Averaging Principle for Quasi-Geostrophic Motions under Rapidly Oscillating Forcing*

Hongjun Gao[1] and Jinqiao Duan[2]

1. Department of Mathematics
Nanjing Normal University
Nanjing 210097, China
2. Department of Applied Mathematics
Illinois Institute of Technology
Chicago, IL 60616, USA
E-mail: duan@iit.edu

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Abstract

In this paper, the averaging principle for quasi-geostrophic motions with rapidly oscillating forcing is proved, both on finite but large time intervals and on the entire time axis. This includes comparison estimate, stability estimate, and convergence result between quasi-geostrophic motions and its averaged motions. Furthermore, the existence of almost periodic quasi-geostrophic motions and attractor convergence are also investigated.

Key words: Quasi-geostrophic fluid flows, almost periodic motions, rapidly oscillating forcing, averaging principle, stable manifolds and unstable manifolds

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1 Introduction

The quasi-geostrophic(QG) equation models large scale geophysical flows. It is derived as an approximation of the rotating Navier-Stokes equations by an asymptotic expansion in a small Rossby number. The barotropic QG equation is written in terms of stream function $\psi(x,y,t)$ ([1], 16, p. 234):

$$\triangle\psi_t + J(\psi,\triangle\psi) + \beta\psi_x = \nu\triangle^2\psi - r\triangle\psi + f(x,y,t), \tag{1.1}$$

where $\beta > 0$ the meridional gradient of the Coriolis parameter, $\nu > 0$ the viscous dissipation constant, $r > 0$ the Ekman dissipation constant, $f(x,y,t)$ the wind forcing, and $J(f,g) = f_x g_y - f_y g_x$ is the Jacobian operator.

Equation (1.1) can be rewritten in terms of the relative vorticity $\omega(x,y,t) = \triangle\psi(x,y,t)$ as

$$\omega_t + J(\psi,\omega) + \beta\psi_x = \nu\triangle\omega - r\omega + f(x,y,t). \tag{1.2}$$

For an arbitrary bounded planar domain D with sufficiently regular (such as, piecewise smooth) boundary ∂D this equation can be supplemented by homogenous Dirichlet boundary conditions for both ψ and ω, namely, the no-penetration and slip boundary conditions proposed by Pedlosky [2], p. 34:

$$\psi(x,y,t) = 0, \;\; \omega(x,y,t) = 0, \text{ on } \partial D, \tag{1.3}$$

together with an appropriate initial condition,

$$\omega(x,y,0) = \omega_0(x,y), \text{ in } D. \tag{1.4}$$

The global well-posedness (i.e., existence and uniqueness of smooth solution) of the dissipative model (1.2)-(1.4) can be obtained similarly as in, for example, [3], [4], [5]; see also [6]. Steady wind forcing has been used in numerical simulations([7]). Brannan et al [8] considered the effect of quasi-geostrophic dynamics under random forcing. Duan et al [9] and [10] obtained the existence of time periodic, time almost periodic quasi-geostrophic response of time periodic and time almost periodic wind forcing respectively.

In this paper, we assume that the right-hand side or the forcing term of the QG flow model (1.1) is rapidly oscillating, i.e., it has the form $f(x,y,t) = f(x,y,\eta t) \overset{\triangle}{=} f(\eta t)$, with parameter $\eta \gg 1$. We also assume that f has a time average. With such forcing, it is desirable to understand the fluid dynamics in some averaged sense, and compare the averaged flows with the original unaveraged flows.

The main result of this paper is the averaging principle for quasi-geostrophic motions with rapidly oscillating forcing, both on finite but large time intervals and on the entire time axis. This includes comparison estimate, stability estimate, and convergence result (as $\eta \to \infty$) between quasi-geostrophic motions and its averaged motions. We also investigate the existence of almost periodic quasi-geostrophic motions under almost periodic forcing, and the convergence of the attractor of the non-autonomous equation (1.1) to the attractor of the averaged autonomous equation as $\eta \to \infty$.

In §2, we study the averaging principle for the QG flow model on finite but large time intervals and in §3, we extend the averaging principle to the entire time axis. In the rest of this section, we briefly review some background and provide some preliminaries for later use.

Starting from the fundamental work of Bogolyubov [11] the averaging theory for ODE has been developed and generalized in a large number of works (see [12]–[14] and the references therein). Bogolyubov's main theorems have been generalized in [15] to the case of differential equations with bounded operator-valued coefficients. Some problems of averaging of differential equations with unbounded operator-valued coefficients have been considered in [16]–[18] in the framework of abstract parabolic equations. In [19], Ilyin considered the averaging principle for an equation of the form

$$\partial_t u = N(u) + f(\eta t), \tag{1.5}$$

where f is a given right-hand side and $\eta \gg 1$ is a large dimensionless parameter, and f satisfies

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(s)ds = f_0. \tag{1.6}$$

Standard abbreviations $L^2 = L^2(D)$, $H_0^k = H_0^k(D), k = 1, 2...$, are used for the common Sobolev spaces, with $< \cdot, \cdot >$ and $\|\cdot\|$ denoting the usual scalar product and norm, respectively, in L^2. We need the following properties and estimates (see [4]) of the Jacobian operator $J : H_0^1 \times H_0^1 \to L^1$:

$$\int_D J(f,g)hdxdy = -\int_D J(f,h)gdxdy, \int_D J(f,g)gdxdy = 0, \tag{1.7}$$

$$|\int_D J(f, gdxdy| \le \|\nabla f\|\|\nabla g\|, \tag{1.8}$$

for all $f, g, h \in H_0^1$, and

$$|\int_D J(\nabla f, g)\triangle hdxdy| \le \sqrt{\frac{2|D|}{\pi}}\|\triangle f\|\|\triangle g\|\|\triangle h\|, \tag{1.9}$$

for all $f, g, h \in H_0^2$. We also recall the *Poincaré inequality* [20]

$$\|g\|^2 \le \frac{|D|}{\pi}\int_D |\nabla g|dxdy = \frac{|D|}{\pi}\|\nabla g\|, \tag{1.10}$$

for $g \in H_0^1$, and the *Young's inequality* [20]

$$AB \le \frac{\epsilon}{2}A^2 + \frac{1}{2\epsilon}B^2, \tag{1.11}$$

where A, B are nonnegative numbers and $\epsilon > 0$.

We can further rewrite the QG flow model (1.2). From

$$\triangle\varphi = \omega, \quad (x, y) \in D, \quad \varphi|_{\partial D} = 0, \tag{1.12}$$

we get $\varphi = \triangle^{-1}\omega$. Thus (1.2) can be rewritten as

$$\omega_t + J(\triangle^{-1}\omega, \omega) + \beta\partial_x\triangle^{-1}\omega = \nu\triangle\omega - r\omega + f(x,y,t). \tag{1.13}$$

Let

$$-\mathcal{A} = \nu\triangle - rI - \beta\partial_x\triangle^{-1}.$$

Then by a result in [16], we know that $\mathcal{A}$ is a sectorial operator, and hence $e^{-\mathcal{A}t}$ generates an analytic semigroup in L^2.

We will give a sufficient condition to ensure the smallest eigenvalue of $\mathcal{A}$ to be positive. Consider the eigenvalue equation $\mathcal{A}u = \lambda u$. We have the following energy estimate

$$\begin{aligned}
\lambda\|u\|^2 &= \nu\|\nabla u\|^2 + r\|u\|^2 - \int_D \triangle^{-1}u\partial_x u dxdy \\
&\geq \nu\|u\|^2 + r\|u\|^2 - \beta\|\triangle^{-1}u\|\nabla u\| \\
&\geq \nu\|\nabla u\|^2 + r\|u\|^2 - \frac{\beta|D|}{\pi}\|u\|\|\nabla u\| \\
&\geq \nu\|\nabla u\|^2 + r\|u\|^2 - \frac{\beta|D|}{\pi}(a_1\|u\|^2 + a_2\|\nabla u\|^2) \\
&\geq (\nu - \frac{\beta|D|}{\pi}a_1)\|\nabla u\|^2 + (r - \frac{\beta|D|}{\pi}a_2)\|u\|^2,
\end{aligned} \tag{1.14}$$

where we used the Poincaré inequality (1.10) after the second inequality sign above, and where arbitrary constants a_1, a_2 satisfy $a_1a_2 = \frac{1}{4}$. Therefore, when

$$4\nu r > \frac{\beta^2|D|^2}{\pi^2}, \tag{1.15}$$

and if we take $a_2 = \frac{\beta|D|}{4r\pi}$, then we have

$$\lambda\|u\|^2 \geq (\nu - \frac{\beta^2|D|^2}{4r\pi^2})\|\nabla u\|^2 \geq (\nu - \frac{\beta^2|D|^2}{4r\pi^2})\frac{|D|}{\pi}\|u\|^2. \tag{1.16}$$

So, when ν, r, β and $|D|$ satisfy the condition (1.15), the smallest eigenvalue of $\mathcal{A}$ is positive. In this case, the QG flow model

$$\omega_t + \mathcal{A}\omega + J(\triangle^{-1}\omega, \omega) = f(x,y,t)$$

is a dissipative dynamical system.

We note that the condition (1.15) is sharper than the corresponding condition in Duan and Kloeden [10]. We define the fractional power of $\mathcal{A}$ as follows [16]:

$$\mathcal{A}^\alpha = (\mathcal{A}^{-\alpha})^{-1}, \text{ where } \mathcal{A}^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-\mathcal{A}t}dt.$$

The corresponding domains $D(\mathcal{A}^\alpha)$ are Banach spaces with norm given by

$$\|x\|_\alpha := \|x\|_{D(\mathcal{A}^\alpha)} = \|\mathcal{A}^\alpha x\|.$$

For the rest of this section, we recall some definitions and useful results for later use.

Theorem 1.1 *[16] The following estimates are valid:*

$$\|e^{-\mathcal{A}t}\|_{L^2 \to L^2} \leq Ke^{-at}, \qquad t \geq 0, \tag{1.17}$$

$$\|\mathcal{A}^{\alpha} e^{-\mathcal{A}t}\|_{L^2 \to L^2} \leq \frac{K_{\alpha}}{t^{\alpha}} e^{-at}, \ \ t > 0, \tag{1.18}$$

where K, K_{α} are positive constants.

Theorem 1.2 *[16] Given two sectorial operators A and B in L^2, let $D(A) = D(B)$, $Re\sigma(A) > 0, Re\sigma(B) > 0$, and for some $\alpha \in [0, 1)$. Let the operator $(A - B)A^{-\alpha}$ be bounded in L^2. Then for every $\gamma \in [0, 1)$, $D(A^{\gamma}) = D(B^{\gamma})$, the two norms being equivalent.*

Definition 1.3 *[17] A continuous function $f : \mathbb{R} \to X$ is called almost periodic(a.p.), if for every $\epsilon > 0$ there exists a number $l = l(\epsilon) > 0$ such that each interval $(T, T + l)$ contains a point $\tau = \tau_{\epsilon}$(called an almost period) satisfying the inequality*

$$Sup_{t \in \mathbb{R}} \|f(t + \tau) - f(t)\| \leq \epsilon.$$

If f depends on other arguments, then the above inequality holds uniformly with respect to norms.

It follows from the theory of a.p. functions that there exists a countable set of number λ_{α} for which

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t) e^{i\lambda_{\alpha} t} dt \neq 0.$$

The numbers $\{\lambda_{\alpha}\}$ are called the Fourier exponents of f[17].

Definition 1.4 *[17] A countable set of numbers $\{\omega_{\alpha}\}$ is called the frequency basis (denoted by $\mathcal{M}_f$) of an a. p. function f if every λ_{α} can be uniquely respected by a linear combination of the numbers ω_{α} with integer coefficients.*

Definition 1.5 *[17] For a given a.p. function f, the sequence $\{t_m\}$ is called $f-$current if*

$$Sup_{t \in \mathbb{R}} \|f(t + t_m) - f(t)\| \leq \epsilon_m \to 0, \ as\ m \to \infty.$$

Theorem 1.6 *[17] Given two almost periodic functions f and g, suppose that every $f-$current sequence is also a $g-$current sequence. Then the frequency basis of g is contained in that of f: $\mathcal{M}_g \subset \mathcal{M}_f$.*

We now turn to the averaging principle for the QG flow model.

2 Averaging Principle on Finite Time Intervals

In this section, we consider the averaging principle for the QG flow model on finite (but large) time intervals. In the next section, we extend the result to the entire time axis. We assume that the right-hand side or the forcing term of the QG flow model (1.1) or (1.2) is rapidly oscillating, i.e., it has the form $f(x,y,t) = f(x,y,\eta t) = f(\eta t)$, with parameter $\eta \gg 1$. Let $\eta \gg 1$ be a large dimensionless parameter. Setting

$$\tau = \eta t, \quad \epsilon = \eta^{-1},$$

we obtain the equation in the so-called standard form

$$\omega_\tau + \epsilon \mathcal{A}\omega + \epsilon J(\triangle^{-1}\omega, \omega) = \epsilon f(x,y,t). \tag{2.1}$$

We assume that f has a time average in $D(\mathcal{A}^\gamma)$; the value of γ will be specified later on. More precisely, let $f(\tau), f_0 \in \mathcal{A}^\gamma$ and suppose that

$$\|\mathcal{A}^\gamma(\frac{1}{T}\int_t^{t+T} f(\tau)d\tau - f_0)\| \leq \min(M_\gamma, \sigma_\gamma(T)), \tag{2.2}$$

where $M_\gamma > 0, \sigma_\gamma(T) \to 0,$ as $T \to \infty$.

We consider the averaged equation

$$\bar{\omega}_\tau + \epsilon \mathcal{A}\bar{\omega} + \epsilon J(\triangle^{-1}\bar{\omega}, \bar{\omega}) = \epsilon f_0(x,y). \tag{2.3}$$

By the method and result of [21], [22] and [23], we obtain the semigroup S_t corresponding to equation (2.3) possesses absorbing sets in the space $H = L^2, V = D(\mathcal{A}^{\frac{1}{2}}) = H_0^1$ and $D(\mathcal{A})$, $\|\cdot\|$ and $\|\cdot\|_{D(\mathcal{A})}$ denote the norm in V and $D(\mathcal{A})$. These sets are certain balls $B(R_0)$ in these spaces, where R_0 is large enough. This means that for every bounded set B

$$S_t B \subset B(R_0), \text{ for } t > t_0(B, R_0).$$

In addition, the semigroup is uniformly bounded in these spaces, that is, given any ball, in particular, the ball $B(R_0)$, there exists a ball $B(R)$ such that

$$S_t B(R_0) \subset B(R), \text{ for } t > 0.$$

By increasing R we may assume that

$$S_t B(R_0) \subset B(R - \rho), \text{ for } t > 0, \rho > 0,$$

where ρ is a positive constant. We first consider the averaging principle in the space V. Given a point ω_0 in $B_V(R_0)$, we compare the trajectories (solutions) $\omega(\tau)$, $\bar{\omega}(\tau)$ of system (2.1) and (2.3) starting from this initial point. Consider their difference on the interval $\tau \in [0, \frac{T}{\epsilon}]$, T being arbitrary but fixed. We suppose for the moment that $\omega(\tau) \in B_V(R)$. Then the difference $z(\tau) = \omega(\tau) - \bar{\omega}(\tau)$ satisfies the equation

$$\partial_t z + \epsilon \mathcal{A} + \epsilon[J(\triangle^{-1}\omega, \omega) - J(\triangle^{-1}\bar{\omega}, \bar{\omega})] = \epsilon(f(\tau) - f_0(\tau)). \tag{2.4}$$

We first have some estimates on the nonlinear terms.

Lemma 2.1 *The nonlinear operator $J(u,v)$ is a bounded Lipschitz map in the following sense:*

$$\|J(u_1,v_1)-J(u_2,v_2)\| \leq$$
$$C_{\frac{1}{2}}(\|u_1\|_{\frac{1}{2}}+\|u_2\|_{\frac{1}{2}}+\|v_1\|_{\frac{1}{2}}+\|v_2\|_{\frac{1}{2}})(\|u_1-v_1\|_{\frac{1}{2}}+\|u_2-v_2\|_{\frac{1}{2}}), \tag{2.5}$$
$$\|J(u_1,v_1)-J(u_2,v_2)\|_{\frac{1}{2}} \leq$$
$$C_0(\|u_1\|_{D(\mathcal{A})}+\|u_2\|_{D(\mathcal{A})}+\|v_1\|_{D(\mathcal{A})}+\|v_2\|_{D(\mathcal{A})})(\|u_1-v_1\|_{D(\mathcal{A})}+\|u_2-v_2\|_{D(\mathcal{A})}), \tag{2.6}$$

where $C_{\frac{1}{2}}$ and C_0 are some positive constants.

Proof. Since

$$\begin{aligned} J(u_1,u_2)-J(v_1,v_2) &= \\ (u_{1x}-v_{1x})u_{2y}+(u_{1y}-v_{1y})v_{1x} &- (u_{2x}-v_{2x})u_{1y}-(u_{2y}-v_{2y})v_{2x}, \end{aligned} \tag{2.7}$$

(2.5) and (2.6) are obtained by direct estimtes. Here the equivalence of norms $\|\cdot\|_{H^2}$ and $\|\cdot\|_{D(\mathcal{A})}$ is used. ∎

Now we get back to equation (2.4). Inverting the linear operator we come to an equivalent integral equation

$$\begin{aligned} z(\tau) &= \epsilon\int_0^\tau e^{-\epsilon\mathcal{A}(\tau-s)}[J(\triangle^{-1}\omega,\omega)-J(\triangle^{-1}\bar{\omega},\bar{\omega})]ds \\ &+ \epsilon\int_0^\tau e^{-\epsilon\mathcal{A}(\tau-s)}(f(s)-f_0)ds. \end{aligned} \tag{2.8}$$

Using (1.18) and (2.5), the $D(\mathcal{A}^{\frac{1}{2}})$-norm of the first term in the right hand side satisfies the inequality

$$\begin{aligned} &\|\epsilon\int_0^\tau \mathcal{A}^{\frac{1}{2}}e^{-\epsilon\mathcal{A}(\tau-s)}[J(\triangle^{-1}\omega,\omega)-J(\triangle^{-1}\bar{\omega},\bar{\omega})]ds\| \\ &\leq \epsilon\int_0^\tau K_{\frac{1}{2}}\epsilon^{-\frac{1}{2}}(\tau-s)^{-\frac{1}{2}}e^{-\epsilon a(\tau-s)}2R\|z(s)\|_{\frac{1}{2}}ds \\ &= 2RK_{\frac{1}{2}}\epsilon^{\frac{1}{2}}\int_0^\tau(\tau-s)^{-\frac{1}{2}}e^{-\epsilon a(\tau-s)}\|z(s)\|_{\frac{1}{2}}ds. \end{aligned} \tag{2.9}$$

Let us estimate the second term in the right hand side of (2.8). Integrating by parts we have

$$\begin{aligned} &\|\epsilon\int_0^\tau e^{-\epsilon\mathcal{A}(\tau-s)}(f(s)-f_0)ds\|_{\frac{1}{2}} \\ &= \|-\epsilon e^{-\epsilon\mathcal{A}(\tau-s)}\int_s^\tau(f(t)-f_0)dt|_0^\tau+\epsilon^2\int_0^\tau \mathcal{A}e^{-\epsilon\mathcal{A}(\tau-s)}\int_s^\tau(f(s)-f_0)ds\|_{\frac{1}{2}} \\ &\leq \|\epsilon\mathcal{A}^{\frac{1}{2}-\gamma}e^{-\epsilon\mathcal{A}\tau}\mathcal{A}^\gamma\int_0^\tau(f(t)-f_0)dt\| \\ &+\|\epsilon^2\int_0^\tau \mathcal{A}^{\frac{3}{2}-\gamma}e^{-\epsilon\mathcal{A}(\tau-s)}\mathcal{A}^\gamma\int_s^\tau(f(s)-f_0)ds\|. \end{aligned} \tag{2.10}$$

Using (1.18) and (2.2), we further have

$$\|\epsilon\mathcal{A}^{\frac{1}{2}-\gamma}e^{-\epsilon\mathcal{A}\tau}\mathcal{A}^{\gamma}\int_0^{\tau}(f(t)-f_0)dt\| \le \epsilon K_{\frac{1}{2}-\gamma}e^{-\epsilon a\tau}(\epsilon\tau)^{\gamma-\frac{1}{2}}\|\frac{1}{\tau}\int_0^{\tau}\mathcal{A}^{\gamma}(f(t)-f_0)dt\|$$

$$= (\epsilon\tau)^{\frac{1}{2}+\gamma}K_{\frac{1}{2}-\gamma}e^{-\epsilon a\tau}\|\frac{1}{\tau}\int_0^{\tau}\mathcal{A}^{\gamma}(f(t)-f_0)dt\|$$

$$\le (\epsilon\tau)^{\frac{1}{2}+\gamma}K_{\frac{1}{2}-\gamma}\min(M_\gamma,\sigma_\gamma(\tau))e^{-\epsilon a\tau} =: L(\tau). \tag{2.11}$$

For any $\delta > 0$, let τ_δ be so large that for $\tau \ge \tau_\delta, \sigma_\gamma \le \delta$. Let ϵ_0 be so small that for $\epsilon < \epsilon_0$ then inequality $\frac{T}{\epsilon} > \tau_\delta$ is valid. Then

$$L(\tau) \le G_{\gamma 1}(T,\epsilon) = e^{-\epsilon a\tau}\begin{cases} T^{\frac{1}{2}+\gamma}K_{\frac{1}{2}-\gamma}\delta, & \text{if } \tau \ge \tau_\delta, \\ (\epsilon\tau)^{\frac{1}{2}+\gamma}K_{\frac{1}{2}-\gamma}M_\gamma, & \text{if } \tau < \tau_\delta. \end{cases}$$

Let $\gamma > -\frac{1}{2}$. Since τ_δ does not depend on ϵ, we let $\delta \to 0$ and then $\epsilon \to 0$. We obtain

$$\|\epsilon e^{-\epsilon\mathcal{A}\tau}\int_0^{\tau}(f(t)-f_0)dt\|_{\frac{1}{2}} \le G_{\gamma 1}(T,\epsilon) \to 0 \text{ when } \epsilon \to 0. \tag{2.12}$$

$$\|\epsilon^2\int_0^{\tau}\mathcal{A}^{\frac{3}{2}-\gamma}e^{-\epsilon\mathcal{A}(\tau-s)}\mathcal{A}^{\gamma}\int_s^{\tau}(f(s)-f_0)ds\| \le K_{\frac{3}{2}-\gamma}\epsilon^{\frac{1}{2}+\gamma}\int_0^{\tau}\min(M_\gamma,\sigma_\gamma(u))u^{\gamma-\frac{1}{2}}du$$

$$\le K_{\frac{3}{2}-\gamma}M_\gamma\epsilon^{\frac{1}{2}+\gamma}\int_0^{\tau_\mu}u^{\gamma-\frac{1}{2}}du + K_{\frac{3}{2}-\gamma}\epsilon^{\frac{1}{2}+\gamma}\mu\int_0^{\frac{T}{\epsilon}}u^{\gamma-\frac{1}{2}}du$$

$$= K_{\frac{3}{2}-\gamma}(\gamma+\frac{1}{2})^{-1}((\epsilon\tau_\mu)^{\frac{1}{2}+\gamma}+\mu T^{\frac{1}{2}+\gamma}) =: G_{\gamma 2}(T,\epsilon), \tag{2.13}$$

where for any $\mu > 0$ we have chosen τ_μ so large that $\sigma_\gamma(\tau) < \mu$ when $\tau > \tau_\mu$. Letting $\mu \to 0$ and then $\epsilon \to 0$ we obtain

$$G_{\gamma 2}(T,\epsilon) \to 0, \ \ \epsilon \to 0.$$

Thus, by (2.8)–(2.13) we obtain the following inequality:

$$\|z(\tau)\|_{\frac{1}{2}} \le K\epsilon^{\frac{1}{2}}\int_0^{\tau}(\tau-s)^{-\frac{1}{2}}\|z(s)\|_{\frac{1}{2}}ds + G_\gamma(T,\epsilon), \tag{2.14}$$

where $K = 2RK_{\frac{1}{2}}$ and $G_\gamma = G_{\gamma 1} + G_{\gamma 2} \to 0, \epsilon \to 0$.

We need the following fact.

Lemma 2.2 *[16] Let $\gamma \in (0,1]$ and for $t \in [0,T]$*

$$u(t) \le a + b\int_0^t (t-s)^{\gamma-1}u(s)ds.$$

Then

$$u(t) \le aE_\gamma((b\Gamma(\gamma))^{\frac{1}{\gamma}}t),$$

where the function $E_\gamma(z)$ is monotone increasing and $E_\gamma(z) \sim \gamma^{-1}e^z$ as $z \to \infty$.

Applying this lemma to the inequality (2.14) on $\tau \in [0, \frac{T}{\epsilon}]$, we obtain

$$\|z(t)\|_{\frac{1}{2}} \leq G_\gamma(T,\epsilon) E_{\frac{1}{2}}(\epsilon\tau\pi K^2) \leq G_\gamma(T,\epsilon) E_{\frac{1}{2}}(T\pi K^2) = \eta_T^1(\epsilon). \tag{2.15}$$

Using (1.18) and (2.6), we can do the same in $D(\mathcal{A})$ assuming $\gamma > 0$ in (2.2) and obtain

$$\|z(t)\|_{D(\mathcal{A})} \leq F_\gamma(T,\epsilon) E_{\frac{1}{2}}(\epsilon\tau\pi K^2) = \eta_T^2(\epsilon),\ F_\gamma(T,\epsilon) \to 0 \text{ as } \epsilon \to 0. \tag{2.16}$$

We thus have proved the proximity of solutions of (2.1) and (2.3) in V and $D(\mathcal{A})$, assuming that the trajectory $\omega(t)$ with initial condition $\omega(0) \in B_{V,D(\mathcal{A})}(R_0)$ stays in the ball $B(R)$ on the interval $[0, \frac{T}{\epsilon}]$.

Let ϵ be so small that the right-hand side of (2.15) and (2.16) are less than $\frac{\rho}{2}$, where ρ is defined earlier in this section when we discuss absorbibg sets. Suppose that the trajectory $\omega(t)$ leaves the ball $B(R)$ during the interval $[0, \frac{T}{\epsilon}]$ and let τ^* be the first momemt where $\|\omega(\tau^*) = R$. However, on the interval $\tau \in [0, \tau^*]$ both trajectories stay in the ball $B(R)$ and what we have proved so far shows that the inequality $\|\omega(\tau) - \bar{\omega}(\tau)\| \leq \frac{\rho}{2}$ is valid. In particular, it is valid for $\tau = \tau^*$. This together with the inequality $\|\bar{\omega}(\tau^*)\| \leq R - \rho$, which holds by the hypothesis of the following theorem and the property of the semigroup $S(t)$, gives the contradiction

$$\|\omega(\tau^*)\| \leq \|\omega(\tau^*) - \bar{\omega}(\tau^*)\| + \|\bar{\omega}(\tau^*)\| \leq R - \frac{\rho}{2}.$$

Therefore we have the following main result in this section.

Theorem 2.3 *(Averaging principle on finite time intevals) Let the right-hand side of equation (2.1) has an average in the sense of (2.2). Let $T > 0$ be arbitrary and fixed.*

If $\gamma > -\frac{1}{2}$ and $\omega(0) = \bar{\omega}(0) \in B_V(R_0)$, that is, the initial values coincide and belong to the absorbing ball, then for $\tau \in [0, \frac{T}{\epsilon}]$,

$$\|\omega(\tau) - \bar{\omega}(\tau)\|_{\frac{1}{2}} \leq \eta_T^1(\epsilon) \to 0 \text{ as } \epsilon \to 0.$$

If $\gamma > 0$, and $\omega(0) = \bar{\omega}(0) \in B_{D(\mathcal{A})}(R_0)$, then for $\tau \in [0, \frac{T}{\epsilon}]$,

$$\|\omega(\tau) - \bar{\omega}(\tau)\|_{D(\mathcal{A})} \leq \eta_T^2(\epsilon) \to 0 \text{ as } \epsilon \to 0,$$

where $\eta_T^1(\epsilon)$ and $\eta_T^2(\epsilon)$ are defined in (2.15), (2.16), respectively.

This theorem gives comparison estimate and convergence result (as $\eta \to 0$) between the QG flows and averaged QG flows, on finite but large time intevals.

3 Averaging Principle on the Entire Time Axis

Now we turn to averaging principle for the QG flows on the entire time axis. Consider

$$\omega_\tau + \epsilon\mathcal{A}\omega + \epsilon J(\triangle^{-1}\omega, \omega) = \epsilon f(x,y,t). \tag{3.1}$$

All the hypotheses concerning the data of the problem are the same as those in §2; in particular, the average $f_0(x,y)$ exists in the sense of (2.2).

We first consider stationary *averaged* QG flows:

$$\mathcal{A}\omega_0 + J(\triangle^{-1}\omega_0, \omega_0) = f_0. \tag{3.2}$$

Note that this is *not* the stationary QG flow model, but the time-independent version of the *averaged* QG flow model.

Under the condition of (1.15) and using Leray-Schauder fixed point theorem and elliptic regularity([20]), we know the stationary *averaged* QG flow model (3.2) has a unique stationary solution. Note that this unique stationary solution is denoted as $\omega_0(x,y)$ and it is not to be confused with an initial datum for the time-dependent QG flow model.

We change the dependent variable (from ω to z) in equation (3.1) via:

$$\omega = \omega_0 + z.$$

Then by (3.2), we find that z satisfies the equation

$$\partial_t z = \epsilon(-\mathcal{A}z - J(\triangle^{-1}\omega, \omega) + J(\triangle^{-1}\omega_0, \omega_0) + f(\tau) - f_0).$$

Since

$$J(\triangle^{-1}\omega, \omega) - J(\triangle^{-1}\omega_0, \omega_0) = J(\triangle^{-1}\omega, z) + J(\triangle^{-1}z, \omega_0),$$

we have

$$\partial_t z = \epsilon(-\mathcal{A}z - J(\triangle^{-1}\omega, z) - J(\triangle^{-1}z, \omega_0) + f(\tau) - f_0).$$

Changing the dependent variable once again (from z to h) via,

$$z = h - \epsilon v(\tau, \epsilon),$$

we obtain

$$\partial_t z = \partial_t h - \epsilon\partial_t v = \epsilon(-\mathcal{A}h + \epsilon\mathcal{A}v - J(\triangle^{-1}\omega, h) - J(\triangle^{-1}h, \omega_0)$$
$$+(\epsilon J(\triangle^{-1}\omega, v) + \epsilon J(\triangle^{-1}v, \omega_0) + f - f_0).$$

We chose the auxiliary function $v(\tau,\epsilon)$ to satisfy the equation

$$\partial_\tau v = -\epsilon\mathcal{A}v + f_0 - f. \tag{3.3}$$

Then we obtain the following equation for the new dependent variable h:

$$\partial_\tau h = -\epsilon(\mathcal{A}h - J(\triangle^{-1}\omega_0, h) - J(\triangle^{-1}h, \omega_0)) +$$
$$\epsilon(-J(\triangle^{-1}h, h) + J(\epsilon v, h) + \epsilon J(\triangle^{-1}(\omega_0 - \epsilon v), v) + \epsilon J(\triangle^{-1}h, v) + \epsilon J(\triangle^{-1}v, \omega_0)). \tag{3.4}$$

For the rest of the section, we study the equation (3.4) for the new dependent variable h. Now, we first consider equation (3.3) for the auxiliary function $v(\tau,\epsilon)$.

Lemma 3.1 *Assume that the function f has an average in the sense of (2.2). If $\alpha-\gamma<1$, then equation (3.3) has a unique solution $v(\tau,\epsilon)$ bounded in $D(\mathcal{A}^\alpha)$ uniformly in $\tau\in R$. Moreover,*

$$\|\epsilon v(\tau,\epsilon)\|_\alpha \to 0, \ \text{as } \epsilon\to 0. \tag{3.5}$$

If f is almost periodic with values in $D(\mathcal{A}^\gamma)$, then v is almost periodic in $D(\mathcal{A}^\alpha)$ with frequency basis contained in that of f.

Proof. The desired solution is given by the formula

$$v(\tau,\epsilon)=\int_{-\infty}^{\tau} e^{-\epsilon\mathcal{A}(\tau-s)}(f_0-f))ds. \tag{3.6}$$

The uniqueness will be proved in a more general context in Lemma 3.2. below. Now we prove (3.5). Integrating by parts and using (2.2) and (1.18), we obtain

$$\begin{aligned}
\|\epsilon v(\tau,\epsilon)\|_\alpha &= \|\epsilon\int_{-\infty}^{\tau} e^{-\epsilon\mathcal{A}(\tau-s)}(f_0-f))ds\|_\alpha \\
&= \|\int_0^\infty e^{-\epsilon\mathcal{A}s}(f_0-f(\tau-s))ds\|_\alpha \\
&= \|\epsilon^2\int_0^\infty \mathcal{A}e^{-\epsilon\mathcal{A}s}\int_0^s(f_0-f(\tau-t))dtds\|_\alpha \\
&= \|\epsilon^2\int_0^\infty \mathcal{A}^{1+\alpha-\gamma}e^{-\epsilon\mathcal{A}s}s(s^{-1}\mathcal{A}^\gamma\int_0^s(f_0-f(\tau-t))dt)ds\| \\
&\le \epsilon^{1+\alpha-\gamma}K_{1+\alpha-\gamma}\int_0^\infty s^{\gamma-\alpha}e^{-\epsilon as}\min(M_\gamma,\sigma(s))ds \\
&\le K_{1+\alpha-\gamma}(\epsilon^{1+\alpha-\gamma}M_\gamma(1+\alpha-\gamma)^{-1}s_0^{\gamma-\alpha+1}+\delta a^{1+\alpha-\gamma}\Gamma(1+\alpha-\gamma)),
\end{aligned}$$

where δ is small and $s_0=s_0(\delta)$ is so large that $\sigma(s)\delta$ when $s>s_0$. Letting $\delta\to 0$ and then $\epsilon\to 0$, we obtain (3.5).

Finally, let us prove the last statement of Lemma 3.1. By Theorem 1.6, it is sufficient to show that every $f-$recurrent sequence $\{\tau_m\}$ is also $v-$recurrent. By (3.6),

$$v(\tau+\tau_m)-v(\tau)=\int_0^\infty e^{-\epsilon\mathcal{A}s}(f(\tau-s)-f(\tau+\tau_m-s))ds.$$

Therefore

$$\sup_{\tau\in R}\|v(\tau+\tau_m)-v(\tau)\|_\alpha\le\sup_{\tau\in R}\|f(\tau+\tau_m)-f(\tau)\|_\gamma K_{\alpha-\gamma}\Gamma(\gamma-\alpha+1)(\epsilon a)^{\alpha-\gamma}.$$

Thus $\{\tau_m\}$ is indeed $v-$recurrent. The proof of Lemma 3.1 is complete. ■

Now we go back to study the equation (3.4) for the new dependent variable h. We consider the operator

$$\mathcal{L}h=\mathcal{A}h-J(\triangle^{-1}h,\omega_0)-J(\triangle^{-1}\omega_0,h). \tag{3.7}$$

The operator $\mathcal{L}+\lambda I$ has a compact inverse in $H=L^2$ for $\lambda>\lambda_0=\frac{\pi}{2\lambda_1^3|D|}\|f_0\|^2$, where $\lambda_1\overset{\triangle}{=}\nu-\frac{\gamma^2|D|^2}{4r\pi^2}$. In fact, we consider the equation

$$\mathcal{A}h-J(\triangle^{-1}h,\omega_0)-J(\triangle^{-1}\omega_0,h)+\lambda h=f.$$

Multiplying by h in L^2 and observing that $(J(\triangle^{-1}\omega_0, h), h) = 0$, we obtain

$$(\mathcal{A}h, h) - (J(\triangle^{-1}h, \omega_0), h) + \lambda\|h\|^2 = (f, h).$$

Using $H^2 \hookrightarrow L^\infty$, we find that

$$-(J(\triangle^{-1}h, \omega_0), h) \leq \|\nabla h\| \|\nabla \omega_0\|_{\frac{1}{2}} \|h\| \leq \frac{\lambda_1}{2}\|\nabla h\|^2 + \frac{1}{2\lambda_1}\|h\|^2 \|\nabla \omega_0\|^2.$$

By (1.16) and (3.2) we have

$$(\mathcal{A}h, h) \geq \lambda_1 \|\nabla h\|^2, \ \ \|\nabla \omega_0\|^2 \leq \frac{\pi}{\lambda_1^2 |D|}\|f_0\|^2.$$

For $\lambda \geq \lambda_0$, $((\mathcal{L} + \lambda)h, h)$ is coercive and by the Lax-Milgram lemma the equation

$$((\mathcal{L} + \lambda)h, h) = f$$

has a unique solution $h \in D(\mathcal{A}^{\frac{1}{2}})$. Note that the embedding $D(\mathcal{A}^{\frac{1}{2}}) \to L^2$ is compact. Hence $\mathcal{L}$ has a compact resolvent. Using (1.9), we can estimate $J(\triangle^{-1}h, \omega_0)$ as follows.

$$|J(\triangle^{-1}h, \omega_0) + J(\triangle^{-1}\omega_0, h)| \leq \|h\| \|\nabla \omega_0\| + \|\omega_0\| \|\nabla h\|.$$

The operator $\mathcal{A}$ is sectorial, and due to the above estimates , we see that the operator $\mathcal{L}$ is also sectorial ([16]). Moreover, if $Re\lambda > \lambda_0$, then the operator $\mathcal{L}_\lambda = \mathcal{L} + \lambda I$ is invertible; hence

$$Re\ \sigma(\mathcal{L}_{\lambda_0}) > 0.$$

We also noe that

$$(\mathcal{L}_{\lambda_0} - \mathcal{A})\mathcal{A}^{-\alpha} = \lambda_0 \mathcal{A}^{-\alpha} - J(\mathcal{A}^{-\alpha}\triangle^{-1}\cdot, \omega_0) - J(\triangle^{-1}\omega_0, \mathcal{A}^{-\alpha}\cdot).$$

If $\alpha > 0$, the operator $\mathcal{L}_{\lambda_0}$ is bounded from $L^2 \to L^2$ (using (1.9)). The fact that $D(\mathcal{A}) = D(\mathcal{L}_{\lambda_0})$ follows from the theorem on the regularity of the 2-order elliptic problem[([20]]. It now follows from Theorem 1.2 that $D(\mathcal{A}^\alpha) = D(\mathcal{L}_{\lambda_0}^\alpha), \alpha \in [0, 1]$, that is

$$c_{1\alpha}\|\mathcal{L}_{\lambda_0}^\alpha\| \leq \|\mathcal{A}^\alpha\| \leq c_{2\alpha}\|\mathcal{L}_{\lambda_0}^\alpha\|, \ \forall h \in D(\mathcal{A}^\alpha). \tag{3.8}$$

We have thus shown that $\mathcal{L}$ has a discrete spectrum. Let the stationary solution ω_0 be such that $Re(\mathcal{L}) \neq 0$ (which depends on the choice of f_0). In other words, we suppose that

$$\sigma(\mathcal{L}) = \sigma_+(\mathcal{L}) \cup \sigma_-(\mathcal{L}), \ \sigma_+(\mathcal{L}) \cap \sigma_-(\mathcal{L}) = \emptyset \text{ and } Re\sigma_+(\mathcal{L}) < -a, Re\sigma_-(\mathcal{L}) > a, a > 0.$$

We observe that $\sigma_+(\mathcal{L})$ is a finite set of eigenvalues. The $+$ sign indicates the unstable modes and $-$ sign indicates the stable modes.

Let γ_+ be a coutour in the left half-plane enclosing $\sigma_+(\mathcal{L})$. We set

$$P_+ = \frac{1}{2\pi i}\int_{\gamma_+} (\lambda I - \mathcal{A})^{-1} d\lambda, \ P_- = I - P_+.$$

The operator $\mathcal{L}$ can be decomposed as

$$\mathcal{L} = \mathcal{L}_+ + \mathcal{L}_-,\ \mathcal{L}_+ = P_+\mathcal{L},\ \mathcal{L}_- = P_-\mathcal{L},$$

where $P_+\mathcal{L} \subset \mathcal{L}P_+,\ P_-\mathcal{L} \subset \mathcal{L}P_-$.

Setting $H_+ = P_+H, \dim H_+ = N < \infty$ and $H_- = P_-H$, we see that

$$\mathcal{L}_+H_+ \subset H_+,\ \ \mathcal{L}_-H_- \subset H_-.$$

The operator $\mathcal{L}_+ \in \mathcal{L}(H_+)$ (bounded linear operator space on H_+) and $-\mathcal{L}_-$ generates an analytic semigroup in H_-. Note that P_+ and P_- commute with $\mathcal{L}$ and the semigroup $T(t)$ (which is generated by $-\mathcal{L}$), in the sense that $P_+\mathcal{L} \subset \mathcal{L}P_+, P_-\mathcal{L} \subset \mathcal{L}P_-, P_+T(t) = T(t)P_+$ and $P_T(t) = T(t)P_-$ for $t \geq 0$([24]).

We consider the following equation for $t \in \mathbb{R}$:

$$\partial_t h + \mathcal{L}h = f(t). \tag{3.9}$$

Lemma 3.2 *[19] Let $f(t) \in L_\infty(\mathbb{R}; D(\mathcal{L}^\gamma)), \gamma \geq 0$. If $\alpha - \gamma < 1$, then the equation (3.9) has a unique solution h bounded in $D(\mathcal{L}^\alpha)$:*

$$\|h\|_{C_b(\mathbb{R};D(\mathcal{L}^\alpha))} \leq K(\alpha,\gamma)\|f\|_{L_\infty(\mathbb{R};D(\mathcal{L}^\gamma))}. \tag{3.10}$$

We continue to study equation (3.4) . Let $F(h,\epsilon,\tau) = -J(\triangle^{-1}h, h) + J(\epsilon\ v, h) + \epsilon J(\triangle^{-1}(\omega_0 - \epsilon\ v), v) + \epsilon J(\triangle^{-1}h, v) + \epsilon J(\triangle^{-1}v, \omega_0)$, then for $h_i \in D(\mathcal{L}^{\frac{1}{2}})$ and assuming that $\|h_i\|_{\frac{1}{2}} \leq \rho (i = 1, 2)$, we have

$$\|F(h_1,\epsilon,\tau) - F(h_2,\epsilon,\tau)\| \leq N_{\frac{1}{2}}(\epsilon,\rho)\|h_1 - h_2\|_{\frac{1}{2}}, \tag{3.11}$$

$$\|F(0,\epsilon,\tau)\| \leq N_{\frac{1}{2}}(\epsilon), \tag{3.12}$$

where $N_{\frac{1}{2}}(\epsilon,\rho),\ N_{\frac{1}{2}}(\epsilon) \to 0$ as $\epsilon,\rho \to 0$.
The proof of (3.11) and (3.12) can be obtained using **Lemma 2.1** and **Lemma 3.1**. Moreover, if $h_i \in D(\mathcal{L})(i = 1, 2)$, using **Lemma 2.1** and **Lemma 3.1**, we can prove $F : D(\mathcal{A}) \to D(\mathcal{A}^{\frac{1}{2}})$ is a bounded Lipschitz map (using **Theorem 1.2**).

Converting equation (3.4) to the original time variable $t = \epsilon\tau$, we obtain

$$\partial_t h + \mathcal{L}h = Q(h,\epsilon,t), \tag{3.13}$$

where $Q(h,\epsilon,t) = F(h,\epsilon,\frac{t}{\epsilon})$. Obviously, $Q(h,\epsilon,t)$ satisfies (3.11) and (3.12).

Lemma 3.3 *Assume that Q satisfies (3.11) and (3.12). Then if $\epsilon < \epsilon_0$ for some $\epsilon_0 > 0$ small enough, (3.13) has a unique bounded solution h^* with the following properties:*

1. $\|h^*\|_{C_b(\mathbb{R};D(\mathcal{A}^{\frac{1}{2}})} \leq \delta(\epsilon) \to 0$, as $\epsilon \to 0$.

2. There exists an initial manifold $\mathcal{M}_-$ in $D(\mathcal{A}^{\frac{1}{2}})$, $\mathrm{codim}\mathcal{M}_- = N$, such that if $h_0 \in \mathcal{M}_-$ and $\|h_0\|_{\frac{1}{2}} \leq \rho$ for ρ small enough, then the solution $h(t)$ of equation (3.13) with $h(0) = h_0$ satisfies the estimate

$$\|h(t) - h^*(t)\|_{\frac{1}{2}} \leq Ke^{-a|t|}\|h_0 - h^*(0)\|_{\frac{1}{2}},\ a > 0,\ as\, t \to \infty. \tag{3.14}$$

In particular, if $N = 0$, then the solution h^* is asymptotically stable. There exists an initial manifold $\mathcal{M}_+$ in $D(\mathcal{A}^{\frac{1}{2}})$, $\dim \mathcal{M}_+ = N$, such that (3.14) holds as $t \to -\infty$.

3. If the function $Q(h, \epsilon, \cdot) : \mathbb{R} \to D(\mathcal{A}^{\frac{1}{2}})$ is almost periodic, then the solution h^* is almost periodic with frequency basis contained in that of Q.

This lemma is a special case of [19]. Its proof is based on the contraction map principle, stability argument, **Lemma 3.2** and **Theorem 1.6**.

We now have the following main result in this section.

Theorem 3.4 *(Averaging principle on the entire time axis) Assume that the forcing f on the right-hand side of the quasi-geostrophic flow model (2.1) has an average in the sense of (2.2) uniformly in $t \in \mathbb{R}$. Assume also that the spectrum of the linear operator $\mathcal{L}$ in (3.7) does not intersect with the imaginary axis. Then for $\epsilon < \epsilon_0$ small enough:*

1. In a small neighbourhood of the stationary averaged *quasi-geostrophic flow ω_0, the full quasi-geostrophic flow model (2.1) has a unique solution $\omega^*(\tau)$, which is bounded on the entire time axis and satisfies:*

$$\|\omega * (\tau) - \omega_0\|_{\frac{1}{2}} \leq \delta(\epsilon), \ \text{as } \epsilon \to 0,$$

where $\delta(\epsilon)$ is in Lemma 3.3.

2. In the ball $B_V(\rho) \in H_-$ with ρ small enough and $\omega(0) \in B_V(\rho)$, there exists a stable manifold $\mathcal{M}_-$ in $D(\mathcal{A}^{\frac{1}{2}})$, $\operatorname{codim}\mathcal{M}_- = N$, such that if the initial condition $\omega(0) \in \mathcal{M}_-$ and $t \to \infty$, then

$$\|\omega(t) - \omega^*(t)\|_{\frac{1}{2}} \leq ce^{-a|t|}\|\omega(0) - \omega^*(0)\|_{\frac{1}{2}}. \tag{3.15}$$

There also exists an unstable manifold $\mathcal{M}_+$ in $D(\mathcal{A}^{\frac{1}{2}})$, $\dim \mathcal{M}_+ = N$, such that if the initial condition $\omega(0) \in \mathcal{M}_+$, then the inequality (3.15) holds as $t \to -\infty$. In particular, if $N = 0$, then the unsteady quasi-geostrophic flow ω^* is asymptotically stable.

3. If the forcing f is almost periodic in $D(\mathcal{A}^\gamma), \gamma > -\frac{1}{2}$, then $\omega^*(\tau)$ is almost periodic in $D(\mathcal{A}^{\frac{1}{2}})$ with frequency basis contained in that of f.

Proof. These assertions of the theorem follow from the representation

$$\omega = \omega_0 + h(\tau, \epsilon) - \epsilon v(\tau, \epsilon)$$

and **Lemmas 3.1, 3.3**. ■

This theorem gives stability estimate of stationary *averaged* QG flows; stability conclusion of unsteady QG flows near the stationary *averaged* QG flow; comparison estimate between unsteady QG flows; and the existence of almost time periodic QG motions under almost time periodic wind forcing, on the entire time axis.

Combining this theorem and earlier discussion in (1.15) and (1.16), we have

Corollary 3.5 *Under the assumption of* **Theorem 3.4** *and*

$$4\nu r > \frac{\beta^2 |D|^2}{\pi^2}, \ \ \|f_0\| < \sqrt{\frac{2|D|}{\pi}}\lambda_1^2,$$

where $\lambda_1 = \nu - \frac{\beta^2|D|^2}{4r\pi^2}$ *as defined before, then for every* $R > 0$ *such that the stationary* averaged *QG flow* ω_0 *has norm* $\|\omega_0\|_{\frac{1}{2}} \leq R$*, the following inequality is valid:*

$$\|\omega(\tau) - \omega^*(\tau)\|_{\frac{1}{2}} \leq C(R)e^{-\epsilon a\tau}\|\omega(0) - \omega^*(0)\|_{\frac{1}{2}}, \tau \to \infty,$$

where $\omega(\tau) = \omega(\tau, \epsilon)$ *is the solution of the full quasi-geostrophic flow model (2.1) with initial condition* $\omega(0) = \omega_0$ *and* $\epsilon < \epsilon_0(R)$*.*

Combining **Theorem 3.4** and **Corollary 3.5**, we conclude that there exists an a. p. solution for (2.1) when f is a. p., under the assumptions of **Corollary 3.5**. Here we give the restriction for $\|f_0\|$, not for $\|f\|$. This result is obtained under different conditions from those of Duan and Kloeden [10].

Let f be a.p. with values in H. Then by the results of [25]–[27], also see [10], there exists a uniform attractor $A_{\mathcal{H}} = A_{\mathcal{H}(f(\eta\cdot))}$ of the non-autonomous dynamical system (2.1), with $\mathcal{H} = \{f^\tau, f^\tau(t) = f(t+\tau), \tau \in \mathbb{R}\}_{C_b(\mathbb{R};H)}$, and this attractor approaches to the attractor of the averaged dynamical system (2.3):

Theorem 3.6 *(Attractor convergence) Suppose that* f *is a.p. in* H*, then*

$$dist_{D(\mathcal{A}^{\frac{1}{2}})}(A_{\mathcal{H}(f(\eta\cdot))}, \bar{A}) \to 0, \quad as\ \eta \to \infty,$$

where $A_{\mathcal{H}}$ *is the attractor for the quasi-geostrophic flow model (2.1), and* $\bar{A}$ *is the attractor of the averaged quasi-geostrophic flow model (2.3).*

This theorem claims that the uniform attractor of the full quasi-geostrophic flow model approaches to the attractor of the averaged quasi-geostrophic flow model, as the parameter in the oscillating forcing $\eta \to \infty$.

4 Summary

In this paper, we have discussed averaging principle for quasi-geostrophic motions under rapidly oscillating forcing (characterizied by a large dimensionless parameter η), both on finite but large time intervals and on the entire time axis. We have derived comparison estimate, stability estimate and proved convergence result (as $\eta \to \infty$) between quasi-geostrophic motions and its averaged motions.

We also investigated the existence of almost periodic quasi-geostrophic motions under almost periodic forcing, and the convergence of the attractor of the quasi-geostrophic flow model to the attractor of the averaged quasi-geostrophic flow model as $\eta \to \infty$.

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