REPRESENTATION OF THE SPECTRAL SHIFT FUNCTION AND SPECTRAL ASYMPTOTICS FOR TRAPPING PERTURBATIONS

VINCENT BRUNEAU AND VESSELIN PETKOV

ABSTRACT. We obtain in the semi-classical setup of "black box" long-range perturbations a representation for the derivative of spectral shift function $\xi(\lambda)$ related to two self-adjoint operators $L_j(h)$, $j = 1, 2$. We show that the derivative $\xi'(\lambda)$ is estimated by the norms of the cut-off resolvents of the operators $L_j(h)$. Finally, we establish a Weyl type formula for the spectral shift function $\xi(\lambda)$ generalizing the results of Robert [21] and Christiansen [5].

1. INTRODUCTION

The purpose of this paper is to obtain a semi-classical representation for the derivative of the spectral shift function (SSF) related to two self-adjoint operators $L_1 = L_1(h)$, $L_2 = L_2(h)$ on $\mathbb{R}^n$, $n \geq 2$, depending on $h \in]0, h_0]$, and satisfying long-range "black box" assumptions (2.1)- (2.10) given in Section 2. Let $f \in \mathcal{S}(\mathbb{R})$ and let $\chi \in C_0^\infty(\{x : |x| \leq \rho_1\})$ be equal to 1 for $|x| \leq 2\rho_0 < \rho_1$. The constants $\rho_0,\ \rho_1,\ 0 < R_0 < \rho_0$ depends on the behaviour of the Hamiltonian trajectories of L_1 and L_2 (see Section 2). Under weaker assumptions J. Sjöstrand [25] proved that the operators

$$\chi f(L_j),\ f(L_j)\chi,\ j = 1, 2,\ (1 - \chi)f(L_2)(1 - \chi) - (1 - \chi)f(L_1)(1 - \chi)$$

are trace class and we can define

$$\operatorname{tr}_{\rm bb}(f(L_2) - f(L_1)) = [\operatorname{tr}\chi f(L_j)\chi + \operatorname{tr}(1 - \chi)f(L_j)\chi + \operatorname{tr}\chi f(L_j)(1 - \chi)]_{j=1}^2$$
$$+\operatorname{tr}[(1 - \chi)f(L_j)(1 - \chi)]_{j=1}^2.$$

This expression is independent of the choice of χ and here and below we write $[a_j]_{j=1}^2 = a_2 - a_1$. We introduce the *spectral shift function* $\xi(\lambda) \in S'(\mathbb{R})$ as the distribution

$$\langle \xi, f' \rangle_{\mathcal{S}', \mathcal{S}} = -\operatorname{tr}_{\rm bb}(f(L_2) - f(L_1)).$$

In the following to simplify the notations we will often omit the subscript $j = 1, 2$ and simply write $L_.$ or L. Given a fixed interval $J =]\mu_0, \mu_1[\subset\subset \mathbb{R}^+$, we can find two self-adjoint operators $\tilde{L}_1$, $\tilde{L}_2$ on $L^2(\mathbb{R}^n)$ such that $\tilde{L}_.$ is a compactly supported perturbation of $L_.$ and J is non trapping and non-critical for $\tilde{L}_.$ (see [2]). The spectral projections $E_.'(\lambda)$ and $\tilde{E}_.'(\lambda)$ are given by the formula

$$E_.'(\lambda) = \frac{1}{2\pi i}(R_.(\lambda + i0) - R_.(\lambda - i0)),\ \tilde{E}_.'(\lambda) = \frac{1}{2\pi i}(\tilde{R}_.(\lambda + i0) - \tilde{R}_.(\lambda - i0)),\ \lambda \in J,$$

where the operators $R(\lambda \pm i0)$ for fixed $h > 0$ are determined as the limits of the resolvents

$$R_.(\lambda \pm i0) = \lim_{\epsilon \to 0,\, \epsilon > 0} R_.(\lambda \pm i\epsilon) = \lim_{\epsilon \to 0,\, \epsilon > 0} (L_. - \lambda \mp i\epsilon)^{-1}$$

in the spaces of bounded operators $\mathcal{L}(\mathcal{H}^{0,s}, \mathcal{H}^{0,-s})$, $s > 1/2$, (see [2] and the definitions of $\mathcal{H}^{0,s}$ in Section 2) and similarly the resolvents $\tilde{R}_.(\lambda \pm i0)$ are determined as bounded operators in $\mathcal{L}(L_s^2(\mathbb{R}^n), L_{-s}^2(\mathbb{R}^n))$, L_s^2 being the weight space $L^2(\mathbb{R}^n, \langle x \rangle^s dx)$. Given two real functions $f,\ g \in C_0^\infty(\mathbb{R}^n)$, we write $f \prec g$, if supp$f$ is contained in the interior of the region, where $g = 1$.

Under the assumptions (2.1)-(2.10) Sjöstrand obtained [25] a trace formula for $\operatorname{tr}_{\rm bb}(f(L_2) - f(L_1))$ related to the resonances of L_j, $j = 1, 2$, plus a remainder $\mathcal{O}_f(h^{-n^\sharp})$ depending on f. Following this way it seems difficult to obtain directly from the results in [25] a representation for the derivative $\xi'(\lambda)$. On the other hand, for long-range perturbations $P_j = P_j(h)$ acting in a dense set of $L^2(\mathbb{R}^n)$, Robert [21] established a representation formula for $\xi'(\lambda)$ related to P_j, $j = 1, 2$. The representation in [21] involves some remainder terms related to the constructions of long time parametrices for the propagators $U_j(t) = e^{ith^{-1}P_j}$ and some precise estimates of the difference $U_2(t) - U_1(t)$. This construction is very technical and in the case of "black-box" scattering the situation becomes more complicated.

Under suitable assumptions the derivative $\xi'(\lambda)$ coincides modulo some factor with the derivative of the *scattering phase*

$$\sigma(\lambda, h) = \frac{1}{2\pi i} \log \det S(\lambda, h), \ \lambda \in J,$$

where $S(\lambda, h)$ is the scattering operator related to P_j, $j = 1, 2$. The existence of the scattering operator $S(\lambda, h)$ and that of $\det S(\lambda, h)$ in the general setup of "black box" scattering is far from transparent. On the other hand, in the case of "black box" compactly supported perturbations the asymptotic behaviour of the scattering phase $\sigma(\lambda, h)$ is important for the analysis of the semi-classical Breit-Wigner approximation (see [17]).

In this paper we treat the problem of the representation of the derivative $\xi'(\lambda)$ in great generality covering the case of "black box" semi-classical scattering ([25], [26], [29]). Our arguments are not based neither on the existence of the scattering phase $\sigma(\lambda, h)$ nor on the existence of the scattering operator $S(\lambda, h)$. Under the general "black box" scattering assumptions it seems quite difficult to construct a parametrix for the propagators $U_j(t) = e^{ith^{-1}P_j}$ and for this reason we follow a stationary argument as in [25], [26] combined with the recent results in [2] concerning the semi-classical resolvent estimates for trapping perturbations. Moreover, we obtain a more precise information for the remainder terms in the formula for $\xi'(\lambda)$ and we prove a Weyl type asymptotic for $\xi(\lambda)$ generalizing the results of [21] and [5].

Our main result is the following.

Theorem 1. *Let $J =]\mu_0, \mu_1[\subset\subset \mathbb{R}^+$ be fixed and let L_j, $j = 1, 2$ satisfy the assumptions* (2.1) − (2.10). *Let $\rho_0 > R_0$ be as in Proposition 1 and let $\chi \in C_0^\infty(\{x : |x| \leq \rho_1\})$ be equal to 1 for $|x| \leq 2\rho_0$, $\rho_1 > 2\rho_0$. Then for any $\lambda \in J$, the operators $\chi E_{\cdot}'(\lambda)\chi$ and $\chi \tilde{E}_{\cdot}'(\lambda)\chi$ are trace class ones and we have*

$$\begin{aligned} \xi'(\lambda) = &\Big[\mathrm{tr}\Big(\chi E_j'(\lambda)\chi\Big)\Big]_{j=1}^2 + \mathrm{tr} T^+(\lambda + i0) - \mathrm{tr} T^-(\lambda - i0) \\ &+ \tilde{\xi}'(\lambda) - \Big[\mathrm{tr}\Big(\chi \tilde{E}_j'(\lambda)\chi\Big)\Big]_{j=1}^2 + \mathcal{O}(e^{-\frac{C}{h}}), \end{aligned} \tag{1.1}$$

where $C > 0$, $\tilde{\xi}(\lambda)$ is the spectral shift function related to $\tilde{L}_1$ and $\tilde{L}_2$, while $T^\pm(z)$ are trace class operators for $\pm \operatorname{Im} z > 0$ with

$$\|T^\pm(z)\|_{\mathrm{tr}} \leq \mathcal{O}(h^\infty) \Big(1 + \|\tilde{\chi} R_1(z)\tilde{\chi}\|_{\mathcal{H}\to\mathcal{H}} + \|\tilde{\chi} R_2(z)\tilde{\chi}\|_{\mathcal{H}\to\mathcal{H}}\Big), \quad \tilde{\chi} \succ \chi$$

uniformly with respect to $z \in \mathcal{B}_\pm = \{z \in \mathbb{C} : (\operatorname{Re} z, \pm \operatorname{Im} z) \in J \times]0, 1]\}$ and $h \in]0, h_0]$. Moreover, the application $\mathcal{B}_\pm \ni z \mapsto \mathrm{tr}(T^\pm(z))$ is holomorphic on $\mathcal{B}_\pm$.

Remark. Notice that the fact that J is a non-trapping and non-critical interval for $\tilde{L}_j$, $j = 1, 2$, implies the representation

$$\tilde{\xi}'(\lambda) - \Big[\mathrm{tr}\Big(\chi \tilde{E}_j'(\lambda)\chi\Big)\Big]_{j=1}^2 = h^{-n} \sum_{k=0}^N d_k(\lambda) h^k + \mathcal{O}_\chi(h^{-n+N+1}), \ \forall N \in \mathbb{N}, \tag{1.2}$$

where $d_k(\lambda)$ are C^∞ functions on J (see [21]).

Our representation yields an estimate for the derivative of $\xi(\lambda)$ by the norms of the cut-off resolvents.

Corollary 1. *Under the assumptions of Theorem 1 we have the estimate*

$$|\xi'(\lambda)| \leq C\Big(h^{-n} + \|\sum_{\pm} \tilde{\chi} R_1(\lambda \pm i0)\tilde{\chi}\|_{\mathcal{H}\to\mathcal{H}} + \|\sum_{\pm} \tilde{\chi} R_2(\lambda \pm i0))\tilde{\chi}\|_{\mathcal{H}\to\mathcal{H}}\Big),\ \lambda \in J,\ \ h \in]0, h_0].$$

The above estimate shows that the blow up of the derivative $\xi'(\lambda)$ is related to the estimates of the cut-off resolvents which depend on the distribution of the resonances (see [3], [29], [4], [28]). Denote by $\operatorname{Res} L_j$ the set of resonances of L_j, $j = 1, 2$ (see [25], [26] for a precise definition). The representation (1.1) combined with the result of Stefanov [28] imply the following.

Corollary 2. *Assume that L_j, $j = 1, 2$ satisfy the assumptions* (2.1) $-$ (2.10)*. Let $f \in C_0^\infty(J)$ and let $\chi(x)$ be as in Theorem 1. Suppose that for some $\epsilon > 0$, $C > 0$, $q \geq 1$ we have*

$$\operatorname{dist}\{\operatorname{Res} L_j(h),\ J_1\} \geq \epsilon \exp(-Ch^{-q}),\ \ j = 1, 2,\ h \in]0, h_0], \tag{1.3}$$

$J_1 \subset \mathbb{R}^+$ being an open interval containing $\overline{J}$. Then for $|t| \geq \delta > 0$ and $h \in]0, h_0]$ we have

$$\operatorname{tr}_{\mathrm{bb}}\Big(e^{-ith^{-1}L_2} f(L_2) - e^{-ith^{-1}L_1} f(L_1)\Big) = \Big[\operatorname{tr}\Big(\chi e^{-ith^{-1}L_j} f(L_j)\chi\Big)\Big]_{j=1}^{2} + \mathcal{O}(h^\infty).$$

Remark. Applying the recent result of Burq and Zworski [4] for $t > h^{-L}$, we can express the term

$$\Big[\operatorname{tr}\Big(\chi e^{-ith^{-1}L_j} f(L_j)\chi\Big)\Big]_{j=1}^{2}$$

by the resonances of L_j lying in a rectangle close to the real axis. On the other hand, the assumption (1.3) is a technical one and in many cases this assumption is satisfied (see [3]). As it was mentioned in [2], (1.3) implies the estimate

$$\|\chi R_j(\lambda \pm i\tau)\chi\|_{\mathcal{H}\to\mathcal{H}} \leq C\exp(Ch^{-p}), j = 1, 2,\ p \geq 1 \tag{1.4}$$

uniformly with respect to $\lambda \in J$, $\ \tau \in]0, 1]$ and $h \in]0, h_0]$.

The idea of the proof of Theorem 1 is to use an approximation of $\xi(\lambda)$ by a sequence of locally integrable functions. Let $\chi_0 \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 near $\overline{B(0, R_0)}$. For $\rho > 1$, introduce $\chi_\rho(x) = \chi_0(x/\rho)$ and define the temperate distribution ξ_ρ by the equality

$$\langle \xi_\rho, f'\rangle_{\mathcal{S}',\mathcal{S}} = -\operatorname{tr}_{\mathrm{bb}}\Big(\chi_\rho\Big(f(L_2) - f(L_1)\Big)\chi_\rho\Big) = -\Big[\operatorname{tr}\Big(\chi_\rho f(L_j)\chi_\rho\Big)\Big]_{j=1}^{2}. \tag{1.5}$$

The multiplication operator defined by χ_ρ converges strongly as $\rho \longrightarrow \infty$ to the identity in $L^2(\mathbb{R}^n)$ as well as in $\mathcal{H}_.$. Consequently, we have in the sense of distributions (see [12])

$$\lim_{\rho\to+\infty} \xi'_\rho = \xi'. \tag{1.6}$$

In Section 3 we show that

$$\xi'_\rho(\lambda) = \Big[\operatorname{tr}\Big(\chi_\rho E'_j(\lambda)\chi_\rho\Big)\Big]_{j=1}^{2}$$

and the main difficulty is to investigate the limit $\rho \longrightarrow \infty$. The operators $E'_j(\lambda)$ are not trace class so we decompose $\xi'_\rho(\lambda)$ in a sum of three terms and we examine the corresponding limits separately. The terms $\operatorname{tr}(T^\pm(z))$ are obtained as the limits

$$(\lambda - z_0)^m \lim_{\rho\to+\infty} \Big[\operatorname{tr}\Big(\chi_\rho(L_j - z_0)^{-m} R_j(z)[\tilde{L}_j, \chi]\tilde{R}_j(z)(1 - \chi_2^2)\chi_\rho\Big)\Big]_{j=1}^{2},\ \operatorname{Re} z = \lambda,$$

where $z_0 \in \mathbb{C}$, $\operatorname{Im} z_0 > 0$ is fixed, $m > n/2$ and $\chi_2 \in C_0^\infty(\mathbb{R}^n)$ is such that $\chi_2 = 1$ on the support of χ. The analysis of this limit is based on the localization in the incoming and outgoing regions and the ideas of Robert [21] to combine the semi-classical estimates in these regions with the cyclicity of the trace. Moreover, we use the existence of non-trapping perturbations $\tilde{L}_.$ such that $L_.(1 - \chi) = \tilde{L}_.(1 - \chi)$, where the support of χ is related to the existence of a compact set $K(J) \subset \mathbb{R}^n$ containing the bounded trajectories of the

Hamiltonian fields related to $L_.$ and to the energy level $\lambda \in J$ (see [2] and section 2).

The representation (1.1) is used in Section 6 for the proof of a Weyl type asymptotic for $\xi(\lambda)$. The analysis of the integrals $\int_{\mu_0}^{\mu} \operatorname{tr}\Big(T^{\pm}(\lambda \pm i0)\Big) d\lambda$ follows closely the argument in [21]. On the other hand, we deal with the general setup of "black box" semi-classical scattering and in this situation the investigation of $\int_{\mu_0}^{\mu} \operatorname{tr}\Big(\chi E_j'(\lambda)\chi\Big) d\lambda$ leads to some difficulties related to the analysis of $U_j(t)$ for small t. To overcome these difficulties we use the estimates for the trace in [25] combined with a suitable Duhamel formula based on the results in [2].

The paper is organized as follows. In Section 2 we present some known results for non-trapping and trapping perturbations. In Section 3 we prove Theorem 1 and Corollary 1 and 2. Section 4 is devoted to trace class estimates, while the analysis of the operators $T^{\pm}(z)$ is given in Section 5. In Section 6 we study the Weyl type asymptotics for the spectral shift function.

2. Preliminaries

We start by the abstract "black box" scattering assumptions introduced in [27], [25] and [26]. The operator $L_.$ is defined in a domain $\mathcal{D}_. \subset \mathcal{H}_.$ of a complex Hilbert space $\mathcal{H}_.$ with an orthogonal decomposition

$$\mathcal{H}_. = \mathcal{H}_{R_0,.} \oplus L^2(\mathbb{R}^n \setminus B(0,R_0)),\ B(0,R_0) = \{x \in \mathbb{R}^n : |x| \leq R_0\},\ \ n \geq 2.$$

Introduce the spaces

$$\mathcal{H}_.^{0,s} = \mathcal{H}_{R_0,.} \oplus L^2(\mathbb{R}^n \setminus B(0,R_0), \langle x\rangle^s dx),\ \langle x\rangle = (1+|x|^2)^{1/2},$$

and denote by $\|.\|_{s,s'}$ the norm in $\mathcal{L}(\mathcal{H}_.^{0,s}, \mathcal{H}_.^{0,s'})$. We suppose that $\mathcal{D}_.$ satisfies

$$1\!\!1_{\mathbb{R}^n \setminus B(0,R_0)} \mathcal{D}_. = H^2(\mathbb{R}^n \setminus B(0,R_0)), \tag{2.1}$$

uniformly with respect to h in the sense of [25]. More precisely, equip $H^2(\mathbb{R}^n \setminus B(0,R_0))$ with the norm $\| < hD >^2 u\|_{L^2}$, $< hD >^2 = 1 + (hD)^2$, and equip $\mathcal{D}_.$ with the norm $\|(L_. + i)u\|_{\mathcal{H}_.}$. Then we require that $1\!\!1_{\mathbb{R}^n \setminus B(0,R_0)} : \mathcal{D}_. \longrightarrow H^2(\mathbb{R}^n \setminus B(0,R_0))$ is uniformly bounded with respect to h and this map has a uniformly bounded right inverse.

Assume that

$$1\!\!1_{B(0,R_0)}(L_. + i)^{-1} \text{is compact} \tag{2.2}$$

and

$$(L_. u)|_{\mathbb{R}^n \setminus \overline{B(0,R_0)}} = Q_. \Big(u|_{\mathbb{R}^n \setminus \overline{B(0,R_0)}}\Big), \tag{2.3}$$

where $Q_.$ is a formally self-adjoint differential operator

$$Q_. u = \sum_{|\nu| \leq 2} a_{.,\nu}(x;h)(hD_x)^{\nu} u, \tag{2.4}$$

with $a_{.,\nu}(x;h) = a_{.,\nu}(x)$ independent of h for $|\nu| = 2$ and $a_{.,\nu} \in C_b^{\infty}(\mathbb{R}^n)$ uniformly bounded with respect to h.

We assume also the following properties:

There exists $C > 0$ such that

$$l_{.,0}(x,\xi) = \sum_{|\nu|=2} a_{.,\nu}(x)\xi^{\nu} \geq C\langle \xi \rangle^2,\ \ \langle\xi\rangle = (1+|\xi|^2)^{1/2}. \tag{2.5}$$

There exists $\gamma > 0$ such that for every $(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n$, we have

$$|\partial_x^{\alpha}\partial_{\xi}^{\beta}\Big(\sum_{|\nu|\leq 2} a_{.,\nu}(x;h)\xi^{\nu} - |\xi|^2\Big)| \leq C_{\alpha,\beta}\langle x\rangle^{-\gamma-|\alpha|}\langle\xi\rangle^2 \tag{2.6}$$

uniformly with respect to h.

There exists $\overline{n} > n$ such that for every $\alpha \in \mathbb{N}^n$, $|\nu| \leq 2$ we have

$$|\partial_x^\alpha \Big(a_{1,\nu}(x;h) - a_{2,\nu}(x;h)\Big)| \leq C_\alpha \langle x \rangle^{-\overline{n}-|\alpha|} \tag{2.7}$$

uniformly with respect to h. This assumption will guarantee that for every $f \in C_0^\infty(\mathbb{R})$ the operator $f(L_1) - f(L_2)$ is "trace class near infinity".

There exist $\theta_0 \in]0,\pi[$, $\epsilon > 0$ and $R_1 > R_0$ so that the coefficients $a_{.,\nu}(x;h)$ of $Q_.$ can be extended holomorphically in x to

$$\{r\omega;\ \omega \in \mathbb{C}^n,\ \mathrm{dist}\,(\omega, S^{n-1}) < \epsilon,\ r \in \mathbb{C},\ |r| > R_1,\ \arg r \in [-\epsilon, \theta_0 + \epsilon)\} \tag{2.8}$$

and (2.6), (2.7) extend to this larger set.

Let $R > R_0$, $T_{\tilde{R}} = (\mathbb{R}/\tilde{R}\mathbb{Z})^n$, $\tilde{R} > 2R$. Set

$$\mathcal{H}^\#_. = \mathcal{H}_{R_0,.} \oplus L^2(T_{\tilde{R}} \setminus B(0,R_0))$$

and consider a differential operator

$$Q^\#_. = \sum_{|\nu| \leq 2} a^\#_{.,\nu}(x;h)(hD)^\nu$$

on $T_{\tilde{R}}$ with $a^\#_{.,\nu}(x;h) = a_{.,\nu}(x;h)$ for $|x| < R$ satisfying (2.3), (2.4), (2.5) with $\mathbb{R}^n$ replaced by $T_{\tilde{R}}$. Consider a self-adjoint operator $L^\#_. : \mathcal{H}^\#_. \longrightarrow \mathcal{H}^\#_.$ defined by

$$L^\#_. u = L_. \varphi u + Q^\#_.(1-\varphi)u,\ u \in \mathcal{D}^\#_.,$$

with domain

$$\mathcal{D}^\#_. = \{u \in \mathcal{H}^\#_. :\ \varphi u \in \mathcal{D}_.,\ (1-\varphi)u \in H^2\},$$

where $\varphi \in C_0^\infty(B(0,R);[0,1])$ is equal to 1 near $\overline{B(0,R_0)}$.

Denote by $N(L^\#_., [-\lambda^2, \lambda^2])$ the number of eigenvalues of $L^\#_.$ in the interval $[-\lambda^2, \lambda^2]$. Then we assume that

$$N(L^\#_., [-\lambda^2, \lambda^2]) = \mathcal{O}(\Big(\frac{\lambda}{h}\Big)^{n^\#_.}),\ n^\#_. \geq n,\ \lambda \geq 1. \tag{2.9}$$

Finally, we suppose that

$$\sigma_{pp}(L_.(h)) \cap [\mu_0, \mu_1] = \emptyset,\ h \in]0, h_0]. \tag{2.10}$$

We will say that $\lambda \in \mathbb{R}$ is a *non-critical energy level* for Q if for all $(x,\xi) \in \Sigma_\lambda = \{(x,\xi) \in \mathbb{R}^{2n} : l(x,\xi) = \lambda\}$ we have $\nabla_{x,\xi} l(x,\xi) \neq 0$, $l(x,\xi)$ being the principal symbol of Q. Below we collect some known results for non-trapping perturbations concerning resolvent estimates, microlocal estimates and asymptotic behavior of the spectral shift function for non-trapping perturbations. Given a Hamiltonian $l(x,\xi)$, denote by

$$\exp(tH_l)(x_0,\xi_0) = (x(t,x_0,\xi_0),\ \xi(t,x_0,\xi_0))$$

the trajectory of the Hamilton flow $\exp(tH_l)$ passing through $(x_0,\xi_0) \in \Sigma_\lambda$. Recall that $\lambda \in J$ is a *non-trapping energy level* for a classical Hamiltonian $l(x,\xi)$ if for every $R > 0$ there exists $T(R) > 0$ such that for $(x_0,\xi_0) \in \Sigma_\lambda$, $|x_0| < R$, the x-component of the trajectory of $\exp(tH_l)$ passing through (x_0,ξ_0) satisfies

$$|x(t,x_0,\xi_0)| > R,\ \ \forall |t| > T(R).$$

We say that $J \subset\subset \mathbb{R}^+$ is non-trapping for $\tilde{L}(h)$ if every $\lambda \in J$ is a non-trapping energy level for the principal symbol of $\tilde{L}(h)$. Following [2], for a fixed $J \subset\subset \mathbb{R}^+$, we can construct two self-adjoint operators $\tilde{L}_1$, $\tilde{L}_2$ with domains in $L^2(\mathbb{R}^n)$ so that $\tilde{L}_.$ is a compactly supported perturbation of $L_.$ and J is non trapping for $\tilde{L}_.$. In the following, $R_.(z) = (L_.(h) - z)^{-1}$ and $\tilde{R}_.(z) = (\tilde{L}_.(h) - z)^{-1}$ denote the resolvent of the operators $L(h)$ and $\tilde{L}(h)$, respectively.

Proposition 1. *([2]) Let $L(h)$ satisfy the "black box" assumptions* $(2.1)-(2.6)$. *Then there exist $\rho_0 > R_0$ and a self-adjoint differential operator $\tilde{L} = \tilde{L}(h)$ on $L^2(\mathbb{R}^n)$, satisfying $(2.1)-(2.6)$, such that each $\lambda \in J$ is a non-trapping and non-critical energy level for $\tilde{L}(h)$ and*

$$L(h)\psi = \tilde{L}(h)\psi$$

for any $\psi \in C^\infty(\mathbb{R}^n)$ supported in $\{x \in \mathbb{R}^n : |x| > \rho_0\}$.

Denote by $\|.\|_{s,s'}$ the norm of $\mathcal{L}(L^{2,s}, L^{2,s'})$, where $L^{2,s}$ is the weight space $L^2(\mathbb{R}^n, \langle x\rangle^s dx)$. Following the Mourre theory (see [15], [16], [14], [13], [20], [21], [9], [2]), we have the following.

Lemma 1. *Let $s > 1/2$. Then for $\lambda \in J$, the limits*

$$(\tilde{L} - \lambda \mp i0)^{-1} := \lim_{\epsilon \to 0,\ \pm\epsilon > 0} (\tilde{L} - \lambda - i\epsilon)^{-1}$$

exist in $\mathcal{L}(L^{2,s}, L^{2,-s})$, and for any differential operator $P = P(h) = \sum_{|\alpha| \leq p} a_\alpha(x)(hD_x)^\alpha$, $p \leq 2$, there exist $C > 0$ and $h_0 > 0$ such that

$$\|P(\tilde{L} - z)^{-1}\|_{s,-s} \leq Ch^{-1} \tag{2.11}$$

uniformly with respect to $z \in \mathcal{B}_\pm = \{z \in \mathbb{C} : (\operatorname{Re} z, \pm \operatorname{Im} z) \in J \times]0,1]\}$ and $h \in]0, h_0]$.

Moreover, exploiting the constructions of long time approximations in section 4 of [21] for long-range perturbations of the Laplacian, we can establish some microlocal resolvent estimates (see the proof of Lemma 2.3 of [23]). For this purpose we introduce the outgoing and incoming regions of the phase space

$$\Gamma^\pm(R, d, \sigma_\pm) = \{(x, \xi) \in \mathbb{R}^{2n} \ : \ |x| > R,\ d^{-1} < |\xi| < d,\ \frac{\langle x.\xi\rangle}{|x||\xi|} \gtrless \sigma_\pm\}, \tag{2.12}$$

where $d > 1$, $-1 < \sigma_\pm < 1$ and $R \gg R_0$ is large enough. Consider a symbol $\omega_\pm(x, \xi) \in C^\infty(\mathbb{R}^{2n})$ such that $\operatorname{supp} \omega_\pm \subset \Gamma^\pm(R, d, \sigma_\pm)$ and

$$|\partial_x^\alpha \partial_\xi^\beta \omega_\pm(x, \xi)| \leq C_{\alpha,\beta,L} \langle x\rangle^{-|\alpha|} \langle \xi\rangle^{-L},\ \forall\alpha,\ \forall\beta \tag{2.13}$$

for any $L \gg 1$. Then we have the following.

Lemma 2. *For $(\lambda, \tau) \in J \times]0, 1]$ and for any differential operator $P = P(h) = \sum_{|\alpha| \leq p} a_\alpha(x)(hD_x)^\alpha$, $p \leq 2$, the following assertions hold:*

i) For $s > 1/2$, $\delta > 1$ there exist $C > 0$ and $h_0 > 0$ such that for $h \in]0, h_0]$ we have

$$\|P(\tilde{L} - \lambda \mp i\tau)^{-1} \omega_\pm(x, hD_x)\|_{-s+\delta,-s} \leq Ch^{-1}. \tag{2.14}$$

ii) If $\sigma_+ > \sigma_-$, then for any $s \gg 1$ and any $N \in \mathbb{N}$ there exist $C_N > 0$ and $h_0 > 0$ such that for $h \in]0, h_0]$ we have

$$\|\omega_\mp(x, hD_x) P(\tilde{L} - \lambda \mp i\tau)^{-1} \omega_\pm(x, hD_x)\|_{-s,s} \leq C_N h^N. \tag{2.15}$$

iii) If $\chi_1 \in C_0^\infty(B(0, R))$, $\chi_1 = 1$ on $\overline{B(0, R_0)}$, then for any $s \gg 1$ and $N \in \mathbb{N}$ there exist $C_N > 0$ and $h_0 > 0$ such that for $h \in]0, h_0]$ we have

$$\|\chi_1 P(\tilde{L} - \lambda \mp i\tau)^{-1} \omega_\pm(x, hD_x)\|_{-s,0} \leq C_N h^N. \tag{2.16}$$

Now we will recall some results in the *trapping* case.

Proposition 2. *Under the assumptions of Proposition 1 and (2.8) and (2.10) we have*

i) For every $s > \frac{1}{2}$ the limits

$$R(\lambda \pm i0) = \lim_{\epsilon \to 0,\ \epsilon > 0} R(\lambda \pm i\epsilon), \quad \lambda \in J$$

exist in $\mathcal{L}(\mathcal{H}^{0,s}, \mathcal{H}^{0,-s})$ *and* $J \ni \lambda \mapsto R(\lambda \pm i0) \in \mathcal{L}(\mathcal{H}^{0,s}, \mathcal{H}^{0,-s})$ *is continuous on* J*. Moreover, there exist* $C > 0$ *and* $h_0 > 0$ *such that for* $\chi \in C_0^\infty(\{x : |x| \leq \rho_1)$ *with* $\chi(x) = 1$ *for* $|x| \leq 2\rho_0$*, we have*

$$\|R(z)\|_{s,-s} \leq Ch^{-2}\Big(1 + \|\chi R(z)\chi\|_{\mathcal{H}\to\mathcal{H}}\Big)$$

uniformly with respect to $z \in \mathcal{B}_\pm$ *and* $h \in]0, h_0]$.

ii) For any $h \in]0, h_0]$ *the operator* $L(h)$ *has purely absolutely continuous spectrum in* J.

Proof. The assertion i) was established in [2]. To prove $ii)$, it is sufficient to show that

$$\liminf_{\epsilon\to 0,\ \epsilon>0} \sup_{\mu_0\leq\lambda\leq\mu_1} |(R(\lambda \pm i\epsilon)\varphi, \varphi)_{\mathcal{H}}| \leq C(\varphi),$$

for any φ in a dense subset of $\mathcal{H}$ (see Proposition 1.4 of [6]). The set of compactly supported functions is dense in $\mathcal{H}$, and, moreover, for every function $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on $\overline{B(0, R_0)}$ we have

$$\chi R(\lambda + i0)\chi = \chi(L_\theta - \lambda)^{-1}\chi,$$

L_θ being the complex scaling operator related to L (see [27]). The continuity of the cut-off resolvent $\chi R(\lambda + i0)\chi$ implies immediately the above estimate for compactly supported φ equal to 1 on $B(0, R_0)$. $\square$

According to Stone formula (see [18]) and to Proposition 2, i), the spectral projector

$$E(\lambda) := \mathbf{1}_{[\mu_0,\lambda]}(L) = s - \lim_{\epsilon\to 0,\ \epsilon>0} \frac{1}{2i\pi}\int_{\mu_0}^{\lambda} \Big(R(\mu + i\epsilon) - R(\mu - i\epsilon)\Big)d\mu,\ \ \lambda \in J$$

is differentiable in $\mathcal{L}(\mathcal{H}^{0,s}, \mathcal{H}^{0,-s})$, $s > 1/2$, with derivative

$$E'_{.}(\lambda) = \frac{1}{2i\pi}\Big(R_{.}(\lambda + i0) - R_{.}(\lambda - i0)\Big). \tag{2.17}$$

Moreover, for any function $f \in C_b^1(\mathbb{R})$ we have

$$f(L.)E'_{.}(\lambda) = f(\lambda)E'_{.}(\lambda), \quad \lambda \in J, \tag{2.18}$$

provided $f(L_{.})$ is a bounded operator in $\mathcal{L}(\mathcal{H}^{0,-s}, \mathcal{H}^{0,-s})$. To prove this notice that for $\forall\varphi$, $\forall\psi \in \mathcal{H}$ by the spectral theorem we get

$$\Big([f(L) - f(\lambda)](R(\lambda + i\epsilon) - R(\lambda - i\epsilon))\varphi, \psi\Big)_{\mathcal{H}} =$$
$$\int_{-\infty}^{\infty} (f(\mu) - f(\lambda))\Big[\frac{1}{\mu - \lambda - i\epsilon} - \frac{1}{\mu - \lambda + i\epsilon}\Big]d(E_\mu\varphi, \psi) \longrightarrow_{\epsilon\to 0} 0.$$

Since the limit

$$\lim_{\epsilon\to 0}[f(L) - f(\lambda)]\Big[R(\lambda + i\epsilon) - R(\lambda - i\epsilon)\Big] = M$$

exists in $\mathcal{L}(\mathcal{H}^{0,s}, \mathcal{H}^{0,-s})$, we must have

$$\lim_{\epsilon\to 0}\Big([f(L) - f(\lambda)](R(\lambda + i\epsilon) - R(\lambda - i\epsilon))\psi_s\varphi, \psi_s\psi\Big)_{\mathcal{H}} = (M\psi_s\varphi, \psi_s\psi)_{\mathcal{H}} = 0$$

for every function ψ_s equal to 1 on $\overline{B(0, R_0)}$ and equal to $< x >^{-s}$ for $|x|$ large enough. This implies easily that $M = 0$ in $\mathcal{L}(\mathcal{H}^{0,s}, \mathcal{H}^{0,-s})$.

Proposition 3. *([2]) Let* $L(h)$ *be as in Proposition 2 and let* $\psi \in C^\infty(\mathbb{R}^n)$ *be supported away from* $B(0, \rho_1)$ *with* $\psi(x) = 1$ *for* $|x| \gg 1$*. Let* $\chi \in C_0^\infty(\mathbb{R}^n)$ *be as in Proposition 2. Then for any* $(\lambda, \tau) \in J\times]0, 1]$ *and* $h \in]0, h_0]$ *the following assertions hold:*

i) For any $s > 1/2$*,* $\delta > 1$ *and* $N \in \mathbb{N}$ *there exist* $C_N > 0$ *and* $h_0 > 0$ *such that for* $h \in]0, h_0]$ *we have*

$$\|\psi R(\lambda \pm i\tau)\psi\omega_\pm(x, hD_x)\|_{-s+\delta,-s} \leq C_N h^{-1}(1 + h^N\|\chi R(\lambda \pm i\tau)\chi\|_{\mathcal{H}\to\mathcal{H}}). \tag{2.19}$$

ii) If $\sigma_+ > \sigma_-$, then for any $s \gg 1$ and $N \in \mathbb{N}$, there exist $C_N > 0$ and $h_0 > 0$ such that for $h \in]0, h_0]$ we have

$$\|\omega_\mp(x, hD_x)\psi R(\lambda \pm i\tau)\psi\omega_\pm(x, hD_x)\|_{-s,s} \leq C_N h^N (1 + \|\chi R(\lambda \pm i\tau)\chi\|_{\mathcal{H}\to\mathcal{H}}). \tag{2.20}$$

iii) If $\varphi \in C_0^\infty(B(0, 2\rho_0))$, $\varphi = 1$ on $\overline{B(0, R_0)}$, then for any $s \gg 1$ and $N \in \mathbb{N}$ there exist $C_N > 0$ and $h_0 > 0$ such that for $h \in]0, h_0]$ we have

$$\|\varphi R(\lambda \pm i\tau)\psi\omega_\pm(x, hD_x)\|_{-s,0} \leq C_N h^N (1 + \|\chi R(\lambda \pm i\tau)\chi\|_{\mathcal{H}\to\mathcal{H}}). \tag{2.21}$$

In the following section we need some results about the SSF for the pair $(\tilde{L}_2, \tilde{L}_1)$. The operators $\tilde{L}_1$, $\tilde{L}_2$ are differential operators satisfying the assumptions (2.5), (2.6) and (2.7). Then for any $f \in \mathcal{S}(\mathbb{R})$, the operator $f(\tilde{L}_2) - f(\tilde{L}_1)$ is trace class (see [8], [19], [21]). Following the theory of Birman-Krein ([30], [31]), the spectral shift function $\tilde{\xi} \in \mathcal{S}'(\mathbb{R})$ related to $\tilde{L}_1$, $\tilde{L}_2$, is well defined by the equality

$$\langle \tilde{\xi}, f' \rangle_{\mathcal{S}',\mathcal{S}} = -\mathrm{tr}(f(\tilde{L}_2) - f(\tilde{L}_1)). \tag{2.22}$$

Moreover, $\tilde{\xi}'(\lambda)$ is a C^∞ function on J and $\tilde{\xi}'(\lambda)$ coincides with the derivative of the scattering phase and with the average time delay for the pair $(\tilde{L}_1, \tilde{L}_2)$ (see [21]). Next, J being non-trapping and non-critical interval for $\tilde{L}_1$ and $\tilde{L}_2$, Theorem 1.3 of [21] yields the following.

Proposition 4. *There exist $h_0 > 0$ and C^∞ coefficients $c_k(\lambda)$, $k \geq 0$, such that*

$$\tilde{\xi}'(\lambda) = h^{-n} \sum_{k=0}^{N} c_k(\lambda) h^k + \mathcal{O}(h^{-n+N+1}), \forall N \in \mathbb{N} \tag{2.23}$$

uniformly with respect to $\lambda \in J$, $h \in]0, h_0]$.

3. Representation formula for the derivative of the spectral shift function

In this section we will prove Theorem 1 admitting Propositions 5 and 7 below. In the next three sections z_0 denotes a fixed complex number such that $\mathrm{Im}\, z_0 > 0$. We start by the following.

Proposition 5. *Let L satisfy (2.1) – (2.6) and (2.9). Let $\chi_1 \in C_0^\infty(B(0, 2\rho_0))$ be equal to 1 near $\overline{B(0, \rho_0)}$ and let $\varphi_s \in C^\infty(\mathbb{R}^n)$ be equal to a constant in a neighborhood of $\overline{B(0, R_0)}$ so that $\varphi_s(x) = \langle x \rangle^s$, $s \in \mathbb{R}$, for $|x| \geq R > R_0$. Then for $m > n^\#/2$ and $f \in \mathcal{S}(\mathbb{R})$, the operators*

$$\chi_1(L_\cdot - z_0)^{-m}\varphi_s, \;\; \chi_1 f(L_\cdot)\varphi_s, \;\; \varphi_s\Big((1 - \chi_1)(L_\cdot - z_0)^{-m}(1 - \chi_1) - (1 - \chi_1)(\tilde{L}_\cdot - z_0)^{-m}(1 - \chi_1)\Big)\varphi_s$$

can be extended to trace class ones with trace norm bounded by $\mathcal{O}(h^{-n^\#})$. Moreover, the following assertions hold:

i) for $\chi_2 \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on a sufficiently large region, we have

$$\|\chi_1(L_\cdot - z_0)^{-m}(1 - \chi_2)\varphi_s\|_{\mathrm{tr}} + \|\chi_1 f(L_\cdot)(1 - \chi_2)\varphi_s\|_{\mathrm{tr}} = \mathcal{O}(e^{-C/h}). \tag{3.1}$$

ii) there exists $\rho_1 > \rho_0$ such that for χ_2 equal to 1 near $\overline{B(0, \rho_1)}$, we have

$$\|\varphi_s\Big((1 - \chi_2)(L_\cdot - z_0)^{-m}(1 - \chi_2) - (1 - \chi_2)(\tilde{L}_\cdot - z_0)^{-m}(1 - \chi_2)\Big)\varphi_s\|_{tr} = \mathcal{O}(e^{-C/h}). \tag{3.2}$$

This proposition will be proved in Section 4 exploiting the functional calculus developed by Sjöstrand [25, 26].

Proposition 6. *Let L and χ_1 be as in Proposition 5 and let $\chi_2 \in C_0^\infty(\mathbb{R}^n)$, $\chi_2 \succ \chi_1$ be equal to 1 on $\{x \in \mathbb{R}^n : |x| \leq \rho_2\}$, $\rho_2 > \rho_1$ sufficiently large. Then we have*

$$\lim_{\rho\to+\infty} \Big[\mathrm{tr}\Big(\chi_\rho(L_j - z_0)^{-m}(1 - \chi_1)\tilde{E}_j'(\lambda)(1 - \chi_2^2)\chi_\rho\Big)\Big]_{j=1}^{2}$$

$$= (\lambda - z_0)^{-m}\tilde{\xi}'(\lambda) - (\lambda - z_0)^{-m}\left[\mathrm{tr}\Big(\chi_2 \tilde{E}_j'(\lambda)\chi_2\Big)\right]_{j=1}^2 + \mathcal{O}(e^{-C/h})$$

uniformly with respect to $(\lambda, h) \in J\times]0, h_0]$.

Proof. We write

$$\chi_\rho(L_j - z_0)^{-m}(1-\chi_1)\tilde{E}_j'(\lambda)(1-\chi_2^2)\chi_\rho = \chi_\rho(\tilde{L}_j - z_0)^{-m}\tilde{E}_j'(\lambda)(1-\chi_2^2)\chi_\rho + \chi_\rho\mathcal{T}_j\tilde{E}_j'(\lambda)(1-\chi_2^2)\chi_\rho$$

with $\mathcal{T}_j := \Big((L_j - z_0)^{-m}(1-\chi_1) - (\tilde{L}_j - z_0)^{-m}(1-\chi_1)\Big) - (\tilde{L}_j - z_0)^{-m}\chi_1$ and, according to (3.1) and (3.2), we get

$$\|\varphi_s(1-\chi_2^2)\mathcal{T}_j\varphi_s\|_{tr} = \mathcal{O}(e^{-C/h}) \tag{3.3}$$

for any $s \in \mathbb{R}$. Then, combining the formula (2.17) for $\tilde{E}_j'$ and the resolvent estimate (2.11) for $s > \frac{1}{2}$, we get

$$\|\varphi_s^{-1}\tilde{E}_j'(\lambda)\varphi_s^{-1}\| = \mathcal{O}(h^{-1}).$$

Thus, from the cyclicity of the trace and the estimate (3.3) we deduce

$$\mathrm{tr}\Big(\chi_\rho(L_j - z_0)^{-m}(1-\chi_1)\tilde{E}_j'(\lambda)(1-\chi_2^2)\chi_\rho\Big)$$
$$= \mathrm{tr}\Big(\chi_\rho(\tilde{L}_j - z_0)^{-m}\tilde{E}_j'(\lambda)\chi_\rho\Big) - \mathrm{tr}\Big(\chi_\rho\chi_2(\tilde{L}_j - z_0)^{-m}\tilde{E}_j'(\lambda)\chi_2\chi_\rho\Big) + \mathcal{O}(e^{-C/h}).$$

To conclude, we observe that the relations (2.18) and (1.6) hold for $\tilde{E}_j'$ and $\tilde{\xi}'$. □

Proposition 7. *Let* L *satisfy* (2.1) − (2.6) *and* (2.8) − (2.9). *Let* $\chi_1 \prec \chi_2$ *be as in Proposition 6. For* $\pm\,\mathrm{Im}\, z > 0$ *we have*

$$\lim_{\rho\to+\infty}(\lambda - z_0)^m \mathrm{tr}\Big(\chi_\rho(L_\cdot - z_0)^{-m}R_\cdot(z)[\tilde{L}_\cdot, \chi_1]\tilde{R}_\cdot(z)(1-\chi_2^2)\chi_\rho\Big) = \mathrm{tr}\Big(T_\cdot^{\pm}(z)\Big),\ \mathrm{Re}\, z = \lambda,$$

where $T_\cdot^{\pm}(z)$ *are trace class operators. Then for* $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi_2 \prec \chi$ *the following estimates hold*

$$\|T_\cdot^{\pm}(z)\|_{\mathrm{tr}} = \mathcal{O}(h^\infty)\,\Big(1 + \|\chi R_\cdot(z)\chi\|_{\mathcal{H}\to\mathcal{H}}\Big), \quad \pm\,\mathrm{Im}\, z > 0$$

uniformly with respect to $(\mathrm{Re}\, z, \pm\,\mathrm{Im}\, z, h) \in J\times]0,1]\times]0,h_0]$ *and, moreover,* $\mathcal{B}_\pm \ni z \mapsto \mathrm{tr}\Big(T_\cdot^{\pm}(z)\Big)$ *are holomorphic in* $\mathcal{B}_\pm$.

Proposition 7 follows from Proposition 8 taking $P = [\tilde{L}_\cdot, \chi_1]$. Proposition 8 will be proved in Section 5 by localizations in the outgoing and incoming regions of the phase space combined with the resolvent estimates of Section 2.

Proof of Theorem 1. Applying (2.18) and (2.17) for $f(\lambda) = (\lambda - z_0)^{-m}$, $z_0 \in \mathbb{C}\setminus\mathbb{R}$ fixed, we have

$$E_\cdot'(\lambda) = \frac{(\lambda - z_0)^m}{2i\pi}(L_\cdot - z_0)^{-m}\Big(R(\lambda + i0) - R(\lambda - i0)\Big).$$

Then combining Proposition 5 and Proposition 2, we deduce that for any $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 near $\overline{B(0,R_0)}$, the operator $\chi E_\cdot'(\lambda)\chi$ is trace class one for $\lambda \in J$ with trace norm uniformly bounded with respect to λ. The same argument works for $\tilde{E}'$.

The proof of the semi-classical representation formula (1.1) follows tree steps.

Step 1. We will show that ξ_ρ' coincides on J with the locally integrable function $\mathcal{E}_\rho(\lambda) := \mathcal{E}_{\rho,2}(\lambda) - \mathcal{E}_{\rho,1}(\lambda)$, where $\mathcal{E}_{\rho,j}(\lambda) = \mathrm{tr}\Big(\chi_\rho E_j'(\lambda)\chi_\rho\Big)$, $j = 1, 2$. Let $f \in C_0^\infty(\mathbb{R})$, $\mathrm{supp} f \subset J$. According to (2.18), we have

$$\int_{\mathbb{R}} \mathcal{E}_{\rho,j}(\lambda)f(\lambda)d\lambda = \int_J \mathrm{tr}\Big(\chi_\rho f(L_j)E_j'(\lambda)\chi_\rho\Big)d\lambda.$$

However, for any $s > \frac{1}{2}$, $\chi_\rho f(L_j)\varphi_s$ is a trace class operator (see Proposition 5) and ${\varphi_s}^{-1}E_j'(\lambda)\chi_\rho \in \mathcal{L}(\mathcal{H})$. Then $\chi_\rho f(L_j)E_j'(\lambda)\chi_\rho$ is the derivative of $\chi_\rho f(L_j)E_j(\lambda)\chi_\rho$ in the space of the trace class operators, and

$$\int_{\mathbb{R}} \mathcal{E}_{\rho,j}(\lambda)f(\lambda)d\lambda = \mathrm{tr}\Big(\chi_\rho f(L_j)\mathbf{1}_J(L_j)\chi_\rho\Big) = \mathrm{tr}\Big(\chi_\rho f(L_j)\chi_\rho\Big).$$

Consequently, ξ_ρ' becomes a locally integrable function and

$$\xi_\rho'(\lambda) = \Big[\mathrm{tr}\Big(\chi_\rho E_j'(\lambda)\chi_\rho\Big)\Big]_{j=1}^{2}. \tag{3.4}$$

Step 2. Choose $\chi_1 \prec \chi_2$ as in Proposition 6. Then $\tilde{L}(1-\chi_2) = L(1-\chi_2)$ and we have

$$R_\cdot(z)(1-\chi_2^2) = (1-\chi_1)\tilde{R}_\cdot(z)(1-\chi_2^2) + R_\cdot(z)[\tilde{L}_\cdot, \chi_1]\tilde{R}_\cdot(z)(1-\chi_2^2) \tag{3.5}$$

in $\mathcal{L}(\mathcal{H}_\cdot^{0,s}, \mathcal{H}_\cdot^{0,-s})$, $s > 1/2$, for $z = \lambda \pm i0$ and in $\mathcal{L}(\mathcal{H}_\cdot)$ for $\mathrm{Im}\, z \neq 0$. Applying formula (2.17) once more, we get

$$\begin{aligned} E_\cdot'(\lambda) = {} & E_\cdot'(\lambda)\chi_2^2 + (\lambda - z_0)^m(L_\cdot - z_0)^{-m}(1-\chi_1)\tilde{E}_\cdot'(\lambda)(1-\chi_2^2) \\ & + \frac{(\lambda - z_0)^m}{2i\pi}\left(G_\cdot(\lambda + i0) - G_\cdot(\lambda - i0)\right), \end{aligned} \tag{3.6}$$

where

$$G_\cdot(z) = (L_\cdot - z_0)^{-m}R_\cdot(z)[\tilde{L}_\cdot, \chi_1]\tilde{R}_\cdot(z)(1-\chi_2^2).$$

Step 3. Combining the relations (1.6), (3.4), (3.6) with Propositions 6, 7, we obtain

$$\begin{aligned} \xi'(\lambda) = {} & \tilde{\xi}'(\lambda) + \Big[\mathrm{tr}\Big(\chi_2 E_j'(\lambda)\chi_2\Big)\Big]_{j=1}^{2} - \Big[\mathrm{tr}\Big(\chi_2 \tilde{E}_j'(\lambda)\chi_2\Big)\Big]_{j=1}^{2} \\ & + \mathrm{tr}T^+(\lambda + i0) - \mathrm{tr}T^-(\lambda - i0) + (\lambda - z_0)^m\mathcal{O}(e^{-C/h}). \end{aligned}$$

Finally, J being non-trapping for $\tilde{L}_1$ and $\tilde{L}_2$, the functions $\tilde{\xi}'(\lambda)$ and $\mathrm{tr}\Big(\chi_2\tilde{E}_\cdot'(\lambda)\chi_2\Big)$ have complete asymptotics described by Proposition 4 and we deduce Theorem 1. □

Corollary 1 follows directly from (1.2), the estimate for $\|T^\pm(z)\|_{\mathrm{tr}}$ and the representation (1.1).

Proof of Corollary 2. According to Theorem 1 and (1.2), for $|t| \geq \delta > 0$ we have

$$\mathrm{tr}_{\mathrm{bb}}\Big(e^{-ith^{-1}L_2}f(L_2) - e^{-ith^{-1}L_1}f(L_1)\Big) = \int_J e^{-ith^{-1}\lambda}f(\lambda)\xi'(\lambda)d\lambda$$

$$= \Big[\mathrm{tr}\Big(\chi e^{-ith^{-1}L_j}f(L_j)\chi\Big)\Big]_{j=1}^{2} + \mathcal{F}_h\Big(f(\bullet)\mathrm{tr}\Big(T^+(\bullet + i0)\Big)\Big) - \mathcal{F}_h\Big(f(\bullet)\mathrm{tr}\Big(T^-(\bullet - i0)\Big)\Big) + \mathcal{O}(h^\infty),$$

where $\mathcal{F}_h$ denotes the Fourier transform

$$\Big(\mathcal{F}_h\psi\Big)(t) = \int e^{-ith^{-1}\lambda}\psi(\lambda)d\lambda.$$

Clearly, the estimate of $\|T^\pm(z)\|_{\mathrm{tr}}$ obtained in Theorem 1 yields

$$\Big|\mathcal{F}_h\Big(f(\bullet)\mathrm{tr}\Big(T^+(\bullet + i0)\Big)\Big)\Big| \leq \mathcal{O}(h^\infty)\Big(\int_J [\|\tilde{\chi}R_1(\lambda)\tilde{\chi}\|_{\mathcal{H}\to\mathcal{H}} + \|\tilde{\chi}R_2(\lambda)\tilde{\chi}\|_{\mathcal{H}\to\mathcal{H}}]d\lambda\Big).$$

Applying Proposition 3 in [28], we conclude that the assumption (1.3) implies

$$\int_J \|\chi R_j(\lambda)\chi\| d\lambda \leq Ch^{-M}, \;\; j = 1, 2$$

with some $M > 0$ depending on $n^\sharp$ and the constants in (1.3). Thus the Fourier transform of $f(\lambda)\mathrm{tr}T^\pm(\lambda \pm i0)$ give $\mathcal{O}(h^\infty)$ terms and the proof is complete. □

4. Trace class estimates

This section is devoted to the proof of Proposition 5. The main idea is to generalize the trace class estimates of [25] for operators acting in weight spaces. Let $\Gamma \subset \mathbb{R}^n$ be a sufficiently widely spaced lattice and let $0 \leq \psi_\nu \in C_0^\infty(\mathbb{R}^n)$, $\nu \in \Gamma$, be a translation invariant partition of unity $\{\psi_\nu(x)\}_{\nu\in\Gamma}$, $\psi_\nu(x) = \psi_0(x-\nu)$ with $\psi_0 = 1$ near $\overline{B(0,R_0)}$. Denote by $\mu_j(K)$ the j-th characteristic value of the compact operator K and fix $z_0 \in \mathbb{C}\setminus\mathbb{R}$. According to Lemma 3.1 in [25], we obtain easily the following results.

Lemma 3. *Let L satisfy* (2.1) – (2.6) *and* (2.9) *and let $\tilde{L}$ be defined as in Proposition 1.*

i) For ν, $\mu \in \Gamma$ there exists $C > 0$ such that

$$\mu_j\Big(\psi_\mu(L-z_0)^{-1}\psi_\nu\Big) \leq \frac{Ce^{-\frac{1}{Ch}(|\nu-\mu|-C)_+}}{1+h^2 j^{\frac{2}{n^\#}}}.$$

ii) For ν, $\mu \in \Gamma\setminus\{0\}$ there exists $C > 0$ such that

$$\mu_j\Big(\psi_\mu(L-z_0)^{-1}\psi_\nu - \psi_\mu(\tilde{L}-z_0)^{-1}\psi_\nu\Big) \leq \frac{Ce^{-\frac{1}{Ch}(|\nu|+|\mu|-C)_+}}{1+h^2 j^{\frac{2}{n^\#}}}.$$

Lemma 4. *Let $m > n^\#/2$ and let $\varphi_s \in C^\infty(\mathbb{R}^n)$ be a constant in a neighborhood of $\overline{B(0,R_0)}$ so that $\varphi_s(x) = \langle x\rangle^s$, $s \in \mathbb{R}$, for $|x| \geq R > R_0$.*

i) For ν, $\mu \in \Gamma$ the operator $\psi_\mu(L-z_0)^{-m}\psi_\nu\varphi_s$ is trace class and there exists $C > 0$ such that

$$\| \psi_\mu(L-z_0)^{-m}\psi_\nu\varphi_s \|_{\rm tr} \leq Ch^{-n^\#}\langle\nu\rangle^s e^{-\frac{1}{Ch}(|\nu-\mu|-C)_+}.$$

ii) For ν, $\mu \in \Gamma\setminus\{0\}$ the operator $\varphi_s\Big(\psi_\mu(L-z_0)^{-m}\psi_\nu - \psi_\mu(\tilde{L}-z_0)^{-m}\psi_\nu\Big)\varphi_s$ is trace class and there exists $C > 0$ such that

$$\| \varphi_s\Big(\psi_\mu(L-z_0)^{-m}\psi_\nu - \psi_\mu(\tilde{L}-z_0)^{-m}\psi_\nu\Big)\varphi_s \|_{\rm tr} \leq Ch^{-n^\#}\langle\mu\rangle^s\langle\nu\rangle^s e^{-\frac{1}{Ch}(|\nu|+|\mu|-C)_+}.$$

Proof of Proposition 5. For $\chi_1 \in C_0^\infty(\mathbb{R}^n)$, there exists N_0 such that $\chi_1 = \sum_{|\mu|\leq N_0}\chi_1\psi_\mu$. Applying Lemma 4, and writing

$$\chi_1(L-z_0)^{-m}\psi_\nu\varphi_s = \sum_{|\mu|\leq N_0}\chi_1\psi_\mu(L-z_0)^{-m}\psi_\nu\varphi_s,$$

we conclude that the operator $\chi_1(L-z_0)^{-m}\varphi_s\psi_\nu$ is trace class with

$$\| \chi_1(L-z_0)^{-m}\psi_\nu\varphi_s \|_{\rm tr} \leq Ch^{-n^\#}\langle\nu\rangle^s e^{-\frac{1}{Ch}(|\nu|-C)_+}.$$

On the other hand, the series $\sum_\nu \langle\nu\rangle^s e^{-\frac{1}{Ch}(|\nu|-C)_+}$ converges and for N sufficiently large we have

$$\sum_{|\nu|\geq N}\langle\nu\rangle^s e^{-\frac{1}{Ch}(|\nu|-C)_+} = \mathcal{O}(e^{-\frac{1}{Ch}}). \tag{4.1}$$

Consequently, $\chi_1(L_\cdot - z_0)^{-m}\varphi_s$ can be extended to a trace class operator with trace norm bounded by $\mathcal{O}(h^{-n^\#_\cdot})$. Moreover, for $\chi_2 \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on a sufficiently large region, there exists N such that

$$\chi_1(L-z_0)^{-m}\varphi_s(1-\chi_2) = \sum_{|\nu|\geq N}\chi_1(L-z_0)^{-m}\psi_\nu\varphi_s(1-\chi_2).$$

Thus (3.1) follows from the estimate (4.1). In the same way, for $\operatorname{Im} z \neq 0$, $\chi_1(L_\cdot - z_0)^{-m}(L_\cdot - z)^{-1}\varphi_s$ can be extended to a trace class operator with trace norm bounded by $\mathcal{O}(h^{-n^\#_\cdot}/|\operatorname{Im} z|)$ and, as in [25], using the operator version of the Cauchy-Green-Riemann-Stokes formula, we obtain that the operator $\chi_1 f(L_\cdot)\varphi_s$ can be extended to trace class one with trace norm bounded by $\mathcal{O}(h^{-n^\#_\cdot})$. Moreover if $\chi_2 \in C_0^\infty(\mathbb{R}^n)$ is equal to 1 on a sufficiently large region, for $\operatorname{Im} z \neq 0$, the trace norm of $\chi_1(L_\cdot - z_0)^{-m}(L_\cdot - z)^{-1}\varphi_s(1-\chi_2)$ is bounded by $\mathcal{O}(e^{-\frac{1}{Ch}}/|\operatorname{Im} z|)$ and, using once more the operator version of the Cauchy-Green-Riemann-Stokes formula, we obtain that trace norm of the operator $\chi_1 f(L_\cdot)\varphi_s(1-\chi_2)$ is bounded by $\mathcal{O}(e^{-\frac{1}{Ch}})$.

Now, for χ_1 equal to 1 near $\overline{B(0,R_0)}$ and such that $\chi_1\psi_0 = \psi_0$ we write

$$(1-\chi_1)(L-z_0)^{-m}(1-\chi_1) = \sum_{\nu,\mu\in\Gamma\setminus\{0\}} (1-\chi_1)\psi_\mu(L-z_0)^{-m}\psi_\nu(1-\chi_1).$$

Taking into account Lemma 4, the convergence of the series $\sum_{\nu,\mu}\langle\mu\rangle^s\langle\nu\rangle^s e^{-\frac{1}{Ch}(|\nu|+|\mu|-C)_+}$ implies that for N sufficiently large we have

$$\sum_{|\mu|,|\nu|\geq N} \langle\mu\rangle^s\langle\nu\rangle^s e^{-\frac{1}{Ch}(|\nu|+|\mu|-C)_+} = \mathcal{O}(e^{-\frac{1}{Ch}}). \tag{4.2}$$

Thus we deduce that the operator $\varphi_s\Big((1-\chi_1)(L-z_0)^{-m}(1-\chi_1) - (1-\chi_1)(\tilde{L}-z_0)^{-m}(1-\chi_1)\Big)\varphi_s$ can be extended to a trace class one with trace norm bounded by $\mathcal{O}(h^{-n^\#_\cdot})$. Moreover, for χ_1 equal to 1 on a sufficiently large region, there exists N such that

$$(1-\chi_1)(L-z_0)^{-m}(1-\chi_1) = \sum_{|\nu|,|\mu|\geq N} (1-\chi_1)\psi_\mu(L-z_0)^{-m}\psi_\nu(1-\chi_1).$$

Finally, (3.2) follows from the estimate (4.2) and this completes the proof of Proposition 5. □

5. Limits of traces

The purpose of this section is to prove Proposition 8 which implies Proposition 7.

Proposition 8. *Let P be a p-order h-admissible differential operator $P(h) = \sum_{|\alpha|\leq p} a_\alpha(x)(hD_x)^\alpha$, $p \leq 2$, with coefficients $(a_\alpha(x))_{|\alpha|\leq p}$ compactly supported in $\{x \in \mathbb{R}^n : R_0 < |x| < 2\rho_0\}$. For any $\psi \in C_b^\infty(\mathbb{R}^n)$, supported away from the supports of a_α, there exist trace class operators $T^\pm_\cdot(z)$, defined for $(\operatorname{Re} z, \pm\operatorname{Im} z) \in J\times]0,1]$, such that the following assertions hold:*

$$\lim_{\rho\to+\infty} \operatorname{tr}\Big(\chi_\rho(L_\cdot - z_0)^{-m}R_\cdot(z)P\tilde{R}_\cdot(z)\psi\chi_\rho\Big) = \operatorname{tr}\Big(T^\pm_\cdot(z)\Big),$$

$$\|T^\pm_\cdot(z)\|_{\mathrm{tr}} = \mathcal{O}(h^N)\,\Big(1 + \|\chi R_\cdot(z)\chi\|_{\mathcal{H}\to\mathcal{H}}\Big),\quad \pm\operatorname{Im} z > 0,\ \forall N\in\mathbb{N},$$

uniformly with respect to $(\operatorname{Re} z, \pm\operatorname{Im} z, h) \in J\times]0,1]\times]0,h_0]$, where $\chi \in C_0^\infty(\mathbb{R}^n)$ is equal to 1 for $|x| \leq 2\rho_0$. Moreover, $z \mapsto \operatorname{tr}\Big(T^\pm_\cdot(z)\Big)$ is holomorphic for $(\operatorname{Re} z, \pm\operatorname{Im} z) \in J\times]0,1]$.

In this section, given $s \in \mathbb{R}$, we denote by φ_s the multiplication operator by a $C^\infty(\mathbb{R}^n)$ function which is constant near $\overline{B(0,R_0)}$ and equal to $\langle x\rangle^s$ for $|x| \geq R > R_0$. Notice that for $s' \in \mathbb{R}$, the operator φ_s is invertible in $\mathcal{L}(\mathcal{H}^{0,s+s'}, \mathcal{H}^{0,s'})$. Given two bounded operators A, B, introduce the operators

$$G_\rho(z;B) = \chi_\rho(L_\cdot - z_0)^{-m}R_\cdot(z)P\tilde{R}_\cdot(z)B\chi_\rho, \tag{5.1}$$

$$G(z;A,B) = A(L_\cdot - z_0)^{-m}R_\cdot(z)P\tilde{R}_\cdot(z)B \tag{5.2}$$

which are clearly holomorphic for $\pm\operatorname{Im} z > 0$. The proof of Proposition 8 is based on a partition of the phase space into outgoing and incoming regions and on the following lemma.

Lemma 5. *i) Let $B \in \mathcal{L}(\mathcal{H})$ be such that $P\tilde{R}(z)B \in \mathcal{L}(\mathcal{H}^{0,-s}, \mathcal{H})$ with $s > \frac{1}{2}$ for any $(\operatorname{Re} z, \pm\operatorname{Im} z) \in J\times]0,1]$. Then $G_\rho(z;B)$ is trace class and we have*

$$\lim_{\rho\to+\infty} \operatorname{tr}G_\rho(z;B) = \operatorname{tr}G(z;{\varphi_s}^{-1}, B\varphi_s),$$

$$\|G(z;{\varphi_s}^{-1}, B\varphi_s)\|_{\mathrm{tr}} \leq Ch^{-2-n^\#_\cdot}\|P\tilde{R}(z)B\|_{-s,0}\,\Big(1 + \|\chi R_\cdot(z)\chi\|_{\mathcal{H}\to\mathcal{H}}\Big)$$

uniformly with respect to $(\operatorname{Re} z, \pm\operatorname{Im} z, h) \in J\times]0,1]\times]0,h_0]$. Moreover, the trace $\operatorname{tr}G(z;{\varphi_s}^{-1}, B\varphi_s)$ is holomorphic for $(\operatorname{Re} z, \pm\operatorname{Im} z) \in J\times]0,1]$.

ii) Let $s > \frac{1}{2}$, $\delta \geq 0$. Assume that for $(\operatorname{Re} z, \pm \operatorname{Im} z) \in J \times]0,1]$ we have $B \in \mathcal{L}(\mathcal{H}^{0,s})$ and $BR(z) \in \mathcal{L}(\mathcal{H}^{0,s+\delta}, \mathcal{H}^{0,s})$ with $B\chi_\rho^2 R(z)$ uniformly bounded with respect to ρ in $\mathcal{L}(\mathcal{H}^{0,s+\delta}, \mathcal{H}^{0,s})$. Then $G_\rho(z;B)$ is trace class and we have

$$\lim_{\rho \to +\infty} \operatorname{tr} G_\rho(z;B) = \operatorname{tr} G(z; \varphi_s B, {\varphi_s}^{-1}),$$

$$\|G(z;\varphi_s B, {\varphi_s}^{-1})\|_{\mathrm{tr}} \leq Ch^{-1-n^\#} \|BR(z)\chi\|_{0,s} + \mathcal{O}(e^{-\frac{C}{h}})\|BR(z)\|_{s+\delta,s}$$

uniformly with respect to $(\operatorname{Re} z, \pm \operatorname{Im} z, h) \in J \times]0,1] \times]0,h_0]$, for any $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on $\{x : |x| \leq 2\rho_0\}$ and the trace $\operatorname{tr} G(z;\varphi_s B, {\varphi_s}^{-1})$ is holomorphic for $(\operatorname{Re} z, \pm \operatorname{Im} z) \in J \times]0,1]$.

Proof. $i)$ The coefficients of P being compactly supported, there exists $\chi_1 \in C_0^\infty(\mathbb{R}^n)$ such that $P = \chi_1 P$ and we can write $G_\rho(z;B)$ in the following form

$$G_\rho(z;B) = \chi_\rho R_\cdot(z)\varphi_s^{-1}\varphi_s(L_\cdot - z_0)^{-m}\chi_1 P\tilde{R}_\cdot(z)B\chi_\rho.$$

Then $G_\rho(z;B)$ is trace class because $\chi_\rho R_\cdot(z)\varphi_s^{-1}$ is bounded according to Proposition 2, while Proposition 5 shows that the operator $\varphi_s(L_\cdot - z_0)^{-m}\chi_1$ is trace class one. Clearly, $P\tilde{R}_\cdot(z)B\chi_\rho \in \mathcal{L}(\mathcal{H}^{0,-s}, \mathcal{H}) \subset \mathcal{L}(\mathcal{H})$. Moreover, by the cyclicity of the trace, we have

$$\operatorname{tr} G_\rho(z;B) = \operatorname{tr}\Big(\chi_\rho \varphi_s^{-1} R_\cdot(z)(L_\cdot - z_0)^{-m} P\tilde{R}_\cdot(z)B\varphi_s\chi_\rho\Big) = \operatorname{tr}\Big(\chi_\rho G(z;\varphi_s^{-1}, B\varphi_s)\chi_\rho\Big). \tag{5.3}$$

On the other hand, for $s > \frac{1}{2}$, $G(z;\varphi_s^{-1}, B\varphi_s)$ is obviously trace class with trace norm satisfying

$$\|G(z;\varphi_s^{-1}, B\varphi_s)\|_{\mathrm{tr}} \leq \|R_\cdot(z)\|_{s,-s}\, \|\varphi_s(L_\cdot - z_0)^{-m}\chi_1\|_{\mathrm{tr}}\, \|P\tilde{R}_\cdot(z)B\|_{-s,0}.$$

Consequently, $\operatorname{tr} G_\rho(z;B)$ converges to $\operatorname{tr} G(z;\varphi_s^{-1}, B\varphi_s)$ and applying Propositions 5 and 2, we obtain the claimed estimate. Finally, for fixed h and $\pm \operatorname{Im} z > 0$ the resolvent $R(z)$, $\tilde{R}(z)$ are analytic and we deduce easily the analyticity of the trace $\operatorname{tr} G(z;\varphi_s^{-1})$ for $(\operatorname{Re} z, \pm \operatorname{Im} z) \in J \times]0,1]$.

ii) We observe that $G_\rho(z,B)$ is trace class by writing

$$G_\rho(z;B) = \Big(\chi_\rho R_\cdot(z)\varphi_{s+\delta}^{-1}\Big)\Big(\varphi_{s+\delta}(L_\cdot - z_0)^{-m}\chi_1\Big)\Big(P\tilde{R}_\cdot(z)\varphi_s^{-1}\Big)\Big(\varphi_s B\chi_\rho\Big).$$

The operator $\varphi_{s+\delta}(L_\cdot - z_0)^{-m}\chi_1$ is trace class according to Proposition 5, while the other operators are bounded as it is easy to see applying Proposition 2, Lemma 1 and our assumption. The trace being cyclic, we deduce

$$\operatorname{tr} G_\rho(z;B) = \operatorname{tr}\Big(\varphi_s B\chi_\rho^2 R_\cdot(z)(L_\cdot - z_0)^{-m} P\tilde{R}_\cdot(z)\varphi_s^{-1}\Big).$$

Next, our assumption implies that the operator in the R.H.S. of the above relation is uniformly bounded with respect to ρ in $\mathcal{L}(\mathcal{H})$ and it converges weakly to $G(z;\varphi_s B, \varphi_s^{-1})$ as $\rho \longrightarrow +\infty$. Then $\operatorname{tr} G_\rho(z;B)$ converges to $\operatorname{tr} G(z;\varphi_s B, \varphi_s^{-1})$ (see Chapter 6, [12]). Finally, for any $\chi \in C_0^\infty(\mathbb{R}^n)$ we get

$$G(z;\varphi_s B, \varphi_s^{-1}) = \varphi_s BR_\cdot(z)\chi(L_\cdot - z_0)^{-m}P\tilde{R}_\cdot(z)\varphi_s^{-1} + \varphi_s BR_\cdot(z)(1-\chi)(L_\cdot - z_0)^{-m}P\tilde{R}_\cdot(z)\varphi_s^{-1}. \tag{5.4}$$

Then, taking $\chi_1 \in C_0^\infty(\mathbb{R}^n)$ such that $\chi_1 P = P$, and χ such that $\chi_1 \prec \chi$, χ equal to 1 on a sufficiently large region, from Lemma 1 and Proposition 5 we deduce

$$\|G(z;\varphi_s B, \varphi_s^{-1})\|_{\mathrm{tr}} \leq C\|BR_\cdot(z)\chi\|_{0,s} h^{-n^\#_\cdot - 1} + \mathcal{O}(e^{-C/h})\|BR_\cdot(z)\|_{s+\delta,s}.$$

As above we can establish the analyticity of the trace $\operatorname{tr} G(z;\varphi_s B, {\varphi_s}^{-1})$ and this completes the proof. $\square$

Proof of Proposition 8. Let $f \in C_0^\infty(\mathbb{R}^+)$ be equal to 1 on $\overline{J}$. We have

$$\chi_\rho(L_\cdot - z_0)^{-m}R_\cdot(z)P\tilde{R}_\cdot(z)\psi\chi_\rho = G_\rho(z; f(\tilde{L})\psi) + G_\rho(z;(1-f(\tilde{L}))\psi). \tag{5.5}$$

Exploiting the ellipticity of $\tilde{L}$ and the functional calculus ([8], [19]), we observe that for $\operatorname{Re} z \in J$ the operator $P\tilde{R}_\cdot(z)(1-f(\tilde{L}))$ becomes a h-admissible pseudo-differential operator with compactly supported symbol. Since $a_\alpha\psi = 0$ for any α, it follows easily that for any $s \geq 0$ the operator $P\tilde{R}_\cdot(z)(1-f(\tilde{L}))\psi$ can be extended to a bounded one in $\mathcal{L}(\mathcal{H}^{0,-s},\mathcal{H})$, $s > 1/2$, with norm estimated by $\mathcal{O}(h^\infty)$, uniformly with respect to $(\operatorname{Re} z, \operatorname{Im} z, h) \in J\times]0,1]\times]0,h_0]$. Consequently, applying Lemma 5, i), we deduce

$$\lim_{\rho\to+\infty} \operatorname{tr} G_\rho(z;(1-f(\tilde{L}))\psi) = \operatorname{tr} G(z;{\varphi_s}^{-1},(1-f(\tilde{L}))\psi\varphi_s), \tag{5.6}$$

$$\|G(z;{\varphi_s}^{-1},(1-f(\tilde{L}))\psi\varphi_s)\|_{\mathrm{tr}} = \mathcal{O}(h^\infty)\,\Big(1+\|\chi R_{.}(z)\chi\|_{\mathcal{H}\to\mathcal{H}}\Big)$$

uniformly with respect to $(\operatorname{Re} z, \operatorname{Im} z, h) \in J\times]0,1]\times]0,h_0]$. Next, applying Lemma 5, i), once more we deduce the analyticity of the trace $\operatorname{tr} G(z;{\varphi_s}^{-1},(1-f(\tilde{L}))\psi\varphi_s)$ for $(\operatorname{Re} z, \pm\operatorname{Im} z) \in J\times]0,1]$.

The analysis of the term $G_\rho(z;f(\tilde{L})\psi)$ is more complicated. First, it is convenient to make a localization in the outgoing and incoming regions of the phase space

$$\Gamma^\pm(R,d,\sigma_\pm) = \{(x,\xi)\in\mathbb{R}^{2n} \,:\, |x|>R,\ d^{-1}<|\xi|<d,\ \frac{\langle x.\xi\rangle}{|x||\xi|} \gtrless \sigma_\pm\}. \tag{5.7}$$

Let $R>\rho_1>2\rho_0$ and $-1<\sigma_+<\sigma_-<1$. For any function $\psi\in C_b^\infty(\mathbb{R}^n)$ such that $\psi=0$ on a neighborhood of $\overline{B(0,R)}$, there exist $\omega_\pm(x,\xi)\in C^\infty(\mathbb{R}^{2n})$ supported in $\{(x,\xi)\in\mathbb{R}^{2n} : |x|>R,\ \frac{\langle x.\xi\rangle}{|x||\xi|}\gtrless\sigma_\pm\}$ such that

$$\psi(x) = \omega_+(x,hD_x)+\omega_-(x,hD_x).$$

Here for a symbol $a(x,\xi;h)$ we denote by $a(x,hD_x;h)$ the operator given by the Weyl quantization

$$a(x,hD_x;h)\varphi(x) = \frac{1}{(2\pi)^n}\int \exp(i\langle x-y,\xi\rangle)a\Big(\frac{x+y}{2},h\xi;h\Big)\varphi(y)dyd\xi,\ \ \varphi\in\mathcal{S}(\mathbf{R}^n).$$

From the functional calculus (see Appendix) we know that for any $N>0$,

$$f(\tilde{L}) = a_N(x,hD_x;h)+\tilde{r}_N(x,hD_x;h),$$

where the symbol $a_N(x,\xi;h)$ is localized in $\{(x,\xi)\in\mathbb{R}^{2n} \,:\, \tilde{l}_0(x,\xi)\in\operatorname{supp} f\}$, $\tilde{l}_0$ being the principal symbol of $\tilde{L}$ and the symbol $\tilde{r}_N(x,\xi;h)$ satisfies the following estimates: for any β_1, β_2 in $\mathbb{N}^n$ and $L\in\mathbb{N}$ there exists $C_{\beta_1,\beta_2,L,N}$ such that

$$|\partial_x^{\beta_1}\partial_\xi^{\beta_2}\tilde{r}_N(h,x,\xi)| \le C_{\beta_1,\beta_2,L,N}h^N\langle x\rangle^{-N}\langle\xi\rangle^{-L},$$

uniformly with respect to $(x,\xi)\in\mathbb{R}^{2n}$, $h\in]0,h_0]$. Moreover, under the assumptions (2.5), (2.6), we can choose $R>1$ and $d>1$ so that $\{(x,\xi)\in\mathbb{R}^{2n} \,:\, |x|\ge R,\ \tilde{l}_0(x,\xi)\in\operatorname{supp} f\}$ is contained in $\{(x,\xi)\in\mathbb{R}^{2n} : d^{-1}<|\xi|<d\}$. Then applying the theorem for composition of pseudodifferential operators ([8], [19]), we conclude that there exist sequences $\omega_{\pm,N;h}(x,\xi)\in C^\infty(\mathbb{R}^{2n})$ and $r_N(x,\xi;h)\in C^\infty(\mathbb{R}^{2n})$ satisfying the following conditions:

- $\operatorname{supp}\omega_{\pm,N} \subset \Gamma^\pm(R,d,\sigma_\pm)$,
- for any β_1, β_2 in $\mathbb{N}^n$ and $L\in\mathbb{N}$ there exists $C_{\beta_1,\beta_2,L}$ such that
$$|\partial_x^{\beta_1}\partial_\xi^{\beta_2}\omega_{\pm,N}(x,\xi;h)| \le C_{\beta_1,\beta_2,L}\langle x\rangle^{-|\beta_1|}\langle\xi\rangle^{-L},$$
$$|\partial_x^{\beta_1}\partial_\xi^{\beta_2}r_N(x,\xi;h)| \le C_{\beta_1,\beta_2,L,N}h^N\langle x\rangle^{-N}\langle\xi\rangle^{-L},$$
uniformly with respect to $(x,\xi)\in\mathbb{R}^{2n}$, $h\in]0,h_0]$,
- $f(\tilde{L})\psi = \omega_{+,N}(x,hD_x;h)\tilde{\psi}+\omega_{-,N}(x,hD_x;h)\tilde{\psi}+r_N(x,hD_x;h)\tilde{\psi}$ for any $\tilde{\psi}\succ\psi$.

Thus we have

$$G_\rho(z;f(\tilde{L})\psi) = G_\rho(z;\omega_{+,N}\tilde{\psi})+G_\rho(z;\omega_{-,N}\tilde{\psi})+G_\rho(z;r_N\tilde{\psi}). \tag{5.8}$$

For $s>\frac{1}{2}$ and $N>2s$, by construction, $R_N = r_N(x,hD_x;h)$ can be extended to an operator in $\mathcal{L}(\mathcal{H}^{0,-s},\mathcal{H}^{0,s})$ with norm equal to $\mathcal{O}(h^N)$. Next, the coefficients of P are compactly supported and Lemma 1 implies immediately that $P\tilde{R}(z)R_N\in\mathcal{L}(\mathcal{H}^{0,-s},\mathcal{H})$ with norm equal to $\mathcal{O}(h^{N-1})$ uniformly with respect to $(\operatorname{Re} z,\pm\operatorname{Im} z,h)\in J\times]0,1]\times]0,h_0]$. From Lemma 5, i) we deduce

$$\lim_{\rho\to+\infty}\operatorname{tr} G_\rho(z;r_N\tilde{\psi}) = \operatorname{tr} G(z;{\varphi_s}^{-1},r_N\tilde{\psi}\varphi_s), \tag{5.9}$$

$$\|G(z;{\varphi_s}^{-1}, r_N\tilde{\psi}\varphi_s)\|_{\rm tr} = \mathcal{O}(h^{N-3-n^\#})\ \Big(1+\|\chi R_\cdot(z)\chi\|_{\mathcal{H}\to\mathcal{H}}\Big).$$

uniformly with respect to $(\mathrm{Re}\, z, \pm\,\mathrm{Im}\, z, h) \in J\times]0,1]\times]0,h_0]$.

The coefficients $(a_\alpha(x))_{|\alpha|\leq p}$ being compactly supported in $\{x\in\mathbb{R}^n : R_0<|x|<2\rho_0\}$, an application of Lemma 2, *iii*) shows that $P\tilde{R}(z)\omega_{\pm,N}(x,hD_x)\tilde{\psi}\in\mathcal{L}(\mathcal{H}^{0,-s},\mathcal{H})$ for $\pm\,\mathrm{Im}\, z\geq 0$ with norm $\mathcal{O}(h^\infty)$ uniformly with respect to $(\mathrm{Re}\, z, \pm\,\mathrm{Im}\, z, h) \in J\times]0,1]\times]0,h_0]$. Then using Lemma 5, *i*), once more, we obtain for $\pm\,\mathrm{Im}\, z\geq 0$

$$\lim_{\rho\to+\infty}\mathrm{tr}G_\rho(z;\omega_{\pm,N}\tilde{\psi}) = \mathrm{tr}G(z;{\varphi_s}^{-1},\omega_{\pm,N}\tilde{\psi}\varphi_s), \tag{5.10}$$

$$\|G(z;{\varphi_s}^{-1},\omega_{\pm,N}\tilde{\psi}\varphi_s)\|_{\rm tr} = \mathcal{O}(h^\infty)\ \Big(1+\|\chi R_\cdot(z)\chi\|_{\mathcal{H}\to\mathcal{H}}\Big)$$

uniformly with respect to $(\mathrm{Re}\, z, \pm\,\mathrm{Im}\, z, h) \in J\times]0,1]\times]0,h_0]$. As above we conclude that traces

$$\mathrm{tr}G(z;{\varphi_s}^{-1},\omega_{\pm,N}\tilde{\psi}\varphi_s),\ \mathrm{tr}G(z;{\varphi_s}^{-1},r_N\tilde{\psi}\varphi_s)$$

are analytic for $\mathrm{Re}\, z\in J$, $\pm\,\mathrm{Im}\, z>0$.

To deal with the terms $G_\rho(z;\omega_{\pm,N}\tilde{\psi})$, $\pm\,\mathrm{Im}\, z\leq 0$, we will apply Lemma 5, *ii*). First, $\omega_{\pm,N}$ being a pseudodifferential operator, we conclude that $\omega_{\pm,N}\tilde{\psi}\in\mathcal{L}(\mathcal{H}^{0,s})$ for any $s\in\mathbb{R}$. Secondly, using Proposition 3 and the relation

$$\omega_{\pm,N}(x,hD_x)\tilde{\psi}R_\cdot(z) = \Big(R_\cdot(\overline{z})\tilde{\psi}\overline{\omega}_{\pm,N}(x,hD_x)\Big)^*,$$

we conclude that $\omega_{\pm,N}\tilde{\psi}R_\cdot(z)\in\mathcal{L}(\mathcal{H}^{0,s+\delta},\mathcal{H}^s)$ for $\delta>1$ and $s>\frac{1}{2}$ with norm equal to $\mathcal{O}(h^{-1})+\mathcal{O}(h^\infty)\ \|\chi R_\cdot(z)\chi\|_{\mathcal{H}\to\mathcal{H}}$. The same estimate holds uniformly with respect to ρ replacing $\omega_{\pm,N}$ by $\omega_{\pm,N}\chi_\rho^2$. In fact, the symbol of the pseudo-differential operator $\omega_{\pm,N}\chi_\rho^2$ is uniformly bounded in the same class of symbol as $\omega_{\pm,N}$ and, moreover, it has smaller support. Consequently, for $\pm\,\mathrm{Im}\, z\leq 0$ an application of Lemma 5, *ii*) implies

$$\lim_{\rho\to+\infty}\mathrm{tr}G_\rho(z;\omega_{\pm,N}\tilde{\psi}) = \mathrm{tr}G(z;\varphi_s\omega_{\pm,N}\tilde{\psi},\varphi_s^{-1}), \tag{5.11}$$

$$\|G(z;\varphi_s\omega_{\pm,N}\tilde{\psi},\varphi_s^{-1})\|_{\rm tr} = \mathcal{O}(h^{-1-n^\#})\ \|\omega_{\pm,N}\tilde{\psi}R_\cdot(z)\chi\|_{0,s}+\mathcal{O}(h^\infty)\ \|\omega_{\pm,N}\tilde{\psi}R_\cdot(z)\|_{s+\delta,s}$$

uniformly with respect to $(\mathrm{Re}\, z, \mp\,\mathrm{Im}\, z, h) \in J\times]0,1]\times]0,h_0]$. Taking into account *iii*) and *i*) of Proposition 3, we obtain

$$\|G(z;\varphi_s\omega_{\pm,N}\tilde{\psi},\varphi_s^{-1})\|_{\rm tr} = \mathcal{O}(h^\infty)\ \Big(1+\|\chi R_\cdot(z)\chi\|_{\mathcal{H}\to\mathcal{H}}\Big)$$

uniformly with respect to $(\mathrm{Re}\, z, \mp\,\mathrm{Im}\, z, h) \in J\times]0,1]\times]0,h_0]$. Combining the relations (5.5), (5.6), (5.8), (5.9), (5.10) and (5.11) and applying Lemma 5 for the analyticity of traces, we obtain the assertion of Proposition 8 with

$$\begin{aligned} T^\pm(z) \quad = \quad & \Big[\mathrm{tr}\Big(\varphi_s^{-1}R_j(z)(L_j-z_0)^{-m}P\tilde{R}_j(z)(1-f(\tilde{L}_j))\psi\varphi_s\Big) \\ & +\mathrm{tr}\Big(\varphi_s^{-1}R_j(z)(L_j-z_0)^{-m}P\tilde{R}_j(z)r_N(x,hD_x;h)\tilde{\psi}\varphi_s\Big) \\ & +\mathrm{tr}\Big(\varphi_s^{-1}R_j(z)(L_j-z_0)^{-m}P\tilde{R}_j(z)w_{\pm,N}(x,hD_x;h)\tilde{\psi}\varphi_s\Big) \\ & +\mathrm{tr}\Big(\varphi_s w_{\mp,N}(x,hD_x;h)\tilde{\psi}R_j(z)(L_j-z_0)^{-m}P\tilde{R}_j(z)\varphi_s^{-1}\Big)\Big]_{j=1}^2, \end{aligned} \tag{5.12}$$

for $\pm\,\mathrm{Im}\, z\geq 0$.

□

6. Weyl asymptotics for the spectral shift function

In this section we obtain a Weyl type asymptotic for the spectral shift function. We generalize the results of Christiansen [5] and Robert [21] covering the "black box" long-range perturbations of the Laplacian.

As in Section 2, introduce $\chi \in C_0^\infty(\{x : |x| \leq \rho_1\})$ equal to 1 for $|x| \leq 2\rho_0 < \rho_1$. The constant ρ_0 depends on the behaviour of the Hamiltonian trajectories of L_1 and L_2. In this section the operators $L_1^\sharp$ and $L_2^\sharp$ are defined on the torus $T_{\tilde{R}} = (\mathbb{R}/\tilde{R}\mathbb{Z})^n$ with $\tilde{R} > 2R > 2\rho_1$.

Theorem 2. *Assume that L_j, $j = 1, 2$ satisfy the assumptions* (2.1) $-$ (2.10) *and suppose that for some $\epsilon > 0$, $C > 0$, $q \geq 1$ the assumption* (1.3) *fulfilled. Moreover, assume that μ_0 and μ are non-critical energy levels for Q_j, $j = 1, 2$. Then for $h \in]0, h_0]$ we have the following asymptotic*

$$\xi(\mu) - \xi(\mu_0) = \Big[N(L_j^\sharp, [\mu_0, \mu])\Big]_{j=1}^2 - \Big[\mathrm{tr}\Big(\mathbf{1}_{[\mu_0,\mu]}(Q_j^\sharp)(1-\chi^2)\Big)\Big]_{j=1}^2 + w_\chi(\mu)h^{-n} + \mathcal{O}(h^{-n^\sharp+1}) \tag{6.1}$$

with $w_\chi(\mu) \in C^1(J)$.

Remarks.

- The smooth function $w_\chi(\mu)$ is given by the integral $\int_{\mu_0}^{\mu} d_0(\lambda)d\lambda$ of the main term $d_0(\lambda)$ in the asymptotic (1.2).
- Recall that $Q^\#$ is the differential operator
$$Q^\# = \sum_{|\nu|\leq 2} a_\nu^\#(x; h)(hD)^\nu$$
on the torus $T_{\tilde{R}} = (\mathbb{R}/\tilde{R}\mathbb{Z})^n$ with $\tilde{R} > 2R > 2\rho_1$ and $a_\nu^\#(x; h) = a_\nu(x; h)$ for $|x| < R$, $R > \rho_1 > 2\rho_0$. Then if μ_0 and μ are non critical for Q, it is clear that without loss of generality we can extend $a_\nu^\#$ on $T_{\tilde{R}}$ so that μ_0 and μ remain non-critical for $Q^\#$. Then using the well known method based on the construction of a parametrix for small t for the unitary group $e^{ith^{-1}Q^\#}$, we deduce
$$\mathrm{tr}\Big(\mathbf{1}_{[\mu_0,\mu]}(Q_j^\sharp)(1-\chi^2)\Big) = \Omega_j^\sharp(\mu)h^{-n} + \mathcal{O}(h^{-n+1})$$
with a function $\Omega_j^\sharp(\mu) \in C^1(J)$.
- If L_2 is a compactly supported perturbation of L_1, then for R large enough we have $Q_1(1-\chi^2) = Q_2(1-\chi^2)$ and we can construct $Q_j^\sharp$ and $\tilde{L}_j$ so that $Q_1^\sharp(1-\chi^2) = Q_2^\sharp(1-\chi^2)$ and $\tilde{L}_2 = \tilde{L}_1$. Consequently, the asymptotic (6.1) yields
$$\xi(\mu) - \xi(\mu_0) = \Big[N(L_j^\sharp, [\mu_0, \mu])\Big]_{j=1}^2 + \mathcal{O}(h^{-n^\sharp+1}).$$

Proof. We will examine the asymptotic of the integral

$$\begin{aligned}\int_{\mu_0}^{\mu} \xi'(\lambda)d\lambda = \int_{\mu_0}^{\mu} \Big[\mathrm{tr}(\chi E_j'(\lambda)\chi)\Big]_{j=1}^2 d\lambda + \int_{\mu_0}^{\mu} \mathrm{tr}\Big(T^+(\lambda + i0)\Big)d\lambda \\ - \int_{\mu_0}^{\mu} \mathrm{tr}\Big(T^-(\lambda - i0)\Big)d\lambda + w_\chi(\mu)h^{-n} + \mathcal{O}(h^{-n+1}),\end{aligned} \tag{6.2}$$

where $w_\chi(\mu)$ was defined in the first remark above. First, for fixed $h \in (0, h_0]$ notice that for any $N \in \mathbb{N}$ there exists $\epsilon = \epsilon(N, h) > 0$ such that

$$\left|\int_{\mu_0}^{\mu} \mathrm{tr}\Big(T^\pm(\lambda \pm i0)\Big)d\lambda - \int_{\mu_0}^{\mu} \mathrm{tr}\Big(T^\pm(\lambda \pm i\epsilon)\Big)d\lambda\right| \leq h^N$$

uniformly with respect to $\mu \in \overline{J}$. This follows from the limit

$$\lim_{\epsilon\to 0, \epsilon>0} \mathrm{tr}\Big(T^\pm(\lambda \pm i\epsilon)\Big) = \mathrm{tr}\Big(T^\pm(\lambda \pm i0)\Big)$$

which for fixed $h \in]0, h_0]$ is uniform with respect to $\lambda \in \overline{J}$.

For simplicity we will treat only the term involving T^+. Consider the rectangle

$$\Pi_\epsilon = \{z \in \mathbb{C} : \mu_0 \leq \operatorname{Re} z \leq \mu, \ 0 < \epsilon \leq \operatorname{Im} z \leq 1\}.$$

According to Proposition 8, the analyticity of $T^+(z)$ for $\operatorname{Im} z > 0$ and Stokes formula, we have

$$0 = \int_{\Pi_\epsilon} \partial_{\overline{z}} \Big(\operatorname{tr}(T^+(z))\Big) dz = \int_{\partial \Pi_\epsilon} \operatorname{tr} T^+(z) dz.$$

The integral over the segment $\{z = \mu + it : \epsilon \leq t \leq 1\}$ can be estimated by using the representation (see [21] for a similar argument)

$$\int_\epsilon^1 = \int_\epsilon^{C \exp(-Ch^{-p})} + \int_{C \exp(-Ch^{-p})}^1.$$

The first integral is estimated taking into account the bound

$$|\operatorname{tr}(T^+(z))| \leq C_N h^N \exp(Ch^{-p}), \ \forall N \in \mathbb{N},$$

where we have used (1.3) and the estimates (1.4). For the second integral we exploit the trivial estimate

$$\|\chi R(z) \chi\|_{\mathcal{H} \to \mathcal{H}} \leq \frac{C_1}{\operatorname{Im} z}, \ \operatorname{Im} z > 0.$$

Thus, using Theorem 1 and Lemma 3 of [2], we conclude that

$$\int_\epsilon^1 \operatorname{tr}\Big(T^+(\mu + it)\Big) dt = \mathcal{O}(h^\infty), \ \int_\epsilon^1 \operatorname{tr}\Big(T^+(\mu_0 + it)\Big) dt = \mathcal{O}(h^\infty), \ h \in]0, h_0],$$

where $\mathcal{O}(h^\infty)$ means that we have an estimate by $C_N h^N$, $\forall N \in \mathbb{N}$, uniformly with respect to $\epsilon > 0$. The analysis of the integral over the segment $\{z = \lambda + i, \ \mu_0 \leq \lambda \leq \mu\}$ is trivial and we conclude that the contribution of $\operatorname{tr} T^\pm(\lambda \pm i0)$ to the integral over (μ_0, μ) may be estimated by $\mathcal{O}(h^\infty)$. Similarly, exploiting the analyticity of $\operatorname{tr}(T^-(z))$ for $\operatorname{Im} z < 0$ we obtain the same result for the integral involving $\operatorname{tr}(T^-(\mu - i0))$.

To deal with the first term in the right hand side of (6.2), observe that

$$\int_{\mu_0}^\lambda \operatorname{tr}\Big(\chi E_j'(t) \chi\Big) dt = \operatorname{tr}\Big(\chi \mathbf{1}_{[\mu_0, \lambda]}(L_j) \chi\Big).$$

Let $I_{\mu_0} \ni \mu_0$ and $I_\lambda \ni \lambda$ be two small intervals which are non-critical for L_1 and L_2. We introduce $\varphi_1, \ \varphi_2, \ \varphi_3$ so that $\varphi_1 \in C_0^\infty(I_{\mu_0}; \mathbb{R}^+)$, $\varphi_2 \in C_0^\infty(]\mu_0, \lambda[; \mathbb{R}^+)$, $\varphi_3 \in C_0^\infty(I_\lambda; \mathbb{R}^+)$ and $\varphi_1 + \varphi_2 + \varphi_3 = 1$ on $[\mu_0, \lambda]$. Then the characteristic function $\mathbf{1}_{[\mu_0, \lambda]}$ becomes a sum of the smooth function φ_2 and two functions $\mathbf{1}_{[\mu_0, \lambda]} \varphi_1$, $\mathbf{1}_{[\mu_0, \lambda]} \varphi_3$ localized on non-critical intervals. Thus

$$\operatorname{tr}\Big(\chi \mathbf{1}_{[\mu_0, \lambda]}(L_j) \chi\Big) = \sum_{k=1}^3 \xi_{\varphi_k, \chi}(L_j, \lambda), \tag{6.3}$$

where

$$\xi_{\varphi, \chi}(L, \lambda) = \operatorname{tr}\Big(\chi \mathbf{1}_{[\mu_0, \lambda]}(L) \varphi(L) \chi\Big). \tag{6.4}$$

The analysis of $\xi_{\varphi_2, \chi}(L_j, \lambda)$ is simple since by definition $\xi_{\varphi_2, \chi}(L, \lambda) = \operatorname{tr}\Big(\chi \varphi_2(L) \chi\Big)$ is independent on λ. Then using the functional calculus for smooth functions and the estimates for the trace [25], we have

$$\xi_{\varphi_2, \chi}(L_j, \lambda) = \operatorname{tr}\Big(\chi \varphi_2(L_j^\sharp) \chi\Big) + \mathcal{O}(h^\infty) = \operatorname{tr}\Big(\varphi_2(L_j^\sharp)\Big) - \operatorname{tr}\Big(\varphi_2(Q_j^\sharp)(1 - \chi^2)\Big) + \mathcal{O}(h^\infty). \tag{6.5}$$

To treat $\xi_{\varphi_k, \chi}(L_j, \lambda)$ for $k = 1, 3$, we will exploit a Tauberian argument and for this purpose we will prove the following.

Proposition 9. *Let* $\xi_{\varphi,\chi}(\lambda) = \mathrm{tr}\Big(\chi \mathbf{1}_{[\mu_0,\lambda]}(L)\varphi(L)\chi\Big)$ *with* $\varphi \in C_0^\infty(I;\mathbb{R}^+)$, $I \subset\subset \mathbb{R}^+$ *and let*

$$N^\sharp_\varphi(\lambda) = \mathrm{tr}\Big(\mathbf{1}_{[\mu_0,\lambda]}(L^\sharp)\varphi(L^\sharp)\Big),\ \ M^\sharp_{\varphi,\chi}(\lambda) = \mathrm{tr}\Big(\mathbf{1}_{[\mu_0,\lambda]}(Q^\sharp)\varphi(Q^\sharp)(1-\chi^2)\Big).$$

Then there exists $h_0 > 0$ *such that*

$$\xi_{\varphi,\chi}(\lambda) = N^\sharp(\lambda) - M^\sharp_{\varphi,\chi}(\lambda) + \mathcal{O}(h^{-n^\sharp+1}),\ \forall h \in]0,h_0]. \tag{6.6}$$

End of the proof of Theorem 2. We apply Proposition 9 with $\varphi = \varphi_1, \varphi_3$ and the relations (6.3), (6.5) and conclude that

$$\mathrm{tr}\Big(\chi\mathbf{1}_{[\mu_0,\lambda]}(L_j)\chi\Big) = \mathrm{tr}\Big(\mathbf{1}_{[\mu_0,\lambda]}(L^\sharp_j)\Big) - \mathrm{tr}\Big(\mathbf{1}_{[\mu_0,\lambda]}(Q^\sharp_j)(1-\chi^2)\Big) + \mathcal{O}(h^{-n^\sharp+1}),\ j=1,2.$$

Combining this with the asymptotic expansion of the integrals of other terms in (6.2), we complete the proof. □

Proof of Proposition 9. Consider a function $\theta \in C_0^\infty(]-\epsilon_1,\epsilon_1[)$, $\theta(0)=1$, $\theta(-t)=\theta(t)$ such that the Fourier transform of θ satisfies $\hat{\theta}(\lambda) \geq 0$ on $\mathbb{R}$ and assume that there exist $\epsilon_0 > 0$, $\delta_0 > 0$ so that $\hat{\theta}(\lambda) \geq \delta_0 > 0$ for $|\lambda| \leq \epsilon_0$. Next introduce

$$\Big(\mathcal{F}_h^{-1}\theta\Big)(\lambda) = (2\pi h)^{-1}\int e^{it\lambda/h}\theta(t)dt = (2\pi h)^{-1}\hat{\theta}(-h^{-1}\lambda).$$

Obviously, the function $\xi_{\varphi,\chi}(\lambda)$ is increasing, $\xi_{\varphi,\chi}(\lambda)$ vanishes for $\lambda \leq \inf(\operatorname{supp}\varphi)$ and it coincides with a constant for $\lambda \geq \sup(\operatorname{supp}\varphi)$. We claim that $\xi_{\varphi,\chi}(\lambda)$ satisfies the following conditions

$$\xi_{\varphi,\chi}(\lambda) = \mathcal{O}(h^{-n^\sharp}), \tag{6.7}$$

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * \xi_{\varphi,\chi})(\lambda) = \mathcal{O}(h^{-n^\sharp}). \tag{6.8}$$

The estimate (6.7) is a consequence of the functional calculus (see [25]) and the fact that $\mathbf{1}_{[\mu_0,\lambda]}\varphi(\mu)$ is bounded from above by a smooth function $\Psi(\mu)$ equal to 1 on $\operatorname{supp}\mathbf{1}_{[\mu_0,\lambda]}\varphi(\mu)$. For the second estimate will established the following.

Lemma 6. *There exists* $\epsilon_1 > 0$ *such that for* $\theta \in C_0^\infty(]-\epsilon_1,\epsilon_1[)$ *we have*

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * \xi_{\varphi,\chi})(\lambda) = \frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * N^\sharp_\varphi)(\lambda) - \frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * M^\sharp_{\varphi,\chi})(\lambda) + \mathcal{O}(h^\infty)$$

uniformly with respect to $\lambda \in \overline{I}$.

Proof. For simplicity of the notations, we will omit the subscript φ and write ξ_χ, $N^\sharp$, $M^\sharp_\chi$ for $\xi_{\varphi,\chi}$, $N^\sharp_\varphi$, $M^\sharp_{\varphi,\chi}$. The definition of ξ_χ yields

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * \xi_\chi)(\lambda) = \mathcal{F}_h^{-1}\Big(\mathrm{tr}(\chi\theta(t)e^{-ith^{-1}L}\varphi(L)\chi)\Big).$$

For $\chi_1 \succ \chi$ we have the Duhamel formula

$$\chi_1 e^{-ith^{-1}L^\sharp}\varphi(L^\sharp)\chi = e^{-ith^{-1}L}\varphi(L)\chi + ih^{-1}\int_0^t e^{-i(t-s)h^{-1}L}[L^\sharp,\chi_1]e^{-ish^{-1}L^\sharp}\varphi(L^\sharp)\chi ds + B(t)$$

with $\|B(t)\|_{\mathrm{tr}} = \mathcal{O}(h^\infty)$. Here we have used the fact (see [26], Section 3) that

$$\chi_1\varphi(L^\sharp)\chi = \varphi(L)\chi + B(0), \tag{6.9}$$

$B(0)$ being a trace negligible operator. Let $\tilde{\varphi} \in C_0^\infty(I;\mathbb{R}^+)$ be a function equal to 1 on $\operatorname{supp}\varphi$. Then modulo a trace negligible operator we obtain

$$\chi e^{-ith^{-1}L}\varphi(L)\chi = \chi e^{-ith^{-1}L^\sharp}\varphi(L^\sharp)\chi$$

$$-ih^{-1}\int_0^t \chi e^{-i(t-s)h^{-1}L}\tilde{\varphi}(L)[L^\sharp,\chi_1]e^{-ish^{-1}L^\sharp}\varphi(L^\sharp)\chi ds,$$

since $\|\chi\tilde{\varphi}(L^\sharp)(1-\chi_1)\|_{\mathrm{tr}} = \mathcal{O}(e^{-C/h})$. Taking $\chi_1 \succ \chi_2 \succ \chi_3 \succ \chi$, we observe that

$$(1-\chi_2)[L^\sharp,\chi_1] = [L^\sharp,\chi_1],$$

and applying the Duhamel formula once more, we get

$$\chi e^{-ith^{-1}L}\tilde{\varphi}(L)(1-\chi_2) = \chi\tilde{\varphi}(L)(1-\chi_3)e^{-ith^{-1}Q}(1-\chi_2)$$

$$+ih^{-1}\int_0^t \chi e^{-i(t-s)h^{-1}L}\tilde{\varphi}(L)[Q,\chi_3]e^{-ish^{-1}Q}(1-\chi_2)ds,$$

since $L(1-\chi_3) = Q(1-\chi_3)$. Next the semi-classical Egorov theorem (see [8], [19]) implies for sufficiently small s

$$\|\tilde{\chi_3}e^{-ish^{-1}Q}(1-\chi_2)\|_{L^2\to L^2} = \mathcal{O}(h^\infty),$$

where $\tilde{\chi_3}$ is a cut-off function such that $\chi_2 \succ \tilde{\chi_3} \succ \chi_3$. Moreover,

$$\|\chi\tilde{\varphi}(L)(1-\chi_3)\|_{L^2\to L^2} = \mathcal{O}(h^\infty),$$

and we conclude that

$$\|\tilde{\varphi}(L)[Q,\chi_3]\|_{L^2\to L^2} = \mathcal{O}(1).$$

Finally, using the estimate $\|[L^\sharp,\chi_1]\varphi(L^\sharp)\|_{\mathrm{tr}} = \mathcal{O}(h^{-n^\sharp})$, we have

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * \xi_\chi)(\lambda) = \mathrm{tr}_{L^2(T_{\tilde{R}})}\Big(\chi\mathcal{F}_h^{-1}\theta(\lambda - L^\sharp)\varphi(L^\sharp)\chi\Big) + \mathcal{O}(h^\infty).$$

By exploiting the cyclicity of the trace this implies

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * \xi_\chi)(\lambda) = \frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * N^\sharp)(\lambda) - \mathrm{tr}_{L^2(T_{\tilde{R}})}\Big((1-\chi^2)\mathcal{F}_h^{-1}\theta(\lambda - L^\sharp)\varphi(L^\sharp)\psi\Big) + \mathcal{O}(h^\infty)$$

for any function $\psi \in C^\infty(T_{\tilde{R}})$, $\psi \succ (1-\chi^2)|_{T_{\tilde{R}}}$, $\psi = 0$ on $\overline{B(0,R_0)}$.

To treat the second term we will use the equality

$$\psi e^{-ith^{-1}L^\sharp}\varphi(L^\sharp)(1-\chi^2)|_{T_{\tilde{R}}} = \psi e^{-ith^{-1}Q^\sharp}\varphi(Q^\sharp)(1-\chi^2)|_{T_{\tilde{R}}}$$

$$+ih^{-1}\int_0^t \psi e^{-i(t-s)h^{-1}Q^\sharp}[L^\sharp,\psi_1]e^{-ish^{-1}L^\sharp}\varphi(L^\sharp)(1-\chi^2)|_{T_{\tilde{R}}}ds + B_1,$$

where $\psi_1 \succ \psi \succ (1-\chi_2)|_T$, $\psi_1 \in C^\infty(T_{\tilde{R}})$ on $\overline{B(0,R_0)}$, $\|B_1\|_{\mathrm{tr}} = \mathcal{O}(h^\infty)$. The trace of the second term in the right hand side is $\mathcal{O}(h^\infty)$ for $t-s$ small since by Egorov theorem we have

$$\|\psi e^{-i(t-s)h^{-1}Q^\sharp}\tilde{\psi}_1\|_{L^2\to L^2} = \mathcal{O}(h^\infty),$$

where $\tilde{\psi}_1 = 0$ on supp ψ_1.

Consequently, we deduce

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * \xi_\chi)(\lambda) = \frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * N^\sharp)(\lambda) - \mathrm{tr}_{L^2(T_{\tilde{R}})}\Big((1-\chi^2)\mathcal{F}_h^{-1}\theta(\lambda - Q^\sharp)\varphi(Q^\sharp)\psi\Big) + \mathcal{O}(h^\infty).$$

and this completes the proof of Lemma 6. □

End of the proof of Proposition 9. According to the assumption (2.9), we have

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * N^\sharp)(\lambda) = (2\pi h)^{-1}\int e^{it\lambda h^{-1}}\theta(t)\mathrm{tr}\Big(e^{-ith^{-1}L^\sharp}\varphi(L^\sharp)\Big)dt = \mathcal{O}(h^{-n^\sharp}).$$

Recall that $Q^\sharp$ is a differential operator on the n-dimensional manifolds $T_{\tilde{R}}$, so by a standard argument we obtain

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * M_\chi^\sharp)(\lambda) = \mathcal{O}(h^{-n}).$$

Then, applying a Tauberian theorem (see Theorem IV. 13 in [19]) for $N^\sharp$, $M^\sharp_\chi$ and exploiting Lemma 6 for ξ_χ, we deduce

$$N^\sharp(\lambda) = (\mathcal{F}_h^{-1}\theta * N^\sharp)(\lambda) + \mathcal{O}(h^{-n^\sharp+1}),$$
$$M^\sharp_\chi(\lambda) = (\mathcal{F}_h^{-1}\theta * M^\sharp_\chi)(\lambda) + \mathcal{O}(h^{-n+1}),$$
$$\xi_\chi(\lambda) = (\mathcal{F}_h^{-1}\theta * \xi_\chi)(\lambda) + \mathcal{O}(h^{-n^\sharp+1}).$$

Finally, applying Lemma 6 once more and integrating, we get

$$\xi_\chi(\lambda) = N^\sharp(\lambda) - M^\sharp_\chi(\lambda) + \mathcal{O}(h^{-n^\sharp+1})$$

and the proof of Proposition 9 is complete. □

In the case of compactly supported perturbations of the Laplacian (or if $\gamma > n$ in the assumption (2.6)) Theorem 2 holds without the assumption (1.3). More precisely, we have the following.

Theorem 3. *Assume that $L_1 = L$ satisfy the assumptions (2.1) − (2.10) with $L_2 = L_0 = -h^2\Delta$. Assume that μ_0 and μ are non-critical energy levels for Q_1. Then we have the following asymptotic*

$$\xi(\mu) - \xi(\mu_0) = N(L^\sharp, [\mu_0, \mu]) - \operatorname{tr}\Big(\mathbf{1}_{[\mu_0,\mu]}(Q^\sharp)(1-\chi^2)\Big) - \operatorname{tr}\Big(\mathbf{1}_{[\mu_0,\mu]}(L_0^\sharp)\chi^2\Big) \tag{6.10}$$
$$+ w_\chi(\mu)h^{-n} + \mathcal{O}(h^{-n^\sharp+1}),\ h \in]0, h_0]$$

with a function $w_\chi(\mu) \in C^1(J)$.

Remark. If the perturbation is compactly supported we can construct $L^\sharp$ so that $Q^\sharp = L_0^\sharp = -h^2\Delta|_{T_{\tilde{R}}}$ and $\tilde{L} = L_0$. Then the asymptotic (6.10) becomes

$$\xi(\mu) - \xi(\mu_0) = N(L^\sharp, [\mu_0, \mu]) - c_n \mathrm{vol}(T_{\tilde{R}})(\mu^{\frac{n}{2}} - {\mu_0}^{\frac{n}{2}})h^{-n} + \mathcal{O}(h^{-n^\sharp+1}),\ h \in]0, h_0],$$

where $c_n = (2\pi)^{-n}\ \mathrm{Vol}(B(0,1))$.

Proof. The analysis of this case is simpler because we can express $\xi(\lambda)$ by a sum of monotone functions. This decomposition was established by the first author [1] in the case $\mathcal{H} = L^2(\mathbb{R}^n)$, while the "black box" scattering has been studied by Christiansen [5] for compactly supported perturbations in high energy régime. For $\mu \in J$ we have

$$\xi(\mu) - \xi(\mu_0) = \Big[\sigma_1(L_j, \mu_0, \mu)\Big]_{j=0}^1 + \Big[\sigma_{W_j}(L_j, \mu_0, \mu)\Big]_{j=0}^1, \tag{6.11}$$

where

$$\sigma_1(L_j, \mu_0, \mu) = \operatorname{tr}(\chi \mathbf{1}_{[\mu_0,\mu]}(L_j)\chi),$$

χ being as in Theorem 1, and

$$\sigma_{W_j}(L_j, \mu_0, \mu) = \operatorname{tr}(W_j(h)\mathbf{1}_{[\mu_0,\mu]}(L_j)),\ j = 0, 1.$$

Here

$$W_1(h) = h^{-1}[\chi^2, L]\mathcal{A}_h L^{-1}\varphi(L) + (1-\chi^2)(L - L_0 - h^{-1}[\mathcal{A}_h, L - L_0])L^{-1}\varphi(L)$$
$$W_0(h) = h^{-1}[\chi^2, L_0]\mathcal{A}_h L_0^{-1}\varphi(L_0),$$

$\mathcal{A}_h := -\frac{1}{4}(x.h\nabla + h\nabla.x)$ and $\varphi \in C_0^\infty(\mathbb{R})$ is equal to 1 on J. This representation is due to Robert [20], [22] and it is based on the relation $[\mathcal{A}_h, L_0] = hL_0$ and the cyclicity of the trace.

Now, consider the terms $\sigma_1(L, \mu_0, \mu)$ and $\sigma_{W_1}(L, \mu_0, \mu)$ and remark that a similar representation holds for L replaced by L_0. As in Theorem 2, applying Proposition 9, we have

$$\sigma_1(L, \mu_0, \mu) = N(L^\sharp, [\mu_0, \mu]) - \operatorname{tr}\Big(\mathbf{1}|_{[\mu_0,\mu]}(Q^\sharp)(1-\chi^2)\Big) + \mathcal{O}(h^{-n^\sharp+1}).$$

To treat σ_{W_1}, notice that W_1 is localized outside $\overline{B(0, R_0)}$. Then in the representation of W_1 we can replace L by Q_1 modulo $\mathcal{O}(h^\infty)$ and W_1 becomes a classical h-admissible pseudodifferential operator of order 0 with symbol in the space $S(\langle x\rangle^{-\delta})$, $\delta > n$, localized in the non-trapping region $\{x : |x| \geq 2\rho_0\}$. Here we used the notation

$$S(\langle x\rangle^{-\delta}) = \{s \in C^\infty(\mathbb{R}^{2n}) : | \partial_x^\alpha \partial_\xi^\beta s(x,\xi) | \leq C_{\alpha,\beta}\langle x\rangle^{-\delta-|\alpha|}\langle\xi\rangle^{-|\beta|}, \ \forall\alpha, \ \forall\beta\}. \tag{6.12}$$

Following Lemma 4.2 of [1], we decompose σ_{W_1} as follows

$$\sigma_{W_1}(L,\mu_0,\mu) = \mathrm{tr}(W_+(h)\mathbf{1}_{[\mu_0,\mu]}(L)) - \mathrm{tr}(W_-(h)\mathbf{1}_{[\mu_0,\mu]}(L)) + \mathcal{O}(h^\infty),$$

where $W_\pm(h)$ are classical, non-negative, h-admissible pseudodifferential operators of order 0 with symbols in $S(\langle x\rangle^{-\delta})$, $\delta > n$, supported in the non-trapping region.

By a Tauberian argument for the monotone functions $\sigma_\pm(\mu) := \mathrm{tr}(W_\pm(h)\mathbf{1}_{[\mu_0,\mu]}(L))$, we deduce

$$\sigma_\pm(\mu) = (\mathcal{F}_h^{-1}\theta * \sigma_\pm)(\mu) + \mathcal{O}(h^{-n+1}).$$

On the other hand, we have

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * \sigma_\pm)(\lambda) = \mathcal{F}_h^{-1}\Big(\mathrm{tr}(\theta(t)W_\pm e^{-ith^{-1}L})\Big).$$

Next, as in the proof of Theorem 2, for sufficiently small t we apply Egorov theorem and conclude that

$$\mathrm{tr}(W_\pm e^{-ith^{-1}L}) = \mathrm{tr}(W_\pm e^{-ith^{-1}Q}) + \mathcal{O}(h^\infty).$$

Consequently, exploiting the construction of a parametrix for the propagator $e^{ith^{-1}Q}$ for small t (see Proposition 4.3 in [1]), we get

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * \sigma_\pm)(\mu) = h^{-n}\beta_0(\mu, W_\pm) + \mathcal{O}(h^{-n+1})$$

with $\beta_0(., W_\pm) \in C^\infty(J)$. Finally,

$$\sigma_{W_1}(L,\mu_0,\mu) = h^{-n}w_\chi(\mu) + \mathcal{O}(h^{-n+1})$$

with $w_\chi \in C^\infty(J)$ and the proof is complete. □

7. Appendix

Define the class of symbols $S_\nu^{p,l}$ as the set of smooth functions $b(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for any β_1, β_2 in $\mathbb{N}^n$ we have

$$|\partial_x^{\beta_1}\partial_\xi^{\beta_2} b(x,\xi)| \leq C_{\beta_1,\beta_2}\langle\xi\rangle^p\langle x\rangle^{l-\nu|\beta_1|}.$$

Let $b \in S_\nu^{2,0}$ be an elliptic symbol in the sense that $|b(x,\xi)| \geq C\langle\xi\rangle^2$. Denote by $B = b(x, hD_x)$ the operator which is the Weyl quantization of $b(x,\xi)$, that is

$$b(x,hD_x)\varphi(x) = \frac{1}{(2\pi)^n}\int \exp(i\langle x-y,\xi\rangle) b\Big(\frac{x+y}{2}, h\xi\Big)\varphi(y)dyd\xi, \ \ \varphi \in \mathcal{S}(\mathbf{R}^n).$$

Choosing $m(x,\xi) = \langle\xi\rangle^p\langle x\rangle^l$, for $\nu = 0$ the class $S_0^{p,l}$ coincides with the class $S^0(m)$ introduced in [8]. Then, according to [8], we know that given $f \in C_0^\infty(\mathbb{R})$ and $b \in S_0^{2,0}$, the operator $f(B)$ becomes a pseudodifferential operator with symbol $a \in S_0^{-k,0}$ for any $k \in \mathbb{N}$.

In the proof of Proposition 8, we needed the following more precise estimate concerning the remainder of the symbol of the operator $f\Big(b(x,hD_x)\Big)$ with $b \in S_\nu^{2,0}$, $\nu > 0$.

Lemma 7. *Let $b \in S_\nu^{2,0}$ be elliptic. Then for any $N \in \mathbb{N}$ we have*

$$f\Big(b(x,hD_x)\Big) = a_N(x,hD_x;h) + h^N r_N(x,hD_x;h), \tag{7.1}$$

where for any $k \in \mathbb{N}$ we have $a_N \in S_\nu^{-k,0}$ and $r_N \in S_0^{-k,-\nu N}$ uniformly with respect to $h \in (0,h_0]$.

Proof. Following the calculus in [8] and applying the Helffer-Sjöstrand's formula, it is sufficient to prove the representation (7.1) for the resolvent $(B-z)^{-1}$, $\operatorname{Im} z \neq 0$, $|z| \leq Const$ with an estimate with respect to z. On the other hand, the parametrix construction exposed in [19], [24] (see also [7]) yields

$$(B-z)^{-1} = a_N(x, hD_x; h) + h^N (B-z)^{-1} \delta_N(x, hD_x; h),$$

where for any β_1, β_2 in $\mathbb{N}^n$ we have the estimates

$$|\partial_x^{\beta_1} \partial_\xi^{\beta_2} a_N(x, \xi; h)| \leq C_{\beta_1,\beta_2} \sum_{j=0}^{N-1} h^j \langle x\rangle^{-\nu j - \nu|\beta_1|} |\operatorname{Im} z|^{-(|\beta_1|+|\beta_2|)-2j-1}$$

$$\leq C_{\beta_1,\beta_2,N} \operatorname{Max}\Big(1, \frac{\sqrt{h}}{|\operatorname{Im} z|}\Big)^{(N-1)/2} |\operatorname{Im} z|^{-(|\beta_1|+|\beta_2|)-1} \langle x\rangle^{-\nu|\beta_1|}.$$

Moreover, there exist $\rho \geq 3/2$ and N_0 such that for any β_1, β_2 in $\mathbb{N}^n$ we get

$$|\partial_x^{\beta_1} \partial_\xi^{\beta_2} \delta_N(x, \xi; h)| \leq C_{\beta_1,\beta_2,N} |\operatorname{Im} z|^{-(|\beta_1|+|\beta_2|)-\rho N - N_0} \langle x\rangle^{-\nu N - \nu|\beta_1|}.$$

Next, exploiting the functional calculus of [8], we know that $(B-z)^{-1}$ is the Weyl quantization of a symbol $q(x, \xi; h)$ satisfying

$$|\partial_x^{\beta_1} \partial_\xi^{\beta_2} q(x, \xi; h)| \leq C_{\beta_1,\beta_2} \operatorname{Max}\Big(1, \frac{\sqrt{h}}{|\operatorname{Im} z|}\Big)^{2n+1} |\operatorname{Im} z|^{-(|\beta_1|+|\beta_2|)-1}.$$

We complete the proof by using the composition of pseudodifferential operators. □

References

[1] V. Bruneau, *Semi-classical behaviour of the scattering phase for trapping perturbations of the Laplacian*, Comm. in P.D.E., **24**, 5 & 6, (1999), 1095-1125.

[2] V. Bruneau, V. Petkov, *Semiclassical resolvent estimates for trapping perturbations*, Commun. Math. Phys. **213** (2000), 413-432.

[3] N. Burq, *Absence de résonances près du réel pour l'opérateur de Schrödinger*, Exposé XVII, Séminaire EDP, Ecole Polytechnique, 1997/1998.

[4] N. Burq and M. Zworski, *Resonance expansions in semi-classical propagation*, Preprint, 2000.

[5] T. Christiansen, *Spectral asymptotics for general compactly supported perturbations of the Laplacian on* $\mathbb{R}^n$, Comm. P.D.E. **23** (1998), 933-947.

[6] H. L. Cycon, R. G. Froese, W. Kirsch, B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*, Texts and Monographs in Physics, Berlin, Heidelberg, Springer-Verlag, 1987.

[7] M. Dauge, D. Robert, *Weyl's formula for a class of pseudodifferential operators with negative order in* $L^2(\mathbb{R}^n)$, Lecture Notes in Mathematics, **1256**, Berlin, Springer 1986, 91-122.

[8] M. Dimassi, J. Sjöstrand, *Spectral asymptotics in the semi-classical limit*, London Mathematical Society, Lecture Notes Series **268**, Cambridge University Press, 1999.

[9] C. Gérard, A. Martinez, *Principe d'absorption limite pour des opérateurs de Schrödinger à longue portée*, C. R. Acad. Sci. Paris, **306** (1988), 121-123.

[10] C. Gérard, A. Martinez, *Semi-classical asymptotics of the spectral function of long range Schrödinger operators*, J. Funct. Anal. **84** (1989), 226-254.

[11] C. Gérard, A. Martinez and D. Robert, *Breit-Wigner formulas for the scattering poles and total scattering cross-section in the semi-classical limit*, Commun. Math. Phys. **121** (1989), 323-336.

[12] I.C. Gohberg, M.G. Krein, *Introduction à la théorie des opérateurs linéaires non-autoadjoints*, Paris, Dunod, 1971.

[13] P. Hislop, S. Nakamura, *Semiclassical resolvents estimates*, Ann. Inst. H. Poincaré (Physique théorique), **51** (1989), 187-198.

[14] A. Jensen, E. Mourre, P. Perry, *Multiple Commutator Estimates and Resolvent Smoothness in Quantum Scattering Theory*, Ann. Inst. H. Poincaré (Physique théorique), **41** (1984), 207-225.

[15] E. Mourre, *Absence of singular continuous spectrum for certain selfadjoint operators*, Commun. Math. Phys. **78** (1981), 391-408.

[16] P. Perry, I. M. Sigal, B. Simon, *Spectral analysis of N-body Schrödinger operators*, Ann. Math. **114** (1981), 519-567.

[17] V. Petkov and M. Zworski, *Semi-classical estimates on the scattering determinant*, Preprint, 2000.

[18] M. Reed, B. Simon, *Methods of Modern Mathematical Physics T. 1*, San Diego, Academic Press, 1979.

[19] D. Robert, *Autour de l'approximation semi-classique* PM, **68**, Basel, Birkhäuser, 1987.

[20] D. Robert, *Asymptotique de la phase de diffusion à haute énergie pour des perturbations du second ordre du Laplacien*, Ann. Sci. Ecole Norm. Sup., **25** (1992), 107-134.
[21] D. Robert, *Relative time-delay for perturbations of elliptic operators and semiclassical asymptotics*, J. Funct. Anal. **126** (1994), 36-82.
[22] D. Robert, *On the Weyl formula for obstacles*, Prog. Nonlinear Differential Equations Appl., **21** (1996), 264-285.
[23] D. Robert, H. Tamura, *Asymptotic behavior of the scattering amplitudes in semi-classical and low energy limits*, Ann. Inst. Fourier, **39** (1989), 155-192.
[24] R.T. Seeley, *Complex powers of an elliptic operator*, Proc. Symp. Pure Math. AMS, **10** (1967), 288-307.
[25] J. Sjöstrand, *A trace formula and review of some estimates for resonances*, in *Microlocal analysis and spectral theory (Lucca, 1996),* 377–437, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 490, Dordrecht, Kluwer Acad. Publ.1997.
[26] J. Sjöstrand, *Resonances for bottles and trace formulae*, Math. Nachrichten, (to appear).
[27] J. Sjöstrand and M. Zworski, *Complex scaling and the distribution of scattering poles*, J. Amer. Math. Soc. **4** (1991), 729-769.
[28] P. Stefanov, *Resonances expansions and Rayleigh waves*, Preprint, 2000.
[29] S. Tang and M. Zworski, *From quasimodes to resonances*, Math. Res. Lett. **5** (1998), 261-272.
[30] M. Wollenberger, H. Baumgartel, *Mathematical Scattering Theory*, Basel, Bostons, Stuttgart, Birkhäuser, 1983.
[31] D.R. Yafaev, *Mathematical Scattering Theory, general theory*, Translations of Mathematical Monograph **105**, AMS, 1992.

Département de Mathématiques Appliquées, Université Bordeaux I,
351, Cours de la Libération, 33405 Talence, FRANCE
vbruneau@math.u-bordeaux.fr
petkov@math.u-bordeaux.fr