A Nonlinear Approximation of Operator Equation $V^*QV = Q$: Nonspectral Decomposition of Nonnormal Operator and Theory of Stability

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Abstract

V denote arbitrary bounded bijection on Hilbert space H. We try to describe the sets of V-stable vectors, i.e. $\{x \in H|$ the sequence $\|V^N x\|(N = 1, 2, \ldots)$ is bounded$\}$ and some other analogous sets. We do it in terms of one-parameter operator equation $Q = V^*(Q + tI)(I + tQ)^{-1}V$ (t is real valued parameter $0 \leq t \leq 1$, Q is operator to be found, $0 \leq Q$)

1 Introduction

Throughout this paper H will denote a Hilbert space with scalar product $<,>$, V denotes a linear bounded bijection H onto H,

$$r(T) := \text{spectral radious of } T\ .$$

We will discuss the structure of the next four sets:

$$Stab_+(V) \ := \ \{x \in H | \forall a > 1 \ \exists C \geq 0 \ \forall N \geq 0 \quad \|V^N x\| \leq Ca^N\}.$$

$$Stab(V) := \{x \in H| \quad \|V^n x\| \leq M \text{ for some real } M \text{ and every } n = 0, 1, 2, \ldots\}$$

$$Stab_0(V) := \{x \in H| \quad \|V^n x\| \to 0 \quad (n \to \infty)\}$$

$$l_2(V) := \{x \in H| \quad \|Vx\|^2 + \|V^2x\|^2 + \cdots + \|V^n x\|^2 + \cdots < \infty\}$$

Recall V is **similar to an unitary operator** iff there exists a bounded uniformely positive operator Q such that

$$V^*QV = Q$$

With this equation we shall consider an 'approximation' equation (parametrized by real t)

$$Q = V^* \frac{Q + t}{I + tQ} V, \quad Q \geq 0, \quad 0 < t \leq 1 \tag{*}$$

(hereinafter t denotes always a real number such that $0 < t \leq 1$ and if no confusion can occur we shall often write t instead of tI, I is identity operator).

The interest in this equation can be motivated by the next

Example 1. *Let V be normal. Routine, though tedious calculation shows that*

$$Q_t := \left[-\frac{I - V^*V}{2} + \sqrt{\left(\frac{I - V^*V}{2}\right)^2 + t^2 V^*V} \right] \frac{1}{t}$$

is uniformely positive solution of $(^*)$*, there holds*

$$Q_t^{-1} = \left[-\frac{I-(V^*V)^{-1}}{2} + \sqrt{\left(\frac{I-(V^*V)^{-1}}{2}\right)^2 + t^2(V^*V)^{-1}} \right] \frac{1}{t}$$

and there exist

$$X_0 \ := \ strong-lim \ tQ_t = (V^*V-I)E(1,\infty) \qquad (t \to +0)$$

$$Y_0 \ := \ strong-lim \ tQ_t^{-1} = ((V^*V)^{-1}-1)E(0,1) \qquad (t \to +0)$$

Besides that it is fast evident that $Q_tE([1]) = E([1])$, $Q_tE(1,\infty)$ *is monotone increasing (with* $t \to +0$*),* $Q_tE(0,1)$ *is monotone decreasing and there exists an*

$$R_0 \ := strong-lim(I+Q_t)^{-1} = E(0,1) + E([1])/2 \qquad (t \to +0).$$

here $E(\Delta)$ *denotes the spectral function of the selfadjoint operator* V^*V*. Note:*

$$I - R_0 = E([1])/2 + E(1,\infty)$$

$$Ker(I-R_0) = E(0,1)H$$

$$RanX_0 \subset E(1,\infty)H = \overline{RanX_0}$$

$$RanY_0 \subset E(0,1)H = \overline{RanY_0}$$

□

Also, in the above considered case of the normal V it is estableshed that the operators X_0, Y_0, R_0 define (in essential) the spectral subspaces of V (with V together one can consider $aV-b, b/a \notin spectrum\, V$) . In this article we shall show that the similar situation holds for the arbitrary bounded bijection V.

We follow standards of [RS] when we apply mathematical concepts and sometimes we apply P.A.M. Dirac's 'bra-ket' syntax.

We will often cite some assertions and propositions of [Ch1-4]. For the most convenient and accesible way to do it, we collect them together and resume them here as

Theorem 1.
(i) *the solution of* $(^*)$ *exists and it is unique; denote it by* Q_t
(ii) Q_t *is bounded selfadjoint uniformely positive and there are satisfied inequalities:*

$$tV^*V \le Q_t \le V^*V/t$$

$$tQ_t \le sQ_s \qquad (0 < t \le s \le 1)$$

denote $X_t := tQ_t$
(iii) Q_t^{-1} *is (unique) solution of the analogous equation:*

$$Q_t^{-1} = V^{-1}\frac{Q_t^{-1}+t}{I+tQ_t^{-1}}(V^{-1})^*$$

so there are satisfied inequalities:

$$tV^{-1}(V^{-1})^* \le Q_t^{-1} \le V^{-1}(V^{-1})^*/t$$

$$tQ_t^{-1} \le sQ_s^{-1} \qquad (0 < t \le s \le 1)$$

denote $Y_t := tQ_t^{-1}$.
(iv) *Let*

$$X_0 := strong - lim\ X_t, \quad Y_0 := strong - lim\ Y_t \quad (t \to +0)$$

The operators X_0, Y_0 *are bounded positive and they are maximal solutions of the equations*

$$X = V^*\frac{X}{I+X}V,\ X \ge 0,\ resp.\ VYV^* = \frac{Y}{I+Y},\ Y \ge 0$$

('maximal' denotes 'maximal with respect to usual partial order for bounded operators on Hilbert space')
(v) *There hold the formulae*

$$Y_0 = strong - lim\ (V^*V + V^{*2}V^2 + \cdots + V^{*n}V^n)^{-1},\ (n \to \infty)$$

$$X_0 = strong - lim\ ((V^*V)^{-1} + (V^{*2}V^2)^{-1} + \cdots + (V^{*n}V^n)^{-1})^{-1},\ (n \to \infty)$$

(vi) *Denote* $R_t := (I+Q_t)^{-1}$. *Then* $0 \le R_t \le I$, $Q_t = R_t^{-1} - I$ *and the equation* $(^*)$ *is equivalent to the equation*

$$[R_t + t(I-R_t)](V^{-1})^*(I-R_t) = [(I-R_t) + tR_t]VR_t$$

(vii) *Let* R_0 *be a weak operator limit point of the net* $\{R_t,\, t \to +0\}$ *(it is clear that* R_0 *exists and* $0 \le R_0 \le I$*). Then*

$$V\,Ker\,(I-R_0) = Ker\,(I-R_0)$$

$$V^{*-1}\,Ker\,R_0 = Ker\,R_0$$

In particular,

$$V\overline{Ran\,R_0} = \overline{Ran\,R_0} \qquad \square$$

2 Equation $Q = V^* \dfrac{Q+t}{I+tQ} V$. General Properties.

Hereinafter $\mathbf{F}$ denotes an arbitrary ultra filter, which majorizes usual convergence to $+0$. We will write

$$t \to +0 \quad \text{instead of } t \xrightarrow{\mathbf{F}} +0$$

if no confusion can occur.

Definition 1.

$$Fin\,Q := \{x \in \mathbf{H} | < x, Q_t x > \le M_x \textit{ for some real } M_x \textit{ and for almost every } t \textit{ resp. } \mathbf{F}\}$$

$$Ker\,Q_0 := \{x \in \mathbf{H} | < x, Q_t x > \to 0 \textit{ for } t \to +0\}$$

Theorem 1.

$$Ran\,Y_0^{1/2} = l_2(V) \subset Stab_0(V) \subset Ker\,Q_0 \subset Ker\,(I - R_0) \subset \overline{Ran\,R_0} \subset Ker\,X_0 \quad (*)$$

$$Stab(V) \subset Fin\,Q \subset \overline{Ran\,R_0} \subset Ker\,X_0 \qquad (**)$$

Every set of these series is V-surinvariant. (recall, some L **is said to be** *T-***surinvariant, iff** *$TL = L$).*

In addition

$$Fin\,Q \subset Ran\,R_0^{1/2}$$

Observation 0. *It is well-known and evident that*

$$A^*A \le B^*B \iff \|Ax\| \le \|Bx\| \quad (x \in H) \implies RanA^* \subset RanB^*$$

(A and B are bounded)

Corollary 0. *Let $x \in H$, Y be selfadjoint and let $Y \ge 0$. Then*

$$x \in Ran\,Y^{1/2} \iff x\langle x| \le cY \qquad \textit{for some real } c.$$

Proof.
Proof of $x\langle x| \le cY \implies x \in Ran\,Y^{1/2}$:
Take into account Observation 0. Then obtain

$$x\langle x| \le cY \implies Ran\,x\langle x| \subset Ran\,Y^{1/2} \implies x \in Ran\,Y^{1/2}\,.$$

Now proof of $x \in Ran\,Y^{1/2} \implies x\langle x| \le cY$:
By the definition of *Ran*

$$x \in Ran\,Y^{1/2} \iff x = Y^{1/2}y \quad \text{for an } y.$$

Hence

$$x \in Ran\,Y^{1/2} \implies x\langle x| \equiv Y^{1/2}y\langle Y^{1/2}y| = Y^{1/2}y\langle y|Y^{1/2} \le \|y\|^2 Y\,.$$

Now denote $\|y\|^2$ by c.

□

Observation 1. *Let $\{A_t\}_t$ be a net of selfadjoint positive bounded operators. Suppose $A_t \le aI$ for some positive number a (and every t), A_0 be a weak operator limit point of this net (clear: A_0 exists and $0 \le A_0 \le aI$).*
Then

$$< x, A_t x > \to 0 \Leftrightarrow \|A_t x\| \to 0 \Leftrightarrow x \in Ker\ A_0$$

Proof. Clear (see e.g. [Ch1]).

□

Observation 2. *Let Q be bounded selfadjoint positive,$0 < t \le 1$. Then*

$$\frac{Q}{I + t\|Q\|} \le \frac{Q + t}{I + tQ} \le Q + t$$

Corollary.

$$\boxed{VFin\,Q = Fin\,Q\,, \quad VKerQ_0 = KerQ_0\,.}$$

Observation 2'. *Let Q be bounded selfadjoint positive,$0 < t \le 1$. Then*

$$\frac{1-t}{1+t}Q + \frac{2t}{1+t} - \frac{Q+t}{I+tQ} = \frac{t(1-t)}{1+t}\frac{(Q-I)^2}{I+tQ}$$

In particular

$$\frac{Q+t}{I+tQ} \le \frac{1-t}{1+t}Q + \frac{2t}{1+t}$$

Denote $z := (1-t)/(1+t)$. Then $1 - z = 2t/(1+t)$ and with these denotations

$$\frac{Q+t}{I+tQ} \le zQ + (1-z)$$

Note

$$t \to +0 \iff z \to 1 - 0\,.$$

Proof. Clear.

□

Observation 3 . *For Q_t the just mentioned inequality gives*

$$Q_t \;\le\; V^*(zQ_t + (1-z))V$$

and with iterating this inequality one can obtain

$$Q_t \le (1-z)[V^*V + zV^{*2}V^2 + \cdots + z^{n-1}V^{*n}V^n] + z^n V^{*n} Q_t V^n$$

In particular, given numbers z, M_0 and an $x \in H$ such that

$$0 < z < 1,\ M_0 \ge 0,\ \|V^n x\|^2 \le M_0\ (n = 1, 2, 3, ...)$$

then

$$< x, Q_t x > \le (1-z)(\|Vx\|^2 + z\|V^2x\|^2 + z^2\|V^3x\|^2 + \cdots) \le M_0\,.$$

In particular

$$\boxed{Stab(V) \subset Fin\,Q}$$

Now suppose $\|V^n x\| \to 0$ $(n \to \infty)$ for some $x \in H$. Let $t \to +0$. Then $z \to 1 - 0$ and hence $<x, Q_t x> \to 0$ since $\|V^n x\| \to 0$ and $< x, Q_t x > \ge 0$.
In particular

$$\boxed{Stab_0(V) \subset KerQ_0}$$

□

Observation 4. *Let $x \in H$, $\|x\| = 1$. Then*
a)

$$x \in Fin\, Q \Rightarrow \exists M \geq 0 \forall \alpha \geq 0 : \qquad x\langle x| \leq (M+\alpha)(Q_t+\alpha)^{-1}$$

b) Given some $M, \alpha \geq 0$, such that

$$x\langle x| \leq (M+\alpha)(Q_t+\alpha)^{-1}$$

then

$$x \in Fin\, Q$$

Proof

$$\begin{aligned}
&< x, Q_t x > \leq M \\
&\iff \quad < x, (Q_t+\alpha)x > \leq M+\alpha \\
&\iff \quad \|(Q_t+\alpha)^{1/2}x\|^2 \leq M+\alpha \\
&\iff \quad (Q_t+\alpha)^{1/2}x\langle (Q_t+\alpha)^{1/2}x| \leq M+\alpha \\
&\iff \quad (Q_t+\alpha)^{1/2}x\langle x|(Q_t+\alpha)^{1/2} \leq M+\alpha \\
&\iff \quad x\langle x| \leq (M+\alpha)(Q_t+\alpha)^{-1}
\end{aligned}$$

□

Corollary. $\boxed{Fin\, Q \subset Ran\, R_0^{1/2} \subset \overline{Ran\, R_0}}$

Proof First apply Observation 4 for $\alpha := 1$ and let $t \to +0$. Then obtain

$$x\langle x| \leq (M+1)R_0;$$

Now apply Observation 0 or Corollary 0.

□

Observation 5. *Let $x \in H$. Then*

$$x \in Ran\, Y_0^{1/2} \iff x\langle x| \leq cY_0 \iff < x, Q_t x > \leq ct$$

Proof

$$x \in Ran\, Y_0^{1/2} \iff x = Y_0^{1/2}y \Longrightarrow x\langle x| \equiv Y_0^{1/2}y\langle Y_0^{1/2}y| \leq \|y\|^2 Y_0$$

Besides,

$$x\langle x| \leq cY_0 \Longrightarrow Ran\, x\langle x| \subset Ran\, Y_0^{1/2} \Longrightarrow x \in Ran\, Y_0^{1/2}$$

Now recall that

$$Y_0 \leq Y_t \text{ und } Y_t \xrightarrow{s} Y_0\,.$$

Hence

$$\begin{aligned}
x \in Ran\, Y_0^{1/2} \quad &\iff \quad x\langle x| \leq cY_0 \\
&\iff \quad x\langle x| \leq cY_t \\
&\iff \quad x\langle x| \leq ctQ_t^{-1} \\
&\iff \quad Q_t^{1/2}x\langle Q_t^{1/2}x| \leq ct \\
&\iff \quad \|Q_t^{1/2}x\|^{1/2} \leq ct \\
&\iff \quad < x, Q_t x > \leq ct
\end{aligned}$$

$\square$

Proof of $\boxed{KerQ_0 \subset Ker(I - R_0).}$
Note

$$I - R_t = Q_t(I + Q_t)^{-1} \leq Q_t$$

Now suppose $x \in KerQ_0$ i.e. $< x, Q_t x > \to 0$ and apply the Observation 1 to $A_t = I - R_t$.

$\square$

Proof of $\boxed{Ker(I - R_0) \subset \overline{Ran\, R_0}.}$
Note Ker is closed and use definitions of Ker , Ran .

$\square$

Proof of $\boxed{\overline{Ran\, R_0} \subset Ker\, X_0\ .}$
Note

$$R_t X_t = t(I + Q_t)^{-1} = X_t R_t$$

Then

$$R_t X_t = X_t R_t \xrightarrow{\|\cdot\|} 0.$$

Recall R_t, X_t are selfadjoit, positive, bounded and $X_t \xrightarrow{s} X_0$, $R_t \xrightarrow{w} R_0$. Hence $R_0 X_0 = 0 = X_0 R_0$.

The rest is evident.

$\square$

Proof of $\boxed{Ran\, Y_0^{1/2} \ \subset \ l_2(V)\,.}$
We have

$$V Y_0 V^* = \frac{Y_0}{I + Y_0} \qquad (*)$$

Hence $\|Y_0^{1/2} V^* x\| = \|(I+Y_0)^{1/2} Y_0^{1/2} x\| \ \leq \ \|Y_0^{1/2} x\|$ and the relation

$$W :\, Y_0^{1/2} x \ \to \ Y_0^{1/2} V^* x$$

defines a contraction $Ran\, Y_0^{1/2} \ \to \ Ran\, Y_0^{1/2}$. This contraction has extension to a contraction $\overline{Ran\, Y_0^{1/2}} \ \to \ \overline{Ran\, Y_0^{1/2}}$. It will be denote by W too. Clear, there hold

$$(i)\ W Y_0^{1/2} = Y_0^{1/2} V^* \qquad (iii)\ W^* W = (I+Y_0)^{-1} | \overline{Ran\, Y_0^{1/2}}$$

$$(ii)\ Y_0^{1/2} W^* P = V Y_0^{1/2} \quad (iv)\ Y_0 P = ((W^* W)^{-1} - I) P$$

and $WP = PWP, \quad W^* P = PW^* P$ .Here is

$$P := orthoprojection\ onto\ \overline{Ran\, Y_0^{1/2}}$$

Hence

$$\begin{aligned} Y_0^{1/2} V^{*n} V^n Y_0^{1/2} \ &= \ W^n Y_0 W^{*n} P \ = \ P W^n Y_0 W^{*n} P \\ &= \ P W^n ((W^* W)^{-1} - I) W^{*n} P \\ &= \ P (W^{n-1} W^{*n-1} - W^n W^{*n}) P. \end{aligned}$$

and

$$\sum_{n=2}^{N} \|V^n Y_0^{1/2} x\|^2 = \|WPx\|^2 - \|W^N P x\|^2$$

Hence

$$\sum_{0}^{\infty} \|V^n Y_0^{1/2} x\|^2 < \infty \qquad \square$$

$\square$

Remark to this proof.
It follows from $(*)$ *and* $Y_0 \geq 0$ *that*

$$V Ran\, Y_0 \ = \ Ran\, Y_0, \quad V^* Ker\, Y_0 \ = \ Ker\, Y_0, \quad V \overline{Ran\, Y_0} \ = \ \overline{Ran\, Y_0}$$

Besides that,

$$\begin{aligned} (iii) \quad &=> \quad \|Wx\| \ \geq \ \|x\|/(1 + \|Y_0\|) \quad (x \in D_W) \\ (ii) \quad &=> \quad V\ Ran\, Y_0^{1/2} \ \subset \ Ran\, Y_0^{1/2} \end{aligned}$$

$$(i,\ or\ definition\ of\ W) \ => \ W Ran\, Y_0^{1/2} \ = \ Ran\, Y_0^{1/2} \qquad \square$$

Proof of $\boxed{l_2(V) \ \subset \ Ran\, Y_0^{1/2}.}$
Let $x \in l_2(V)$ and set

$$\begin{aligned} c &:= \|Vx\|^2 + \|V^2x\|^2 + \|V^3x\|^2 + \cdots \\ P &:= \textit{orthoprojection mapping onto span}\{x\} \end{aligned}$$

Then

$$<x, Y_t^{-1}x> = <x, \frac{1}{t}Q_t x> \leq \ \frac{1-z}{t}c \ = \ \frac{2c}{1+t} \ \leq \ 2c$$

Hence

$$\|x\|^2 P \ \leq \ 2cY_t, \quad \|x\|^2 P \ \leq \ 2cY_0, \quad Ran\, P \ \subset \ Ran\, Y_0^{1/2}$$

$\square$

Remark to this proof.

$$(i) \quad Y_t \gg 0, \ Y_0 \ \geq \ 0, \ P = P^{1/2} \ \geq \ 0$$

$$(ii) \quad \|Ax\| \ \leq \ \|Bx\| \quad (x \in H) \ => \ RanA^* \ \subset \ RanB^*$$

(A and B are bounded). $\square$

Corollary. $\boxed{l_2(V) = Ran\, Y_0^{1/2}.}$

3 Nonspectral Decomposition

Observation 1.

It follows from the definitions of X_t, Y_t *that* $X_tY_t = Y_tX_t = t^2$. *Hence* $X_0Y_0 = Y_0X_0 = 0$ *and* $\overline{Ran\ X_0} \subset Ker\ Y_0$, $\overline{Ran\ Y_0} \subset Ker\ X_0$. *But* X_0, Y_0 *are selfadjoint. Thus we obtain an orthogonal decomposition*

$$H = \overline{Ran\ Y_0} + (Ker\ X_0 \cap Ker\ Y_0) + \overline{Ran\ X_0}$$

such that

1) first component is V*-surinvariant;*

2) third component is V^{*-1}*-surinvariant.*

(recall, some L **is said to be** T**-surinvariant, iff** $TL = L$*). In particular second component reduces* V *Moreover, denote* $j_t := (I + tQ_t^{-1})^{1/2}/(I + tQ_t)^{1/2}$ *,then* j_t *is uniformely positive, bounded, there hold*

$$(1 + \|V\|^2)^{-1/2} \leq j_t \leq (1 + \|V^{-1}\|^2)^{1/2} ,$$

and $(j_tV)^*Q_t(j_tV) = Q_t$*. In particular* j_tV *is similar to an unitary operator.*

For a moment suppose $dim\, H < \infty$*. It is clear that now the restriction of* V *onto* $\overline{Ran\, Y_0}$ *and the restriction of* V^{*-1} *onto* $\overline{Ran\, X_0}$ *are similar to uniform contractions (see theorems 2.1 with remarks to the proof of* $l_2(V) = Ran\, Y_0^{1/2}$*). In addition, if*

$$RanY_0 = \{0\} = RanX_0 ,$$

then the restriction of V *onto* $Ker\, X_0 \cap Ker\, Y_0$ *has the unit spectrum.* □

This motivates the

Definition 1. *We shall say, a linear bounded operator* T *is* **near similar to uniform contraction** *, iff there exists a bounded operator* $Y > 0$ *such that*

$$TYT^* = \frac{Y}{I + Y}$$

We shall say, T *is* **s-approximately similar to an unitary** *, iff there exists a net* $\{j_t\}_t$ *of bounded uniformely positive operators such that*

1) $1/M \leq j_t \leq M$ *for some real* M *(and every* t *),*

2) $strong - lim\, j_t = I$ *(hence* $strong - lim\, j_t^{-1} = I$*),*

3) for every fixed t *the operator* j_tT *is similar to an unitary operator.* □

With this definition one can resume the section as follows:

Theorem 1. *There exists an orthogonal decomposition*

$$H = H_< + H_= + H_>$$

such that

1) $H_<$ *is* V*- surinvariant and the restriction of* V *onto* $H_<$ *is near similar to an uniform contraction;*

2) $H_>$ *is* V^{*-1} *- surinvariant and the restriction of* V^{*-1} *onto* $H_<$ *is near similar to an uniform contraction;*

3) If $H_< = \{0\} = H_>$ *then* V *is s-aproximately similar to an unitary;*

4) $H_< \subset \overline{l_2(V)}$, $H_> \subset \overline{l_2(V^{*-1})}$.

Proof. Set $H_< := \overline{Ran\, Y_0}$, $H_= := Ker\, X_0 \cap Ker\, Y_0$, $H_> := \overline{Ran\, X_0}$
and apply the text.

□

4 Stability of U in terms of Q_t

Proposition 1. *Let*

$$\frac{1}{M}I \leq Q_t \leq MI$$

for some number $M > 0$ (and every t). Then V is similar to an unitary operator.

Proof. Let Q_0 be a weak limit point of the net $Q_t, t \to +0$. It exists and

$$\frac{1}{M}I \leq Q_0 \leq MI$$

$$Q_0 = V^* Q_0 V$$

(see Observation 2.2). Hence, V is similar to an unitary operator.

□

Proposition 2. *Let V be similar to an unitary operator. Then*

$$\frac{1}{M}I \leq Q_t \leq MI$$

for some number $M > 0$ (and every t).

Proof. For assumed V there exists a number $M > 0$ such that for every natural n there hold

$$\|V^n\|^2 \leq M, \quad \|(V^*)^{-n}\| \leq M$$

Take arbitrary $x \in H$, number z, $0 < z < 1$ and apply the Observation 2.3:

$$\begin{aligned} < x, Q_t x > \; &\leq \; (1-z)(\|Vx\|^2 + z\|V^2 x\|^2 + \cdots + z^{n-1}\|V^n x\|^2 + \cdots) \\ &\leq \; (1-z)\frac{M}{1-z} < x, x >= M < x, x > \end{aligned}$$

Hence, $Q_t \leq MI$.

Now apply the Observation 2.3 to Q_t^{-1} and V^{*-1}:

$$\begin{aligned} < x, Q_t^{-1} x > \; &\leq \; (1-z)(\|V^{*-1}x\|^2 + z\|(V^{*-1})^2 x\|^2 + \cdots) \\ &\leq \; M < x, x > \end{aligned}$$

Hence, $Q_t^{-1} \leq M$ and $\frac{1}{M} \leq Q_t$.

□

5 When Spectrum has Dichotomy.

Return us to the Example 1.1, which was called motivating . We remarked there, that R_0 is reminiscent of one of the spectral projector of the operator V (this operator was taken there to be normal).

Now we will show that somewhat similar situation holds always, especially when the spectrum of the operator V does not intersect the unit circle.

Observation 1. $\boxed{Y_0 = Y_0 R_0 = R_0 Y_0 = R_0^{1/2} Y_0 R_0^{1/2}}$ *In addition* R_0 *acts on* $\overline{Ran\, Y_0}$*, hence on* $\overline{l_2(V)}$*, as identity operator*

Proof. Recall $Y_t = tQ_t^{-1}$, $Y_0 = s - \lim_{t\to+0} Y_t$, $R_t = (I + Q_t)^{-1}$, $R_0 \in w - lim\ pt(R_t)$, all these operators are selfadjoint. What is more, the straightforward calculation shows that

$$Y_t(1 - R_t) = \frac{t}{I + Q_t} = (1 - R_t)Y_t\,.$$

The rest is obvious.

□

Theorem 1. *Suppose there is an* V*-invariant subspace,* L *say, such that* $|spectrum(V|L)| < 1$*; let* P *denote orthoprojector onto* L *,*

Then

a)

$$L \subset Ran\, Y_0^{1/2} \subset Ker(I - R_0)$$

b)

$$\|(I - R_t)P\| \to 0 \quad (t \to +0)$$

Proof.

a) Note $L \subset l_2(V)$ and apply theorem 2.1.

b)

Return to the Observation 2.3:

$$0 \le Q_t \le (1 - z)[V^*V + zV^{*2}V^2 + \cdots + z^{n-1}V^{*n}V^n] + z^n V^{*n} Q_t V^n$$

Let V_P denote $PV|L$. Since $VP = PVP$, we can deduce that

$$0 \le PQ_tP \le (1 - z)[V_P^*V_P + zV_P^{*2}V_P^2 + \cdots + z^{n-1}V_P^{*n}V_P^n] + z^n V_P^{*n} PQ_tPV_P^n$$

Next we adopt the spectrum argument. We make it in the same manner that standard practice suggests: there are some real positive ϵ, M such that

$$r(V_P) + \epsilon < 1\,, \quad \|V_P^n\| \le M(r(V) + \epsilon)^n \qquad (n = 1, 2, \ldots)\,,$$

Hence

$$\|z^n V_P^{*n} PQ_tPV_P^n\| \to 0 \quad (n \to \infty)\,.$$

and it is routine matter to verify that

$$\|PQ_tP\| \le \frac{(1 - z)}{1 - z(r(V_P) + \epsilon)^2} M^2 (r(V_P) + \epsilon)^2\,.$$

Note that M and ϵ does not depend on z . So, we obtain $\|PQ_tP\| \to 0$ for $t \to +0$.

(One can show moreover : the serie

$$(1-z)[V_P^*V_P + zV_P^{*2}V_P^2 + \cdots + z^{n-1}V_P^{*n}V_P^n \cdots]$$

is norm-convergent.)

Now note that

$$PR_tP \equiv P(I+Q_t)^{-1}P \geq P(I+PQ_tP)^{-1}P \overset{\|\ \|}{\to} 0$$

$$P(I-R_t)P \equiv PQ_t(I+Q_t)^{-1}P \leq PQ_tP(I+PQ_tP)^{-1} \overset{\|\ \|}{\to} 0$$

Recall

$$0 \leq I - R_t \leq I$$

Hence

$$0 \leq P(I-R_0)^2P = P(I-R_0)^{1/2}(I-R_0)(I-R_0)^{1/2}P \leq P(I-R_0)P$$

So, we can now establish that

$$\|(I-R_t)P\|^2 = \|P(I-R_t)(I-R_t)P\| \leq \|P(I-R_t)P\| \to 0\,,$$

It was to be proved.

□

Corollary. *Suppose that the spectrum of the operator V does not intersect the unit circle; let P denote orthoprojector onto spectral subspace L corresponded to the set spectrum$(V) \cap \{z \in \mathbf{C} \,|\, |z| < 1\}$.*

Then

$$R_0 = P\,.$$

Proof. By Theorem 1

$$(I-R_0)P = 0\,.$$

Hence

$$P = R_0P\,.$$

Next note that the equation

$$Q = V^*\frac{Q+t}{I+tQ}V, \quad Q \geq 0, \quad 0 < t \leq 1 \tag{*}$$

is equivalent to the equation

$$Q^{-1} = V^{-1}\frac{Q^{-1}+t}{I+tQ^{-1}}V^{*-1}, \quad Q \geq 0, \quad 0 < t \leq 1\,. \tag{**}$$

(for details see [Ch1,2])

For a moment introduce for the (unique) solution of (*) a longer denotation:

$$Q_t(V).$$

It is straightforward to deduce now that

$$Q_t(V^{*-1}) = Q_t(V)^{-1}\,, \quad R_t(V) = I - R_t(V^{*-1})\,, \quad Y_t(V) = X_t(V^{*-1}) \ldots \text{etc.}$$

Last recall the Standard Spectrum Theorems (see e.g. [RS]) and apply Theorem 1 to the operator V^{*-1}. Then obtain

$$R_0(I-P) = 0\,.$$

Hence

$$R_0 = R_0P\,.$$

To complete the proof let compare the second displayed formula with the last one in the current period.

□

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