Fluctuation bounds for subharmonic functions

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Jan. 14 2000, (revised)

Abstract

We obtain bounds for the angular fluctuations of a subharmonic function in terms of the distribution of the angular mean.

As an application, we establish positive Lyapunov exponents for matrix cocycles $A_{E,\lambda}(x) = \begin{pmatrix} E-\lambda\, g(x) & -1 \\ 1 & 0 \end{pmatrix}$ over a measure m preserving dynamical system $T : X \mapsto X$, where $g \in L^\infty(X)$ satisfies $C = \int_X \log|g(x)| dm(x) > -\infty$. For every $\delta > 0$, the set of parameters $\lambda > 0$ for which the cocycle $A_{E,\lambda}$ has Lyapunov exponent $\mu(A_{E,\lambda}) \geq \log(\lambda) + C - \delta$ approaches full Lebesgue density at $\lambda = \infty$.

1 Introduction

In this article we prove a theorem on subharmonic functions in the complex plane which are potentials $f(z) = \int \log|z-w| d\mu(w)$ determined by a Borel probability measure μ. We look at the problem to estimate the fluctuations $B_f(r) = \max_{|z|=r} f(z) - \min_{|z|=r} f(z)$ in terms of the measure μ. The main theorem tells that $B_f(r)$ is related to the regularity of the map $r \mapsto \int_{|z|<r} d\mu(z)$. The proof could be adapted to prove generalizations for subharmonic functions in higher dimensions.

Let us first state a pure mathematical consequence of the main theorem: while subharmonic functions are upper-semicontinuous and therefore $\{w|f(w) < f(z)+\epsilon\}$ is open for every $\epsilon > 0$, they are not continuous in general. How discontinuous can they become? A classical result says that if $f(z) > -\infty$, then for almost all θ, one has $\lim_{r\to 0} f(z + r\exp(i\theta)) \to f(z)$ (see [12] Corollary 5.4.4). This implies that $Y_{z,\epsilon} = \{w|f(w) > f(z) - \epsilon\}$ has full Lebesgue density at z. An other result following an inequality of Milloux-Schmidt [7] states that for every θ, there exist a sequence $r_n \to 0$ such that $\lim_{n\to\infty} f(z + r_n \exp(i\theta)) = f(z)$. In other words, $Y_{z,\epsilon} \cap \{\arg(z-w) = \theta\}$ contains infinitely many points. We will see that $Y_{z,\epsilon} \cap \{\arg(z-w) = \theta\}$ has full one-dimensional Lebesgue density at z. This improves the Milloux-Schmidt result and implies in a different way again that $Y_{z,\epsilon}$ has full Lebesgue density at z.

The motivation to look at the fluctuation question are situations in dynamical systems theory and spectral theory, where subharmonic functions are relevant. Examples:

1) **Ergodic theory: Entropy of smooth diffeomorphisms**.
For a smooth diffeomorphism T on a two-dimensional Riemannian manifold M which preserves a volume measure ρ, the Kolmogorov-Sinai entropy is by Pesin's formula equal to the Lyapunov

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exponent $\lim_{n\to\infty} \int_M n^{-1} \log ||dT^n(x)||\, d\rho(x)$ (see i.e. [9]). The Jacobian $(x,n) \mapsto dT^n(x)$ is called a cocycle. If a complex parameter z is introduced into the problem such that $z \mapsto [A(x,z)]_{ij}$ is analytic, T stays fixed and $A(x,z_0) = dT(x)$, then $z \mapsto f(z) = \lim_{n\to\infty} \int_M n^{-1} \log ||dT^n(x,z)||\, d\rho(x)$ is subharmonic as an infimum of a monotonically decreasing sequence of subharmonic functions [8]. One often knows $f(z)$ at some point i.e. $z = 0$ and has therefore with the submean inequality a lower bound on $\int_{|z|=r} f(r\exp(i\theta))\, d\theta$. Because good upper bounds on $f(r\exp(i\theta))$ are usually known, estimating the entropy $f(z_0)$ from below leads to the problem of estimating the fluctuations.

2) **Complex dynamics: Green function of fractals**.
In computer generated pictures of Julia sets J of a polynomial map (see [2]) in the complex plane, one usually plots the potential $f(z) = \int_{\mathbf{C}} \log|z-w|\, d\mu$, where μ is the equilibrium measure supported on the usually fractal set J. This is a subharmonic function. The fluctuations of this function near the boundary of J gives information about the nature of the Julia set J.

3) **Spectral theory: the existence of bound states**.
A bounded selfadjoint Schrödinger operator L on a Hilbert space H defines a quantum mechanical unitary evolution $\psi \mapsto \exp(itL)\psi$. The spectral measure μ_ψ is a Borel measure in the complex plane with Fourier transform $\hat{\mu}(t) = (\psi, \exp(itL)\psi)$. The square $|\hat{\mu}(t)|^2$ is a quantum mechanical probability density. By a theorem of Wiener [10], if $\lim_{n\to\infty} f_n > 0$, where $f_n = n^{-1} \sum_{k=1}^n |\hat{\mu}(k)|^2$, then L has some eigenvalue. If $z \mapsto L(z)$ is a complex parameterization of the operator $L = L(z_0)$, the Cesaro averages $z \mapsto f_n(z)$ are subharmonic function. Consider for example the perturbation problem of an operator $L(z)\psi_n = z(\Delta\psi)_n + V_n\psi_n$ on $l^2(\mathbf{Z}^d)$, where Δ is a discrete Laplacian. We know that $\lim_{n\to\infty} f_n(0) > 0$ because $L(0)$ is a diagonal operator. We would like to understand $f_n(\epsilon)$ for small ϵ because $\lim_{n\to\infty} f_n(\epsilon) > 0$ would imply that the operator $\Delta + (1/\epsilon)V$ had bound states. Estimating the fluctuations of f_n could shed light on that perturbation problem.

4) **Ergodic operators: the absolutely continuous spectrum**.
The density of states dk of an ergodic one-dimensional Jacobi operator L is a Borel measure in the complex plane which is supported by the almost sure spectrum of L. By the Thouless formula, the subharmonic potential $f(z) = \int_{\mathbf{C}} \log|z-w|\, dk(w)$ is equal to the Lyapunov exponent of the transfer cocycle of the complex parameterized operator $L(z) = L - z$, where z is the energy. Ishii-Pastur-Kotani theory (see [4, 3, 11]) relates the Lyapunov exponent with the absolutely continuous spectrum of L. For example, if the $f(z) > 0$ on some interval I in the real line, then there is no absolutely continuous spectrum on I with probability one.

While it has to be seen, to what extent the theory of subharmonic functions will lead to progress in these questions, we give here the following applications of the fluctuation theorem:

If g is a bounded measurable function on a probability space (X,m) such that $\int \log|g(x)|d\, m(x) < \infty$ and if T is a measure preserving transformation on (X,m), then the Lyapunov exponent $\mu(A)$ of

$$A(x) = \begin{pmatrix} E - \lambda g(x) & -1 \\ 1 & 0 \end{pmatrix}$$

is positive for 'most' large λ. This result must be compared with other results in that area: for realanalytic g [8], [13], for smooth g see [15, 16], nondeterministic g see [5]. In our result, no assumption on T is made and essentially no regularity for g assumed. On the other hand, we don't know whether the Lyapunov exponent is positive for all large enough λ, only for most large λ. For a similar result but with E as a parameter see [14].

An other application is a criterium for the existence of singular continuous spectrum for one-dimensional Jacobi matrices which is based on the Ishii-Pastur-Kotani theorem (see [4]): if the spectrum has positive Lebesgue density at some point E, where the Lyapunov exponent is positive, then there is some singular spectrum.

2 The homogenization theorem

Let $z \mapsto f(z)$ be a subharmonic function in the complex plane $\mathbf{C}$. By the Riesz decomposition theorem it is the sum of a potential $\int_{\mathbf{C}} \log|z-w| \, dk_f(w)$ and a harmonic function h_f. The maximal function $M_f(z) = \sup_{|w|=|z|} f(w)$ is a radial subharmonic function by Hadamard's three circle theorem. Also the mean function $C_f(z) = (2\pi)^{-1} \int_0^{2\pi} f(e^{i\theta} z) \, d\theta$ is a radial subharmonic function. Define $m_f(z) = \inf_{|w|=|z|} f(w)$. Define the measure $\rho_f(r) = (2\pi)^{-1} \int_0^{2\pi} dk_f(re^{i\theta}) \, d\theta$ on $\mathbf{R}^+$. By the Poisson-Jensen's formula (for a short proof see [6]), $C_f(r) = f(0) + \int_0^r \log(r/t) \, d\rho_f(t)$. (We use a notation, where $d\rho_f(t) = \rho(t) \, dt$ if ρ is absolutely continuous with respect to Lebesgue measure). While estimates for the **upper angular fluctuation** $B_f(z) := M_f(z) - C_f(z) \in \mathbf{R}^+$ can be achieved usually quite well due to its boundedness, finding bounds for the **lower angular fluctuation** $A_f(z) := C_f(z) - m_f(z) \in \mathbf{R}^+ \cup \{\infty\}$ can be difficult. In general, there are no finite upper bounds on $A_f(z)$ as for example when the Riesz measure $dk = (2\pi)^{-1} \Delta f$ of f has atoms. The Homogenization Theorem below gives an upper bound of A_f in terms of the radial behavior of the mean function C_f.
Define $\mathcal{K}_l(\rho_f)(r) = 2^{-l} \sum_{k=l}^{\infty} \int_0^{\infty} \exp(-|\log(r/t)|2^k) \, d\rho_f(t)$, where we understand $\mathcal{K}_l(\rho_f) = \infty$ at points, where the sum does not have a finite limit e.g. when the measure ρ_f has an atom at r. Write $\mathcal{K} = \mathcal{K}_0$. We say f is a **potential** if the harmonic part of f in the Riesz decomposition theorem is a constant.

Theorem 2.1 (Homogenization theorem) *If f is a potential, then $A_f(z) \leq \mathcal{K}_0(\rho_f)(|z|)$. If additionally $f(z) = f(-z)$ for all $z \in \mathbf{C}$, then $A_f(z) \leq \mathcal{K}_1(\rho_f)(|z|)$.*

Remark. For a general subharmonic function, we additionally could estimate the fluctuations of the harmonic function $h(z)$ on $|z| = r$ using Harnack's inequality.

3 Homogenization

Let $\xi_k = \exp(2\pi i/2^k)$ denote the 2^k'th primitive root of unity.

Definition. For a subharmonic function f, define **the k'th homogenized subharmonic function** $f^{(k)}(z) = \sum_{j=1}^{2^k} f(\xi_k^j z)$.

Lemma 3.1 (Homogenization) *For a subharmonic function f one has $A_f(z) \leq \sum_{k=0}^{\infty} B_{f^{(k)}}(z)$. If $f(z) = f(\xi_l z)$ for all $z \in \mathbf{C}$, then $A_f(z) \leq \frac{1}{2^l} \sum_{k=l}^{\infty} B_{f^{(k)}}(z)$.*

Proof. Define $\xi_{k,m} = \xi_k \xi_{k-1} \cdots \xi_m$ for $m \leq k$. We apply recursively the identity $f^{(k)}(z) = f^{(k-1)}(z) + f^{(k-1)}(\xi_k z)$, to get

$$\begin{aligned} f^{(k)}(z) &= f^{(k-1)}(z) + f^{(k-1)}(\xi_k z) \\ &= f^{(k-1)}(z) + f^{(k-2)}(\xi_k z) + f^{(k-2)}(\xi_{k,k-1} z) \\ &= f^{(k-1)}(z) + f^{(k-2)}(\xi_k z) + f^{(k-3)}(\xi_{k,k-1} z) + f^{(k-3)}(\xi_{k,k-2} z) \end{aligned}$$

$$\begin{aligned} &\cdots \\ &= \quad f^{(k-1)}(z) + f^{(k-2)}(\xi_k z) + \cdots + f^{(1)}(\xi_{k,3} z) + f^{(0)}(\xi_{k,2} z) + f^{(0)}(\xi_{k,1} z) \end{aligned}$$

so that

$$f^{(0)}(\xi_{k,1} z) = f^{(k)}(z) - \sum_{j=0}^{k-1} f^{(j)}(\xi_{k,j+1} z) \; .$$

Subtracting the angular means on each side gives

$$\begin{aligned} f(\xi_{k,1} z) - C_f(z) &= f^{(k)}(z) - 2^k C_f(z) - \sum_{j=0}^{k-1} (f^{(j)}(\xi_{k,j+1} z) - 2^j C_f(z)) \\ &\geq (f^{(k)}(z) - 2^k C_f(z)) - \sum_{j=0}^{k-1} (M_{f^{(j)}}(z) - C_{f^{(j)}}(z)) \; . \end{aligned}$$

We apply this to the subharmonic functions $f_n(z) = 2^{n-1} \int_{-2^{-n}}^{2^{-n}} f(ze^{i2\pi x})\, dx$ for which $M_{f_n^{(j)}}(z) \leq M_{f^{(j)}}(z), C_{f_n}^{(j)} = C_f^{(j)}$. and so $B_{f_n^{(j)}} \leq B_{f^{(j)}}$. This gives

$$f_n(\xi_{k,1} z) - C_f(z) \geq (f_n^{(k)}(z) - 2^k C_f(z)) - \sum_{j=0}^{k-1} B_{f^{(j)}}(z) \; .$$

Taking the limit $k \to \infty$ gives, using $f_n^{(k)}(z) - 2^k C_f(z) = 0$ for $k \geq n$ and $\xi_{k,1} \to 1$ for $k \to \infty$

$$(f_n(z) - C_{f_n}(z)) \geq - \sum_{j=0}^{\infty} B_{f^{(j)}}(z) \; .$$

Since the right hand side is n-independent, we obtain

$$\inf_n (f_n(z) - C_f(z)) \geq - \sum_{j=0}^{\infty} B_{f^{(j)}}(z) \; .$$

Because subharmonic functions are upper semicontinuous, we have for given $\alpha > 0$ and $z \in \mathbf{C}$ that $f_n(z) < f(z) + \alpha$ for large n and therefore $\inf_n f_n(z) \leq f(z)$ and obtain

$$(f(z) - C_f(z)) \geq - \sum_{j=0}^{\infty} B_{f^{(j)}}(z) \; .$$

which is equivalent to the claim.

The case with symmetry $f(z) = f(\xi_l z)$ goes similar but starting with

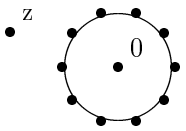
$$2^l f(\xi_{k,1} z) = f^{(k)}(z) - \sum_{j=l}^{k-1} f^{(j)}(\xi_{k,j+1} z) \; .$$

□

Remark. The case with more symmetry applies when $f(z) = f(z^{2^l})$ for all $z \in \mathbf{C}$. The case $f(z) = f(-z)$ is especially relevant in applications.

4 Proof of the homogenization theorem

We estimate first the upper fluctuation bounds $B_{f^{(k)}}(z)$, when f is the potential $f(z) = \log|z-a|$. In this case, $f^{(k)}(z)$ has a Riesz measure representing a circular equidistributed chain of charges with total charge 2^k.



The crucial fact is that for m charges located on the m'th roots of unity of the unit circle, the upper fluctuation $B_f(z)$ of the potential $f(z) = \log|z^m - 1|$ stays bounded above for $m \to \infty$ and decays exponentially $B_f(z) \leq \exp(-|\log(|z|)|m)$ for $|z| \neq 1$.

Lemma 4.1 (Potential of a circular chain of charges) *If $f(z) = \log|z-a|$ with $|a| = r$, there are sharp estimates*

$$B_{f^{(k)}}(z) \leq \begin{cases} \log|1 + (\frac{|z|}{r})^{2^k}| \,, & \text{for } |z| \leq r \,, \\ \log|1 + (\frac{r}{|z|})^{2^k}| \,, & \text{for } |z| \geq r \end{cases} .$$

Proof. Factoristion gives for every $m \in \mathbf{N}$

$$(z^m - 1) = \prod_{j=1}^{m} (z - \xi^j) \,,$$

where ξ is the primitive complex m-th root of unity.

Therefore, $f^{(k)}(z) = \log|\prod_{j=1}^{2^k}(z-\xi^j)| = \log|z^{2^k} - 1|$ for $f(z) = \log|z-1|$.

Assume now that $f(z) = \log|z-a|$ with $|a| = r$.

For $|z| < r = |a|$ and $f^{(k)}(z) = 2^k \log(r) + \log|(z/r)^{2^k} - 1|$, one gets

$$B_{f^{(k)}}(z) \leq \log|1 + (\frac{|z|}{r})^{2^k}| \,.$$

For $|z| > r = |a|$ and with $f^{(k)}(z) = 2^k \log|z| + \log|1 - (r/z)^{2^k}|$

$$B_{f^{(k)}}(z) \leq \log|1 + (\frac{r}{|z|})^{2^k}| \,.$$

□

The proof of Theorem (2.1) is obtained along the following steps:

(i) If $f(z) = \log|z-a|$ then $B_{f^{(k)}}(z) \leq \exp(-|\log(r/|z|)|2^k)$. Proof: Because $\log(1+x) \leq x$ for all $x \geq 0$, we obtain $\log|1 + (|z|/r)^{2^k}| \leq (|z|/r)^{2^k} = \exp(-|\log(r/|z|)|2^k)$ for $|z| > r$ and $\log|1 + r/|z|)^{2^k}| \leq (r/|z|)^{2^k} = \exp(-|\log(r/|z|)|2^k)$ for $|z| < r$. Apply Lemma (4.1).

(ii) "Averaging subharmonic functions does not increase fluctuations". If $\sum_i \alpha_i = 1, \alpha_i > 0$ and $f = \sum_i \alpha_i f_i$, where f_i are subharmonic functions which are potentials, then $M_f(z) \leq \sum_i \alpha_i M_{f_i}(z)$

and $C_f(z) = \sum_i \alpha_i C_{f_i}(z)$ so that $B_f(z) \le \sum_i \alpha_i B_{f_i}(z)$. More generally, if $x \mapsto f_x(z)$ is a random variable over a probability space $(X, \mathcal{A}, \mu)$ for all z and $z \mapsto f_x(z)$ is subharmonic, then $f(z) = \int_X f_x(z)\, d\mu(x)$ is subharmonic and $B_f(z) \le \int_X B_{f_x}(z)\, d\mu(x)$.

(iii) Especially, if $f(z) = \int_{\mathbf{C}} f_w\, dk(w)$ where dk is a finite, positive Borel measure in $\mathbf{C}$ and f_w are subharmonic potentials for which B_{f_w} is bounded above, then f is a subharmonic potential and $B_f(z) \le \int_{\mathbf{C}} B_{f_w}\, dk(w)$.

(iv) If $\mathrm{supp}(dk) \subset \{|z| = r\}$, then $B_{f^{(k)}}(z) \le \exp(-|\log(r/|z|)|2^k)$.
Proof. From (i), the result follows for measures on $\{|z| = r\}$ which are finite point measures. The general case follows from (iii) by integration over the angles.

(v) For a general finite Borel measure ρ on $\mathbf{R}^+$, we have

$$B_{f^{(k)}}(z) \le \int_0^\infty \exp(-|\log(\frac{t}{|z|})|2^k)\, d\rho_f(t)\ .$$

Proof. By (ii) and (iv) by integration over the radius.

(vi) The claim of the theorem: from Lemma (3.1) and (v)

$$A_f(z) \le \sum_{k=0}^\infty B_{f^{(k)}}(z) \le \sum_{k=0}^\infty \int_0^\infty \exp(-|\log(\frac{t}{|z|})|2^k)\, d\rho(t) = \mathcal{K}(\rho)(z)$$

allowing the right hand side to be $+\infty$.

5 The homogenization transformation

The **homogenization transformations**

$$\mathcal{K}_l : \rho \mapsto \frac{1}{2^l} \sum_{k=l}^\infty \int_{\mathbf{R}^+} \exp(-|\log(\frac{r}{t})|2^k)\, d\rho(t)$$

are defined on the linear space of all finite Borel measures ρ on $\mathbf{R}^+ = [0, \infty)$. The image of $\mathcal{K}_l(\rho)$ is a $\mathbf{R}^+ \cup \{+\infty\}$-valued function on $\mathbf{R}^+$.

Proposition 5.1 ($\mathcal{K}_1$ for $\rho \in L^\infty(\mathbf{R}^+)$.) *If $\rho \in L^\infty(\mathbf{R}^+)$ then $\mathcal{K}_1(\rho)(z) \le 2||\rho||_\infty |z|$. Especially, $\mathcal{K}_1(\rho) \in L^\infty_{loc}(\mathbf{R}^+ \setminus \{0\})$.*

Proof. Using $\int_0^r (t/r)^m\, dt = r/(1+m)$ and $\int_r^\infty (r/t)^m\, dt = r/(m-1)$, we estimate

$$\begin{aligned} 2\mathcal{K}_1(\rho)(r) &= \sum_{k=1}^\infty \int_0^r (\frac{t}{r})^{2^k} \rho(t)dt + \sum_{k=1}^\infty \int_r^\infty (\frac{r}{t})^{2^k} \rho(t)dt \\ &\le ||\rho||_\infty (\sum_{k=1}^\infty \int_0^r (\frac{t}{r})^{2^k} dt + \sum_{k=1}^\infty \int_r^\infty (\frac{r}{t})^{2^k} dt) \\ &= 2||\rho||_\infty r \sum_{k=1}^\infty \frac{2^k}{2^{2k}-1} \le 2||\rho||_\infty r \sum_{k=1}^\infty \frac{1}{(2^k - 2^{-k})} \le 4||\rho||_\infty r\ . \end{aligned}$$

□

The next proposition can be applied if C_f is Lipschitz on some interval $[r/a, ra]$ in which case ρ_f is bounded on that interval.

Proposition 5.2 ($\mathcal{K}$ for $\rho \in L^\infty_{loc}(\mathbf{R}^+)$) *Assume $\rho(s) \leq M$ for almost all $s \in [r/a, ar]$, with $a > 1$ and let $c = \rho(\mathbf{C})$ be the total mass of ρ. Then $\mathcal{K}(\rho)(z) \leq 2Mr(1-1/a)+(c/2)/(1-1/a)$. Especially, $\mathcal{K}(\rho) \in L^\infty_{loc}(\mathbf{R}^+)$. Taking $a = 2$, we get $\mathcal{K}_1(\rho)(r) \leq Mr + c$ for $r \in [r/2, 2r]$.*

Proof. We get from $\int_{r/a}^r (t/r)^m \, dt = (1-1/a^{m+1})r/(1+m) \leq (1-1/a)r/(1+m)$, $\int_r^{r+a} (r/t)^m \, dt = (1-1/a^{m-1})r/(m-1) \leq (1-1/a)r/(m-1)$ the first term. The second term comes by assuming the worst case, where we have a Dirac measure of maximal mass c at r/a. □

Proposition 5.3 ($\mathcal{K}_1$ for $\rho \in L^p(\mathbf{R}^+)$) *Given $1 < p < \infty$ define q by $1/p + 1/q = 1$. For $\rho \in L^p(\mathbf{R}^+)$,*

$$2\mathcal{K}_1(\rho)(z) \leq ||\rho||_p |z|^{1/q} (2/q)^{1/q} 1/(1-2^{-1/q}) \; .$$

Especially, $\mathcal{K}_1(\rho) \in L^q_{loc}(\mathbf{R}^+ \setminus \{0\})$.

Proof. Write $r = |z|$. With Hölder's inequality one has

$$\int_0^r (\frac{t}{r})^m \rho(t) dt \leq ||(\frac{t}{r})^m||_{L^q(0,r)} ||\rho||_p = (\frac{r}{qm+1})^{1/q} ||\rho||_p$$

and

$$\int_r^\infty (\frac{r}{t})^m \rho(t) dt \leq ||(\frac{r}{t})^m||_{L^q(r,\infty)} ||\rho||_p = (\frac{r}{qm-1})^{1/q} ||\rho||_p \; ,$$

so that

$$\int_0^r (\frac{t}{r})^m \rho(t) dt + \int_r^\infty (\frac{r}{t})^m \rho(t) dt \leq r^{1/q} (\frac{2qm}{q^2m^2-1})^{1/q} ||\rho||_p \leq r^{1/q} (2/qm)^{1/q} ||\rho||_p \; .$$

Summing up the right hand side for $m = 2^k$ gives

$$2\mathcal{K}_1(\rho)(z) \leq ||\rho||_p |z|^{1/q} (2/q)^{1/q} \sum_{k=1}^\infty (2^{1/q})^{-k} = ||\rho||_p |z|^{1/q} (2/q)^{1/q} 1/(1-2^{-1/q}) \; .$$

□

Examples. For $p = 2$, we get $\mathcal{K}_1(\rho)(z) \leq (2+\sqrt{2})||\rho||_2 \sqrt{|z|}$. The limiting case $p \to \infty$ becomes the L^∞ case treated above. For $p = 1$, the estimate breaks down.

Definition. Given a subharmonic function f, the potential $f^*(r) = f_{\rho_f}(r) = \int_0^\infty \log|r-s| \, d\rho_f(s)$ is called the **$*$-function** of f. Define $C_\rho(r) = \frac{1}{2\pi} \int_0^{2\pi} f^*(r\exp(i\theta)) \, d\theta$. The function f_ρ is obtained from f by sweeping all masses to a positive (or negative) real axes. The $*$ function appears in results on growth estimates for subharmonic functions.

Proposition 5.4 (Bounds by potential of $*$-function)

$$C_\rho(r) - f_\rho(r) \leq \mathcal{K}(\rho)(r) \leq 2(C_\rho(r) - f_\rho(r)) \; .$$

If $f(z) = f(\xi_l z)$ for all $z \in \mathbf{C}$, then $\mathcal{K}_l(\rho)(r) \leq 2^{-(l-1)}(C_\rho(r) - f_\rho(r))$.

Proof. We can assume that ρ has no atom at r because in that case, all three terms are $+\infty$.

(i)

$$\begin{aligned}\int_0^r \log|r-s|\ d\rho(s) &= \int_0^r \log|1-\frac{s}{r}|\ d\rho(s) + \int_0^r \log|r|\ d\rho(s)\\ \int_r^\infty \log|r-s|\ d\rho(s) &= \int_r^\infty \log|1-\frac{r}{s}|\ d\rho(s) + \int_r^\infty \log|s|\ d\rho(s)\ .\end{aligned}$$

(ii)

$$\begin{aligned}\int_0^r \log|1-\frac{s}{r}|\ d\rho(s) &= -\sum_{k=1}^\infty \int_0^r (\frac{s}{r})^k/k\ d\rho(s)\\ \int_r^\infty \log|1-\frac{r}{s}|\ d\rho(s) &= -\sum_{k=1}^\infty \int_r^\infty (\frac{r}{s})^k/k\ d\rho(s)\ .\end{aligned}$$

(iii) For $0<a<1$,

$$\sum_{k=0}^\infty a^{2^k} \le 2\sum_{k=1}^\infty \frac{a^k}{k}\ .$$

Proof. Match each term on the left with a sum on the right hand side: $a \le 2a/1, a^2 \le 2a^2/2, a^4 \le 2(a^3/3+a^4/4), a^8 \le 2(a^5/5+a^6/6+a^7/7+a^8/8)$, etc.

(iv) From (iii), we get

$$\begin{aligned}\mathcal{K}(\rho)(r) &= \sum_{k=0}^\infty \int_{\mathbf{R}^+} \exp(-|\log(r/t)|2^k)\ d\rho(t)\\ &\le 2\sum_{k=0}^\infty \int_{\mathbf{R}^+} \exp(-|\log(r/t)|k)/k\ d\rho(t)\\ &= -2\left(\int_0^r \log|1-\frac{s}{r}|\ d\rho(s) + \int_r^\infty \log|1-\frac{r}{s}|\ d\rho(s)\right)\\ &= -2\left(\int_0^\infty \log|r-s|\ d\rho(s) - \int_0^r \log|r|\ d\rho(s) - \int_r^\infty \log|s|\ d\rho(s)\right)\end{aligned}$$

so that

$$\mathcal{K}(\rho)(r) \le 2\left(-f_\rho(r) + \int_0^r \log|r|\ d\rho(s) + \int_r^\infty \log|s|\ d\rho(s)\right)\ .$$

(v) Subtracting and adding $\int_0^r \log(s)\ d\rho(s)$ in the second and third term respectively gives, using $\rho(\{r\})=0$

$$\mathcal{K}(\rho)(r) \le 2\left(-f_\rho(r) + \int_0^r \log(\frac{r}{s})\ d\rho(s) + f_\rho(0)\right)\ .$$

(vi) With the Poisson-Jensen formula [1] $C_\rho(r) = \int_0^r \log(\frac{r}{s})\ d\rho(s) + f_\rho(0)$, the estimate in (v) can be rewritten as

$$\mathcal{K}(\rho)(r) \le 2(C_\rho(r) - f_\rho(r))\ .$$

(vii) The inequality $-(f_\rho(r) - C_\rho(r)) \le \mathcal{K}(\rho)(r)$ is a consequence of the Homogenization theorem applied to the case, when dk has support in the positive real line.

(viii) If $f(z) = f(\xi_l z)$, things are similar. □

Remarks.
1) The inequalities can be rewritten as $A_{f^*}(r) \leq \mathcal{K}(\rho_f)(r) \leq 2A_{f^*}(r)$. If $f(z) = f(-z)$, then $\mathcal{K}_1(\rho_f)(r) \leq A_{f^*}(r)$. While the lower bound $A_{f^*}(r) \leq \mathcal{K}(\rho_f)(r)$ is sharp as the example shows, when $f = f^*$, we don't expect the upper bound to be sharp. Is a general estimate $\mathcal{K}(\rho_f)(r) \leq aA_{f^*}(r)$ with $a < 2$ possible?
2) It follows that $\mathcal{K}(\rho)(r)$ is finite if ρ is absolutely continuous with respect to the $\alpha > 0$-dimensional Hausdorff measure in an open interval containing r.
3) The potential $f(x) = \int_{\mathbf{R}^+} \log|x-w| \, d\rho(w)$ is as a function on the real line equal to the Hilbert transform of the function $h(x) = \int_0^x d\rho(y)$. The Hilbert transform maps α-Hölder spaces or L^p spaces into itself. We see especially that if ρ is bounded then h is Lipschitz and f α-Hölder for all $\alpha \in (0,1)$. Because also C_f is Lipschitz if ρ is bounded, Proposition (5.4) implies that if ρ is bounded, then $\mathcal{K}(\rho)$ is sandwiched between two Hölder continuous maps.

Corollary 5.1 *There exists a real sequence $r_n \to 0$ such that $\limsup_{n\to\infty} \mathcal{K}(\rho)(r_n) = 0$.*

Proof. It is a consequence of the Milloux-Schmidt inequality [7] that for any subharmonic function f with $f(0) > -\infty$ one has $\limsup_{r\to 0} \inf_\theta f(r\exp(i\theta)) = f(0)$. Therefore $\limsup_{r\to 0} f_\rho(r) = f_\rho(0)$. Together with $\lim_{r\to 0} C_\rho(r) = C_\rho(0)$ which follows from the upper-semicontinuity and the submean inequality for subharmonic functions, we get $\limsup_{r\to 0} 2(f_\rho(r) - C_\rho(r)) = 0$. The claim follows from Proposition (5.4). □

6 Radially mollified subharmonic functions

The homogenization theorem is especially useful if $d\rho_g$ is bounded. It can be applied to cases, where ρ is smeared out radially in such a way that the mollified ρ comes from a radially mollified subharmonic function.

Definition. Let ϕ be smooth positive function on $\mathbf{R}^+$ of compact support $K_\phi \subset \mathbf{R}^+$ satisfying $\phi(x) = \phi(1/x)$. If f is a subharmonic function, define a **radially mollified function** as $f_\phi(z) = \int_{\mathbf{R}} f(\beta z) \, d\phi(\beta)$.

Lemma 6.1 *If f is a subharmonic potential, then $f_\phi(z)$ is a subharmonic potential, ρ_{f_ϕ} has the same smoothness as ϕ and $|\rho_{f_\phi}(r)| \leq ||\phi_\infty||\rho_f(K_\phi r)$.*

Proof. A positive average of subharmonic functions $f_\beta(z) = f(\beta z)$ is subharmonic.

Taking the Laplacian in the distributional sense on $f_\phi(z) = \int_{\mathbf{R}} f(\beta z) \; d\phi(\beta)$ shows $dk_{f_\phi} = \int_0^\infty dk(\beta z) \, d\phi(\beta)$ so that f_ϕ is a potential.

The function $\rho_{f_\phi}(r) = \int_0^\infty \rho(\beta r) \, d\phi(\beta) = \int_0^\infty \rho(\beta^{-1} r) \, d\phi(\beta) = \int_{-\infty}^\infty \tilde{\rho}(r'-\beta')\tilde{\phi}(\beta') \, d\beta' = (\tilde{\phi} \star \tilde{\rho})(r')$ is in logarithmic coordinates $r' = \log(r)$ a convolution with $\tilde{\phi}(r') = \phi(\exp(r'))$ and has therefore the same smoothness as ϕ.

The estimate $|\rho_{f_\phi}(r)| \leq ||\phi_\infty||\rho(K_\phi r)$ is obtained by assuming ρ to have an atom of mass $\rho(K_\phi r)$ at r in which case we have equality. □

Remark. While we could also smooth f using the standard smoothing theorem in the theory of subharmonic functions, the radial smoothing will be more convenient for the applications later on.

Notation. Denote by $|Y|$ the Lebesgue measure of a measurable subset Y of $\mathbf{R}$. For $Y \subset \mathbf{R}$ and $x \in \mathbf{R}$ call $\limsup_{\epsilon \to 0} |Y \cap [x-\epsilon, x+\epsilon]|/(2\epsilon)$ the **Lebesgue density** of Y at x. (It is equal to $M1_Y(x)$, where Mf denotes the Hardy-Littlewood maximal function of f).

We have already used the consequence of Milloux-Schmidt [7] that for any subharmonic function f $\limsup_{\lambda \to 0} \inf_\theta f(\lambda \exp(i\theta)) = f(0)$. The following Corollary is stronger and provides also a new proof to this fact:

Theorem 6.1 *Let f be an arbitrary subharmonic function and let $z \in \mathbf{C}$ given for which $f(z) > -\infty$. For every $\delta > 0$ and every line $z + \lambda e^{i\theta}$ through z, the Lebesgue density of $Y = \{\lambda \in \mathbf{R} \mid f(z + \lambda e^{i\theta}) > f(z) - \delta \}$ at $\lambda = 0$ is equal to 1.*

Proof. Because the claim is obviously true for harmonic functions, it is enough to prove the statement for potentials $f(z) = \int_{\mathbf{C}} \log|z-w| \, dk(w)$. After a translation, we can assume $z = 0$. If K is a compact subset of $\mathbf{C}$, then $dk(K) < \infty$, otherwise, f would be constant $-\infty$. We have $B_f(z) = M_f(z) - C_f(z) \to 0$ for $z \to 0$. Let I be an open ball around 0 such that $B_f(z) \leq \delta/3$ for $z \in I$ and therefore $M_f(z) - f(0) \leq \delta/3$ for $z \in I$. Define $g(z) = (f(z) + f(-z))/2$. We will show that the Lebesgue density of $Y = \{\lambda \mid g(\lambda e^{i\theta}) \geq g(0) - \delta/3\}$ is 1 near 0. This implies the claim because in $Y \cap I$, $g(\lambda e^{i\theta}) > g(0) - \delta/3 = f(0) - \delta/3$, implies together with $M_f(-\lambda e^{i\theta})) - f(0) \leq \delta/3$ that $f(\lambda e^{i\theta}) = 2g(\lambda e^{i\theta}) - f(-\lambda e^{i\theta}) \geq 2f(0) - 2\delta/3 - (f(0) - \delta/3) = f(0) - \delta$.

Let ϕ be a radial mollifier function with support in $[1/2, 2]$ and let g_ϕ be the corresponding radially mollified subharmonic potential. Applying Corollary (6.1) and the Homogenization theorem, we get

$$A_{g_\phi}(z) \leq \mathcal{K}_1(\rho_\phi)(r) \leq 2||\rho_g||\rho([r/2, 2r])r$$

and the right hand side is $\leq \delta/3$ for small enough r.

Because $\sup_{\beta \in [1/2,2]} B_g(\beta r) \to 0$ for $r \to 0$ and $A_g(\beta r) \leq \delta/3$ for a set of $\beta \in [1/2, 2]$ which reaches Lebesgue measure $3/2$ for $r \to 0$, the normalized Lebesgue measure of parameters $s \in [r/2, 2r]$ with $A_g(s) \leq \delta/3$ approaches 1 for $r \to 0$. □

Remark. It was already known that if $f(z) > -\infty$, then for almost all θ $\lim_{r \to 0} f(z + r\exp(i\theta)) \to f(z)$.

This result shows that for "most" points w near z, $f(w)$ is near $f(z)$. A subharmonic function f behaves in a probabilistic sense like a continuous function: if a sequence $z_n \to z$ is chosen at 'random' in the product probability space of a nested sequence of open intervals $I_n \subset \mathbf{R}$ around 0 (all equipped with the normalized Lebesgue measure and $\bigcap_n I_n = \{0\}$) , then with probability 1, a point $x = \{x_n\}$ in the product probability space satisfies $f(z + e^{i\theta} x_n) \to f(z)$. For example, if f is the at 0 discontinuous subharmonic function $f = \sum_{n=1}^{\infty} 2^{-n} \log|z - 2^{-n}|$ and x_n is an independent, identically distributed sequence of random variables with uniform distribution in $[-1, 1]$ then $f(x_n/n) \to f(0)$ with probability 1.

7 Lyapunov exponents of matrix cocycles

Let $(X, \mathcal{A}, m)$ be a probability space and let $T : X \mapsto X$ be a measure preserving invertible transformation. Given $g \in L^\infty(X)$ for which $\int_X \log|g(x)|\ dm(x)$ is finite. Given $E, \lambda \in \mathbf{R}$, the matrix-valued map $A_\lambda(x) = \begin{pmatrix} E - \lambda g(x) & -1 \\ 1 & 0 \end{pmatrix}$ defines a cocycle $(n, x) \mapsto A_\lambda^n(x) = A_\lambda(T^{n-1}x) \cdots A_\lambda(x)$. Denote by $\mu(A_\lambda) = \lim_{n\to\infty} n^{-1} \int_X \log||A_\lambda^n(x)||\ dm(x)$ the Lyapunov exponent of A_λ. Define $f(\epsilon) = \mu(B_\epsilon)$, where $\epsilon = 1/\lambda$ and $B_\epsilon(x) := \begin{pmatrix} \epsilon E + g(x) & -\epsilon \\ \epsilon & 0 \end{pmatrix}$. We have $\mu(A_\lambda) = f(\epsilon) - \log|\epsilon| = f(\epsilon) + \log|\lambda|$. The aim is to estimate the subharmonic function $\epsilon \mapsto f(\epsilon)$ from below, using $f(0) = \int_X \log|g(x)|\ dm(x) > -\infty$. The later will be a standing assumption in this section.

Lemma 7.1 *f is a subharmonic function which is a potential: there exists a constant C such that $f(\epsilon) = C + \int_{\mathbf{C}} \log|\epsilon - w|\ dk(w)$. The total mass of the Riesz measure dk is 1. If there exists a measure preserving involution S on X such that $g(x) = -g(S(x))$ for almost all $x \in X$, then $f(\epsilon) = f(-\epsilon)$.*

Proof. It is an observation of Herman that the Lyapunov exponent of an analytically parameterized cocycle A is subharmonic as the lim inf of subharmonic functions $f_n(\epsilon) = 2^{-n} \int_X \log||A_\epsilon^{2^n}(x)||\ dm(x)$ satisfying $f_n \leq f_m$ for $n \leq m$.

By the Riesz decomposition theorem, the Lyapunov exponent can be written as $f(\epsilon) = \int_{\mathbf{C}} \log|\epsilon - w|\ dk(w) + h(\epsilon)$, where $\epsilon \mapsto h(\epsilon)$ is harmonic.

With the constant $C = \mu(B_0)$, we have for $\epsilon \to \infty$ $f(\epsilon) = C + \log|\epsilon| + O(1)$. This shows that the order of f (defined as $\limsup_{r\to\infty} \log|M_f(r)|/\log|r|$) is equal to 1 implying that the harmonic function h in the Riesz decomposition theorem must be constant (see [7]).
If $g(x) = -g(S(x))$, the pointwise Lyapunov exponent $\mu(\lambda, x) = \lim_{n\to\infty} n^{-1} \log||A_\lambda^n(x)||$ (which exists according to Kingman's subadditive ergodic theorem m-almost everywhere), satisfies $\mu(-\lambda, S(x)) = \mu(\lambda, x)$ so that the average over X satisfies $\mu(-\lambda) = \mu(\lambda)$ implying $f(\epsilon) = f(-\epsilon)$. □

Theorem 7.1 *For a fixed dynamical system, there exists a sequence $\lambda_m \to \infty$ such that*

$$\limsup_{m\to\infty} \mu(A_{\lambda_m}) - \log|\lambda_m| - \int_X \log|g(x)|\ dx \geq 0\ .$$

Proof. This is consequence of Corollary (5.1) and the homogenization theorem. □

Example. For $g(x) = 2x + 2\cos(x)$ we obtain $\limsup_{m\to\infty} \mu(A_{\lambda_m}) - \log(|\lambda_m|/2) \geq 0$.

Theorem 7.2 *Let ϕ be a radial mollifier function. Then*

$$\int_0^\infty \mu(A_{\lambda\beta})\ d\phi(\beta) \geq \log|\lambda| + \int_X \log|g(x)|\ dx - 2||\phi||_\infty/\lambda\ .$$

Let ϕ be a radial mollifier function such that $\rho_\phi(s) \leq M_\phi$ for almost all $s \in [r/2, 2r]$. Then

$$\int_0^\infty \mu(A_{\lambda\beta})\ d\phi(\beta) \geq \log(|\lambda|) + \int_X \log|g(x)|\ dx - 2M_\phi/\lambda - 2c\ ,$$

where $c = \rho(\mathbf{C})$.

Proof. The statements follow from the Homogenization theorem, Corollary (6.1) and the estimates in Proposition (5.1) rsp. Proposition (5.2) which can be applied using Lemma (7.1). □

Example. For $g(x) = 2\cos(x)$, where $S(x) = x + \pi$ produces the symmetry $f(\epsilon) = f(-\epsilon)$, this gives $\int_{1/2}^{2} \mu(A_{\lambda\beta})\, d\phi(\beta) \geq \log(|\lambda|/2) - 2|\phi|_\infty/|\lambda|$.

The set of parameters for which the cocycle has positive Lyapunov exponent reaches full Lebesgue density at $\lambda \to \infty$. This follows from Theorem (6.1). We state this here in a more quantitative way:

Corollary 7.1 *There exists a constant C such that for all λ_0 the Lebesgue measure of parameters $\lambda \in [\lambda_0/2, 2\lambda_0]$ for which $\mu(A_\lambda) = 0$ is $\leq C(3\lambda_0/2)/\log|\lambda_0|$.*

Proof. For $\lambda \in [\lambda_0/2, 2\lambda_0]$ we have $\mu(A_\lambda) \leq c_1 + \log(|\lambda_0|) + C_1/|\lambda_0|$ and $\int_{1/2}^{2} \mu(A_{\lambda\cos})\, d\phi(\beta) \geq c_2 + \log(|\lambda_0|) - C_2/|\lambda_0|$. Therefore, for each λ_0 the normalized Lebesgue measure x of the subset $\{\lambda \in [\lambda_0/2, 2\lambda_0] \mid \mu(A_\lambda) = 0\ \}$ satisfies $(1-x) \geq (c_1 + \log(|\lambda_0|) - C_2/|\lambda_0|)/(c_2 + \log(|\lambda_0|) + C_1/|\lambda_0|)$ which implies that there exists a constant C such that $x \leq C/\log(\lambda_0)$ for some C. To get the Lebesgue measure, we have to multiply this by $|[\lambda_0/2, 2\lambda_0]| = 3\lambda/2$. □

Theorem (6.1) has an application in spectral theory:

Corollary 7.2 *Assume the transfer cocycle A_E of a discrete ergodic selfadjoint Jacobi matrix has a positive Lyapunov exponent at some energy $E \in \mathbf{R}$ for which the spectrum $\sigma(L)$ has positive Lebesgue density $\limsup_{\epsilon\to 0} |\sigma(L) \cap \{[E-\epsilon, E+\epsilon]\}|/(2\epsilon) > 0$. Then L has some spectrum which is not absolutely continuous.*

Proof. We apply Theorem (6.1) to the complex parameterization $E \mapsto A_E$. Let $S = \{E \in \mathbf{R} \mid \mu(A_E) > 0\}$. If $\mu(A_E) > 0$, then $\limsup_{\epsilon\to 0} |S \cap \{[E-\epsilon, E+\epsilon]\}|/(2\epsilon) = 1$. Therefore $S \cap \sigma(L)$ must have positive Lebesgue measure. Ishii-Pastur-Kotani theory tells that on the set $S \cap \sigma(L)$ the spectrum is not absolutely continuous. □

Acknowledgments. This research was supported by the the Swiss National Science Foundation. Thanks both to the department of Mathematics and the Institute for Fusion Studies at the University of Texas for hospitality.

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