\tau_{y,v} := \tau_0 \cdot \Pi_{y,v} : D(\Box) \to H^{-\frac12}(\mathbb{R})\,, \qquad \tau_{y,v} \in B(D(\Box), H^{-\frac12}(\mathbb{R}))$$

one has

$$\forall \Phi \in C_0^\infty(\mathbb{R}^4), \qquad \tau_{y,v}\phi(s) := \phi(l(s)) \ .$$

We begin studing the self-adjoint extensions given by τ_0. By Fourier transform (here and below $z \in \mathbb{C}\backslash\mathbb{R}$) one obviously has

$$\mathcal{F}\cdot(\Box + z)^{-1}\Phi(h,k) = \frac{\mathcal{F}\Phi(h,k)}{-h^2+|k|^2+z} \ .$$

So, since, as $R \uparrow \infty$,

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty dh \, \frac{1}{\left|-h^2+R^2+z\right|^2} \sim \frac{c}{4R^3} \ ,$$

by Hölder inequality and Riemann–Lebesgue lemma there follows that

$$\forall\, \Phi = \phi\otimes\varphi \in L^2(\mathbb{R})\otimes H^s(\mathbb{R}^3),\ s>0, \qquad (\Box + z)^{-1}\Phi \in C_b^0(\mathbb{R}^4)$$

and, by Fubini theorem,

$$\begin{aligned}
&\left[(-\Box + z)^{-1}\Phi\right](t,x)\\
=&\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} dh\, dk\, e^{iht}e^{ik\cdot x}\,\frac{\mathcal{F}\Phi(h,k)}{-h^2+|k|^2+z}\\
=&\frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}} dh\, e^{iht}\mathcal{F}\phi(h)\left(\left(-\Delta - h^2 + z\right)^{-1}\varphi\right)(x)\\
=&\frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}} dh\, e^{iht}\mathcal{F}\phi(h)\int_{\mathbb{R}^3} dy\,\varphi(y)\,\frac{e^{-\sqrt{-h^2+z}\,|x-y|}}{4\pi\,|x-y|}\\
=&\int_{\mathbb{R}^3} dy\,\frac{\varphi(y)}{4\pi\,|x-y|}\,\left(e^{-|x-y|\,\sqrt{\Delta_{(1)}+z}}\,\phi\right)(t) \ .
\end{aligned}$$

Here $\mathrm{Re}\sqrt{-h^2+z} > 0$; this choice will be always assumed in the sequel without further specification. The above calculation then gives

$$\breve{G}(z) : L^2(\mathbb{R})\otimes H^s(\mathbb{R}^3) \to C_b^0(\mathbb{R}), \quad s>0,$$

$$\left(\breve{G}(z)\phi\otimes\varphi\right)(t) := \int_{\mathbb{R}^3} dy\,\frac{\varphi(y)}{4\pi\,|y|}\,\left[e^{-|y|\,\sqrt{\Delta_{(1)}+z}}\phi\right](t)$$

and

$$G(z) : H^{1/2}(\mathbb{R}) \to L^2(\mathbb{R}^4)$$

$$(G(z)\phi)(t,x) := \frac{1}{4\pi\,|x|}\,\left[e^{-|x|\,\sqrt{\Delta_{(1)}+z}}\phi\right](t) \ .$$

Let us note that $\breve{G}(z)$ extends to a continuous linear operator from $L^2(\mathbb{R}^4)$ to $H^{-1/2}(\mathbb{R})$ since

$$\begin{aligned}
&\|\breve{G}(z)\phi\otimes\varphi\|^2_{H^{-1/2}}\\
\le&\int_{\mathbb{R}} dt\left(\int_{\mathbb{R}^3} dy\,\frac{|\varphi(y)|}{4\pi\,|y|}\,\left|\left[(-\Delta_{(1)}+1)^{-1/4}\cdot e^{-|y|\sqrt{\Delta_{(1)}+z}}\phi\right](t)\right|\right)^2\\
\le&\|\varphi\|^2_{L^2}\left\langle\left(-\Delta_{(1)}+1\right)^{-1/2}\cdot\int_0^\infty dR\,e^{-2R\,\mathrm{Re}\left(\sqrt{\Delta_{(1)}+z}\right)}\phi,\phi\right\rangle\\
=&\frac{1}{2}\,\|\varphi\|^2_{L^2}\left\langle\left(-\Delta_{(1)}+1\right)^{-1/2}\cdot\left(\mathrm{Re}\left(\sqrt{\Delta_{(1)}+z}\right)\right)^{-1}\phi,\phi\right\rangle\\
\le&\frac{1}{2}\,\|\phi\otimes\varphi\|^2_{L^2}\left\|\left(\mathrm{Re}\left(\sqrt{\Delta_{(1)}+z}\right)\right)^{-1/2}\right\|_{L^2,H^{-1/2}}\,.
\end{aligned}$$

Similarly $G(z)$ is a continuous linear operator from $H^{1/2}(\mathbb{R})$ to $L^2(\mathbb{R}^4)$ since

$$\begin{aligned}
&\|G(z)\phi\|^2_{L^2}\\
=&\int_0^\infty dR\,\left\|e^{-R\sqrt{\Delta_{(1)}+z}}\phi\right\|^2_{L^2}\\
=&\left\langle\int_0^\infty dR\,e^{-2R\,\mathrm{Re}\left(\sqrt{\Delta_{(1)}+z}\right)}\phi,\phi\right\rangle\\
=&\frac{1}{2}\left\langle\left(\mathrm{Re}\left(\sqrt{\Delta_{(1)}+z}\right)\right)^{-1}\phi,\phi\right\rangle\\
\le&\frac{1}{2}\,\|\phi\|^2_{H^{1/2}}\left\|\left(\mathrm{Re}\left(\sqrt{\Delta_{(1)}+z}\right)\right)^{-1/2}\right\|_{H^{1/2},L^2}\,.
\end{aligned}$$

We now look for the map $\Gamma(z)$. Since

$$\begin{aligned}
&(z-w)\,\mathcal{F}\cdot\breve{G}(w)\cdot G(z)\phi(h)\\
=&\frac{z-w}{(2\pi)^3}\,\mathcal{F}\phi(h)\int_{\mathbb{R}^3} dk\,\frac{1}{(-h^2+|k|^2+w)(-h^2+|k|^2+z)}\\
=&\frac{z-w}{2\pi^2}\,\mathcal{F}\phi(h)\int_0^\infty dr\,\frac{r^2}{(-h^2+r^2+w)(-h^2+r^2+z)}\\
=&\frac{1}{4\pi}\left(\sqrt{-h^2+z}-\sqrt{-h^2+w}\right)\mathcal{F}\phi(h)
\end{aligned}$$

one defines

$$\Gamma(z):=\frac{1}{4\pi}\sqrt{\Delta_{(1)}+z}\ :H^{1/2}(\mathbb{R})\to H^{-1/2}(\mathbb{R})\ .$$

Of course we can view $\Gamma(z)$ as a (unbounded) closed and densely defined linear operator on the Hilbert space $H^{-1/2}(\mathbb{R})$. Therefore, by Proposition 1, $\Gamma_\Theta(z)$ satisfies (h1) (with $Z_\Theta=\rho(\Box)$) for any symmetric $\Theta: H^{1/2}(\mathbb{R})\to H^{-1/2}(\mathbb{R})$ which is $\Gamma(z)$–bounded. It is immediate, by Fourier transform, to check the validity of (h2). Therefore the trace τ_0 gives rise to the family of self-adjoint extensions $\Box^0_\Theta$ with resolvent

$$(-\Box^0_\Theta + z)^{-1} = (-\Box + z)^{-1} + G(z) \cdot \left(\Theta + \frac{1}{4\pi} \sqrt{\Delta_{(1)} + z} \right)^{-1} \cdot \breve{G}(z) \ .$$

By our definition of $\tau_{y,v}$ we have, since $\Pi_{y,v}$ commutes with $\Box$,

$$\breve{G}_{y,v}(z) := \tau_{y,v} \cdot R^0(z) = \breve{G}(z) \cdot \Pi_{y,v}$$

and

$$C_{L^2}^{-1} \cdot \breve{G}_{y,v}(z^*)' = \Pi^*_{y,v} \cdot G(z) \ .$$

This immediately implies that one can use the same $\Gamma_\Theta(z)$ as before and so the trace $\tau_{y,v}$ gives rise to the family of self-adjoint extensions $\Box^{y,v}_\Theta$ with resolvent

$$(-\Box^{y,v}_\Theta + z)^{-1} = (-\Box + z)^{-1} + \Pi^*_{y,v} \cdot G(z) \cdot \left(\Theta + \frac{1}{4\pi} \sqrt{\Delta_{(1)} + z} \right)^{-1} \cdot \breve{G}(z) \cdot \Pi_{y,v} \ ,$$

so that the following kind of Poincaré–invariance holds:

$$D(\Box^{y,v}_\Theta) = \Pi^*_{y,v}(D(\Box^0_\Theta)) \quad \text{and} \quad \Box^{y,v}_\Theta = \Pi^*_{y,v} \cdot \Box^0_\Theta \cdot \Pi_{y,v} \ .$$

Example 8. *Singular perturbation given by traces on the Hörmander spaces $H_\varphi(\mathbb{R}^n)$.*

Given any positive functions φ on $\mathbb{R}^n$, $\varphi \in \mathcal{K}$ will mean

$$\exists\, c,\, N > 0 \ : \ \forall\, x,\, y \in \mathbb{R}^n, \qquad \varphi(x+y) \le (\, 1 + c\,|x|\,)^N \varphi(y) \ .$$

Then we define, following [H, §II.2.2] (also see [VP]), the Hilbert space $H_\varphi(\mathbb{R}^n)$, $\varphi \in \mathcal{K}$, as the set of tempered distribution f such that $\mathcal{F}f$ is a functions and

$$\|f\|^2_\varphi := \int_{\mathbb{R}^n} |\varphi(k)\mathcal{F}f(k)|^2 \, dk < +\infty \ .$$

By [H, example 2, §II.2.2], for any (complex–valued) polynomial p one has $\tilde{p} \in \mathcal{K}$, where

$$\tilde{p}(x) := \left(\sum_{|\alpha| \ge 0} |\, D^\alpha \, p(x)|^2 \right)^{1/2}$$

and $D^\alpha := \partial^{|\alpha|}/\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. Moreover, by [H, thm. 2.1.1],

$$\varphi_1 + \varphi_2,\ \varphi_1\varphi_2,\ \varphi_1 \wedge \varphi_2,\ \varphi_1 \vee \varphi_2,\ \varphi^s,\ \mu * \varphi \ \in \mathcal{K}$$

for any φ, φ_1, φ_2 $\in \mathcal{K}$, for any $s \in \mathbb{R}$ and for any positive measure μ such that $\mu * \varphi(x) \neq \infty$ for some (and hence any) $x \in \mathbb{R}^n$.

In connection with the previous examples note that

$$\varphi(x) = (1 + |x|^2)^{s/2}, \ s \in \mathbb{R} \quad \Longrightarrow \quad H_\varphi(\mathbb{R}^n) = H^s(\mathbb{R}^n)$$

and

$$\varphi(t,x) = (1 + (-t^2 + |x|^2)^2)^{1/2}, \ t \in \mathbb{R}, \ x \in \mathbb{R}^3 \quad \Longrightarrow \quad H_\varphi(\mathbb{R}^4) = D(\Box) \ .$$

The dual space of H_φ can be explicitely characterized (see [H, thm. 2.2.9]) as

$$H'_\varphi \simeq H_{1/\varphi} \ .$$

As regards the relation between different spaces, by [H, thm. 2.2.2] one has

$$\varphi_1 \leq c\,\varphi_2, \quad \varphi_1, \varphi_2 \in \mathcal{K} \quad \Longrightarrow \quad H_{\varphi_1} \subseteq H_{\varphi_2} \ ,$$

the embedding being continuous. Therefore, for any $\varphi \in \mathcal{K}$ such that $\varphi \geq c > 0$, one has $H_\varphi \subseteq L^2(\mathbb{R}^n)$; H_φ is then dense in $L^2(\mathbb{R}^n)$ since $C_0^\infty(\mathbb{R}^n)$ is dense in $H_\varphi(\mathbb{R}^n)$ for any $\varphi \in \mathcal{K}$ (see [H, thm. 2.2.1]). The regularity of elements in $H_\varphi(\mathbb{R}^n)$ is given by [H, thm. 2.2.7]:

$$(1 + |x|)^k/\varphi(x) \in L^2(\mathbb{R}^n) \quad \Longrightarrow \quad H_\varphi(\mathbb{R}^n) \subset C^k(\mathbb{R}^n) \ ,$$

the embedding being continuous.

Let us now come to the trace operator on $H_\varphi(\mathbb{R}^n)$ (see [H, thm. 2.2.8], [VP, §6]). We write $\mathbb{R}^n = \mathbb{R}^d \oplus \mathbb{R}^{n-d}$, $1 \leq d \leq n-1$, $x = (\tilde{x}, \hat{x})$, $\tilde{x} \in \mathbb{R}^d$, $\hat{x} \in \mathbb{R}^{n-d}$. Suppose that

$$\left(\int_{\mathbb{R}^{n-d}} \frac{1}{\varphi^2(0,\hat{x})}\, d\hat{x} \right)^{-1/2} < +\infty \ .$$

Then there exists an unique surjective linear operator $\tau_{(d)}$

$$\tau_{(d)} : H_\varphi(\mathbb{R}^n) \to H_{\tilde{\varphi}}(\mathbb{R}^d) \,, \qquad \tau_{(d)} \in B(H_\varphi(\mathbb{R}^n), H_{\tilde{\varphi}}(\mathbb{R}^d)) \ ,$$

$$\tilde{\varphi}(\tilde{x}) := \left(\int_{\mathbb{R}^{n-d}} \frac{1}{\varphi^2(\tilde{x},\hat{x})}\, d\hat{x} \right)^{-1/2} \ ,$$

such that

$$\forall\, \phi \in C_0^\infty(\mathbb{R}^n) \,, \qquad \tau_{(d)}(\phi)(\tilde{x}) = \phi(\tilde{x}, 0) \ .$$

The reader can check that the case $\varphi(t,x) = (1 + (-t^2 + |x|^2)^2)^{1/2}$, $d = 1$, reproduces the trace τ_0 given in the previous example.

The trace $\tau_{(d)}$ can be generalized to cover the case of non-linear subsets in the following way: let $\mu \in H'_\phi(\mathbb{R}^n)$ (for example μ could be the Hausdorff measure of some subset of $\mathbb{R}^n$ but more general distributions are allowed) for which there exists $\tilde{\phi} \in \mathcal{K}$ such that

$$\int_{\mathbb{R}^n} \frac{\phi^2(x-y)}{\varphi^2(x)\,\tilde{\phi}^2(y)}\,dy < c < +\infty \ .$$

Then, by [VP, §2.4],

$$\forall f \in H_{\tilde{\phi}}(\mathbb{R}^n), \qquad f\mu \in H'_{\varphi}(\mathbb{R}^n) \ ,$$

where

$$f\mu := (2\pi)^{-n/2}\mathcal{F}^{-1}(\mathcal{F}f * \mathcal{F}\mu) \ .$$

So we can define

$$\tilde{\tau}_\mu : H_{\tilde{\phi}}(\mathbb{R}^n) \to H'_{\varphi}(\mathbb{R}^n) \qquad \tilde{\tau}_\mu(f) := f\mu \ ,$$

and then we have a trace generalizing $\tau_{(d)}$ by

$$\tau_\mu : H_{\varphi}(\mathbb{R}^n) \to H'_{\tilde{\phi}}(\mathbb{R}^n) \qquad \tau_\mu := \tilde{\tau}'_\mu \cdot J_{H_\varphi} \ .$$

Let us now consider the self-adjoint pseudo-differential operator ($\varphi \geq c > 0$)

$$\psi(D) : H_{\varphi}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\,, \qquad \psi(D)\Phi := \mathcal{F}^{-1}(\psi\,\mathcal{F}\Phi) \ ,$$

where ψ is a real-valued Borel function such that

$$\frac{1}{c}\,\varphi(x) \leq 1 + |\psi(x)| \leq c\,\varphi(x) \ .$$

By Fourier transform one has, if τ_μ is defined as above,

$$G(z)f = G^\mu(z)f := \frac{1}{(2\pi)^{n/2}}\,\mathcal{F}^{-1}\left(\frac{\mathcal{F}f * \mathcal{F}\mu}{-\psi + z^*}\right) \ .$$

Therefore (h2) is equivalent to $\mathcal{F}f * \mathcal{F}\mu \notin L^2(\mathbb{R}^n)$, i.e. $f\mu \notin L^2(\mathbb{R}^n)$. This condition is surely satisfied when the support of μ is a set of zero Lebesgue measure.

By lemma 2 we have then, for any $f_1, f_2 \in H_{\tilde{\phi}}(\mathbb{R}^n)$,

$$\hat{\Gamma}(z)f_1(f_2) = f_2\mu((\tilde{G}^\mu - G^\mu(z))f_1) \ , \tag{22}$$

where

$$\tilde{G}^\mu := \frac{G^\mu(z_0) + G^\mu(z_0^*)}{2}\,, \qquad z_0 \in \rho(\psi(D)) \ .$$

In the case

$$\int_{\mathbb{R}^n} \frac{\phi^2(x-y)}{\tilde{\phi}^2(x)\,\varphi^2(y)}\,dy < c < +\infty \ ,$$

by [VP, §2.4] as above, we have

$$\forall\Phi \in H_\varphi(\mathbb{R}^n), \qquad \Phi\mu \in H'_\varphi(\mathbb{R}^n)\ ,$$

and so, by (22), $\hat\Gamma(z) = \hat\Gamma_0^\mu(z)$, where

$$\hat\Gamma^\mu(z) : H''_{\tilde\phi}(\mathbb{R}^n) \simeq H_{\tilde\phi}(\mathbb{R}^n) \to H'_{\tilde\phi}(\mathbb{R}^n)\,, \qquad \hat\Gamma^\mu(z)f := ((\tilde G^\mu - G^\mu(z))f)\mu\ .$$

In the case the range of τ_μ is dense (this is certainly true when the support of μ is a regular submanifold), by Remark 9, Proposition 1 and Theorem 1, τ_μ gives rise to the family of self-adjoint operators $\psi(D)^\mu_\Theta$ with resolvents

$$(-\psi(D)^\mu_\Theta + z)^{-1} = (-\psi(D) + z)^{-1} + G^\mu(z)\cdot(\Theta + \hat\Gamma^\mu(z))^{-1}\cdot \breve G^\mu(z)\ ,$$

where Θ is any closed and densely defined operator in $\tilde S(H_{\tilde\phi}, H'_{\tilde\phi})$.

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