# MINIMALITY OF THE DATA IN WAVELET FILTERS

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with an Appendix by Brian Treadway

ABSTRACT. Orthogonal wavelets, or wavelet frames, for $L^2(\mathbb{R})$ are associated with quadrature mirror filters (QMF), a set of complex numbers which relate the dyadic scaling of functions on $\mathbb{R}$ to the $\mathbb{Z}$-translates. In this paper, we show that generically, the data in the QMF-systems of wavelets is minimal, in the sense that it cannot be nontrivially reduced. The minimality property is given a geometric formulation in the Hilbert space $\ell^2(\mathbb{Z})$, and it is then shown that minimality corresponds to irreducibility of a wavelet representation of the algebra $\mathcal{O}_2$; and so our result is that this family of representations of $\mathcal{O}_2$ on the Hilbert space $\ell^2(\mathbb{Z})$ is irreducible for a generic set of values of the parameters which label the wavelet representations.

## 1. Introduction

Let $L^2(\mathbb{R})$ be the Hilbert space of all $L^2$-functions. For $\psi \in L^2(\mathbb{R})$, set

$$\psi_{n,k}(x) := 2^{\frac{n}{2}}\psi(2^n x - k) \qquad \text{for } x \in \mathbb{R}, \text{ and } n,k \in \mathbb{Z}. \tag{1.1}$$

We say that $\psi$ is a wavelet (in *the strict sense*) if $\{\psi_{n,k}\,;\, n,k \in \mathbb{Z}\}$ constitutes an orthonormal basis in $L^2(\mathbb{R})$; and we say that $\psi$ is a wavelet in *the frame sense* (tight frame) if

$$\|f\|^2_{L^2(\mathbb{R})} = \sum_{n,k\in\mathbb{Z}} |\langle \psi_{n,k} \,|\, f\rangle|^2 \tag{1.2}$$

holds for all $f \in L^2(\mathbb{R})$, where $\langle\cdot\,|\,\cdot\rangle$ is the usual $L^2(\mathbb{R})$-inner product, i.e., $\langle \psi_{n,k} \,|\, f\rangle = \int_{\mathbb{R}} \overline{\psi_{n,k}(x)} f(x)\,dx = c_{n,k}$. The numbers $c_{n,k}$ are the wavelet coefficients. It is known [Dau92, Hör95] that a given wavelet $\psi$ in the sense of frames is a (strict) wavelet if and only if $\|\psi\|_{L^2(\mathbb{R})} = 1$. We shall have occasion to consider scaling on $\mathbb{R}$ other than the dyadic one, say $x \mapsto Nx$ where $N \in \mathbb{N}$, $N > 2$. Then the analogue of (1.1) is

$$\psi_{n,k}(x) := N^{\frac{n}{2}}\psi(N^n x - k), \qquad x \in \mathbb{R},\ n,k \in \mathbb{Z}. \tag{1.3}$$

However, in that case, it is generally not enough to consider only one $\psi$ in $L^2(\mathbb{R})$: If the wavelet is derived from an $N$-subband wavelet filter as in [BrJo00], then we

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construct $\psi^{(1)}, \psi^{(2)}, \dots, \psi^{(N-1)}$ in $L^2(\mathbb{R})$ such that the functions in (1.3) have the basis property, either in the strict sense, or in the sense of frames. Then the system

$$\left\{\psi_{n,k}^{(i)} \,;\, 1 \le i < N,\ n,k \in \mathbb{Z}\right\} \tag{1.4}$$

constitutes an orthonormal basis of $L^2(\mathbb{R})$, or, alternatively, a tight frame, as in (1.2) but with the $\psi_{n,k}^{(i)}$ functions in place of $\psi_{n,k}$.

Our main point is to show how the notion of *irreducibility* for representations of the Cuntz algebra $\mathcal{O}_N$ corresponds to *optimality* of the corresponding wavelet filters. Since we are addressing two different audiences (wavelets vs. representation theory), a few more details are included in this paper than might otherwise be customary. Our main result is that the irreducibility of the representation (equivalently, minimality of the filter) is *generic* for the wavelet representations; see Theorems 5.9 and 6.7. In addition, we show that generically, two different filters yield inequivalent representations, i.e., the corresponding two representations are not unitarily equivalent. This was known earlier only in very restrictive special cases [BrJo00], and the general case treated here has not previously been discussed in the literature. Moreover, the methods used for the special cases in fact do *not at all* carry over to the general case. We are concerned with the wavelet filters which enter into the construction of $\psi^{(1)}, \psi^{(2)}, \dots, \psi^{(N-1)}$ in (1.4). These filters (see (1.5)–(1.7) and (3.7) below) are just a finite set of numbers which relate the $\mathbb{Z}$-translates of these functions to the corresponding scalings by $x \mapsto Nx$. Hence the analysis may be discretized via the filters, but the question arises whether or not the data which go into the wavelet filters are *minimal*. Representation theory is ideally suited to make the minimality question mathematically precise. (This is a QMF-multiresolution construction, and it is its minimality and efficiency which concern us here. While it is true, see, e.g., [Gab98], [FPT99], [Bag00], [BaMe99], and [DaiL98], that there are other and different possible wavelet constructions, it is not yet clear how our present techniques might adapt to the alternative constructions, although the approach in [DaiL98] is also based on operator-theoretic considerations.)

To explain the *minimality issue* for multiresolution quadrature mirror (QMF) wavelet filters, we recall the *scaling function* $\varphi$ of a resolution in $L^2(\mathbb{R})$. Let $g \in \mathbb{N}$, and let $a_0, a_1, \dots, a_{2g-1}$ be given complex numbers such that

$$\sum_{k=0}^{2g-1} a_k = 2, \tag{1.5}$$

and

$$\sum_k a_{k+2l}\bar{a}_k = \begin{cases} 2 & \text{if } l = 0, \\ 0 & \text{if } l \neq 0. \end{cases} \tag{1.6}$$

In the summation (1.6), and elsewhere, we adopt the convention that terms are defined to be zero when the index is not in the specified range. Hence, in (1.6), it is understood that $a_{k+2l} = 0$ whenever $k$ and $l$ are such that $k + 2l$ is not in $\{0, 1, \dots, 2g-1\}$. It is known [BrJo00, BEJ00, Mal99] that there is a $\varphi \in L^2(\mathbb{R}) \setminus \{0\}$ of compact support, unique up to a constant multiple, such that

$$\varphi(x) = \sum_{k=0}^{2g-1} a_k \varphi(2x - k), \qquad x \in \mathbb{R}; \tag{1.7}$$

in fact, supp $(\varphi) \subset [0, 2g-1]$. (If $H$ denotes the Hilbert transform of $L^2(\mathbb{R})$, and $\varphi$ solves (1.7), then $H\varphi$ does as well; but $H\varphi$ will not be of compact support if $\varphi$ is.) In finding $\varphi$ in (1.7), there are methods based on iteration (see Appendix), on random matrix products, and on Fourier transform, see [BrJo00], [BEJ00], [BrJo99b], [Coh92b], [CoRy95], and [Dau92]; and the various methods intertwine in the analysis of $\varphi$, i.e., in deciding when $\varphi(x)$ is continuous, or not, or if it is differentiable. This issue will be resumed in the Appendix below, which is based on [BrJo99b]. But the next two sections will deal with the minimality question alluded to above.

Let $\varphi$ be as in (1.7), and let $\mathcal{V}_0$ be the closed subspace in $\mathcal{H}$ $(:= L^2(\mathbb{R}))$ spanned by $\{\varphi(x-k)\,;\, k \in \mathbb{Z}\}$, i.e., by the integral translates of the scaling function $\varphi$. Let $U$ $(:= U_N)$ be

$$Uf(x) := N^{-\frac{1}{2}} f\left(\frac{x}{N}\right), \qquad f \in L^2(\mathbb{R}), \tag{1.8}$$

the unitary scaling operator in $\mathcal{H} = L^2(\mathbb{R})$. Then if $N = 2$,

$$U\mathcal{V}_0 \subset \mathcal{V}_0 \tag{1.9}$$

is a proper subspace, and

$$\bigwedge_n U^n \mathcal{V}_0 = \{0\}\,; \tag{1.10}$$

see [BEJ00] and [Dau92, Ch. 5]. Setting $\mathcal{V}_n := U^n\mathcal{V}_0$ and

$$\mathcal{W}_n := \mathcal{V}_{n-1} \ominus \mathcal{V}_n, \tag{1.11}$$

we arrive at the resolution

$$\mathcal{V}_0 = \sum_{n\geq 1}^{\oplus} \mathcal{W}_n, \tag{1.12}$$

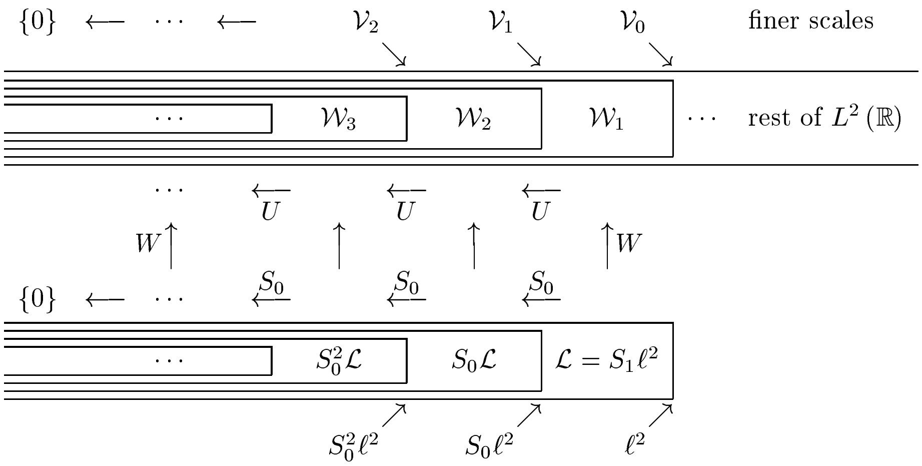
and the wavelet function $\psi$ is picked in $\mathcal{W}_0$; see Table 1. We will set up an isomorphism between the resolution subspace $\mathcal{V}_0$ and $\ell^2(\mathbb{Z})$, and associate operators in $\ell^2(\mathbb{Z})$ with the wavelet operations in $\mathcal{V}_0 \subset L^2(\mathbb{R})$. This is of practical significance given that the operators in $\ell^2(\mathbb{Z})$ are those which are defined directly from the wavelet filters, and it is the digital filter operations which lend themselves to algorithms. Generalizing (1.11), in the case of scale $N$ $(> 2)$ the space $\mathcal{V}_0 \ominus U_N\mathcal{V}_0$ splits up as a sum of orthogonal spaces $\mathcal{W}_1^{(i)}$, $i = 1, 2, \dots, N-1$; see (3.16)–(3.17).

## 2. Representations of $\mathcal{O}_N$ and Table 1 (Discrete vs. continuous wavelets)

The computational significance of the operator system in Table 1 (scale $N = 2$) is that the operators which generate wavelets in $L^2(\mathbb{R})$ become modeled by an associated system of operators in the *sequence space* $\ell^2$ $(:= \ell^2(\mathbb{Z}) \cong L^2(\mathbb{T}))$. (We will do the discussion here in Section 2 just for $N = 2$, but this is merely for simplicity; it easily generalizes to arbitrary $N$.) Then the algorithms are implemented in $\ell^2$ by basic discrete operations, and only in the end are the results then "translated" back to the space $L^2(\mathbb{R})$. The space $L^2(\mathbb{R})$ is not amenable (in its own right) to *discrete* computations. This is made precise by the frame operator $W\colon \ell^2$ $(\cong L^2(\mathbb{T})) \to \mathcal{V}_0$ $(\subset L^2(\mathbb{R}))$ defined as

$$W\colon \ell^2 \ni (\xi_k) \longmapsto \sum_{k\in\mathbb{Z}} \xi_k \varphi(x-k) \in L^2(\mathbb{R}). \tag{2.1}$$

TABLE 1. Discrete vs. continuous wavelets, i.e., $\ell^2$ vs. $L^2(\mathbb{R})$




If $\varphi$ has orthogonal translates, then $W$ will be an isometry of $\ell^2$ onto $\mathcal{V}_0$ $(\subset L^2(\mathbb{R}))$. Even if the functions $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ formed from $\varphi$ by $\mathbb{Z}$-translates only constitute a frame in $\mathcal{V}_0$, then we will have the following estimates:

$$c_1^{1/2} \cdot \|\xi\|_{\ell^2} \leq \|W\xi\|_{L^2(\mathbb{R})} \leq c_2^{1/2} \cdot \|\xi\|_{\ell^2}\ , \tag{2.2}$$

where $c_1$ and $c_2$ are positive constants depending only on $\varphi$.

**Lemma 2.1.** *If the coefficients* $\{a_k\ ;\ k=0,1,\dots,2g-1\}$ *from* (1.7) *satisfy the conditions in* (1.6), *then the corresponding operator* $S_0\colon \ell^2 \to \ell^2$, *given by*

$$(S_0\xi)_k = \frac{1}{\sqrt{2}}\sum_{l\in\mathbb{Z}} a_{k-2l}\xi_l = \frac{1}{\sqrt{2}}\sum_{\substack{p\in\mathbb{Z}:\\ p\equiv k \bmod 2}} a_p\xi_{\frac{k-p}{2}}, \qquad k\in\mathbb{Z}, \tag{2.3}$$

*is* isometric *and satisfies the following* intertwining identity:

$$WS_0 = UW, \tag{2.4}$$

*where $U$ is the dyadic scaling operator in $L^2(\mathbb{R})$ introduced in* (1.8). (*Here we restrict attention to $N=2$, but just for notational simplicity!*) *Setting* $b_k := (-1)^k\,\bar{a}_{2g-1-k}$, *and defining a second isometric operator* $S_1\colon \ell^2\to\ell^2$ *by formula* (2.3) *with the only modification that* $(b_k)$ *is used in place of* $(a_k)$, *we get*

$$S_j^*S_k = \delta_{j,k}\mathbb{1}_{\ell^2} \tag{2.5}$$

*and*

$$\sum_j S_jS_j^* = \mathbb{1}_{\ell^2}\ , \tag{2.6}$$

*which are the Cuntz identities from operator theory* [Cun77], *and the operators* $S_0$ *and* $S_1$ *satisfy the identities indicated in Table* 1.

**Remark 2.2.** For understanding the second line in Table 1, note that $S_0$ is a *shift* as an isometry, in the sense of [SzFo70], and $\mathcal{L} := S_1\ell^2$ is a wandering subspace for $S_0$, in the sense that the spaces $\mathcal{L}$, $S_0\mathcal{L}$, $S_0^2\mathcal{L}$, $\dots$ are mutually orthogonal in $\ell^2$. To see this, note that (2.6) implies that $(\mathcal{L} :=)\ S_1\ell^2 = \ell^2 \ominus S_0\ell^2 = \ker(S_0^*)$. As a result, we get the following:

**Corollary 2.3.** *The projections onto the orthogonal subspaces in the second line of Table* 1 *corresponding to the* $\mathcal{W}_1, \mathcal{W}_2, \dots$ *subspaces of the first line* (*see* (1.11)) *are*

$$\begin{aligned} \operatorname{proj} \mathcal{L} &= S_1 S_1^* = I - S_0 S_0^*, \\ &\vdots \quad \vdots \\ \operatorname{proj} S_0^{n-1} \mathcal{L} &= S_0^{n-1} S_0^{*n-1} - S_0^n S_0^{*n}. \end{aligned}$$

*Proof.* Immediate from Lemma 2.1, Remark 2.2, and (1.12). □

**Remark 2.4.** Any system of operators $\{S_j\}$ satisfying (2.5)–(2.6) is said to be a *representation* of the $C^*$-algebra $\mathcal{O}_2$, and there is a similar notion for $\mathcal{O}_N$ when $N > 2$, with $\mathcal{O}_N$ having generators $S_0, S_1, \dots, S_{N-1}$, but otherwise also satisfying the operator identities (2.5)–(2.6). The power and the usefulness of the multiresolution subband filters for the analysis of wavelets and their algorithms was first demonstrated forcefully in [CoWi93] and [Wic93]; see especially [CoWi93, p. 140] and [Wic93, p. 157], where the $\mathcal{O}_N$-relations (2.5)–(2.6) are identified, and analyzed in the case $N = 2$. Around the same time, A. Cohen [Coh92b] identified and utilized the interplay between $\ell^2$ and $L^2(\mathbb{R})$ which, as noted in Section 2 above, is implied by the $\mathcal{O}_N$-relations and their representations. But neither of those prior references takes up the construction of $\mathcal{O}_N$-representations in a systematic fashion. Of course the quadrature mirror filters (QMF's) have a long history in electrical engineering (speech coding problems), going back to long before they were used in wavelets, but the form in which we shall use them here is well articulated, for example, in [CEG77]. Some more of the history of and literature on wavelet filters is covered well in [Mey93] and [Ben00].

**Definition 2.5.** A representation of $\mathcal{O}_N$ on the Hilbert space $\ell^2$ is said to be *irreducible* if there are no closed subspaces $\{0\} \subsetneqq \mathcal{H}_0 \subsetneqq \ell^2$ which reduce the representation, i.e., which yield a representation of (2.5)–(2.6) on each of the two subspaces in the decomposition

$$\ell^2 = \mathcal{H}_0 \oplus \left(\ell^2 \ominus \mathcal{H}_0\right), \tag{2.7}$$

where $\ell^2 \ominus \mathcal{H}_0 = (\mathcal{H}_0)^\perp = \left\{\xi \in \ell^2 \,;\, \langle \xi \,|\, \eta \rangle = 0,\ \forall\, \eta \in \mathcal{H}_0 \right\}$.

*Proof of Lemma* 2.1. Most of the details of the proof are contained in [BrJo97b] and [BrJo00], so we only sketch points not already covered there. The essential step (for the present applications) is the formula (2.4), which shows that $W$ intertwines the isometry $S_0$ with the restriction of the unitary operator $U\colon f \mapsto \frac{1}{\sqrt{2}} f(x/2)$ to the resolution subspace $\mathcal{V}_0 \subset L^2(\mathbb{R})$. We have:

$$\begin{aligned}
(UW\xi)(x) &= \frac{1}{\sqrt{2}}(W\xi)\left(\frac{x}{2}\right) \\
&= \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} \xi_k \varphi\left(\frac{x}{2}-k\right) && \text{(by (2.1))} \\
&= \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \xi_k a_l \varphi(x-2k-l) && \text{(by (1.7))} \\
&= \frac{1}{\sqrt{2}}\sum_{p\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}} \xi_k a_{p-2k}\right)\varphi(x-p) \\
&= \sum_{p\in\mathbb{Z}} (S_0\xi)_p\, \varphi(x-p) && \text{(by (2.3))} \\
&= (WS_0\xi)(x) && \text{(by (2.1))}
\end{aligned}$$

for all $\xi \in \ell^2$, and all $x \in \mathbb{R}$. This proves (2.4). □

For later use, we record the operators on the respective Hilbert spaces $L^2(\mathbb{T}) \cong \ell^2$ and $L^2(\mathbb{R})$, and the corresponding transformation rules with respect to the operator $W$. Let $N$ be the scale number, and let $(a_k)_{k=0}^{Ng-1}$ be given satisfying

$$\sum_{k\in\mathbb{Z}} a_{k+Nl}\bar{a}_k = \delta_{0,l}N \tag{2.8}$$

and set $m_0(z) := \frac{1}{\sqrt{N}}\sum_{k=0}^{Ng-1} a_k z^k$, $z \in \mathbb{T}$. The following summary table of transformation rules may clarify the proof.

(2.9)

| | **SCALING** | **TRANSLATION** | |
|---|---|---|---|
| $L^2(\mathbb{R})$ : | $F \longmapsto \frac{1}{\sqrt{N}}F\left(\frac{x}{N}\right)$ | $F(x) \longmapsto F(x-1)$ | real wavelets |
| $\uparrow W$ | | | |
| $\ell^2$: | $\xi \longmapsto \sum_l a_{k-Nl}\xi_l$ | $(\xi_k) \longmapsto (\xi_{k-1})$ | discrete model |
| $\uparrow$ Fourier transform | | | |
| $L^2(\mathbb{T})$ : | $f \longmapsto m_0(z) f(z^N)$ | $f(z) \longmapsto zf(z)$ | periodic model, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ |

**Remark 2.6.** The significance of irreducibility (when satisfied) is that the *wavelet subbands* which are indicated in Table 1 are then the *only subbands* of the corresponding multiresolution. We will show that in fact irreducibility holds *generically,* but it does not hold, for example, for the Haar wavelets. In the simplest case, the Haar wavelet has $g = 2 = N$, and the numbers from Lemma 2.1 are

$$\begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{2.10}$$

Hence, for this representation of $\mathcal{O}_2$ on $\ell^2$, we may take $\mathcal{H}_0 = \ell^2(0,1,2,\dots)$, and therefore $\mathcal{H}_0^\perp = \ell^2(\dots,-3,-2,-1)$. Returning to the multiresolution diagram in Table 1, this means that we get additional subspaces of $L^2(\mathbb{R})$, on top of the standard ones which are listed in Table 1. Specifically, in addition to

$$\mathcal{V}_n = U^n\mathcal{V}_0 = WS_0^n\ell^2 \quad \text{and} \quad \mathcal{W}_n = \mathcal{V}_{n-1} \ominus \mathcal{V}_n = WS_0^{n-1}S_1\ell^2,$$

we get a new system with "twice as many", as follows: $\mathcal{V}_n^{(\pm)}$ and $\mathcal{W}_n^{(\pm)}$, where

$$\mathcal{V}_n^{(+)} = WS_0^n\left(\mathcal{H}_0\right), \qquad \mathcal{W}_n^{(+)} = WS_0^{n-1}S_1\left(\mathcal{H}_0\right);$$

and

$$\mathcal{V}_n^{(-)} = WS_0^n\left(\mathcal{H}_0^{\perp}\right), \qquad \mathcal{W}_n^{(-)} = WS_0^{n-1}S_1\left(\mathcal{H}_0^{\perp}\right).$$

For the case of the Haar wavelet, see (2.10),

$$\mathcal{V}_0^{(+)} \subset L^2\left(0,\infty\right), \qquad \mathcal{V}_0^{(-)} \subset L^2\left(-\infty,0\right),$$

or rather, $\mathcal{V}_0$ consists of finite linear combinations of $\mathbb{Z}$-translates of

$$\varphi\left(x\right) = \begin{cases} 1 & \text{if } 0 \le x < 1, \\ 0 & \text{if } x \in \mathbb{R} \setminus [0,1), \end{cases} \tag{2.11}$$

i.e., functions in $L^2\left(\mathbb{R}\right)$ which are constant between $n$ and $n+1$ for all $n \in \mathbb{Z}$; and

$$\mathcal{V}_0^{(+)} = \mathcal{V}_0 \cap L^2\left(0,\infty\right), \qquad \mathcal{V}_0^{(-)} = \mathcal{V}_0 \cap L^2\left(-\infty,0\right). \tag{2.12}$$

Hence we get two separate wavelets, but with translations built on $\{0,1,2,\dots\}$ and $\{\dots,-3,-2,-1\}$. In view of the graphics in the Appendix below, it is perhaps surprising that other wavelets (different from the Haar wavelets) do not have the corresponding additional "positive vs. negative" splitting into subbands within the Hilbert space $L^2\left(\mathbb{R}\right)$.

**Remark 2.7.** There are other dyadic Haar wavelets (mock Haar wavelets), in addition to (2.11). For example, let

$$\varphi_k\left(x\right) = \begin{cases} \frac{1}{\sqrt{2k+1}} & \text{if } 0 \le x < 2k+1, \\ 0 & \text{if } x \in \mathbb{R} \setminus [0,2k+1). \end{cases} \tag{2.13}$$

Then it follows that there is a splitting of $\mathcal{V}_0$ into orthogonal subspaces which is analogous to (2.12), but it has many more subbands than the two, "positive vs. negative", which are special to the standard Haar wavelet (2.11). For details on these other Haar wavelets, and their decompositions, see [BrJo99a, Proposition 8.2]. They are only tight frames, and the $m$-functions of (2.13) are

$$m_0\left(z\right) = \frac{1}{\sqrt{2}}\left(1+z^{2k+1}\right), \qquad m_1\left(z\right) = \frac{1}{\sqrt{2}}\left(1-z^{2k+1}\right), \qquad z \in \mathbb{T}. \tag{2.14}$$

Hence, after adjusting the $\mathcal{O}_2$-representation $T$ with a rotation $V \in \mathrm{U}_2\left(\mathbb{C}\right)$, we have

$$T_0 f\left(z\right) = f\left(z^2\right), \qquad T_1 f\left(z\right) = z^{2k+1} f\left(z^2\right), \qquad f \in L^2\left(\mathbb{T}\right) \cong \ell^2, \tag{2.15}$$

and the two new operators $T_0, T_1$ will satisfy the $\mathcal{O}_2$-identities (2.5)–(2.6); the representation will have the same reducing subspaces as the one defined directly from $m_0$ and $m_1$. The explicit decomposition of the multiresolution subspaces corresponding to (2.12) may be derived, via $W$ in Table 1, from the decomposition into sums of irreducibles for the $\mathcal{O}_2$-representation on $\ell^2$ which corresponds to (2.12). This means that the decomposition (2.7) associated with (2.13) and (2.15) has *more than two* terms in its subspace configuration.

## 3. Wavelet filters and subbands

The operators of wavelet filters may be realized on either one of the two Hilbert spaces $\ell^2(\mathbb{Z})$ or $L^2(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and $L^2(\mathbb{T})$ defined from the normalized Haar measure $\mu$ on $\mathbb{T}$. But, of course, $\ell^2(\mathbb{Z}) \cong L^2(\mathbb{T})$ via the Fourier series. For a given sequence $a_0, a_1, \dots, a_{Ng-1}$, consider the operator $S_0$ in $\ell^2(\mathbb{Z})$ given by

$$\xi \longmapsto S_0\xi \quad \text{and} \quad (S_0\xi)_k = \frac{1}{\sqrt{N}} \sum_l a_{k-lN}\xi_l. \tag{3.1}$$

Setting $m_0(z) = \frac{1}{\sqrt{N}} \sum_{k=0}^{Ng-1} a_k z^k$ and

$$(\hat{S}_0 f)(z) = m_0(z) f(z^N), \qquad f \in L^2(\mathbb{T}),\ z \in \mathbb{T}, \tag{3.2}$$

we note that $S_0$ and $\hat{S}_0$ are really two versions of the same operator, i.e., that $(\hat{S}_0 f)\hat{}= S_0(\hat{f})$ when $\hat{f} = (\xi_k)$ from the Fourier series. (The first one is the discrete model, and the second, the periodic model, referring to the diagram (2.9).) Hence, we shall simply use the same notation $S_0$ in referring to this operator in either one of its incarnations. It is the (3.1) version which is used in algorithms, of course.

Let $\varphi \in L^2(\mathbb{R})$ be the compactly supported scaling function solving

$$\varphi(x) = \sum_{k=0}^{Ng-1} a_k \varphi(Nx - k). \tag{3.3}$$

Then define the operator $W\colon \ell^2(\mathbb{Z}) \to L^2(\mathbb{R})$ by (2.1). The conditions on the wavelet filter $\{a_k\}$ in (1.5)–(1.6) and (2.8) may now be restated in terms of $m_0(z)$ in (3.2) as follows:

$$\sum_{k=0}^{N-1} \left| m_0(z e^{i\frac{k2\pi}{N}}) \right|^2 = N, \tag{3.4}$$

and

$$m_0(1) = \sqrt{N}. \tag{3.5}$$

It then follows from Lemma 2.1 that $W$ in (2.1) maps $\ell^2(\mathbb{Z})$ onto the resolution subspace $\mathcal{V}_0$ ($\subset L^2(\mathbb{R})$), and that

$$U_N W = W S_0 \tag{3.6}$$

where $U_N f(x) = N^{-1/2} f(x/N)$, $f \in L^2(\mathbb{R})$, $x \in \mathbb{R}$. We showed in [BrJo00] that there are $L^\infty$-functions $m_1, \dots, m_{N-1}$ such that the $N$-by-$N$ complex matrix

$$\frac{1}{\sqrt{N}} \left( m_j(e^{i\frac{k2\pi}{N}} z) \right)_{j,k=0}^{N-1} \tag{3.7}$$

is unitary for all $z \in \mathbb{T}$. If we define

$$S_j f(z) = m_j(z) f(z^N), \qquad f \in L^2(\mathbb{T}), \qquad z \in \mathbb{T}, \tag{3.8}$$

then

$$S_j^* S_k = \delta_{j,k} I_{L^2(\mathbb{T})}, \tag{3.9}$$

and

$$\sum_{j=0}^{N-1} S_j S_j^* = I_{L^2(\mathbb{T})}. \tag{3.10}$$

((3.5) is not needed for this, only for the algorithmic operations of the Appendix.)

**Lemma 3.1.** *The solutions* $(m_j)_{j=0}^{N-1}$ *to* (3.7) *are in* 1–1 *correspondence with the semigroup of all polynomial functions*

$$A\colon \mathbb{T} \longrightarrow \mathrm{U}_N(\mathbb{C}), \tag{3.11}$$

*where* $\mathrm{U}_N(\mathbb{C})$ *denotes the unitary* $N \times N$ *matrices.*

*Proof.* The correspondence is $m \leftrightarrow A$ with

$$m_j(z) = \sum_{k=0}^{N-1} A_{j,k}(z^N) z^k, \tag{3.12}$$

and in the reverse direction,

$$A_{j,k}(z) = \frac{1}{N} \sum_{w^N = z} w^{-k} m_j(w) \tag{3.13}$$

does the job, as can be checked by direct substitution. □

We also showed in [BrJo00] that if $m_0$ is given, and if it satisfies (3.4), then it is possible to construct $m_1, \dots, m_{N-1}$ such that the extended system $m_0, m_1, \dots, m_{N-1}$ will satisfy (3.7). As a consequence, $A$ in (3.13) will be a $\mathrm{U}_N(\mathbb{C})$-loop, and the original $m_0$ is then recovered from (3.12) for $j = 0$. To stress the dependence of the operators in (3.8) on the loop group element $A$ we will denote the corresponding operators $T_i^{(A)}$, and it follows that, if $A = \mathbb{1}_N$, then the operators $S_i$ of (3.8) are

$$f(z) \longmapsto z^i f\left(z^N\right), \qquad \text{where } i = 0, 1, \dots, N-1, \tag{3.14}$$

and we will reserve the notation $S_i$ for those special ones, i.e., $S_i := T_i^{(\mathbb{1}_N)}$.

Let $s_j \mapsto T_j^{(A)}$ be an arbitrary wavelet representation. By virtue of (3.9)–(3.10), $L^2(\mathbb{T})$, or equivalently $\ell^2(\mathbb{Z})$, splits up as an orthogonal sum

$$T_j^{(A)}(\ell^2(\mathbb{Z})), \qquad j = 0, 1, \dots, N-1. \tag{3.15}$$

We saw that the wavelet transform $W$ of (2.1) maps $\ell^2(\mathbb{Z})$ onto $\mathcal{V}_0$, and from (3.6) we conclude that $W$ maps $T_0^{(A)}(\ell^2(\mathbb{Z}))$ onto $U_N(\mathcal{V}_0)$ $(=: \mathcal{V}_1)$. Hence, in the $N$-scale wavelet case, $W$ transforms the spaces $T_j^{(A)}(\ell^2(\mathbb{Z}))$ $(\subset \ell^2(\mathbb{Z}))$ onto orthogonal subspaces $\mathcal{W}_1^{(j)}$, $j = 1, \dots, N-1$ in $L^2(\mathbb{R})$, and

$$\mathcal{W}_1 = \mathcal{V}_0 \ominus \mathcal{V}_1 = \sum_{j=1}^{N-1}{}^{\oplus}\, \mathcal{W}_1^{(j)}, \tag{3.16}$$

where

$$\mathcal{W}_1^{(j)} = T_j^{(A)} \ell^2, \qquad j = 1, \dots, N-1. \tag{3.17}$$

Each of the spaces $\mathcal{V}_1$ and $\mathcal{W}_1^{(j)}$ is split further into orthogonal subspaces corresponding to iteration of the operators $T_0^{(A)}, T_1^{(A)}, \dots, T_{N-1}^{(A)}$ of (3.9)–(3.10). It is the system $\{T_j^{(A)}\}_{j=0}^{N-1}$ which is called a wavelet representation, and it follows that the wavelet decomposition may be recovered from the representation. Moreover, the variety of all wavelet representations is in 1–1 correspondence with the semigroup of polynomial functions $A$ in (3.11). Operators $\{T_j^{(A)}\}$ satisfying (3.9)–(3.10) are said to constitute a representation of the $C^*$-algebra $\mathcal{O}_N$, the Cuntz algebra [Cun77], and it is the irreducibility of the representations from (3.8) which will concern us. If a representation (3.8) is reducible (Definition 2.5), then there is a subspace

$$0 \subsetneqq \mathcal{H}_0 \subsetneqq L^2(\mathbb{T}) \tag{3.18}$$

which is invariant under all the operators $T_j^{(A)}$ and $T_j^{(A)\,*}$, and so the data going into the wavelet filter system $\{m_j\}$ are then not minimal.

## 4. A lemma about projections

Our main result is that for a generic set within the class of all wavelet representations, we do have irreducibility, i.e., there is no reduction as indicated in (3.18) in Section 3. In proving this, we will first reduce the question to a *finite-dimensional matrix problem*. We will also, using [BJKW00], show that every wavelet representation, if it is reducible, decomposes into a *finite* orthogonal sum of irreducible representations, i.e., if the $S_j$ operators from (3.8) are given, then there is a finite orthogonal splitting

$$\ell^2(\mathbb{Z}) = \sum_p{}^{\oplus} \mathcal{H}_p \tag{4.1}$$

such that each of the subspaces $\mathcal{H}_p$ reduces the representation, each of the restricted representations of $\mathcal{O}_N$ is irreducible, and moreover that the irreducible subrepresentations which do occur are *mutually inequivalent* (and therefore disjoint). It is this last property of inequivalence of the irreducible subrepresentation which amounts to the fact that the commutant of the original representation from (3.8) is *abelian*. Let $\mathcal{H} := \ell^2(\mathbb{Z})$, let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded operators in $\mathcal{H}$, and let $s_i \mapsto T_i = T_i^{(A)}$ be an arbitrary wavelet representation. Then the commutant is

$$\mathcal{O}_N' = \{X \in \mathcal{B}(\mathcal{H})\,;\, T_i X = X T_i \;\forall\, i\} = \left\{ X \in \mathcal{B}(\mathcal{H})\,;\, \sum_{i=0}^{N-1} T_i X T_i^* = X \right\}. \tag{4.2}$$

Of course, there are many representations of $\mathcal{O}_N$ such that the corresponding commutant $\mathcal{O}_N'$ is not abelian, see for example [DKS99], but the *abelian property* (i.e., that the decomposition into irreducibles is *multiplicity-free*), is specific to the wavelet representations; see Sections 6 and 8 below. The proof of the abelian property is based on a lemma regarding a certain matrix which turns out to be diagonal with respect to a basis which is a finite subset of the Fourier basis

$$\{z^n\,;\, n \in \mathbb{Z}\} \qquad \text{(also denoted } e_n(z) := z^n\text{)}, \tag{4.3}$$

or equivalently the canonical basis vectors in $\ell^2(\mathbb{Z})$. This lemma in turn depends on a sublemma about a finite set of projections $P_1, \dots, P_g$ in Hilbert space $\mathcal{H}$. Recall $P \in \mathcal{B}(\mathcal{H})$ is a projection iff $P = P^* = P^2$. However, there are more details to the full argument, and they will be taken up in Sections 6 and 8 below.

**Lemma 4.1.** *Let $P_1, \dots, P_g$ be projections. Suppose the operator*

$$R = P_g P_{g-1} \cdots P_2 P_1 P_2 \cdots P_{g-1} P_g \tag{4.4}$$

*is nonzero. Then $R$ is a projection if and only if the $P_i$'s are mutually commuting.*

*Proof.* It is clear that the operator $R$ in (4.4) is a projection if the family $P_1, \dots, P_g$ consists of mutually commuting projections. We now prove the converse by induction starting with two given projections $P_1, P_2$ such that $R := P_2 P_1 P_2$ is given to be a projection. Then the commutator $S := P_1 P_2 - P_2 P_1$ satisfies $S^* = -S$. Using that $R^2 = R$ we conclude that $S^3 = 0$, and therefore $S = 0$; in other words, the two projections $P_1, P_2$ commute.

Suppose the lemma holds for fewer than $g$ projections. If $R$ is given as in (4.4), then

$$R = P_g T P_g \tag{4.5}$$

where

$$T = P_{g-1} \cdots P_2 P_1 P_2 \cdots P_{g-1}. \tag{4.6}$$

Writing the operator $T$ in matrix form relative to the two projections $P_g$ and $P_g^\perp = I - P_g$, we get

$$T = \begin{pmatrix} R & P_g T P_g^\perp \\ P_g^\perp T P_g & P_g^\perp T P_g^\perp \end{pmatrix} = (T_{ij})_{i,j=0}^1 \tag{4.7}$$

with $T_{0,0} = R$, etc. But then (4.8)–(4.9) yield the conclusion:

$$(T^2)_{0,0} = (T_{0,0})^2 + T_{0,1} T_{1,0} = R + T_{0,1} (T_{0,1})^*, \tag{4.8}$$

and

$$(T^2)_{0,0} \le T_{0,0} = R \tag{4.9}$$

imply $T_{0,1}(T_{0,1})^* = 0$, and therefore $T_{0,1} = 0$. As a result, the block matrix in (4.7) reduces to

$$T = \begin{pmatrix} R & 0 \\ 0 & P_g^\perp T P_g^\perp \end{pmatrix}.$$

A further calculation shows that $T$ must then itself be a projection. From the definition of $T$ in (4.6), and the induction hypothesis, we then conclude that the family $\{P_i\}_{i=1}^g$ is indeed commutative. □

**Remark 4.2.** In the special case when all the projections $\{P_i\}_{i=1}^g$ are one-dimensional, i.e., $P_i = |v_i\rangle \langle v_i|$ in the Dirac notation, and $\|v_i\| = 1$, there is a simpler proof based on the Schwarz inequality, as follows: Let $R$ in (4.4) be given to be a projection, i.e., $R^2 = R \neq 0$. We also have $R = |\lambda_{1,2}\lambda_{2,3} \cdots \lambda_{g-1,g}|^2 P_g$ with $\lambda_{i,j} := \langle v_i \,|\, v_j \rangle$. We then conclude that $|\lambda_{1,2}\lambda_{2,3} \cdots \lambda_{g-1,g}| = 1$, and therefore by Schwarz, there are constants $\zeta_i \in \mathbb{C}$, $|\zeta_i| = 1$, such that $v_2 = \zeta_1 v_1$, $v_3 = \zeta_2 v_2, \dots,$ and the commutativity of the family $\{P_i\}_{i=1}^g$ is immediate. But in this case we find, in addition, that the projections all coincide.

## 5. Minimality and representations

The representations of the $C^*$-algebra $\mathcal{O}_N$ are used in other parts of mathematics, in addition to wavelet analysis. While it is known that in general the irreducible representations of $\mathcal{O}_N$ cannot be given a measurable labeling, see, e.g., [BrJo97a], [BrJo97b], [Cun77], and [BJKW00], there are various families of $\mathcal{O}_N$-representations which do admit labeling of their irreducibles, and their decomposition into sums of irreducibles. We show that the decomposition into sums of irreducibles occurs only for the special (permutative) representations [BrJo99a] which generalize those derived from the Haar wavelets. When decompositions do occur, the irreducibles have multiplicity at most one; see Section 8 below. The basis for our analysis is the presence of certain finite-dimensional subspaces $\mathcal{K}$ which are invariant under the operators $S_i^*$ when the representation is defined from the $S_i$'s with relations (3.9)–(3.10). For related $\mathcal{O}_N$-representations which arise in statistical mechanics, see [FNW92], [FNW94], and Section 6 below. These finite-dimensional subspaces have the the significance of labeling the correlations of the sites in the quantum spin chain model. If it is an infinite spin model on a one-dimensional lattice, then $\mathcal{K}$ describes the correlations of spin observables $\sigma_0, \sigma_1, \dots$ with those on the other side, $\dots, \sigma_{-2}, \sigma_{-1}$.

We say that a representation of $\mathcal{O}_N$ in a Hilbert space $\mathcal{H}$ is a *wavelet representation* if $\mathcal{H} = L^2(\mathbb{T})$ $(\cong \ell^2(\mathbb{Z}))$ and if the corresponding operators $S_i$ are given by (3.8) for some QMF functions $\{m_i\}_{i=0}^{N-1}$. By (3.12)–(3.13) that is equivalent to using polynomial functions $A\colon \mathbb{T} \to \mathrm{U}_N(\mathbb{C})$ for labeling the representations. We will let $\mathcal{P}(\mathbb{T}, \mathrm{U}_N(\mathbb{C}))$ be the semigroup of such *polynomial loops*, loops because they may be viewed as loops in the unitary group $\mathrm{U}_N(\mathbb{C})$, see [PrSe86]. We will use the notation $A(z) = (A_{i,j}(z))_{i,j=0}^{N-1}$ for the loop-group element $A\colon \mathbb{T} \to \mathrm{U}(N)$. Since the Fourier expansion is finite, there is a $g$ such that $A(z)$ has the form

$$A(z) = \sum_{k=0}^{g-1} z^k A^{(k)} \qquad (A^{(g-1)} \neq 0) \tag{5.1}$$

where $A^{(k)} \in \mathcal{B}(\mathbb{C}^N)$ for $k = 0, \dots, g-1$. The factorization in [BrJo00, Lemma 3.3] motivates the name *genus* for $g$.

**Lemma 5.1.** *If $A(z)$ is a general polynomial of $z$ with values in $\mathcal{B}(\mathbb{C}^N)$ of the form (5.1), the following four conditions (5.2)–(5.5) are equivalent:*

(5.2) $A(z)^* A(z) = \mathbb{1}_N$, $z \in \mathbb{T}$, *i.e., $A$ takes values in* $\mathrm{U}(N)$;

(5.3) $\sum_k A^{(k)\,*} A^{(k+n)} = \begin{cases} \mathbb{1}_N & \text{if } n = 0, \\ 0 & \text{if } n \in \mathbb{Z} \setminus \{0\}, \end{cases}$ *with the convention that* $A^{(m)} = 0$ *if* $m \notin \{0, 1, \dots, g-1\}$;

(5.4) *there are projections $P_1, \dots, P_s$ in $\mathcal{B}(\mathbb{C}^N)$, positive integers $r_1, \dots, r_s$, and a unitary $W \in \mathrm{U}(N)$ such that* $A(z) = \left(\prod_{j=1}^{s} (\mathbb{1}_N - P_j + z^{r_j} P_j)\right) W$;

*and*

(5.5) *there are projections $Q_0, Q_1, \dots, Q_{g-2}$ and a unitary $V \in \mathrm{U}(N)$ such that*

$$
\begin{aligned}
A^{(0)} &= V \prod_{j=0}^{g-2} \left( \mathbb{1}_N - Q_j \right), \\
A^{(1)} &= V \sum_{j=0}^{g-2} \left( \mathbb{1}_N - Q_0 \right) \cdots \\
&\qquad \cdots \left( \mathbb{1}_N - Q_{j-1} \right) Q_j \left( \mathbb{1}_N - Q_{j+1} \right) \cdots \\
&\qquad\qquad \cdots \left( \mathbb{1}_N - Q_{g-2} \right), \\
&\vdots \qquad \vdots \\
A^{(g-1)} &= V \prod_{j=0}^{g-2} Q_j .
\end{aligned}
$$

*Proof.* We refer the reader to [BrJo00, Proposition 3.2]. ☐

**Remark 5.2.** The case $g = 2 = N$ includes the family of wavelets introduced by Daubechies [Dau92] and studied further in [BEJ00]. Note that $g = 2$ yields the representation

$$A^{(0)} = V \left( \mathbb{1}_N - Q \right), \qquad A^{(1)} = VQ, \tag{5.6}$$

by (5.5). But then (5.3) takes the form

$$A^{(0)\,*} A^{(0)} = \mathbb{1}_N - Q, \qquad A^{(1)\,*} A^{(1)} = Q, \tag{5.7}$$

which will be used in the Sections 7 and 8 below.

In the general case, we will need the operators (alias matrices) $R\left(k,l\right) := A^{(l)\,*} A^{(k)}$, and the representation (5.5) then yields

$$
\begin{aligned}
R\left(0,0\right) &= Q_{g-2}^{\perp} \cdots Q_1^{\perp} Q_0^{\perp} Q_1^{\perp} \cdots Q_{g-2}^{\perp}, \\
&\vdots \qquad \vdots \\
R\left(g-1,g-1\right) &= Q_{g-2} \cdots Q_1 Q_0 Q_1 \cdots Q_{g-2},
\end{aligned}
\tag{5.8}
$$

which were introduced in Lemma 4.1 above.

A loop $A \in \mathcal{P}(\mathbb{T}, \mathrm{U}_N(\mathbb{C}))$ is viewed as an entire analytic matrix function, $\mathbb{C} \to M_N(\mathbb{C})$, and we consider (5.1) also as a representation for this extended (entire) function. The (unique) entire extension will be denoted $A(z)$ as well. The estimates in the next corollary translate into a *stability property* for the corresponding wavelet filters, the significance of which will be established in Section 8 below.

**Corollary 5.3.** *If $g$ is the genus, then we have the following estimate relative to the order on the positive operators on $\mathbb{C}^N$:*

$$\left( \min\left(1, \left|z\right|^2\right) \right)^{g-1} \cdot \mathbb{1}_N \leq A(z)^* A(z) \leq \left( \max\left(1, \left|z\right|^2\right) \right)^{g-1} \cdot \mathbb{1}_N,$$

*valid for all $z \in \mathbb{C}$, where $\mathbb{1}_N$ is the identity matrix.*

*Proof.* The corollary is applied in Section 8 below, so we postpone its proof to Section 8. The argument is in Observation 8.9, and it is based on the ordered factorizations (5.4)–(5.5) in Lemma 5.1. ☐

It follows from the corollary and (3.12) that the system $m_0, m_1, \dots, m_{N-1}$ of polynomials that makes up the multiresolution filter cannot have any other common zeroes than $z = 0$, i.e., if some $z_0 \in \mathbb{C}$ satisfies $m_i(z_0) = 0$ for all $i$, then $z_0 = 0$.

We now turn to some representation theory for the $C^*$-algebra $\mathcal{O}_N$ which will be needed in the following sections. Some background references for this are [BrJo99a], [Eva80], [Pop92], and [ReWe98]. Our references for wavelets and filters are [Hör95], [Pol90], and [Vai93].

Let $P$ $(\in \mathcal{B}(\mathcal{H}))$ be a projection. We say that it is *co-invariant* for some (fixed) representation $\{T_i\}_{i=0}^{N-1}$ of $\mathcal{O}_N$ if

$$T_i^* P = P T_i^* P \qquad \text{for all } i. \tag{5.9}$$

Let $\mathcal{H}_-$ be the closed span of $\{z^{-n}\,;\, n = 0, 1, \dots\}$, and let $P_-$ be the projection onto $\mathcal{H}_-$. Then (5.9) is satisfied for $P_-$ and all wavelet representations $T^{(A)}$, as follows from (3.8), (3.12), and the formula

$$T_i^{(A)\,*} = \sum_{j=0}^{N-1} \overline{A_{i,j}(z)}\, S_j^*, \tag{5.10}$$

where $S_j^*$ are the adjoints of the respective operators $S_j$ in (3.14). Specifically,

$$\left(S_j^* f\right)(z) = \frac{1}{N} \sum_{w^N = z} w^{-j} f(w), \qquad f \in L^2(\mathbb{T}). \tag{5.11}$$

**Lemma 5.4.** *Let $E$ and $P$ be co-invariant projections for a fixed representation $T^{(A)}$. Suppose $E \le P \le P_-$, and further that for some $r \in \mathbb{N}$,*

$$P\mathcal{H} = \operatorname{span}\left\{z^{-k}\,;\, 0 \le k \le r\right\}. \tag{5.12}$$

*Finally assume that*

$$T_i^{(A)\,*} E = E T_i^{(A)\,*} P \qquad \textit{for all } i = 0, 1, \dots, N-1. \tag{5.13}$$

*Then we have the following identities:*

$$S_j^* E S_k P = \sum_{i=0}^{N-1} A_{i,j} E \bar{A}_{i,k} P_k \tag{5.14}$$

*for all $j, k = 0, 1, \dots, N-1$, where the functions $A_{i,j}$ are the matrix entries of the given loop, a function is identified with the corresponding multiplication operator in $\mathcal{H} = L^2(\mathbb{T})$, and $P_k := S_k^* P S_k$ are projections.*

*Proof.* It is given that both $E$ and $P$ satisfy (5.9) relative to $T^{(A)}$, and further that $T_i^{(A)\,*} E = E T_i^{(A)\,*} P$. Equivalently, by (5.10), $\sum_l \bar{A}_{i,l} S_l^* E = E \sum_k \bar{A}_{i,k} S_k^* P$. Using $\sum_i A_{i,j} \bar{A}_{i,l} = \delta_{j,l}$, we get

$$S_j^* E = \sum_{i,k} A_{i,j} E \bar{A}_{i,k} S_k^* P. \tag{5.15}$$

Now multiplying through from the right with $S_k P$ on both sides in (5.15), the conclusion of the lemma follows. To see this, notice first from (5.11)–(5.12) that

$$S_k^* P S_l P = 0 \qquad \text{if } k \ne l. \tag{5.16}$$

The proof of (5.16) is based on the observation that the representation $S$ $(= T^{(\mathbb{1}_N)})$ in (3.14) is permutative, see [BrJo99a]. Specifically, $S_k^*\left(z^{j-nN}\right) = \delta_{k,j} z^{-n}$ if $0 \le j < N$, and $n \in \mathbb{Z}$, and $S_l(z^{-n}) = z^{l-nN}$. The desired formula (5.14) now follows from this and $S_k^* P S_k P = S_k^* P S_k = P_k$, since $P$ is relatively co-invariant for the representation $S = T^{(\mathbb{1}_N)}$ by assumption. □

**Remark and Terminology 5.5.** The proof shows more generally that the implication (5.13) ⇒ (5.14) holds for any operator $E \in \mathcal{B}(P\mathcal{H})$ when $P$ is specified as in the statement of the lemma. In $\mathcal{B}(P\mathcal{H})$, we may then introduce the basis $e_{-k,-l} := \left|z^{-k}\right\rangle \left\langle z^{-l}\right|$, and coordinates

$$E = \sum_{k,l} X_{k,l} e_{-k,-l}. \tag{5.17}$$

If the loop $A(z)$ is given by (5.1), then the operators

$$R(k,l) := A^{(l)\,*} A^{(k)} \tag{5.18}$$

of Lemma 5.1 go into the calculation of the right-hand side in (5.14) as follows: The $(r,s)$-matrix entry of the matrix $(S_j^* E S_k =) \sum_{i=0}^{N-1} A_{i,j} E \bar{A}_{i,k} P$ is given by the following matrix product:

$$\sum_{\substack{p,q \\ p \geq r,\, q \geq s}} X_{p,q} R(p-r, q-s)_{k,j}\,, \tag{5.19}$$

again with the convention that the summation indices restrict to the range where the terms in the sum are defined and nonvanishing.

**Definition 5.6.** Let $\{T_i\}_{i=0}^{N-1}$ be a representation of $\mathcal{O}_N$ on a Hilbert space $\mathcal{H}$, and let $\mathcal{K}$ be a finite-dimensional subspace which satisfies

$$T_i^* \mathcal{K} \subset \mathcal{K} \qquad \text{for all } i. \tag{5.20}$$

Hence the projection $P$ onto $\mathcal{K}$ satisfies

$$P T_i = P T_i P \qquad \text{for all } i. \tag{5.21}$$

We say that $\mathcal{K}$ is *cyclic* if it is cyclic for the $\mathcal{O}_N$-representation, i.e., if

$$\bigvee_{i_1, i_2, \dots, i_n} T_{i_1} T_{i_2} \cdots T_{i_n} \mathcal{K} = \mathcal{H}. \tag{5.22}$$

For the wavelet representations, $\mathcal{H} = L^2(\mathbb{T})$, the Fourier basis $\{z^n \,;\, n \in \mathbb{Z}\}$ has the following property: There is an $r_0 \in \mathbb{N}$ such that, for all $n \in \mathbb{Z}$, there is a $p \in \mathbb{N}$ satisfying

$$T_{i_q}^* \cdots T_{i_2}^* T_{i_1}^* (z^n) \in \operatorname{span}\left\{z^{-k} \,;\, 0 \leq k \leq r_0\right\} \tag{5.23}$$

for all multi-indices $i_1, \dots, i_q$ and $q \geq p$. We showed [BrJo00] that $r_0$ may be taken

$$r_0 = \left\lfloor \frac{gN-1}{N-1} \right\rfloor \tag{5.24}$$

where $g$ is the genus, $N$ is the scale, and $\lfloor x \rfloor$ is the largest integer $\leq x$. We also showed that, whenever (5.23) holds, then

$$\mathcal{K} := \operatorname{span}\left\{z^{-k} \,;\, 0 \leq k \leq r_0\right\} \tag{5.25}$$

is cyclic. It is known in general that, if some $\mathcal{K}$ is *minimal* with respect to the two properties, (5.20) and $\mathcal{O}_N$-cyclicity, then

$$\mathcal{B}(\mathcal{K})^{\sigma^{(A)}} := \left\{ X \in \mathcal{B}(\mathcal{K}) \,;\, \sum\nolimits_i P T_i^{(A)} X T_i^{(A)\,*} P = X \right\} \tag{5.26}$$

is an *algebra.* The set (5.26) is the fixed-point set for the completely positive map

$$\sigma_{\mathcal{K}}^{(A)}(\,\cdot\,)=\sum_i V_i(\,\cdot\,)V_i^*,\qquad \text{where } V_i:=PT_i^{(A)}. \tag{5.27}$$

We further showed in [BJKW00] that $T^{(A)}$ is irreducible if and only if $\mathcal{B}(\mathcal{K})^{\sigma^{(A)}}=\mathbb{C}\mathbb{1}_{\mathcal{K}}$. In general, this set is *not* an algebra, but the above minimality on $\mathcal{K}$ forces it to be an algebra, see [DKS99].

We shall need, in the later proofs, the following two results from [BJKW00]. We include the statements here since they seem not to be well known in the wavelet community. Let $\pi$ be a representation of $\mathcal{O}_N$ on a Hilbert space $\mathcal{H}$.

**Theorem 5.7.** [BJKW00, Section 6] *There is a positive norm-preserving linear isomorphism between the commutant algebra*

$$\pi(\mathcal{O}_N)'=\{A\in\mathcal{B}(\mathcal{H})\,;\,A\pi(x)=\pi(x)A \text{ for all } x\in\mathcal{O}_N\} \tag{5.28}$$

*and the fixed-point set*

$$\mathcal{B}(\mathcal{K})^{\sigma}=\{A\in\mathcal{B}(\mathcal{K})\,;\,\sigma(A)=A\} \tag{5.29}$$

*given by*

$$\pi(\mathcal{O}_N)'\ni A\longrightarrow PAP, \tag{5.30}$$

*where $P$ is the projection of $\mathcal{H}$ onto $\mathcal{K}$. In particular, $\pi$ is irreducible if and only if $\sigma$ is ergodic (where $\sigma$ is the mapping $\mathcal{B}(\mathcal{K})\to\mathcal{B}(\mathcal{K})$ defined in* (5.27)*).*

*More generally, if $\mathcal{K}_1$, $\mathcal{K}_2$ (with corresponding projections $P^{(1)}$ and $P^{(2)}$) are $T^*$-invariant cyclic subspaces for two representations $\pi_1$, $\pi_2$ of $\mathcal{O}_N$ on $\mathcal{H}_1$, $\mathcal{H}_2$, and*

$$V_i^{(j)}=P^{(j)}\pi_j(s_i)|_{\mathcal{K}_j} \tag{5.31}$$

*for $j=1,2$, $i=0,\dots,N-1$, define $\rho$ on $\mathcal{B}(\mathcal{K}_1,\mathcal{K}_2)$ by*

$$\rho(A)=\sum_i V_i^{(2)}AV_i^{(1)\,*}. \tag{5.32}$$

*Then there is an isometric linear isomorphism between the set of intertwiners*

$$\{A\in\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)\,;\,A\pi_1(x)=\pi_2(x)A \text{ for all } x\in\mathcal{O}_N\} \tag{5.33}$$

*and the fixed-point set*

$$\{B\in\mathcal{B}(\mathcal{K}_1,\mathcal{K}_2)\,;\,\rho(B)=B\} \tag{5.34}$$

*given by*

$$A\longrightarrow B=P^{(2)}AP^{(1)}. \tag{5.35}$$

**Theorem 5.8.** [BJKW00, Theorem 3.5] *Let $\varphi=\sum_i V_i\,\cdot\,V_i^*$ be a normal unital completely positive map of $\mathcal{B}(\mathcal{K})$. Then*

$$\{V_i,V_i^*\}'\subset\mathcal{B}(\mathcal{H})^{\varphi}.$$

*Furthermore, the space $\mathcal{B}(\mathcal{H})^{\varphi}$ contains a largest $*$-subalgebra, and this algebra is $\{V_i,V_i^*\}'$.*

We are now ready to state and prove the main result of the present section. Its significance becomes more clear when it is seen in the light of the two previous Theorems 5.7 and 5.8. In particular, we will show in Section 8 below that Theorem 5.8 is applicable in verifying irreducibility, as we will show that $\mathcal{B}(\mathcal{K})^{\sigma}$ is generically an algebra for the wavelet representations.

**Theorem 5.9.** *Let $T^{(A)}$ be a wavelet representation of $\mathcal{O}_N$ on $\mathcal{H} = L^2(\mathbb{T})$, and assume the genus of $A$ is $g$. Let $r_0$ be as in* (5.24)*, and let $P$ be the projection onto $\mathcal{K} := \operatorname{span}\{z^{-k}\,;\, 0 \leq k \leq r_0\}$. Suppose there is a second projection $E \in \mathcal{B}(\mathcal{K})$ such that $0 \neq E \neq P$, and $E$ commutes with $T_i^{(A)\,*}P$ for all $i = 0, 1, \dots, N-1$. Then it follows that $E$ is diagonal with respect to the basis $\{z^{-k}\,;\, k = 0, 1, \dots, r_0\}$ in $\mathcal{K}$. Moreover, $A(z)$ has a matrix corner of the form*

$$V\begin{pmatrix} z^{n_0} & 0 & \cdots & 0 \\ 0 & z^{n_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z^{n_{M-1}} \end{pmatrix}, \tag{5.36}$$

*where $V \in \mathrm{U}_M(\mathbb{C})$, and where the exponents $n_i$ of the diagonal corner in $A(z)$ satisfy $0 \leq n_i \leq g-1$ for all $i = 0, 1, \dots, M-1$.*

**Remark 5.10.** The loops $A\colon \mathbb{T} \to \mathrm{U}_N(\mathbb{C})$ which do admit nontrivial projections $E$ as in the statement of the theorem are described in detail in Definition 6.6 below, to which we refer. Hence, the existence of such a projection $E$ means that it is possible to "split off" a matrix block in $A(z)$ which is in diagonal form.

*Proof of Theorem* 5.9. Let $V_i := PT_i^{(A)}$. Suppose $E \in \mathcal{B}(P\mathcal{H})$ satisfies $EV_i^* = V_i^*E$, or equivalently $ET_i^{(A)\,*}P = T_i^{(A)\,*}E$. Then by Lemma 5.4 and Remark 5.5, we have

$$\left(S_j^*ES_k\right)_{r,s} = \sum_{\substack{p,q \\ p \geq r,\, q \geq s}} X_{p,q} R(p-r, q-s)_{k,j} = \sum_{\substack{p,q \\ p \geq 0,\, q \geq 0}} X_{p+r,q+s} R(p,q)_{k,j}\,. \tag{5.37}$$

The $j,k$-indices are in $\{0, 1, \dots, N-1\}$. For the matrices $R(p,q)$, we have the identities

$$\sum_p R(p,p) = \mathbb{1}_N \quad \text{and} \quad \sum_p R(p, p+l) = 0 \qquad \text{if } l \neq 0. \tag{5.38}$$

See Lemma 5.1 above. The terms on the left-hand side in (5.37) are

$$\left(S_j^*ES_k\right)_{r,s} = X_{rN-j, sN-k}\,, \tag{5.39}$$

again with the convention that the terms are defined to be zero when the subscript indices are not in the prescribed range.

If $E \neq 0$, consider the lexicographic order on the subscript indices of the corresponding matrix entries $X_{p,q}$ (in (5.17)). The range on both indices $p, q$ is $\{0, 1, 2, \dots, r_0\}$ where $r_0$ is determined as in Lemma 5.4, see also (5.24). Then pick the last (relative to lexicographic order) nonzero $X_{r,s}$, i.e., $r, s$ are determined such that

$$X_{p+r,q+s} = 0 \qquad \text{if } p > 0 \text{ or } q > 0. \tag{5.40}$$

It follows that there are only the following possibilities for this $(r,s)$:

$$\begin{aligned}
&(0,0) \qquad E = X_{0,0}\left|z^0\right\rangle\left\langle z^0\right|, \\
&(1,1) \qquad E = X_{0,0}\left|z^0\right\rangle\left\langle z^0\right| + X_{1,1}\left|z^{-1}\right\rangle\left\langle z^{-1}\right|, \\
&(2,2) \qquad E = X_{0,0}\left|z^0\right\rangle\left\langle z^0\right| + X_{1,1}\left|z^{-1}\right\rangle\left\langle z^{-1}\right| + X_{2,2}\left|z^{-2}\right\rangle\left\langle z^{-2}\right|, \\
&\;\;\vdots \qquad\quad \vdots\,.
\end{aligned}$$

If $(r,s)=(0,0)$, then, using (5.37) and (5.39), we arrive at the matrix identity

$$X_{0,0}R(0,0)=\left(\begin{array}{c|ccc} X_{0,0} & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{array}\right)\in M_N(\mathbb{C}), \qquad \text{where } X_{0,0}\neq 0, \tag{5.41}$$

and therefore $R(0,0)=|\varepsilon_0\rangle\langle\varepsilon_0|$ where $\varepsilon_0$ is the first canonical basis vector in $\mathbb{C}^N$. By Lemma 4.1, (5.8), and Remark 4.2, we conclude that $Q_i^{\perp}\geq|\varepsilon_0\rangle\langle\varepsilon_0|$ for all $i$, and therefore

$$A(z)=V\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & B(z) & \\ 0 & & & \end{array}\right) \tag{5.42}$$

for some $V\in\mathrm{U}_N(\mathbb{C})$ and $B\in\mathcal{P}(\mathbb{T},\mathrm{U}_{N-1}(\mathbb{C}))$; see Lemma 5.1.

If $(r,s)=(1,1)$, then, using again (5.37) and (5.39), we arrive at the matrix identity

$$X_{1,1}R(0,0)=\left(\begin{array}{ccc|c} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ \hline 0 & \cdots & 0 & X_{1,1} \end{array}\right)\in M_N(\mathbb{C}), \qquad \text{where } X_{1,1}\neq 0, \tag{5.43}$$

and therefore $R(0,0)=|\varepsilon_{N-1}\rangle\langle\varepsilon_{N-1}|$. Using again Lemma 4.1, (5.8), and Remark 4.2, we conclude that $Q_i^{\perp}\geq|\varepsilon_{N-1}\rangle\langle\varepsilon_{N-1}|$ for all $i$, and therefore

$$A(z)=V\left(\begin{array}{ccc|c} & & & 0 \\ & C(z) & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right) \tag{5.44}$$

for some $V\in\mathrm{U}_N(\mathbb{C})$ and $C\in\mathcal{P}(\mathbb{T},\mathrm{U}_{N-1}(\mathbb{C}))$.

The reason for why the matrix of $E$ has diagonal form relative to the natural Fourier basis is as follows: Let $N=2$, for simplicity. (The argument is the same, *mutatis mutandis,* in the general case.) Then pick the last term $(r,s)$, $r\neq s$, with $X_{r,s}\neq 0$, where again "last" refers to the lexicographic order of the matrix-entry indices, see (5.40). We then get, using (5.37) and (5.39), the following matrix-identity (where we specialize to $(r,s)=(0,1)$):

$$X_{0,1}R(0,0)=\begin{pmatrix} 0 & 0 \\ X_{0,1} & 0 \end{pmatrix}\in M_2(\mathbb{C})$$

and $X_{0,1}\neq 0$, as mentioned. This forces $R(0,0)=\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, which is impossible by Lemma 5.1, since $R(0,0)$ is positive and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ is not.

Let $N\geq 2$, and suppose $(r,s)=(2,2)$, i.e., assume that $X_{2,2}\neq 0$, and $X_{2+p,2+q}=0$ if $p>0$ or $q>0$, referring to the lexicographic order. Then by the same argument which we used in the earlier cases,

$$X_{2,2}\left(R(0,0)\right)_{k,j}=\left(X_{2N-j,2N-k}\right)_{j,k=0}^{N-1}. \tag{5.45}$$

But all the double indices $(2N-j, 2N-k)$ of the matrix on the right are strictly bigger than $(2,2)$ in the lexicographic order, and we conclude that

$$R(0,0) = 0 \qquad \text{in } M_N(\mathbb{C}). \tag{5.46}$$

The formula for $R(0,0)$ then yields $Q_0^\perp Q_1^\perp \cdots Q_{g-1}^\perp = 0$. Moreover, the additional restrictions are:

$$X_{0,0}R(0,0) + X_{1,1}R(1,1) + X_{2,2}R(2,2) = X_{0,0} \left|\varepsilon_0\right\rangle \left\langle\varepsilon_0\right| \in M_N(\mathbb{C})$$

as in (5.41), and

$$X_{1,1}R(0,0) + X_{2,2}R(1,1) = \left( \begin{array}{ccc|cc} 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ \hline 0 & \cdots & 0 & X_{2,2} & 0 \\ 0 & \cdots & 0 & 0 & X_{1,1} \end{array} \right).$$

Now substituting $R(0,0) = 0$, we arrive at

$$X_{1,1}R(1,1) + X_{2,2}R(2,2) = X_{0,0} \left|\varepsilon_0\right\rangle \left\langle\varepsilon_0\right| \tag{5.47}$$

and

$$X_{2,2}R(1,1) = X_{2,2} \left|\varepsilon_{N-2}\right\rangle \left\langle\varepsilon_{N-2}\right| + X_{1,1} \left|\varepsilon_{N-1}\right\rangle \left\langle\varepsilon_{N-1}\right|. \tag{5.48}$$

Since $E = X_{0,0} \left|1\right\rangle \left\langle 1\right| + X_{1,1} \left|z^{-1}\right\rangle \left\langle z^{-1}\right| + X_{2,2} \left|z^{-2}\right\rangle \left\langle z^{-2}\right|$ is a projection, and $X_{2,2} \neq 0$, we must have $X_{2,2} = 1$ and $X_{0,0}$ and $X_{1,1} \in \{0,1\}$. The conclusion of the theorem can then be checked case by case, using (5.47) and (5.48).

In general, let $X_{s,s} \neq 0$ be the last (in lexicographic order) nonzero term, and assume $s \geq 2$. Then by (5.38)–(5.39), we get $X_{s,s}R(0,0) = 0$, and therefore $R(0,0) = 0$ as before. Using this, the equation for the $(s-1, s-1)$ term is then

$$X_{s,s}R(1,1) = \left(X_{(s-1)N-j,(s-1)N-k}\right)_{j,k=0}^{N-1}.$$

If $(s-1)(N-1) \leq N$, then all the entries in the matrix on the right must vanish, and we get $R(1,1) = 0$. If not, we proceed as in (5.48). If $R(1,1) = 0$, we go to the $(s-2, s-2)$ term, viz.,

$$X_{s,s}R(2,2) = \left(X_{(s-2)N-j,(s-2)N-k}\right)_{j,k=0}^{N-1}. \tag{5.49}$$

Eventually the matrix on the right will have nonzero terms, starting with $X_{s,s}$, and terms before that in the lexicographic order. Suppose, for example, that the matrix on the right in (5.49) has nonzero entries. Then the equation for the $(s-3, s-3)$ term is

$$X_{s-1,s-1}R(2,2) + X_{s,s}R(3,3) = \left(X_{(s-3)N-j,(s-3)N-k}\right)_{j,k=0}^{N-1},$$

and the argument is done by a case-by-case check, using that the coordinates $X_{0,0}, X_{1,1}, \ldots$ are in $\{0,1\}$ while $X_{s,s} = 1$.

There is a similar argument, based on the reversed lexicographic order, starting with $(N-1, N-1)$, which will account for a possible lower right matrix corner of diagonal form. This completes the proof of the theorem. $\square$

**Remark 5.11.** (*Permutative Representations*) The form

$$A(z) = V \begin{pmatrix} z^{n_0} & 0 & \cdots & 0 \\ 0 & z^{n_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z^{n_{N-1}} \end{pmatrix}, \tag{5.50}$$

$V \in \mathrm{U}_N(\mathbb{C})$, in the conclusion of Theorem 5.9 corresponds to the representations of $\mathcal{O}_N$ which permute the basis elements $\{z^n ; n \in \mathbb{Z}\}$ for $\mathcal{H} = L^2(\mathbb{T})$; they are studied more generally in [BrJo99a] under the name *permutative representations.* We also met them, in a special case, in Remark 2.7 above, in connection with the "stretched out" Haar wavelets. So the conclusion of Theorem 5.9 is that the wavelet representations which are not of this form are irreducible.

Now for the details: Let $T^{(A)}$ be the representation of $\mathcal{O}_N$ corresponding to $A(z)$ in (5.50). The element $V \in \mathrm{U}_N(\mathbb{C})$ defines an automorphism of $\mathcal{O}_N$, denoted $\alpha_V$ or $\mathrm{Ad}(V)$. Let $D(z) = V^{-1}A(z)$ be the diagonal factor in (5.50). If $\pi^{(A)}(s_i) = T_i^{(A)}$ and $\pi^{(D)}(s_i) = T_i^{(D)}$ are the corresponding representations, then it follows that

$$\pi^{(A)} = \pi^{(D)} \circ \alpha_V, \tag{5.51}$$

which means that $\pi^{(A)}$ and $\pi^{(D)}$ have the same decomposition into sums of irreducibles, corresponding to irreducible subspaces of $L^2(\mathbb{T})$. The formulas for the operators $T_i^{(D)}$ are as follows:

$$T_i^{(D)}\left(z^k\right) = z^{N(n_i+k)+i}, \qquad k \in \mathbb{Z},\ i = 0, \dots, N-1, \tag{5.52}$$

which justifies the "permutative" label; in other words, both the operators $T_i^{(D)}$ and their adjoints permute the basis elements of the Fourier basis $\left\{z^k ; k \in \mathbb{Z}\right\}$ for $L^2(\mathbb{T})$. The decomposition structure of these representations is worked out in [BrJo99a]; see also [DKS99].

**Remark 5.12.** Note that if $N > 2$, then some representation $T^{(A)}$ may be reducible even if $A$ is not itself of the form (5.50); it may only have a matrix corner of this form. Take, for example,

$$A(z) = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1/\sqrt{2} & z/\sqrt{2} \\ 0 & z/\sqrt{2} & -z^2/\sqrt{2} \end{array}\right) = 1\!\!1_0 \oplus \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & z \\ z & -z^2 \end{pmatrix}, \tag{5.53}$$

i.e., $N = 3$, $g = 3$. Then $T^{(A)}$ is a *reducible* representation of $\mathcal{O}_3$ acting on $L^2(\mathbb{T})$, and, in fact, the Hardy subspace $H^2 \subset L^2(\mathbb{T})$ reduces this representation. For more details, see Section 6 below.

## 6. Irreducibility

In this section we consider as in Lemma 5.4 (5.25) the finite-dimensional subspace

$$\mathcal{K} := \mathrm{span}\{z^{-k} ; 0 \le k \le r_0\} \subset L^2(\mathbb{T}), \qquad r_0 = \left\lfloor \frac{gN-1}{N-1} \right\rfloor, \tag{6.1}$$

defined from a polynomial loop $A(z)$ of scale size $N$ and genus $g$, and we show that the irreducibility property for the corresponding representation $T^{(A)}$ of $\mathcal{O}_N$ is *generic*, i.e., it holds for all $A$ except for a subvariety of smaller dimension, once

$N$ and $g$ are fixed. In order to apply the results in Sections 5 and 6, some more details are needed regarding the subspace $\mathcal{K}$, and they are taken up in Section 8.

We begin with some notation and a lemma:

**Notation 6.1.** Let $e_n(z) := z^n, n \in \mathbb{Z}$, denote the Fourier basis in $L^2(\mathbb{T})$. For finite subsets $J \subset \mathbb{Z}$, set $\langle J \rangle := \operatorname{span}\{e_j \,;\, j \in J\} \subset L^2(\mathbb{T})$. If

$$J_0 = \{0, -1, -2, \dots, -r_0\}, \tag{6.2}$$

set $\mathcal{K} := \langle J_0 \rangle$. If $T = T^{(A)}$ is a wavelet representation and $r_0$ is as above, we note [BrJo00, Proposition 5.5] that $\mathcal{K}$ is cyclic.

**Lemma 6.2.** *Let $A \in \mathcal{P}(\mathbb{T}, \mathrm{U}_N(\mathbb{C}))$ be a (polynomial) loop of genus $g$. Then*

$$\mathcal{K} = \langle\{0, -1, \dots, -r_0\}\rangle = \operatorname{span}\left\{z^{-k} \,;\, 0 \le k \le r_0\right\}, \qquad r_0 = \left\lfloor \frac{gN-1}{N-1} \right\rfloor, \tag{6.3}$$

*contains no one-dimensional subspace which is both $T_i^{(A)\,*}$-invariant, and also cyclic for the representation of $\mathcal{O}_N$ on $L^2(\mathbb{T})$.*

*Proof.* To show that a subspace $\mathcal{K}$ is minimal in the sense specified in the lemma, we must check that whenever

$$(0 \neq) \qquad \mathcal{K}_0 \subsetneqq \mathcal{K} \tag{6.4}$$

is a subspace satisfying

$$T_i^{(A)\,*}\mathcal{K}_0 \subset \mathcal{K}_0 \qquad \text{for } i = 0, 1, \dots, N-1, \tag{6.5}$$

then $\mathcal{K}_0$ cannot be *cyclic* for the representation $T^{(A)}$ of $\mathcal{O}_N$, i.e., it generates a cyclic subspace which is a *proper* subspace of $L^2(\mathbb{T})$. The cyclic subspace generated by $\mathcal{K}_0$ is the closed subspace spanned by

$$T_{i_1}^{(A)} \cdots T_{i_n}^{(A)} \mathcal{K}_0 \qquad \text{for } n = 0, 1, \dots, \tag{6.6}$$

and all multi-indices $(i_1, \dots, i_n)$. This follows from (6.5), and we will denote this space $[\mathcal{O}_N \mathcal{K}_0]$. We will prove the assertion by checking that if (6.4)–(6.5) hold, then there is an $m_0 \in \mathbb{Z}$ such that $[\mathcal{O}_N \mathcal{K}_0]$ is contained in the closed span of $\{z^k \,;\, k \ge m_0\}$. Note that this integer $m_0$ might be negative, and also that $[\mathcal{O}_N \mathcal{K}_0]$ might well be a *proper* subspace.

Now let $\mathcal{K}_0$ be given subject to conditions (6.4)–(6.5), and suppose (as in the lemma) that $\dim \mathcal{K}_0 = 1$. Let $\xi \in \mathcal{K}_0$, $\|\xi\| = 1$. Then by (6.5) there are $\lambda_i \in \mathbb{C}$ with

$$T_i^{(A)\,*}\xi = \overline{\lambda_i}\xi, \tag{6.7}$$

or equivalently,

$$\xi(z) = \sum_i \overline{\lambda_i} m_i^{(A)}(z) \xi(z^N). \tag{6.8}$$

Using [Jor99a], [BrJo97b] we conclude that

$$\xi(z) = z^{-k} \qquad \text{for some } k, \tag{6.9}$$

after adjusting with a constant multiple, and moreover that

$$\sum_i \overline{\lambda_i} m_i^{(A)}(z) = z^{k(N-1)}. \tag{6.10}$$

Setting $\alpha(z) := \left(1, z, \dots, z^{N-1}\right)^{\mathrm{tr}}$, this may be rewritten as

$$\left\langle \lambda \,\middle|\, A(z^N)\alpha(z) \right\rangle = z^{k(N-1)}. \tag{6.11}$$

Now pick $j \in \{0, 1, \dots, N-1\}$ such that $-k \equiv j \bmod N$, and apply the operators $S_l^*$, $l = 0, 1, \dots, N-1$, to both sides in (6.11). It follows that there is some $m \in \mathbb{Z}$ such that

$$\langle \lambda \,|\, A(z)\varepsilon_j \rangle = z^m \quad \text{and} \quad \langle \lambda \,|\, A(z)\varepsilon_l \rangle = 0 \quad \text{if } l \neq j. \tag{6.12}$$

Since $|\langle \lambda \,|\, A(z)\varepsilon_j \rangle| \leq \|\lambda\| \, \|A(z)\varepsilon_j\| = 1$, the first part of (6.12) implies equality in a Schwarz inequality. Then (6.12) yields $A(z)^*\lambda = z^{-m}\varepsilon_j$, or equivalently,

$$A(z)\varepsilon_j = z^m \lambda. \tag{6.13}$$

Using the formula in Lemma 5.1 for the coefficients in $A(z)$, and Lemma 4.1, we note that (6.13) implies

$$A(z)\varepsilon_j = z^m V \varepsilon_j, \tag{6.14}$$

and in particular $\lambda = V\varepsilon_j$, where $V \in \mathrm{U}_N(\mathbb{C})$ is as in Lemma 5.1.

For the convenience of the reader, we sketch the argument, but only in the simplest case $m = 0$. With the notation of Lemma 5.1, we get

$$A^{(0)}\varepsilon_j = \lambda, \qquad A^{(k)}\varepsilon_j = 0, \qquad 1 \leq k < g, \tag{6.15}$$

where $j$ is the (fixed) number determined from (6.11) as described. Introducing the projections $Q_0, Q_1, \dots \in \mathcal{B}(\mathbb{C}^N)$ from Lemma 5.1, the first part of (6.15) then reads

$$V Q_0^\perp Q_1^\perp \cdots Q_{g-1}^\perp \varepsilon_j = \lambda, \tag{6.16}$$

and since

$$\|\lambda\| = \left( \sum_i |\lambda_i|^2 \right)^{\frac{1}{2}} = 1, \qquad \left\| Q_0^\perp Q_1^\perp \cdots Q_{g-1}^\perp \varepsilon_j \right\| = 1. \tag{6.17}$$

Hence $Q_1^\perp \cdots Q_{g-1}^\perp \varepsilon_j$ is in the range of $Q_0^\perp$, and $Q_0^\perp (Q_1^\perp \cdots Q_{g-1}^\perp \varepsilon_j) = Q_1^\perp \cdots Q_{g-1}^\perp \varepsilon_j$. We get

$$\left\| Q_1^\perp \cdots Q_{g-1}^\perp \varepsilon_j \right\| = 1, \tag{6.18}$$

and by induction,

$$Q_{g-1}^\perp \varepsilon_j = Q_{g-2}^\perp \varepsilon_j = \cdots = Q_1^\perp \varepsilon_j = Q_0^\perp \varepsilon_j = \varepsilon_j \tag{6.19}$$

and therefore $Q_k \varepsilon_j = 0$, and $A(z)\varepsilon_j = V\varepsilon_j = \lambda$. This proves the claim (6.14). Since we wish to prove that $[\mathcal{O}_N \mathcal{K}_0]$ is contained in $z^m \mathcal{H}_+$ for some $m \in \mathbb{Z}$, where $\mathcal{H}_+ := \overline{\operatorname{span}}\left\{ z^k \,;\, 0 \leq k \right\}$ is the Hardy space in $L^2(\mathbb{T})$, we may assume that $m$ in (6.14) is taken as $m = 0$. Since the invariant subspaces for a representation $\pi$ of $\mathcal{O}_N$ are the same as for $\pi \circ \alpha_V$ where $\alpha_V = \operatorname{Ad} V$, we may replace $A(z)$ with $V^{-1}A(z)$, or equivalently, reduce to the special case $A(z)\varepsilon_j = \varepsilon_j$ of formula (6.14). Since the matrix of the basis permutation $\varepsilon_0 \leftrightarrow \varepsilon_j$ is in $\mathrm{U}_N(\mathbb{C})$, the same argument leaves us with the simpler case $A(z)\varepsilon_0 = \varepsilon_0$, or equivalently,

$$A(z) = \left( \begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & A_{1,1}(z) & \cdots & A_{1,N-1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & A_{N-1,1}(z) & \cdots & A_{N-1,N-1}(z) \end{array} \right). \tag{6.20}$$

Then [BrJo00, Theorem 6.2] implies $\xi(z) = z^0 = 1$, or equivalently $k = 0$ in (6.9), and so $[\mathcal{O}_N \xi] \subset \mathcal{H}_+$. Since $\mathcal{K}_0 = \mathbb{C}\xi$, we have proved that $\mathcal{K}_0$ is not cyclic, i.e., the representation on the single vector $\xi$ does not generate all of $L^2(\mathbb{T})$. This concludes the proof of the lemma. ☐

**Remark 6.3.** If $A \in \mathcal{P}(\mathbb{T}, \mathrm{U}_N(\mathbb{C}))$, then the wavelet representation $T^{(A)}$ may or may not be irreducible, as a representation of $\mathcal{O}_N$ on $L^2(\mathbb{T})$. It is not irreducible, for example, for the Haar wavelet. Nonetheless, we will show that when $N$ and $g$ are given, then irreducibility holds *generically* for $T^{(A)}$ as $A$ ranges over all $\mathcal{P}_g(\mathbb{T}, \mathrm{U}_N(\mathbb{C}))$, i.e., has scale number $N$ and genus $g$.

Now for $T^{(A)}$ irreducible, of course every $\xi \in L^2(\mathbb{T})$, $\xi \neq 0$, will be cyclic, so cyclicity will then not be an issue, but rather the question of when $\xi$ satisfies

$$T_i^{(A)*} \xi \in \mathbb{C}\xi \qquad \text{for all } i. \tag{6.21}$$

The lemma states that (6.21) cannot hold for $\xi \neq 0$ if $T^{(A)}$ is irreducible. The issue is then instead to find *minimal* subspaces $\mathcal{K}_{\mathrm{red}}$ such that

$$T_i^{(A)*} (\mathcal{K}_{\mathrm{red}}) \subset \mathcal{K}_{\mathrm{red}}. \tag{6.22}$$

These subspaces are relevant for the algorithms which are used in the construction of and the analysis of wavelets, as we sketched in Section 1, as the subspaces $\mathcal{K}_{\mathrm{red}}$ in $L^2(\mathbb{T}) \cong \ell^2$ correspond to subspaces in the associated multiresolution subspaces in $L^2(\mathbb{R})$ (See Table 1). We first addressed (6.22) in [BrJo00], but the minimality was not considered there. We also note that (6.22) has applications for different representations of $\mathcal{O}_N$, and is there connected with finitely correlated states in statistical mechanics, see [BrJo97a, FNW94, FNW92]. The minimality issue was also considered in [DKS99] in a different context. We noted in (5.25) that $\mathcal{K} = \operatorname{span}\{z^0, z^{-1}, \dots, z^{-r_0}\}$, with $r_0$ as in (5.24), satisfies (6.22). Let $P$ or $P_{\mathcal{K}}$ denote the projection onto the subspace $\mathcal{K}$. We will consider subspaces of $\mathcal{K}$ which still satisfy (6.22), are cyclic, and minimal with respect to (6.22) and cyclicity. If, for example, $g = 3$ and $N = 2$, we will show that for some $A \in \mathcal{P}_3(\mathbb{T}, \mathrm{U}_2(\mathbb{C}))$ we may have

$$\mathcal{K}_{\mathrm{red}} = \operatorname{span}\left\{z^{-2}, z^{-3}\right\}. \tag{6.23}$$

This is a little surprising since then $r_0 = 5$, and so $\mathcal{K}$ in (5.25) is 6-dimensional.

**Corollary 6.4.** *Let $J_0$ be as above in* (6.2)*, and consider the two finite-dimensional subspaces $\mathcal{K}_0 = \langle J_0 \backslash \{0\} \rangle$ and $\mathcal{K}_1 = \langle J_0 \backslash \{-r_0\} \rangle$ in $L^2(\mathbb{T})$. Then $\mathcal{K}_0$ is non-cyclic if*

$$T_i^{(A)*} e_0 \in \mathbb{C} e_0 \qquad \textit{for all } i. \tag{6.24}$$

*Suppose $N - 1$ divides $gN - 1$. Then $\mathcal{K}_1$ is non-cyclic if*

$$T_i^{(A)*} (e_{-r_0}) \in \mathbb{C} e_{-r_0} \qquad \textit{for all } i. \tag{6.25}$$

*Moreover, $\mathcal{K}_0$ is cyclic if $\lambda_0 = \sum_i \left| A_{i,0}^{(0)} \right|^2 = 0$.*

*Proof.* The two vectors $e_0$ and $e_{-r_0}$, corresponding to the endpoints in $J_0$, are special in that

$$P T_i^{(A)} e_0 \in \mathbb{C} e_0, \tag{6.26}$$

and when $N - 1$ divides $gN - 1$,

$$P T_i^{(A)} e_{-r_0} \in \mathbb{C} e_{-r_0} \qquad \text{for all } i. \tag{6.27}$$

It follows that, when (6.24) holds, then

$$[\mathcal{O}_N \langle \{0\} \rangle] \oplus [\mathcal{O}_N \langle J_0 \backslash \{0\} \rangle] = L^2(\mathbb{T}) \tag{6.28}$$

where $\langle \{0\} \rangle = \mathbb{C}z^0$ = the one-dimensional space of the constants, and neither of the two subspaces in this orthogonal sum is zero. Hence (6.24) implies that $\mathcal{K}_0 = \langle J_0 \backslash \{0\} \rangle$ is not cyclic. The same argument proves that $\mathcal{K}_1 = \langle J_0 \backslash \{-r_0\} \rangle$ is not cyclic if (6.25) holds.

We also note that (6.24) holds if and only if

$$\sum_i \left| A_{i,0}^{(0)} \right|^2 = 1, \qquad \text{i.e., } \lambda_0 (A) = 1, \tag{6.29}$$

or equivalently,

$$R(0,0)_{0,0} \left( = \left( A^{(0)*} A^{(0)} \right)_{0,0} \right) = 1. \tag{6.30}$$

(The issue is resumed in Remark 6.5 below.)

We now turn to the converse implications in the corollary, doing the details only for $\mathcal{K}_0 = \langle J_0 \backslash \{0\} \rangle$. If (6.24) does not hold, then

$$\lambda_0 := \sum_i \left| A_{i,0}^{(0)} \right|^2 \tag{6.31}$$

satisfies $\lambda_0 < 1$. Using the following formula,

$$\begin{aligned} \left\langle e_0 \,\middle|\, T_i^{(A)}(e_{-k}) \right\rangle &= \left\langle T_i^{(A)\,*} e_0 \,\middle|\, z^{-k} \right\rangle = \left\langle \overline{A_{i,0}(z)} \,\middle|\, \overline{z^k} \right\rangle \\ &= \left\langle z^k \,\middle|\, A_{i,0}(z) \right\rangle = A_{i,0}^{(k)}, \end{aligned} \tag{6.32}$$

we therefore have the conclusion: For each $i$, $T_i^{(A)*} e_0$ is in $\mathcal{K}$, and it splits according to the sum $\mathcal{K} = \mathbb{C}e_0 + \mathcal{K}_0$ as follows:

$$T_i^{(A)*} e_0 = \overline{A_{i,0}^{(0)}} e_0 + \xi_i, \qquad 0 \leq i < N, \tag{6.33}$$

where $\xi_i \in \mathcal{K}_0$ is computed according to formula (6.32). Applying $PT_i^{(A)}$ to both sides in (6.33), we conclude that

$$e_0 = \sum_i PT_i^{(A)} T_i^{(A)*} e_0 \in \left( \sum_i \left| A_{i,0}^{(0)} \right|^2 \right) e_0 + P \left[ \mathcal{O}_N \mathcal{K}_0 \right],$$

or equivalently,

$$(1 - \lambda_0)\, e_0 \in P \left[ \mathcal{O}_N \mathcal{K}_0 \right]. \tag{6.34}$$

Since in the second part of the corollary, $\lambda_0 = 0$ by assumption, we conclude from (6.33) that $e_0 \in [\mathcal{O}_N \mathcal{K}_0]$, and the inclusion

$$\langle \underbrace{\{0, -1, \dots, -r_0\}}_{=J_0} \rangle \subset [\mathcal{O}_N \mathcal{K}_0] \tag{6.35}$$

follows. Finally, we get

$$L^2(\mathbb{T}) = [\mathcal{O}_N \langle J_0 \rangle] \subset [\mathcal{O}_N \mathcal{K}_0] \subset L^2(\mathbb{T}), \tag{6.36}$$

which proves that the reduced space $\mathcal{K}_0$ is then cyclic, and the corollary is established. $\square$

**Remark 6.5.** To summarize, $T^{(A)}$ is given by some $A \in \mathcal{P}(\mathbb{T}, \mathrm{U}_N(\mathbb{C}))$, where the coefficients $A^{(0)}, A^{(1)}, \dots \in M_N(\mathbb{C})$ satisfy $A^{(g-1)} \neq 0$ and the conditions in (5.1) and Lemma 5.1. The conclusions about cyclicity in the previous corollary may be restated as follows in terms of these matrices:

$$e_0 \notin [\mathcal{O}_N \langle J_0 \backslash \{0\} \rangle] \Longleftrightarrow A \text{ has the form} \tag{6.37}$$

$$\underbrace{\left|\begin{array}{c:ccc} * & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \\ \vdots & & M & \\ * & \cdot & \cdot & \cdot \end{array}\right|}_{A^{(0)}}\underbrace{\begin{array}{c:ccc} 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ \vdots & & I & \\ 0 & \cdot & \cdot & \cdot \end{array}\Big|}_{A^{(1)}}\begin{array}{c:ccc} 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ \vdots & & S & \\ 0 & \cdot & \cdot & \cdot \end{array}\Big| \quad \cdots \quad C \;\Big|\underbrace{\begin{array}{c:ccc} 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ \vdots & & L & \\ 0 & \cdot & \cdot & \cdot \end{array}\Big|}_{A^{(g-1)}},$$

i.e., all in the first of the first columns; and

$$e_{-r_0} \notin [\mathcal{O}_N \langle J_0 \backslash \{-r_0\} \rangle] \Longleftrightarrow A \text{ has the form} \tag{6.38}$$

$$\underbrace{\left|\begin{array}{ccc:c} \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ & M & & \vdots \\ \cdot & \cdot & \cdot & 0 \end{array}\right|}_{A^{(0)}}\underbrace{\begin{array}{ccc:c} \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ & I & & \vdots \\ \cdot & \cdot & \cdot & 0 \end{array}\Big|}_{A^{(1)}}\begin{array}{ccc:c} \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ & S & & \vdots \\ \cdot & \cdot & \cdot & 0 \end{array}\Big| \quad \cdots \quad \Big|\begin{array}{ccc:c} \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ & C & & \vdots \\ \cdot & \cdot & \cdot & 0 \end{array}\Big|\underbrace{\begin{array}{ccc:c} \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & * \\ & L & & \vdots \\ \cdot & \cdot & \cdot & * \end{array}\Big|}_{A^{(g-1)}},$$

i.e., all in the last of the last columns. It follows from Theorem 5.9 that the representation $T^{(A)}$ is *reducible* if either one of the conditions (6.37) or (6.38) holds.

We can show, using [BrJo00, Theorem 6.2], that a wavelet representation $T^{(A)}$ satisfies (6.37) if and only if there are $V \in \mathrm{U}_N(\mathbb{C})$ and $B \in \mathcal{P}(\mathbb{T}, \mathrm{U}_{N-1}(\mathbb{C}))$ such that $A$ has the form (5.42). There is a similar conclusion concerning the other condition (6.38).

Let us say that, for fixed $N$ and $g$, a property is *generic* if it holds for all loops $A(z)$ of scale $N$ and genus $g$, except for $A$ in a variety of lower dimension. Then we conclude from (6.37)–(6.38) that $\mathcal{K}_{\text{red}} = \langle\{-1, \dots, -(r_0 - 1)\}\rangle$ or $\langle\{-1, \dots, -(r_0 - 1), -r_0\}\rangle$ is cyclic for a generic set of loops, when $g$ and $N$ are fixed. The process described in (6.31) of elimination starting with the elimination of, if possible, 0 and $-r_0$ from $\langle\{0, -1, \dots, -r_0\}\rangle$ to get a smaller space, say $\mathcal{K}_0$ such that

$$T_i^{(A)*}(\mathcal{K}_0) \subset \mathcal{K}_0 \qquad \text{for all } i, \tag{6.39}$$

and

$$\mathcal{K}_0 \text{ is } \mathcal{O}_N\text{-cyclic in } L^2(\mathbb{T}), \tag{6.40}$$

may be continued, subject to certain spectral conditions on the given loop $A$. These conditions are *generic* in the same sense; for example, as noted in (6.31), $e_0$ can be eliminated (so that the smaller $\mathcal{K}_0$ will still satisfy (6.39)–(6.40)) if and only if $R(0,0)_{0,0} < 1$. There is a similar spectral condition for the elimination of two vectors $e_0$ *and* $e_{-1}$, i.e., for getting $\mathcal{K}_0 = \langle\{-2,-3,\dots,-r_0\}\rangle$ to also satisfy (6.39)–(6.40): For the $N = 2$ case, this condition is that the 2-by-2 matrix

$$\begin{pmatrix} R(1,1)_{0,0} & R(0,1)_{0,1} \\ R(1,0)_{1,0} & R(0,0)_{1,1} \end{pmatrix} \tag{6.41}$$

does not have 1 in its spectrum (recall $R(k,l) := A^{(l)*}A^{(k)}$). The argument is the same as before, even if $N > 2$, *mutatis mutandis.* If 1 is not in the spectrum, and if $R(0,0)_{0,0} = 0$, then we show that both the vectors $e_0$ and $e_{-1}$ are in the cut-down of the cyclic space, i.e., in

$$P\left[\mathcal{O}_2\left\langle\{-2,-3,\dots,-r_0\}\right\rangle\right], \tag{6.42}$$

which is then $T_i^{(A)*}$-invariant. (Here we use the symbol $P$ for the projection onto the subspace $\mathcal{K}$ spanned by $\left\{z^{-k}\ ;\ k = 0,1,\dots,r_0\right\}$ where $r_0 = \left\lfloor \frac{Ng-1}{N-1} \right\rfloor$. If $N = 2$, then, of course, $\mathcal{K}$ is of dimension $2g$.) Hence, the smaller subspace $\langle\{-2,-3,\dots,-r_0\}\rangle \subsetneqq \langle\{0,-1,-2,\dots,-r_0\}\rangle$ will also satisfy conditions (6.39)–(6.40).

To illustrate the spectral condition more explicitly, we need $g > 2$. In the case $g = 3, N = 2$, there are $V \in \mathrm{U}_2(\mathbb{C})$, and projections $P, Q$ in $\mathbb{C}^2$, such that

$$A(z) = V(P^\perp + zP)(Q^\perp + zQ), \tag{6.43}$$

and we get

$$\begin{gathered} R(0,0) = Q^\perp P^\perp Q^\perp, \qquad R(0,1) = QP^\perp Q^\perp, \qquad R(1,0) = Q^\perp P^\perp Q, \\ R(1,1) = QP^\perp Q + Q^\perp PQ^\perp, \qquad R(2,2) = QPQ. \end{gathered} \tag{6.44}$$

Hence, $R(0,0)_{0,0} = 0$ holds if and only if $P^\perp Q^\perp \varepsilon_0 = 0$, or equivalently,

$$PQ\varepsilon_0 = P\varepsilon_0 + Q\varepsilon_0 - \varepsilon_0. \tag{6.45}$$

The entries of the matrix (6.41) are then

$$\begin{pmatrix} \left\|P^\perp Q\varepsilon_0\right\|^2 + \left\|PQ^\perp\varepsilon_0\right\|^2 & \left\langle \varepsilon_0 \,\middle|\, P^\perp Q^\perp \varepsilon_1 \right\rangle \\ \left\langle P^\perp Q^\perp \varepsilon_1 \,\middle|\, \varepsilon_0 \right\rangle & \left\|P^\perp Q^\perp \varepsilon_1\right\|^2 \end{pmatrix}. \tag{6.46}$$

Since

$$R(0,0) + R(1,1) + R(2,2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{6.47}$$

the restriction $R(0,0)_{0,0} = 0$ therefore implies that

$$R(1,1)_{0,0} = 1 - R(2,2)_{0,0} = 1 - \left\|PQ\varepsilon_0\right\|^2. \tag{6.48}$$

Using this, we get that 1 is in the spectrum of (6.41), so reduced, if and only if

$$\left\|PQ\varepsilon_0\right\|^2 \cdot \left(1 - \left\|P^\perp Q^\perp \varepsilon_1\right\|^2\right) = \left|\left\langle \varepsilon_0 \,\middle|\, P^\perp Q^\perp \varepsilon_1 \right\rangle\right|^2. \tag{6.49}$$

To solve this, let for example $P$ and $Q$ be the respective projections onto

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos\rho \\ \sin\rho \end{pmatrix} \tag{6.50}$$

and solve for $\theta$ and $\rho$. For these examples (i.e., in the complementary region of the $(\theta,\rho)$-plane), we will then have $\mathcal{K}_{\mathrm{red}} := \langle\{-2,-3\}\rangle$ satisfy the covariance condition as well as the cyclicity. All of the cases $N = 2$, $g = 3$, will be taken up again in the Appendix, where the algorithmic properties of (1.7) for the scaling function $\varphi$ are displayed in detail. This is an iteration based on (1.7), and the regularity of the corresponding $x \mapsto \varphi_{\theta,\rho}(x)$ turns out to depend on the spectral properties of the operators in (6.44).

We now turn to the distinction between the diagonal elements $A(z)$ in $\mathcal{P}(\mathbb{T}, \mathrm{U}_N(\mathbb{C}))$, and the non-diagonal ones. We say that $A$ is *diagonal* if it maps into the diagonal matrices in $\mathrm{U}_N(\mathbb{C})$, except for a constant factor, i.e., if there is some $V \in \mathrm{U}_N(\mathbb{C})$, and $n_0, \dots, n_{N-1} \geq 0$, such that $A$ has the form (5.50). The variety of these diagonal loops will be called $\mathcal{P}_{\mathrm{diag}}(\mathbb{T}, \mathrm{U}_N(\mathbb{C}))$. Note that this definition includes

$$\begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \tag{6.51}$$

in $\mathcal{P}_{\mathrm{diag}}(\mathbb{T}, \mathrm{U}_2(\mathbb{C}))$.

**Definition 6.6.** We say that a loop $A \in \mathcal{P}(\mathbb{T}, \mathrm{U}_N(\mathbb{C}))$ is *purely non-diagonal* if there is not a decomposition $N = d_0 + b + d_1$ with $d_0 > 0$, or $d_1 > 0$, diagonal elements $D_i(z) \in \mathcal{P}_{\mathrm{diag}}(\mathbb{T}, \mathrm{U}_{d_i}(\mathbb{C}))$, $i = 0, 1$, $B(z) \in \mathcal{P}(\mathbb{T}, \mathrm{U}_b(\mathbb{T}))$, and $V \in \mathrm{U}_N(\mathbb{C})$ such that

$$A(z) = V \begin{pmatrix} D_0(z) & 0 & 0 \\ 0 & B(z) & 0 \\ 0 & 0 & D_1(z) \end{pmatrix}. \tag{6.52}$$

Easy examples of loop matrices for $N = 2$ and $g = 3$ which are not diagonal, i.e., do not have the representation (5.50) or (6.52) for any $V$, are

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & z \\ z & -z^2 \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} z & z^2 \\ 1 & -z \end{pmatrix}. \tag{6.53}$$

Both have $\lambda_0(A) = 1/2$. Both correspond to Haar wavelets, and both are exceptional cases in the wider family of the Appendix below. Both have irreducible wavelet representations, by the next theorem. For more details about matrix factorizations in the loop groups, we refer the reader to [PrSe86], and the paper [AlPe99].

We now turn to our first explicit result about minimal subspaces $\mathcal{L} \subset \mathcal{K}$, i.e., subspaces $\mathcal{L}$ which are $T^*$-invariant, cyclic, and which do not contain proper $T^*$-invariant subspaces which are also cyclic. In Theorem 8.2 below, we shall then further give a formula for the (unique) minimal such space $\mathcal{L}$. We stress that these results are special for the wavelet representations, and that they do not hold for other kinds of representations of $\mathcal{O}_N$.

**Theorem 6.7.** (a) *Let $A \in \mathcal{P}(\mathbb{T}, \mathrm{U}_N(\mathbb{C}))$ be given. Suppose it is purely non-diagonal, and let $T^{(A)}$ be the corresponding wavelet representation of $\mathcal{O}_N$ on $L^2(\mathbb{T})$. Then it follows that $T^{(A)}$ is irreducible.*

(b) *Let $r_0$ be as in* (5.24). *Let optimal numbers $p, q$, $0 \leq p \leq q \leq r_0$ be determined by the spectral condition in Remark* 6.5 *such that*

$$\mathcal{K}_{\mathrm{red}} = \langle \{-p, -(p+1), \dots, -q\} \rangle \tag{6.54}$$

*is $T_i^{(A)*}$-invariant for all $i$, and further satisfies*

$$\langle \{0, -1, \dots, -p+1, -q-1, \dots, -r_0\} \rangle \subset [\mathcal{O}_N \mathcal{K}_{\mathrm{red}}]. \tag{6.55}$$

*Then the following three properties hold:*

(i) $T_i^{(A)*}(\mathcal{K}_{\mathrm{red}}) \subset \mathcal{K}_{\mathrm{red}}$ *for all* $i$,
(ii) $\mathcal{K}_{\mathrm{red}}$ *is cyclic* (*for* $L^2(\mathbb{T})$),
(iii) $\mathcal{K}_{\mathrm{red}}$ *is minimal with respect to properties* (i)–(ii).

(c) *The minimal space* $\mathcal{K}_{\text{red}}$ *from* (b) *is reduced from the right if* $N-1$ *divides* $gN-1$*, where* $g$ *is the genus, and if not, it is* $\langle\{-p,\dots,-r_0\}\rangle$*; so it is only "truncated" at one end when* $N-1$ *does not divide* $gN-1$.

*Proof.* Once $\mathcal{K}_{\text{red}}$ has been chosen as in the statement (b) of the theorem, the three properties (i)–(iii) follow from Theorem 5.8 and 5.9. The significance of (i)–(iii) is that they imply that if

$$\sigma(\,\cdot\,) := \sum_i P_{\mathcal{K}_{\text{red}}} T_i^{(A)}(\,\cdot\,) T_i^{(A)*} P_{\mathcal{K}_{\text{red}}}, \tag{6.56}$$

then the fixed-point set $\mathcal{B}(\mathcal{K}_{\text{red}})^\sigma$ is in fact an algebra. This is a result of Davidson et al. [DKS99]. Using Theorem 5.8, we conclude that the projections in $\mathcal{B}(\mathcal{K}_{\text{red}})^\sigma$ are characterized by the condition of Lemma 5.4. Now, by [DKS99], there are projections $E_j \in \mathcal{B}(\mathcal{K}_{\text{red}})$ such that, for each $i,j$, we have the covariance properties

$$E_j V_i^* E_j = V_i^* E_j, \tag{6.57}$$

where $V_i^* = T_i^{(A)*} P_{\mathcal{K}_{\text{red}}}$, or equivalently,

$$V_i = P_{\mathcal{K}_{\text{red}}} T_i^{(A)}; \tag{6.58}$$

and in addition, we have

$$\sum_j E_j = \mathbb{1}_{\mathcal{K}_{\text{red}}}, \tag{6.59}$$

and each subspace $[\mathcal{O}_N E_j \mathcal{K}_{\text{red}}]$ *irreducible*, in the sense that each $[\mathcal{O}_N E_j \mathcal{K}_{\text{red}}]$ reduces the representation $\mathcal{O}_N$ to one which is irreducible on the subspace.

It follows from (6.57)–(6.59) that the complementary projection

$$\mathbb{1}_{\mathcal{K}_{\text{red}}} - E_j = \sum_{l:l\neq j} E_l \tag{6.60}$$

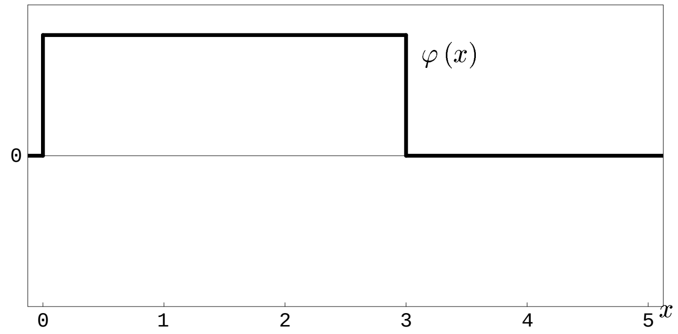
then also satisfies (6.57), and so in particular $E_j$ must commute with each $V_i$ $(= P_{\mathcal{K}_{\text{red}}} T_i^{(A)})$. Then by Theorems 5.8 and 5.9, we conclude that each $E_j$ has a matrix which is diagonal with respect to the Fourier basis $\{z^{-k}\}$. Since the loop $A(z)$ is picked to be purely non-diagonal, we finally conclude that the decomposition $\{E_j\}$ of (6.59) can only have one term, and the proof is concluded. □

**Remark 6.8.** Even if the assumption in Theorem 6.7, to the effect that $A$ be purely non-diagonal, is removed, we have the decomposition into irreducibles, and these irreducibles $[\mathcal{O}_N E_j \mathcal{K}_{\text{red}}]$ are mutually disjoint, i.e., inequivalent representations when $j \neq j'$ for two possible terms $j, j'$ in a decomposition. This follows from the Theorems 5.8 and 5.9, which state that the projections $E_j$ are all diagonal relative to the same basis (see also Theorem 8.2 below!). So in particular, $\mathcal{B}(\mathcal{K}_{\text{red}})^\sigma$ is abelian when $\mathcal{K}_{\text{red}}$ is chosen subject to conditions (i)–(iii) in the statement of Theorem 6.7. This means that the corresponding decomposition of $T^{(A)}$ into a sum of irreducible representations of $\mathcal{O}_N$ is multiplicity-free.

**Example 6.9** (An Application)**.** Even though we list only the scaling functions $\varphi(x)$ in the examples in the Appendix, the *wavelet generator* $\psi(x)$ is significant. But it is not unique: We can have a loop $A$ of genus 2, and a different one $B$

of genus 3, which have the same $\varphi$. Then, of course, there are different wavelet generators, say $\psi_A$ and $\psi_B$. To see this, take $\varphi$ as follows (see also Remark 2.7):

(6.61)

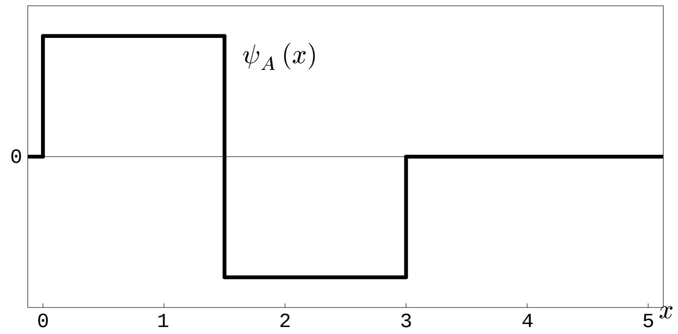



A loop $A$ in diagonal form giving this $\varphi$ is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}, \qquad \text{genus } g = 2.$$

This is of the form (5.50). The corresponding wavelet generator $\psi_A$ is then

(6.62)




But setting

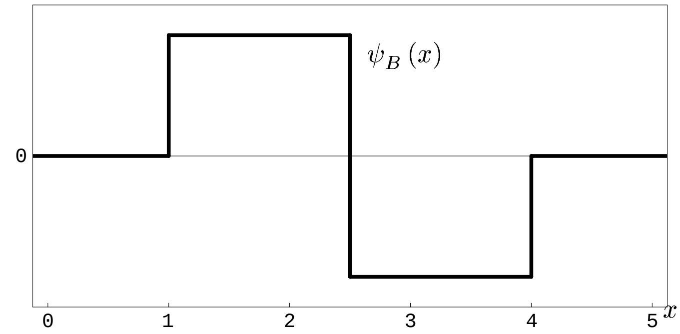
$$B(z) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & z \\ z & -z^2 \end{pmatrix}, \tag{6.63}$$

then this loop has the same $\varphi$. Since

$$m_0^{(B)}(z) = \frac{1}{\sqrt{2}}\left(1 + z^3\right) = m_0^{(A)}(z), \qquad m_1^{(B)}(z) = z^2 m_1^{(A)}(z), \tag{6.64}$$

the corresponding wavelet generator $\psi_B$ is now different from $\psi_A$ only by a translation.

(6.65)




In fact, $\psi_B(x) = \psi_A(x-1)$. However, the most striking contrast between the two loops $A$ and $B$ is that the minimality question comes out differently from one to the next: The representation $T^{(A)}$ of $\mathcal{O}_2$ on $L^2(\mathbb{T})$ is *reducible,* while $T^{(B)}$ is *irreducible,* i.e., there are no nonzero closed subspaces of $L^2(\mathbb{T})$, other than $L^2(\mathbb{T})$, which are invariant under all $T_i^{(B)}$ and $T_i^{(B)\,*}$. (Or, stated equivalently, by (4.2) we have the implication $\sum_i T_i^{(B)} X T_i^{(B)\,*} = X$, $X \in \mathcal{B}(L^2(\mathbb{T}))$, $\Rightarrow X \in \mathbb{C}\mathbb{1}_{L^2(\mathbb{T})}$.) The two conclusions for $T^{(A)}$ and $T^{(B)}$ follow from Lemma 6.2, Corollary 6.4, and Theorems 5.9 and 6.7, respectively; but Theorem 8.2 is also used. What is perhaps more surprising is that the matrix loop

$$\mathcal{A}(z) := B \oplus B = \frac{1}{\sqrt{2}} \left( \begin{array}{cc|cc} 1 & z & 0 & 0 \\ z & -z^2 & 0 & 0 \\ \hline 0 & 0 & 1 & z \\ 0 & 0 & z & -z^2 \end{array} \right) \tag{6.66}$$

(see (6.63)) in $\mathrm{U}_4(\mathbb{C})$, i.e., $N = 4$ and $g = 3$, defines a representation $T^{(\mathcal{A})}$ of $\mathcal{O}_4$ which acts *irreducibly* on $L^2(\mathbb{T})$.

Our general result in this paper is that the wavelet representations are *irreducible,* except for isolated examples of Haar type, such as $\psi_A$ in (6.62). But (6.64) above shows that even for the reducible ones, irreducibility can still be achieved, if only $\mathbb{Z}$-translations are allowed; see (6.65). The following result is a corollary of Theorem 6.7, and it helps to distinguish the wavelet representations $T^{(A)}$ from the more general representations of [FNW92, FNW94] associated with finitely correlated states in statistical mechanics. It is a crucial distinction, and it is concerned with the completely positive maps $\sigma$ which are described in Theorems 5.7 and 5.8. In [FNW94], the representations are determined by maps $\sigma$ which possess *faithful* invariant states, and these states play a role in the proofs of the results there. Our next corollary asserts that such faithful invariant states do *not* exist for the wavelet representations.

**Corollary 6.10.** *Let $T^{(A)}$ be a wavelet representation of $\mathcal{O}_N$ on $L^2(\mathbb{T})$ which satisfies the conditions in Theorem 6.7, and let $\sigma_{\mathcal{K}}^{(A)}(\,\cdot\,) = \sum_i V_i(\,\cdot\,)V_i^*$ be the corresponding completely positive mapping of Theorems 5.7 and 5.8. Then there is no* faithful *state $\rho$ on $\mathcal{B}(\mathcal{K})$ which leaves $\sigma_{\mathcal{K}}^{(A)}$ invariant, i.e., which satisfies*

$$\rho \circ \sigma_{\mathcal{K}}^{(A)} = \rho. \tag{6.67}$$

*Proof.* We will restrict to the case $N=2$, although for $g=2$, we cover arbitrary $N$ in [BrJo00]. (If $g=2$, then $\mathcal{K}=\langle e_0,e_{-1},e_{-2},e_{-3}\rangle$. Setting $E_{-k,-l}:=|e_{-k}\rangle\langle e_{-l}|$, we showed in [BrJo00] that the density matrix $D$ given by $D=\lambda_{N-2}E_{-1,-1}+(1-\lambda_{N-1})\,E_{-2,-2}$ satisfies $\sigma^*(D)=D$, where $\sigma=\sigma_{\mathcal{K}}^{(A)}$ and $\lambda_i:=R\,(0,0)_{i,i}$, and where $\sigma^*$ is the adjoint of $\sigma\colon\mathcal{B}(\mathcal{K})\to\mathcal{B}(\mathcal{K})$ with respect to the trace inner product. Defining the state $\rho$ on $\mathcal{B}(\mathcal{K})$ by

$$\rho\,(X):=\operatorname{trace}(DX)\,,\qquad\forall\,X\in\mathcal{B}(\mathcal{K})\,,\tag{6.68}$$

we check that $\rho$ satisfies (6.67). We know from [BJKW00] that $\ker(\sigma-\mathbb{1})$ and $\ker(\sigma^*-\mathbb{1})$ have the same dimension. But $T^{(A)}$ is irreducible by Theorem 6.7 when $0<\lambda_0<1$. Hence $\ker(\sigma-\mathbb{1})$ is one-dimensional by Theorem 5.7, and there are therefore no other states $\rho$ satisfying (6.67). But the state $\rho$ in (6.68) is clearly not faithful, and the proof is complete, in the special case $g=2$.)

We now turn to the details for $N=2$, $g=3$, and it will be clear that they generalize to arbitrary $g$. If $N=2$, $g=3$, we get $\mathcal{K}=\langle e_0,e_{-1},e_{-2},e_{-3},e_{-4},e_{-5}\rangle\cong\mathbb{C}^6$, and $T_i^{(A)\,*}e_{-k}$ may easily be computed; see, e.g., the details in Section 8 below, especially (8.26)–(8.31). As a result, we get $\sigma^*(E_{-k,-l})=\sum_i|T_i^*e_{-k}\rangle\langle T_i^*e_{-l}|$, and therefore

$$\sigma^*(E_{-1,-1})=\sum_{k,l}R\,(l,k)_{1,1}\,E_{-1-k,-1-l}\,,\tag{6.69}$$

$$\sigma^*(E_{-2,-2})=\sum_{k,l}R\,(l,k)_{0,0}\,E_{-1-k,-1-l}\,,\tag{6.70}$$

$$\sigma^*(E_{-3,-3})=\sum_{k,l}R\,(l,k)_{1,1}\,E_{-2-k,-2-l}\,,\tag{6.71}$$

$$\sigma^*(E_{-4,-4})=\sum_{k,l}R\,(l,k)_{0,0}\,E_{-2-k,-2-l}\,,\tag{6.72}$$

where the $k,l$ summations are both over $\{0,1,2\}$. In addition, by (6.26) and (6.27),

$$\sigma\,(E_{0,0})=\lambda_0E_{0,0},\quad\text{and}\quad\sigma\,(E_{-5,-5})=\lambda_0E_{-5,-5},\tag{6.73}$$

where $\lambda_0=\lambda_0\,(A)=R\,(0,0)_{0,0}$. So the complement of $\langle E_{0,0},E_{-5,-5}\rangle$ in $\mathcal{B}(\mathcal{K})$ is invariant under $\sigma^*$, and the element $D$ which is fixed by $\sigma^*$ must be diagonal in the Fourier basis, by Theorem 5.9. Using Lemma 5.1 and the argument from the previous step, we then check that a density matrix $D$ may be found in the form

$$D=\delta_1E_{-1,-1}+\delta_2E_{-2,-2}+\delta_3E_{-3,-3}+\delta_4E_{-4,-4},\qquad\delta_i\geq0,\ \sum_i\delta_i=1,\tag{6.74}$$

such that the state $\rho(\,\cdot\,)=\operatorname{trace}(D\,\cdot\,)$ on $\mathcal{B}\left(\mathbb{C}^6\right)$ will satisfy (6.67). But if $0<\lambda_0\,(A)<1$, the wavelet representation $T^{(A)}$ is irreducible, and so (6.67) has no other state solutions. Finally, it is clear from (6.74) that $\rho$ is not faithful. $\square$

## 7. Filtrations in $\mathcal{P}(\mathbb{T},\mathrm{U}_2(\mathbb{C}))$ as factorizations of quadrature mirror filters

Since $\mathcal{P}(\mathbb{T},\mathrm{U}_N(\mathbb{C}))$ has multiplicative structure, it has ideals, and since the unimodular polynomials, i.e., $\mathbb{T}\to\mathbb{T}$, are monomials, we may reduce the consideration to the ideals $z^k\mathcal{P}(\mathbb{T},\mathrm{U}_N(\mathbb{C}))$, $k=0,1,2,\dots$.

In view of the examples, we specialize the discussion to the case $N=2$, but the arguments work generally.

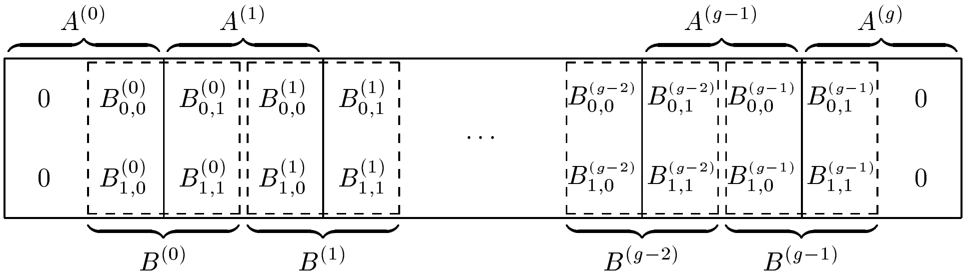



FIGURE 1. $B(z) \in \mathcal{P}_g(\mathbb{T}, \mathrm{U}_2(\mathbb{C}))$ vs. $m_i^{(A)}(z) = zm_i^{(B)}(z) \in \mathcal{P}_{g+1}(\mathbb{T}, \mathrm{U}_2(\mathbb{C}))$

In this section, we explain how the subspace $\mathcal{K} := \langle\{0, -1, \dots, -(2g-1)\}\rangle$ in (5.25) is reduced first to the smaller one $\langle\{-1, -2, \dots, -(2g-2)\}\rangle$, and then further to $\langle\{-2, \dots, -(2g-3)\}\rangle$, in the case $N = 2$. Returning to the semigroup $\mathcal{P}(\mathbb{T}, \mathrm{U}_2(\mathbb{C}))$, we note that it has a natural filtration of ideals:

$$z\mathcal{P}(\mathbb{T}, \mathrm{U}_2(\mathbb{C})) \supset z^2\mathcal{P}(\mathbb{T}, \mathrm{U}_2(\mathbb{C})) \supset \cdots . \tag{7.1}$$

A loop $A(z)$ is in $z^k\mathcal{P}(\mathbb{T}, \mathrm{U}_2(\mathbb{C}))$ if and only if there is some $B(z) \in \mathcal{P}(\mathbb{T}, \mathrm{U}_2(\mathbb{C}))$ such that

$$A(z) = z^k B(z), \qquad z \in \mathbb{T}. \tag{7.2}$$

Since

$$m_i^{(B)}(z) = \sum_j B_{i,j}(z^2)z^j, \tag{7.3}$$

we get $m_i^{(A)}(z) = z^{2k}m_i^{(B)}(z)$, and for the representations

$$T_i^{(A)} = M_{z^{2k}}T_i^{(B)} \tag{7.4}$$

where $M_{z^{2k}}$ denotes multiplication by $z^{2k}$ on the Hilbert space $L^2(\mathbb{T})$. Despite this simple relationship between $T^{(A)}$ and $T^{(B)}$, the irreducibility question can come out differently from one to the other.

If $A(z) = zB(z)$, and $B$ is of genus $g$, then $A$ is of genus $g+1$, but it has vanishing first and last columns in its representation, as is clear from Figure 1. Specifically, suppose $m_i^{(A)}(z) = zm_i^{(B)}(z)$ for all $i$; then we have the following system of identities:

(7.5)
$$A^{(0)}_{i,0} \equiv 0,\ A^{(0)}_{i,1} = B^{(0)}_{i,0},\ A^{(1)}_{i,0} = B^{(0)}_{i,1},\ A^{(1)}_{i,1} = B^{(1)}_{i,0}, \dots, A^{(g)}_{i,0} = B^{(g-1)}_{i,1},\ A^{(g)}_{i,1} \equiv 0$$

for $i = 0, 1$, and so the matrix

$$\begin{pmatrix} R(1,1)_{0,0} & R(0,1)_{0,1} \\ R(1,0)_{1,0} & R(0,0)_{1,1} \end{pmatrix}$$

of Section 6 takes the following form:

$$\begin{pmatrix} R_A(1,1)_{0,0} & R_A(0,1)_{0,1} \\ R_A(1,0)_{1,0} & R_A(0,0)_{1,1} \end{pmatrix} = \begin{pmatrix} R_B(0,0)_{1,1} & R_B(0,0)_{1,0} \\ R_B(0,0)_{0,1} & R_B(0,0)_{0,0} \end{pmatrix}. \tag{7.6}$$

Moreover, a given $A \in \mathcal{P}_{g+1}(\mathbb{T}, \mathrm{U}_2(\mathbb{C}))$ has the form $m_i^{(A)}(z) = zm_i^{(B)}(z)$ for some $B \in \mathcal{P}(\mathbb{T}, \mathrm{U}_2(\mathbb{C}))$ if and only if

$$\lambda_0(A)(:= R_A(0,0)_{0,0}) = 0. \tag{7.7}$$

Putting this together, we get the following result:

**Proposition 7.1.** (a) *Let*

$$A \in \mathcal{P}_{g+1}(\mathbb{T}, \mathrm{U}_2(\mathbb{C})),$$

*and let $P$ be the projection onto the subspace $\mathcal{K}$. Then the following three conditions,* (i), (ii), *and* (iii), *are equivalent:*

(i) $\lambda_0(A) = 0$;

(ii) $m_i^{(A)}(z) = zm_i^{(B)}(z)$;

(iii) $e_0 \in [\mathcal{O}_2 \langle\{-1, -2, \dots, -2(g-1)\}\rangle]$.

(b) *The following two conditions are equivalent:*

(i) $\lambda_0(A) = 1$;

(ii) *there is some*

$$V \in \mathrm{U}_2(\mathbb{C}), \qquad b \in \mathbb{T},$$

*such that*

$$A(z) = V \begin{pmatrix} 1 & 0 \\ 0 & bz^g \end{pmatrix} \tag{7.8}$$

*for all $z \in \mathbb{T}$.*

(c) *The following two conditions are equivalent:*

(i) $\lambda_0(A) < 1$;

(ii)

$$\begin{aligned} e_0 &\in P[\mathcal{O}_2 \langle\{-1, -2, \dots, -2g\}\rangle], \qquad \textit{and} \\ e_{-(2g+1)} &\in P[\mathcal{O}_2 \langle\{-1, -2, \dots, -2g\}\rangle]. \end{aligned} \tag{7.9}$$

(d) *Suppose $\lambda_0(A) = 0$. Then the following three conditions are equivalent:*

(i) 1 *is* not *in the spectrum of the matrix* (7.6);

(ii)

$$\begin{aligned} e_0, e_{-1} &\in P[\mathcal{O}_2 \langle\{-2, \dots, -2g+1\}\rangle], \qquad \textit{and} \\ e_{-2g-1}, e_{-2g} &\in P[\mathcal{O}_2 \langle\{-2, \dots, -2g+1\}\rangle]; \end{aligned} \tag{7.10}$$

(iii) *the loop $B$ in $m_i^{(A)}(z) = zm_i^{(B)}(z)$ has the two vectors $\left(B_{i,0}^{(0)}\right)_i$ and $\left(B_{i,1}^{(0)}\right)_i$ linearly independent in $\mathbb{C}^2$. (Hence, given the factorization $m_i^{(A)}(z) = zm_i^{(B)}(z)$, cyclicity of the reduced subspace*

$$\langle\{-2, -3, \dots, -2g+1\}\rangle,$$

*i.e.,*

$$\operatorname{span}\left\{z^{-k} \,;\, 2 \leq k \leq 2g-1\right\},$$

*holds for a generic subfamily $\{B\}$ in $\mathcal{P}_g(\mathbb{T}, \mathrm{U}_2(\mathbb{C}))$.)*

*Proof.* (a), (i) $\Rightarrow$ (ii): If $\lambda_0(A)=0$, then $A_{i,0}^{(0)}\equiv 0$, and therefore

$$A_{0,1}^{(g)}=\overline{A_{1,0}^{(0)}},\qquad A_{1,1}^{(g)}=-\overline{A_{0,0}^{(0)}}, \tag{7.11}$$

i.e., $A_{i,1}^{(g)}\equiv 0$. This means that the coefficient matrices in the expansion

$$A(z)=A^{(0)}+A^{(1)}z+\cdots+A^{(g)}z^g$$

satisfy the conditions in Figure 1; and, if we define matrices $B^{(0)},B^{(1)},\dots,B^{(g-1)}$ by (7.5) above, then it follows that $A(z)=zB(z)$ where

$$B(z)=B^{(0)}+B^{(1)}z+\cdots+B^{(g-1)}z^{g-1}. \tag{7.12}$$

Hence (ii) holds.

(ii) $\Rightarrow$ (i): This is clear from reading (7.5) in reverse.

The equivalence (i) $\Leftrightarrow$ (iii) follows from the observation that the following sum representation

$$e_0=\sum_{i_1,\dots,i_n}T_{i_1}T_{i_2}\cdots T_{i_n}l_{i_1,\dots,i_n}$$

holds for some $n$ and $l_{i_1,\dots,i_n}\in\langle\{-1,-2,\dots\}\rangle$ if and only if

$$T_{i_n}^*\cdots T_{i_2}^*T_{i_1}^*e_0\in\langle\{-1,-2,\dots\}\rangle\qquad\text{for all } i_1,\dots,i_n.$$

The conclusion can therefore be read off from the following general fact:

$$T_{i_n}^*\cdots T_{i_2}^*T_{i_1}^*e_0\in\overline{A_{i_1,0}^{(0)}}\cdots\overline{A_{i_n,0}^{(0)}}e_0+\langle\{-1,-2,\dots\}\rangle.$$

(b), (i) $\Leftrightarrow$ (ii): If $\lambda_0(A)=1$, then $A_{i,0}^{(k)}\equiv 0$ for $k>0$, and conversely. This follows from the identity

$$A^{(0)*}A^{(0)}+\cdots+A^{(g)*}A^{(g)}=\begin{pmatrix}1&0\\0&1\end{pmatrix}, \tag{7.13}$$

which is part of the defining axiom system for $A$. Hence, the result follows from [BrJo00, Theorem 6.2], once we note that the only polynomials $b(z)$ such that $|b(z)|=1$ for all $z\in\mathbb{T}$ are the monomials; see also [BrJo00, Lemma 3.1].

(c): We already showed in Section 6 that (i) *implies* the first of the conditions in (ii). The second one then follows from (7.11), i.e., the second line in (7.10) follows from the first one.

(ii) $\Rightarrow$ (i): This follows from (b) above. For if $\lambda_0(A)=1$, then it follows from (b) that

$$L^2(\mathbb{R})=[\mathcal{O}_2(\mathbb{C}e_0)]\oplus[\mathcal{O}_2\langle\{-1,-2,\dots\}\rangle], \tag{7.14}$$

and so $e_0$ is not in the subspace

$$[\mathcal{O}_2\langle\{-1,-2,\dots\}\rangle];$$

and the same argument, based on (7.11), shows that $e_{-(2g+1)}$ is not in

$$[\mathcal{O}_2\langle\{\dots,-2g+1,-2g\}\rangle],$$

concluding the proof of (c).

(d): We already saw that if $\lambda_0(A)=0$, then the conditions in (7.10) hold if and only if 1 is not in the spectrum of the matrix from (7.6). Having now

$m_i^{(A)}(z) = zm_i^{(B)}(z)$ from (a) above, we can use the identity (7.6) relating the $R_A(\,\cdot\,,\,\cdot\,)$-numbers to the $R_B(\,\cdot\,,\,\cdot\,)$-numbers. But 1 is in the spectrum of the matrix

$$\begin{pmatrix} d_1 & c \\ \bar{c} & d_0 \end{pmatrix}$$

if and only if

$$(1-d_0)(1-d_1) = |c|^2 . \tag{7.15}$$

The matrix on the right-hand side in (7.6) is of this form, and

$$|c|^2 \leq d_0 d_1 \tag{7.16}$$

by Schwarz's inequality. Here

$$c := \sum_i \overline{B_{i,1}^{(0)}} B_{i,0}^{(0)}, \qquad d_0 := \sum_i \left| B_{i,0}^{(0)} \right|^2 \leq 1, \qquad d_1 := \sum_i \left| B_{i,1}^{(0)} \right|^2 \leq 1. \tag{7.17}$$

But using (7.11) and (7.13), we also get $d_0 + d_1 \leq 1$. Now (7.15)–(7.16) yield $1 \leq d_0 + d_1$, and therefore $d_0 + d_1 = 1$. Substituting this back into formula (7.15) then yields $d_0 d_1 = |c|^2$, which amounts to "equality" in Schwarz's inequality (7.16); and so the corresponding vectors (d)(iii) are proportional. We already noted the equivalence of (i) and (ii) in (d); and we just established that the negation of (i) amounts to linear dependence of the vectors in (d)(iii). So (i) is equivalent to the linear independence, as claimed in (d)(iii). This completes the proof of the proposition. □

**Remark 7.2.** Let loops $A$ and $B$ be as in Proposition 7.1(a), see also Figure 1, and let $T^{(A)}$ and $T^{(B)}$ be the corresponding wavelet representations. Then, as a result of the theorems in Sections 6 and 8, we conclude that $T^{(A)}$ is irreducible if and only if $T^{(B)}$ is. Since the factorization in Proposition 7.1(a) corresponds to $\lambda_0(A) = 0$, we conclude that the general irreducibility question has therefore been reduced to the case $\lambda_0(A) > 0$, which is the subject of the next section.

## 8. An explicit formula for the minimal subspace

Given $N = 2$, and $A \in \mathcal{P}_g(\mathbb{T}, \mathrm{U}_2(\mathbb{C}))$, we considered the wavelet representation $T^{(A)}$ of $\mathcal{O}_2$ on $L^2(\mathbb{T}) \cong \ell^2$. We showed that

$$\mathcal{K} := \left\langle z^0, z^{-1}, \dots, z^{-(2g-1)} \right\rangle \tag{8.1}$$

is $T^{(A)\,*}$-invariant and cyclic (in $L^2(\mathbb{T})$) for the representation $T^{(A)}$. But we also showed that the first one of the basis vectors, $z^0$, *is* in the cyclic space generated by $z^{-1}, z^{-2}, \dots, z^{-2(g-1)}$ and the representation, if and only if $A_{i,0}^{(0)} \equiv 0$ for all $i$. Specifically, setting

$$\lambda_0(A) := \sum_i \left| A_{i,0}^{(0)} \right|^2 , \tag{8.2}$$

we showed in Corollary 6.4 and Proposition 7.1(a) that

$$e_0 \in \left[\mathcal{O}_2 \left\langle e_{-1}, e_{-2}, \dots, e_{-2(g-1)} \right\rangle\right] \tag{8.3}$$

if and only if $\lambda_0(A) = 0$. Hence, it follows that $\mathcal{K}$ is *not minimal* (in the sense of the following definition) if $\lambda_0(A) = 0$. In other words, if $\lambda_0(A) = 0$, $\mathcal{K}$ then contains a strictly smaller subspace which is both $T^{(A)\,*}$-invariant and cyclic. In this section, we show the converse implication. But first a definition:

**Definition 8.1.** We say that a subspace $\mathcal{L} \subset \mathcal{K}$ is *minimal* if it is $T^{(A)\,*}$-invariant, cyclic for the representation $T^{(A)}$, and minimal with respect to the two properties, i.e., it does not contain a proper subspace which is also $T^{(A)\,*}$-invariant and cyclic.

We will now prove the converse to the above-mentioned result, showing, in particular, that if $\lambda_0(A) > 0$, then $\mathcal{K}$ is *generically* minimal; see Corollary 8.7.

**Theorem 8.2.** *Let $A \in \mathcal{P}_g(\mathbb{T}, \mathrm{U}_2(\mathbb{C}))$, and let $T^{(A)}$ be its wavelet representation. Let $\mathcal{K} = \langle z^0, z^{-1}, \dots, z^{-(2g-1)} \rangle$ as in* (8.1)*, and assume $\lambda_0(A) > 0$. Then $\mathcal{K}$ contains a unique minimal subspace $\mathcal{L}$, i.e., $\mathcal{L}$ is $T^{(A)\,*}$-invariant, cyclic, and minimal. It is spanned by the complex conjugates of the following family of $4g$ functions:*

$$A_{i,j}(z)\, z^{k+j}, \qquad \text{where } i,j \in \{0,1\} \text{ and } k \in \{0,1,\dots,g-1\}. \tag{8.4}$$

(*Here $k$ varies independently of* both *$i$ and $j$.*) *Moreover, within the class*

$$\mathcal{P}_g(\mathbb{T}, \mathrm{U}_2(\mathbb{C})), \qquad \lambda_0(A) > 0, \tag{8.5}$$

*the dimension of $\mathcal{L}$ is $2g$, for a generic subfamily, and so, for this subfamily, $\mathcal{L} = \mathcal{K}$, and $\mathcal{K}$ itself is minimal.*

**Remark 8.3.** An immediate consequence of the definition of the subspace $\mathcal{L} \subset \mathcal{K} \subset L^2(\mathbb{T})$ is the following formula for the "deficiency space":

$$\mathcal{K} \ominus \mathcal{L} = \bigwedge_i \ker(PT_iP) \tag{8.6}$$

where $P$ denotes the projection onto $\mathcal{K}$. We will show below, using (8.6) and Corollary 5.3, that $\mathcal{L} = \mathcal{K}$ if $\lambda_0(A) > 0$. This means that $\mathcal{K}$ itself is then the unique minimal subspace when the loop $A$ does not have $m_i^{(A)}(z) \in z\mathbb{C}[z]$, as in Section 7. But before arriving at the conclusion, we must first derive several *a priori* properties of $\mathcal{L}$.

*Proof of Theorem* 8.2. The details are somewhat technical, and it seems more practical to first do them for the special case when $g = 3$, and then comment at the end on the (relatively minor) modifications needed in the proof for the case when $g$ is arbitrary $g \geq 2$.

Using the terminology of (5.1), we then get

$$A(z) = A^{(0)} + A^{(1)} z + A^{(2)} z^2, \tag{8.7}$$

where $A^{(2)} \neq 0$ and $A^{(0)}$, $A^{(1)}$, and $A^{(2)}$ are 2-by-2 complex matrices satisfying

$$\sum_{k=0}^{2} A^{(k)\,*} A^{(k+l)} = \delta_{0,l} \mathbb{1}_2. \tag{8.8}$$

When $A$ is given, we denote that corresponding wavelet representation by $T^{(A)}$, or just $T$ for simplicity. Recall

$$(T_i f)(z) = \sum_j A_{i,j}(z^2)\, z^j f(z^2), \tag{8.9}$$

or simply

$$T_i f(z) = m_i^{(A)}(z)\, f(z^2), \tag{8.10}$$

where

$$m_i^{(A)}(z) = \sum_j A_{i,j}(z^2) z^j. \tag{8.11}$$

As we saw in (8.1), the subspace

$$\mathcal{K} = \langle z^0, z^{-1}, z^{-2}, z^{-3}, z^{-4}, z^{-5} \rangle \tag{8.12}$$

is then $T^*$-invariant, and also cyclic for the representation. But the issue is when $\mathcal{K}$ is *minimal* with respect to these two properties. The minimality of some subspace $\mathcal{L} \subset \mathcal{K}$ then means that $\mathcal{L}$ is $T^*$-invariant, and cyclic, and that no proper $T^*$-invariant subspace of $\mathcal{L}$ is cyclic.

In working out details on $\mathcal{L}$, we use (8.9)–(8.11) in conjunction with (5.10)–(5.11), and it is more helpful to work with the complex conjugates

$$\mathcal{M} := \overline{\mathcal{L}} = \left\{ \overline{f(x)} \,;\, f \in \mathcal{L} \right\}, \tag{8.13}$$

and so $\mathcal{M}$ consists of polynomials of degree at most 5. It follows from (8.9)–(8.11) that $\mathcal{M}$ is then spanned by the functions (polynomials) in the following list:

$$A_{i,j}(z) z^{k+j}, \qquad i = 0, 1,\ j = 0, 1,\ k = 0, 1, 2. \tag{8.14}$$

Hence, by (8.13), $\mathcal{L}$ consists of the space spanned by the complex conjugates of these functions. By (8.12) it is clear that $\mathcal{L} \subset \mathcal{K}$.

The proof of Theorem 8.2 will now be split up into several lemmas:

**Lemma 8.4.** *The space $\mathcal{L}$ is $T^*$-invariant.*

*Proof.* Now, the functions $A_{i,j}(z)$ in (8.14) are the matrix elements of the loop $A(z)$, and so it follows from (8.7) that each of them is a polynomial of degree at most 2, say

$$a(z) = c_0 + c_1 z + c_2 z^2 \tag{8.15}$$

(since $A(z)$ itself has degree 2 when $g = 3$). Hence,

$$T_i^*(\bar{a}) = \bar{c}_0 \overline{A_{i,0}(z)} + \bar{c}_1 \overline{A_{i,1}(z)} z^{-1} + \bar{c}_2 \overline{A_{i,0}(z)} z^{-1}, \tag{8.16}$$

or equivalently,

$$\overline{T_i^*(\bar{a})} = c_0 A_{i,0}(z) + c_1 A_{i,1}(z) z^1 + c_2 A_{i,0}(z) z^1. \tag{8.17}$$

Now set $b(z) := z^{2p} a(z)$ where $a$ is as in (8.15). From (8.11), we then get

$$T_i^*(\bar{b}) = z^{-p} T_i^*(\bar{a}),$$

or equivalently,

$$\overline{T_i^*(\bar{b})} = z^p (c_0 A_{i,0}(z) + c_1 A_{i,1}(z) z + c_2 A_{i,0}(z) z), \tag{8.18}$$

using (8.17). In view of (8.14), we need then only to compute the following:

$$T_i^*\left(\overline{za(z)}\right) = \bar{c}_0 \overline{A_{i,1}(z)} z^{-1} + \bar{c}_1 \overline{A_{i,0}(z)} z^{-1} + \bar{c}_2 \overline{A_{i,1}(z)} z^{-2},$$

or equivalently,

$$\overline{T_i^*\left(\overline{za(z)}\right)} = c_0 A_{i,1}(z) z + c_1 A_{i,0}(z) z + c_2 A_{i,1}(z) z^2.$$

Now putting the formulas together, we get the value of $T_i^*$ on each of the functions (8.14) which go into the definition of $\mathcal{L}$, and the conclusion of the lemma follows. □

**Lemma 8.5.** *The space $\mathcal{L}$ is cyclic.*

*Proof.* Since both $\mathcal{L}$ and $\mathcal{K}$ are $T^*$-invariant, the conclusion will follow if we check the inclusion

$$\mathcal{K} \subset \operatorname*{span}_i (T_i \mathcal{L}). \tag{8.19}$$

For the space on the right-hand side in (8.19), we shall use the terminology $\left[\mathcal{O}_2^1 \mathcal{L}\right]$, and similarly, the space spanned by all the spaces

$$T_{i_1} T_{i_2} \cdots T_{i_n} \mathcal{L} \tag{8.20}$$

will be denoted $[\mathcal{O}_2^n \mathcal{L}]$. In (8.20), we vary the multi-index $(i_1, i_2, \dots, i_n)$ over all the $2^n$ possibilities. It follows from the $T^*$-invariance of $\mathcal{K}$ (in (8.1)) and $\mathcal{L}$ (in (8.4)) that we get different families of nested finite-dimensional subspaces:

$$\mathcal{K} \subset \left[\mathcal{O}_2^1 \mathcal{K}\right] \subset \left[\mathcal{O}_2^2 \mathcal{K}\right] \subset \left[\mathcal{O}_2^3 \mathcal{K}\right] \subset \cdots \subset \left[\mathcal{O}_2^n \mathcal{K}\right] \subset \left[\mathcal{O}_2^{n+1} \mathcal{K}\right] \subset \cdots, \tag{8.21}$$

and a similar sequence for $\mathcal{L}$. Since $\mathcal{K}$ is cyclic, we have

$$\bigvee_n [\mathcal{O}_2^n \mathcal{K}] = L^2(\mathbb{T}) \qquad (\cong \ell^2). \tag{8.22}$$

But $\mathcal{K} \subset [\mathcal{O}_2^n \mathcal{L}]$, for some $n$, so $\mathcal{L}$ is also cyclic. The conclusion of the lemma follows from (8.21) and (8.22), once we check that

$$\mathcal{K} \subset \left[\mathcal{O}_2^1 \mathcal{L}\right], \tag{8.23}$$

and so $n = 1$ works, and$[\mathcal{O}_2^p \mathcal{K}] \subset \left[\mathcal{O}_2^{p+1} \mathcal{L}\right]$ for all $p$.

Turning now to the details: Since $\mathcal{K}$ is spanned by $z^{-k}$, $k = 0, 1, \dots, 5$, we must check that each of these basis functions has the representation

$$z^{-k} = \sum_i T_i l_i \tag{8.24}$$

for $l_0, l_1 \in \mathcal{L}$, where we refer to (8.14) (see also (8.4)) for the characterization of the space $\mathcal{L}$, or rather $\mathcal{M} := \overline{\mathcal{L}}$.

But (8.24) is equivalent to the assertion that

$$T_i^* \left(z^{-k}\right) \in \mathcal{L} \tag{8.25}$$

for all $i = 0, 1$ and $k = 0, 1, \dots, 5$; and (8.25) can be checked by a direct calculation, which is very similar to the one going into the proof of Lemma 8.4. Specifically, using (5.10)–(5.11) we get the following:

$$T_i^* \left(z^0\right) = \overline{A_{i,0}(z)} \in \mathcal{L}, \tag{8.26}$$

$$T_i^* \left(z^{-1}\right) = \overline{A_{i,1}(z)} z^{-1} \in \mathcal{L}, \tag{8.27}$$

$$T_i^* \left(z^{-2}\right) = \overline{A_{i,0}(z)} z^{-1} \in \mathcal{L}, \tag{8.28}$$

$$T_i^* \left(z^{-3}\right) = \overline{A_{i,1}(z)} z^{-2} \in \mathcal{L}, \tag{8.29}$$

$$T_i^* \left(z^{-4}\right) = \overline{A_{i,0}(z)} z^{-2} \in \mathcal{L}, \tag{8.30}$$

and finally

$$T_i^* \left(z^{-5}\right) = \overline{A_{i,1}(z)} z^{-3} \in \mathcal{L}. \tag{8.31}$$

Recall that the complex conjugates of the functions on the right-hand side in this list are precisely the ones from (8.14), or equivalently, (8.4). This proves (8.23), and therefore the cyclicity of $\mathcal{L}$, which was claimed in the lemma. As a bonus, we get from (8.26)–(8.31) that the inclusion $\mathcal{L} \subset \langle \{-1,-2,-3,-4\} \rangle$ holds if and only if $\lambda_0(A) = 0$. To see this, use the fact (for $g = 3$) that

$$A_{i,1}^{(2)} = (-1)^i \overline{A_{1-i,0}^{(0)}}.$$

□

**Lemma 8.6.** *The space $\mathcal{L}$ is minimal in the sense of Definition* 8.1.

*Proof.* We will establish the conclusion by proving that if $\mathcal{L}_1$ is any subspace of $\mathcal{K}$ which is both $T^*$-invariant and cyclic, then $\mathcal{L} \subset \mathcal{L}_1$. So in particular, $\mathcal{L}$ does not contain a *proper* subspace which is both $T^*$-invariant and cyclic.

Now suppose that some space $\mathcal{L}_1$ has the stated properties. Since it is cyclic, we must have

$$\mathcal{K} \subset [\mathcal{O}_2^n \mathcal{L}_1] \tag{8.32}$$

satisfied for *some* $n \in \mathbb{N}$. As noted in the proof of Lemma 8.5, this is equivalent to

$$T_{i_n}^* \cdots T_{i_2}^* T_{i_1}^* \left(z^{-k}\right) \in \mathcal{L}_1 \tag{8.33}$$

for all $i_1, \dots, i_n \in \{0,1\}$, and all $k \in \{0,1,\dots,5\}$. But we also saw in the proof of Lemma 8.4 that the functions on the left-hand side in (8.33) are precisely those which are listed in (8.14). Note that the functions in (8.14), or (8.4), are those given by

$$T_i^* \left(z^{-k}\right), \qquad i = 0,1, \qquad k = 0,1,\dots,5. \tag{8.34}$$

But $\lambda_0(A) > 0$ by assumption, so for some $i$, we have $A_{i,0}^{(0)} \neq 0$, and the calculation in the proof of Lemma 8.4, and in the previous two sections, then shows that the families of functions in (8.34) and (8.33) are the same, i.e., we get the same functions in (8.33) for $n > 1$ as the ones which are already obtained for $n = 1$ in (8.34). This is the step which uses the assumption $\lambda_0(A) > 0$. Since $\mathcal{L}$ is spanned by the vectors in (8.34), the desired inclusion $\mathcal{L} \subset \mathcal{L}_1$ follows. More details are worked out in Remark 8.8 below. □

*Proof of Theorem* 8.2 *concluded.* The result in the theorem is now immediate from the three lemmas, and we need only comment on the size of the genus $g$. We argued the case $g = 3$; but, for the general case, $\mathcal{K}$ is spanned by $z^{-k}$, $k = 0,1,\dots,2g-1$, and the functions from the list (8.14), or equivalently (8.4), will then be

$$A_{i,j}(z) z^{k+j}, \qquad i = 0,1,\ j = 0,1,\ k = 0,1,\dots,g-1. \tag{8.35}$$

Otherwise, all the arguments from the proofs of the lemmas carry over. See Remark 8.8 for more details. □

**Corollary 8.7.** *When $g$ is given, and $\lambda_0 := \sum_i \left|A_{i,0}^{(0)}\right|^2 = R(0,0)_{0,0} > 0$, then $\mathcal{L} = \mathcal{K}$ for a generic set of loops $A$ in $\mathcal{P}_g(\mathbb{T}, \mathrm{U}_2(\mathbb{C}))$.*

*Proof.* The proof comes down to a dimension count. Since $\mathcal{K} = \left\langle z^0, \dots, z^{-(2g-1)} \right\rangle$ is of dimension $2g$, we just need to check that the space $\mathcal{L}$ ($\subset \mathcal{K}$), spanned by the $4g$ functions in (8.35), is of dimension $2g$ for a generic set of loops $A$ in $\mathcal{P}_g(\mathbb{T}, \mathrm{U}_2(\mathbb{C}))$, and that can be checked by a determinant argument based on the conditions for the matrices $A^{(0)}, A^{(1)}, \dots, A^{(g-1)}$ defining $A(z)$; see (8.7)–(8.8) above.

The above-mentioned dimension count is based on the following consideration (which we only sketch in rough outline). A possible linear relation among the functions from (8.4) takes the form

$$\sum_i \sum_j \sum_k C_{i,j,k} A_{i,j}(z)\, z^{j+k} \equiv 0, \tag{8.36}$$

where the $i,j$ summation indices are $0,1$, and the $k$ summation is over $0,1,\dots,g-1$. As a result, we get the following system of relations:

$$\sum_{i=0}^{1} C_{i,j,k} A_{i,j}(z) \equiv 0 \qquad \left(\operatorname{mod} z^{g-j-k}\right) \tag{8.37}$$

for all $j=0,1$, and all $k=0,1,\dots,g-1$. Note that (8.37) is a matrix multiplication. Using finally

$$\sum_i \left| A_{i,0}^{(0)} \right|^2 > 0, \tag{8.38}$$

we see that the dimension of the space spanned by $\left\{ A_{i,j}(z)\, z^{j+k} \right\}$ is $2g$, as claimed. See Remark 8.8 and Observation 8.9 for details. ☐

**Remark 8.8.** A more detailed study of the space $\mathcal{L}$ will be postponed to a later paper, but one point is included here: The function $z^0$ $(= e_0 \equiv 1)$ is in $\mathcal{L}$ if and only if the polynomials

$$\left\{ A_{i,0}(z), A_{j,1}(z)\, z \right\}_{i,j} \tag{8.39}$$

do not have a common divisor. This follows from (8.26)–(8.31). Indeed, for $e_0$ to be in $\mathcal{L}$, we must have the existence of $h_{i,j}(z) \in \mathbb{C}[z]$ such that

$$1 = \sum_i h_{i,0}(z)\, A_{i,0}(z) + \sum_j h_{j,1}(z)\, z A_{j,1}(z). \tag{8.40}$$

But by algebra, this amounts to the assertion that the family of polynomials listed in (8.39) is mutually prime within the ring $\mathbb{C}[z]$. Also note that, by the result in Section 7, monomials such as $d(z)=z$ are not common divisors in the polynomials of (8.39) if $d_0(A) > 0$. In fact, a possible common divisor $d(z) \in \mathbb{C}[z]$ for (8.39) yields the following factorization:

$$A_{i,0}(z) = d(z)\, k_{i,0}(z), \qquad A_{j,1}(z)\, z = d(z)\, k_{j,1}(z) \tag{8.41}$$

$(k_{i,0}(z), k_{j,1}(z) \in \mathbb{C}[z])$. Hence:

**Observation 8.9.** If $d_0(A) > 0$, then $e_0 \in \mathcal{L}$.

*Proof.* For if not, the greatest common divisor $d(z)$ of the family (8.39) would have a root $\gamma \in \mathbb{C} \setminus \{0\}$, i.e., $d(\gamma) = 0$. By (8.41), we would then have

$$A(\gamma) = 0, \tag{8.42}$$

where $A \in \mathcal{P}(\mathbb{T}, \mathrm{U}_2(\mathbb{C}))$ is the originally given loop. Recall that, by (5.1), we may view $A(z)$ as an entire analytic matrix function, i.e., an entire analytic function, $\mathbb{C} \to M_2(\mathbb{C})$, whose restriction to $\mathbb{T}$ takes values in $\mathrm{U}_2(\mathbb{C})$. (These are also called *inner* matrix functions [PrSe86].) But (8.42) is impossible (for $\gamma \neq 0$) in view of Lemma 5.1 and its corollary. We will give the details for $g = 3$, but they apply with the obvious modifications to the general case of $g \geq 2$. If (8.42) holds, then by Lemma 5.1,

$$V^{-1}A(\gamma) = \left(Q_0^{\perp} + \gamma Q_0\right)\left(Q_1^{\perp} + \gamma Q_1\right) = 0, \tag{8.43}$$

where we use the projections $Q_0, Q_1$ in $\mathbb{C}^2$ from (5.5). Setting $A_j(z) = Q_j^{\perp} + zQ_j$, $j = 0, 1$, (8.43) then yields the following estimate:

$$\begin{aligned} 0 &= A_1(\gamma)^* A_0(\gamma)^* A_0(\gamma) A_1(\gamma) = A_1(\gamma)^* \left(Q_0^{\perp} + |\gamma|^2 Q_0\right) A_1(\gamma) \\ &\geq \min\left(1, |\gamma|^2\right) \cdot \left(A_1(\gamma)^* A_1(\gamma)\right) = \min\left(1, |\gamma|^2\right) \cdot \left(Q_1^{\perp} + |\gamma|^2 Q_1\right) \\ &\geq \left(\min\left(1, |\gamma|^2\right)\right)^2 \cdot \mathbb{1}_2, \end{aligned} \tag{8.44}$$

where the order $\geq$ is that of positive operators on $\mathbb{C}^2$. But this is impossible, since $\gamma \neq 0$. The latter is from the assumption $d_0(A) > 0$. □

This last argument in this proof also serves as a proof of Corollary 5.3, and this corollary is again the basis for the following stronger result, which we now sketch:

**Theorem 8.10.** *If $\lambda_0(A) > 0$, it follows that $\mathcal{L} = \mathcal{K}$.*

*Proof.* By Corollary 5.3 and (3.12), we have the estimate

$$\sum_{i=0}^{1} |m_i(z)|^2 \geq \left(\min\left(1, |z|^2\right)\right)^{g-1}\left(1 + |z|^2\right) \qquad \text{for all } z \in \mathbb{C}. \tag{8.45}$$

If $k \in \mathcal{K} \ominus \mathcal{L}$, then by (8.6), we get

$$Pm_i(z)\, k\left(z^2\right) = 0, \qquad i = 0, 1,\ z \in \mathbb{T}. \tag{8.46}$$

If $\lambda_0(A) > 0$, then the value $z = 0$ is not a common root of the two complex polynomials $m_0(z)$, $m_1(z)$, and so by (8.45) the two polynomials $m_0$ and $m_1$ have no common roots at all, by Proposition 7.1(a). Therefore, when (8.45) and (8.46) are combined, we get $k = 0$. Hence $\mathcal{K} \ominus \mathcal{L} = 0$, and the proof is completed. □

**Remark 8.11.** Not everything that works out easily in the case $g = 3$ generalizes immediately to $g > 3$: The case $g = 3$, and arbitrary $N$, amounts to a choice of two projections, say $P$ and $Q$, in $\mathbb{C}^N$, and the finite-dimensional representations of the algebra generated by two projections are completely known by folklore (see, e.g., [JSW95]). In fact, it can be easily checked that this is the same as displaying the finite-dimensional representations of the Clifford algebra with two generators, $A$, $B$, say. The relations between $A$, $B$ are:

$$A^* = A, \qquad B^* = B, \qquad A^2 + B^2 = \mathbb{1}_N, \quad \text{and} \quad AB + BA = 0. \tag{8.47}$$

For any such pair, set

$$P = \frac{1}{2}(\mathbb{1} + A - B), \qquad Q = \frac{1}{2}(\mathbb{1} - A - B). \tag{8.48}$$

Then it is immediate that $P$ and $Q$ are projections, i.e., $P = P^* = P^2$, etc. Conversely, if $P$, $Q$ are any projections, set

$$A = P - Q, \qquad B = \mathbb{1}_N - P - Q, \tag{8.49}$$

and an easy calculation shows that (8.47) is then satisfied. Since the finite-dimensional representations of (8.47), the Clifford algebra $\mathcal{C}_2$, are known [JSW95], we then get a useful classification of $\mathcal{P}_3(\mathbb{T}, \mathrm{U}_N(\mathbb{C}))$. But these comments do not carry over to the case $g > 3$. (A good reference on the Clifford algebra and its representations is [LaMi89, Ch. 1, §5].)

It is perhaps a little early to identify regions in the parameters $\theta$, $\rho$ where the scaling function $x \mapsto \varphi_{\theta,\rho}(x)$ is regular, and where it is not, but a primitive test would be the vanishing-moment condition of Daubechies [Dau92, Ch. 6]. We would look for values $\theta$, $\rho$ such that $m_0^{(\theta,\rho)}(z)$ is divisible by $(1+z)^2$ and by $(1+z)^3$. These are the conditions which ensure that

$$\left(\frac{d}{d\xi}\right)^k \hat{\psi}(\xi)\,|_{\xi=0} = 0, \qquad k = 0, 1, \dots, \tag{8.50}$$

starting with

$$0 = \hat{\psi}(0) = \int_{\mathbb{R}} \psi(x)\, dx, \qquad \dots, \tag{8.51}$$

$$0 = \int_{\mathbb{R}} x^k \psi(x)\, dx. \tag{8.52}$$

Here (8.51) is automatic since $m_0(-1) = 0$. Recall the coordinates $z = e^{-i\xi}$, $\xi \in \mathbb{R}$. The second one (8.52) corresponds to $\left(\frac{d}{d\xi}\right)^k \hat{\psi}(\xi)\,|_{\xi=0} = 0$, or alternatively, $\left(\frac{d}{dz}\right)^k m_0(z)\,|_{z=-1} = 0$, or in yet another form, the condition that $(1+z)^{k+1}$ is a factor of $m_0(z)$, etc.

**Proposition 8.12.** (a) *The polynomial $m_0^{(\theta,\rho)}(z)$ is divisible by $(1+z)^2$ (see (8.52)) if and only if*

$$\cos(2\theta) + \cos(2\rho) = \frac{1}{2} \tag{8.53}$$

(*shown as curves in the four corners of Figure* 4 *in the Appendix*).

(b) *The polynomial $m_0^{(\theta,\rho)}(z)$ is divisible by $(1+z)^3$ if and only if*

$$\cos 2\theta + \cos 2\rho = \frac{1}{2} \quad \textit{and} \quad \sin 2\theta + \sin 2\rho = 2\sin(2\theta - 2\rho), \tag{8.54}$$

*i.e., when*

$$\theta = \cos^{-1} \sqrt[4]{\frac{5}{32}} \approx 0.89 \approx 0.28\,\pi, \qquad \rho = \cos^{-1} \sqrt{\frac{5}{4} - \sqrt{\frac{5}{32}}} \approx 0.39 \approx 0.12\,\pi$$

*or when $(\theta,\rho)$ is related to this pair by $(\theta,\rho) \to (\theta + m\pi, \rho + n\pi)$, $m, n \in \mathbb{Z}$, or $(\theta,\rho) \to (-\theta,-\rho)$, or both.* (*The points in $[0,\pi] \times [0,\pi]$ are shown in Figure* 4.)

*Proof.* (a) In this example, $N = 2$ and $g = 3$. Let $m(z) := \begin{pmatrix} m_0(z) \\ m_1(z) \end{pmatrix}$ be the usual QMF-polynomials in $z$, viewed as a column vector. By Lemma 5.1, we have

$$m(z) = V\left(P^{\perp} + z^2 P\right)\left(Q^{\perp} + z^2 Q\right)\alpha(z), \tag{8.55}$$

where $V \in \mathrm{U}_2(\mathbb{C})$, $P$ and $Q$ are projections, and as before, $\alpha(z) := \begin{pmatrix} 1 \\ z \end{pmatrix}$. Here $V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $P$, $Q$ are the $\theta$, $\rho$ projections specified in (6.50) (see also (A.11) below). We need only check that $\frac{d}{dz}m\left(z\right)|_{z=-1}=\left(\begin{smallmatrix}0\\ \xi_0\end{smallmatrix}\right)$ for some number $\xi_0$. But

$$\frac{d}{dz}m\left(z\right)=V2zP\left(Q^{\perp}+z^2Q\right)\alpha\left(z\right)+V\left(P^{\perp}+z^2P\right)2zQ\alpha\left(z\right) + V\left(P^{\perp}+z^2P\right)\left(Q^{\perp}+z^2Q\right)\begin{pmatrix}0\\ 1\end{pmatrix}. \tag{8.56}$$

Substitution of $z=-1$ yields

$$m'\left(-1\right)=-2V\left(P+Q\right)\begin{pmatrix}1\\ -1\end{pmatrix}+V\begin{pmatrix}0\\ 1\end{pmatrix}. \tag{8.57}$$

Hence

$$2\left(P+Q\right)\begin{pmatrix}1\\ -1\end{pmatrix}-\begin{pmatrix}0\\ 1\end{pmatrix}=\begin{pmatrix}\xi_1\\ -\xi_1\end{pmatrix}, \tag{8.58}$$

for some number $\xi_1$, and therefore, with

$$P+Q=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}\cos 2\theta+\cos 2\rho & \sin 2\theta+\sin 2\rho\\ \sin 2\theta+\sin 2\rho & -\left(\cos 2\theta+\cos 2\rho\right)\end{pmatrix} \tag{8.59}$$

we have

$$\begin{pmatrix}2\\ -2\end{pmatrix}+\begin{pmatrix}\cos 2\theta+\cos 2\rho & \sin 2\theta+\sin 2\rho\\ \sin 2\theta+\sin 2\rho & -\left(\cos 2\theta+\cos 2\rho\right)\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}-\begin{pmatrix}0\\ 1\end{pmatrix}=\begin{pmatrix}\xi_1\\ -\xi_1\end{pmatrix}.$$

The result (a) follows.

Part (b) follows upon solving

$$\frac{d^2}{dz^2}m\left(z\right)|_{z=-1}=\begin{pmatrix}0\\ \xi_2\end{pmatrix}, \tag{8.60}$$

for some number $\xi_2$, in addition to the conditions in (a). But we only need to work out the next derivative, using (8.53) to eliminate the cosine terms where possible:

$$\begin{aligned} V^{-1}\frac{d^2}{dz^2}m\left(z\right)|_{z=-1} &= \left(2\left(P+Q\right)+8PQ\right)\begin{pmatrix}1\\ -1\end{pmatrix}-4\left(P+Q\right)\begin{pmatrix}0\\ 1\end{pmatrix}\\ &= \begin{pmatrix}\frac{5}{2}-3\left(\sin 2\theta+\sin 2\rho\right)\\ -\frac{9}{2}+\left(\sin 2\theta+\sin 2\rho\right)\end{pmatrix}+\begin{pmatrix}3-2\left(\sin 2\theta+\sin 2\rho\right)\\ -1+2\left(\sin 2\theta+\sin 2\rho\right)\end{pmatrix}\\ &\quad +2\begin{pmatrix}\cos 2\left(\theta-\rho\right)+\sin 2\left(\theta-\rho\right)\\ \sin 2\left(\theta-\rho\right)-\cos 2\left(\theta-\rho\right)\end{pmatrix}\\ &= V^{-1}\begin{pmatrix}0\\ \xi_2\end{pmatrix}=\begin{pmatrix}\xi_3\\ -\xi_3\end{pmatrix}, \end{aligned} \tag{8.61}$$

where we used

$$PQ=\frac{1}{4}\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}\cos 2\theta+\cos 2\rho & \sin 2\theta+\sin 2\rho\\ \sin 2\theta+\sin 2\rho & -\left(\cos 2\theta+\cos 2\rho\right)\end{pmatrix} + \begin{pmatrix}\cos 2\left(\theta-\rho\right) & -\sin 2\left(\theta-\rho\right)\\ \sin 2\left(\theta-\rho\right) & \cos 2\left(\theta-\rho\right)\end{pmatrix}\right). \tag{8.62}$$

Noting again that the right-hand side of (8.61) takes the form $\left(\begin{smallmatrix}\xi\\ -\xi\end{smallmatrix}\right)$, we arrive at

$$-2\left(\sin 2\theta+\sin 2\rho\right)+4\sin 2\left(\theta-\rho\right)=0, \tag{8.63}$$

which directly gives the second part of (8.54).

The specific solution is found by transforming (8.63) into

$$(1-2\cos 2\rho)\sin 2\theta = -(1+2\cos 2\theta)\sin 2\rho. \tag{8.64}$$

Squaring and using (8.53) to eliminate $\rho$ yields

$$2\cos^2 2\theta + 4\cos 2\theta + \frac{3}{4} = 0, \tag{8.65}$$

which gives

$$\cos 2\theta = -1 + \sqrt{\frac{5}{8}} \tag{8.66}$$

by the quadratic formula, using $|\cos 2\theta| \leq 1$ to choose the positive radical. Working this back through (8.53) yields

$$\cos 2\rho = \frac{3}{2} - \sqrt{\frac{5}{8}}, \tag{8.67}$$

and substitution of (8.66) and (8.67) into (8.64) then shows that $\sin 2\theta$ and $\sin 2\rho$ must have the same sign, so that the solutions stated in the proposition are the only ones possible, the numerically exhibited pair being those for which $\sin 2\theta$ and $\sin 2\rho$ are both positive. □

**Remark 8.13.** The Hölder-Sobolev exponent is at least as good as 0.84 when we have a $(1+z)^2$ factor of $m_0(z)$, and at least as good as 1.136 if $(1+z)^3$ is a factor [LaSu00] [Vol95]; see Figure 2.

## Appendix (by Brian Treadway)

In this appendix we show some cascade approximations of wavelet scaling functions for the example with $g = 3$ discussed in Sections 6 and 8 above. The local method of cascade iteration works here just as it did in [BrJo99b]. It is just a matter of enumerating terms.

The direct method of iterating the relation (1.7), i.e.,

$$\varphi(x) = \sum_{k=0}^{2g-1} a_k \varphi(2x-k), \tag{A.1}$$

proceeds by translating a distance $k$ to the right, multiplying by $a_k$, summing over $k$, and scaling down by 2. This takes an expression in which every term has $n$ factors, each of which is an $a_k$ for some $k$, and turns it into an expression in which every term has $n+1$ such factors. In fact, every ordered product of $n$ $a_k$'s occurs exactly once, at some dyadic point in the $n$'th stage. Multiplication of these factors is commutative, but we forget that for the moment and take care to add the new factors at the left.

For example: With $g = 3$ there are 6 coefficients, $a_0, a_1, \dots, a_5$.

$n = 0$: 1 point, $1 = 6^0$ term, no factors of $a_k$, so the vacuous term is just 1, and we have the Haar function:

$$x = 0 \quad \varphi = 1$$

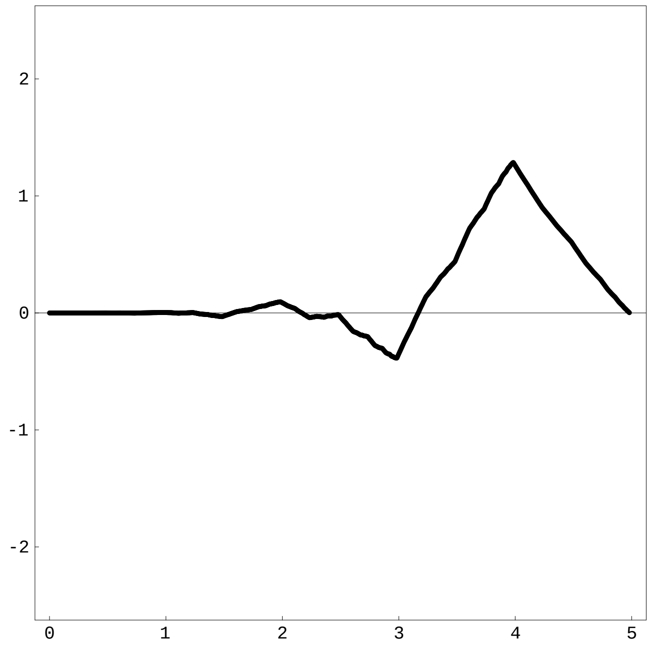



FIGURE 2. The ultra-smooth wavelet scaling function: $\varphi_{\theta,\rho}(x)$ for $\theta \approx 0.284\,\pi$, $\rho \approx 0.124\,\pi$, with two vanishing moments of $\psi$ (see Proposition 8.12(b) and the discussion preceding it). The Hölder-Sobolev exponent of this $\varphi$ is $\geq 1.136$; see Remark 8.13.

$n = 1$: 6 points, $6 = 6^1$ terms, as follows:

$$\begin{array}{llll} x = 0 & \varphi = a_0 & x = 3/2 & \varphi = a_3 \\ x = 1/2 & \varphi = a_1 & x = 2 & \varphi = a_4 \\ x = 1 & \varphi = a_2 & x = 5/2 & \varphi = a_5 \end{array}$$

$n = 2$: 16 points, $36 = 6^2$ terms, as follows:

$$\begin{array}{llll} x = 0 & \varphi = a_0 a_0 & x = 2 & \varphi = a_4 a_0 + a_3 a_2 + a_2 a_4 \\ x = 1/4 & \varphi = a_0 a_1 & x = 9/4 & \varphi = a_4 a_1 + a_3 a_3 + a_2 a_5 \\ x = 1/2 & \varphi = a_1 a_0 + a_0 a_2 & x = 5/2 & \varphi = a_5 a_0 + a_4 a_2 + a_3 a_4 \\ x = 3/4 & \varphi = a_1 a_1 + a_0 a_3 & x = 11/4 & \varphi = a_5 a_1 + a_4 a_3 + a_3 a_5 \\ x = 1 & \varphi = a_2 a_0 + a_1 a_2 + a_0 a_4 & x = 3 & \varphi = a_5 a_2 + a_4 a_4 \\ x = 5/4 & \varphi = a_2 a_1 + a_1 a_3 + a_0 a_5 & x = 13/4 & \varphi = a_5 a_3 + a_4 a_5 \\ x = 3/2 & \varphi = a_3 a_0 + a_2 a_2 + a_1 a_4 & x = 7/2 & \varphi = a_5 a_4 \\ x = 7/4 & \varphi = a_3 a_1 + a_2 a_3 + a_1 a_5 & x = 15/4 & \varphi = a_5 a_5 \end{array}$$

$n = 3$: 36 points, $216 = 6^3$ terms, grouping left as an exercise for the reader.

One could write down all $6^n$ terms at the outset of each stage, and then just ask which ones go with which of the values of $x$. The answer is:

(A.2)

The indices of the $a$'s, interpreted as digits at the right of the fraction point in a non-unique positional binary number system with six digits instead of two, give the $x$ to which a given term should be assigned.

As an example from the list above, the term $a_4a_1$ goes with $x = 4\left(2^{-1}\right) + 1\left(2^{-2}\right) = 9/4$. All the others (at any cascade stage) can be assigned in the same way.

In the ordering used above, if we go from one stage to the next by the "translate by $k$, multiply by $a_k$, sum over $k$, and scale down by 2" method, terms will be built up by adding factors from the left, while if we use the "local linear combination" method from [BrJo99b, Appendix] and [Dau92, Section 6.5, pp. 204–206], terms will be built up by adding factors from the right. In either method, the full set of $6^n$ terms (or $4^n$ in [BrJo99b], where $g = 2$) will be obtained, without duplication, and each term will be assigned to the same $x$ regardless of which method is used.

At cascade stage $n$ there are values of $\varphi$ assigned to values $x_i$ of $x$ that are consecutive integer multiples of $2^{-n}$ ranging from 0 to $\left(q^{(n)} - 1\right)\cdot 2^{-n}$. The iteration begins with the ordered list of values of $\varphi$ at those points on the $x$-axis:

$$\left(\varphi^{(n)}(0), \varphi^{(n)}\left(1\cdot 2^{-n}\right), \varphi^{(n)}\left(2\cdot 2^{-n}\right), \dots, \varphi^{(n)}\left(\left(q^{(n)} - 1\right)\cdot 2^{-n}\right)\right). \tag{A.3}$$

Sets of $g$ adjacent values in this list go into the computation of each point at the next cascade stage, which is on a finer grid of $q^{(n+1)}$ consecutive integer multiples of $2^{-(n+1)}$. Each set of $g$ adjacent values of $\varphi$ from the list (A.3) yields 2 values on the $2^{-(n+1)}$ grid by a linear combination with two sets of alternate $a_k$'s as coefficients. This can be expressed as a matrix product (shown here in the case $g = 3$):

$$\begin{pmatrix} \varphi^{(n+1)}(0) & \varphi^{(n+1)}\left(1\cdot 2^{-(n+1)}\right) \\ \varphi^{(n+1)}\left(2\cdot 2^{-(n+1)}\right) & \varphi^{(n+1)}\left(3\cdot 2^{-(n+1)}\right) \\ \varphi^{(n+1)}\left(4\cdot 2^{-(n+1)}\right) & \varphi^{(n+1)}\left(5\cdot 2^{-(n+1)}\right) \\ \varphi^{(n+1)}\left(6\cdot 2^{-(n+1)}\right) & \varphi^{(n+1)}\left(7\cdot 2^{-(n+1)}\right) \\ \vdots & \vdots \\ \varphi^{(n+1)}\left(\left(q^{(n+1)}-6\right)\cdot 2^{-(n+1)}\right) & \varphi^{(n+1)}\left(\left(q^{(n+1)}-5\right)\cdot 2^{-(n+1)}\right) \\ \varphi^{(n+1)}\left(\left(q^{(n+1)}-4\right)\cdot 2^{-(n+1)}\right) & \varphi^{(n+1)}\left(\left(q^{(n+1)}-3\right)\cdot 2^{-(n+1)}\right) \\ \varphi^{(n+1)}\left(\left(q^{(n+1)}-2\right)\cdot 2^{-(n+1)}\right) & \varphi^{(n+1)}\left(\left(q^{(n+1)}-1\right)\cdot 2^{-(n+1)}\right) \end{pmatrix} \tag{A.4}$$

$$= \begin{pmatrix} 0 & 0 & \varphi^{(n)}(0) \\ 0 & \varphi^{(n)}(0) & \varphi^{(n)}\left(1\cdot 2^{-n}\right) \\ \varphi^{(n)}(0) & \varphi^{(n)}\left(1\cdot 2^{-n}\right) & \varphi^{(n)}\left(2\cdot 2^{-n}\right) \\ \varphi^{(n)}\left(1\cdot 2^{-n}\right) & \varphi^{(n)}\left(2\cdot 2^{-n}\right) & \varphi^{(n)}\left(3\cdot 2^{-n}\right) \\ \vdots & \vdots & \vdots \\ \varphi^{(n)}\left(\left(q^{(n)}-3\right)\cdot 2^{-n}\right) & \varphi^{(n)}\left(\left(q^{(n)}-2\right)\cdot 2^{-n}\right) & \varphi^{(n)}\left(\left(q^{(n)}-1\right)\cdot 2^{-n}\right) \\ \varphi^{(n)}\left(\left(q^{(n)}-2\right)\cdot 2^{-n}\right) & \varphi^{(n)}\left(\left(q^{(n)}-1\right)\cdot 2^{-n}\right) & 0 \\ \varphi^{(n)}\left(\left(q^{(n)}-1\right)\cdot 2^{-n}\right) & 0 & 0 \end{pmatrix}$$

$$\bullet \begin{pmatrix} a_4 & a_5 \\ a_2 & a_3 \\ a_0 & a_1 \end{pmatrix}.$$

Thus a $\left(q^{(n)} + (g-1)\right) \times g$ matrix (the first factor on the right in (A.4) above), partitioned from the list (A.3) of $q^{(n)}$ points at stage $n$, yields a $\left(q^{(n)} + (g-1)\right) \times 2$ matrix (the left-hand side of (A.4)), which is then flattened out to give the list of values for stage $n+1$. The number of points $q^{(n+1)}$ in the new list is the total number of entries in that $\left(q^{(n)} + (g-1)\right) \times 2$ matrix,

$$q^{(n+1)} = 2\left(q^{(n)} + (g-1)\right), \tag{A.5}$$

and with $q^{(0)} = 1$, this recursively yields the number $q^{(n)}$ of points at each stage as

$$q^{(n)} = (2g-1) \cdot 2^n - 2\,(g-1)\,. \tag{A.6}$$

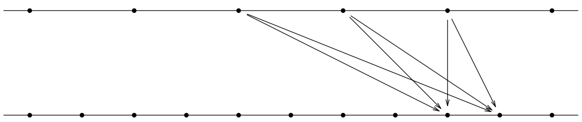
In the case $g = 3$, this is

$$q^{(n)} = 5 \cdot 2^n - 4. \tag{A.7}$$

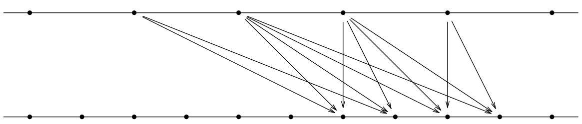
The three steps in the local method, (1) partitioning a list into rows of $g = 3$ points (first adding $g-1$ zeroes at each end), (2) matrix multiplication, and (3) flattening the resulting matrix back into a single list, are easily implemented in *Mathematica* [Wol96]. All that then remains to compute a cascade approximation of a wavelet scaling function $\varphi$ is to specify the numerical values of the coefficients $a_k$ and repeat the procedure $n$ times, starting with the one-element list

$$\begin{pmatrix} 1 \end{pmatrix}. \tag{A.8}$$

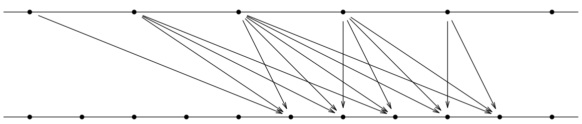
The same local cascade relation expressed in (A.4) as giving two values of $\varphi^{(n+1)}$ from three values of $\varphi^{(n)}$ can also be set up to give four values of $\varphi^{(n+1)}$ from four values of $\varphi^{(n)}$, or five values of $\varphi^{(n+1)}$ from five values of $\varphi^{(n)}$, by combining overlapping ranges of the initial and final lists. The terms involved on the successive $x$-grids are shown in the diagrams below, and the corresponding matrices are given.



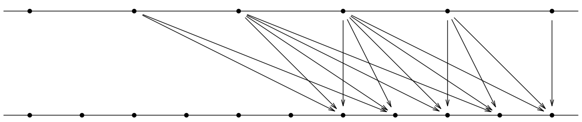
$$\begin{pmatrix} a_4 & a_5 \\ a_2 & a_3 \\ a_0 & a_1 \end{pmatrix} \quad \text{(as in (A.4))}$$



$$\begin{pmatrix} a_4 & a_5 & 0 & 0 \\ a_2 & a_3 & a_4 & a_5 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & a_0 & a_1 \end{pmatrix}$$



$$\begin{pmatrix} a_5 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & a_5 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & a_0 & a_1 \end{pmatrix}$$



$$\begin{pmatrix} a_4 & a_5 & 0 & 0 & 0 \\ a_2 & a_3 & a_4 & a_5 & 0 \\ a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & a_0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & a_0 \end{pmatrix}$$

All of these represent the *same* calculation of the $(n+1)$'st cascade stage fron the $n$'th stage: they merely collect different locally related sets of points in the successive stages. The advantage of the matrices that are square is that they allow successive stages to be expressed as powers of the matrix.

An eigenvector decomposition yielding an explicit expression for the $n \to \infty$ limit of the cascade stages, like that done in [BrJo99b, Appendix], could be done here using the $4 \times 4$ matrix above (here the matrices are written to act on the left). The detailed calculation is much more extensive than in the $2 \times 2$ case of [BrJo99b], and we will not present it here, but note only that the two $5\times 5$ matrices above have simple eigenvalues and left eigenvectors (in addition to the row $\begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}$ that all the matrices have): for the first, eigenvalue $a_5$, eigenvector $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}$, and for the second, eigenvalue $a_0$, eigenvector $\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. Since the starting list for the cascade computation is $\begin{pmatrix} 1 \end{pmatrix}$, to be "padded" with zeroes on the left and right, these two eigenvectors occur explicitly at the first cascade stage and continue thereafter. This shows that $\varphi^{(n)}$ diverges at one end of its support interval or the other when one of these eigenvalues is greater than 1, growing like $a_5^n$ when $a_5 > 1$ or like $a_0^n$ when $a_0 > 1$. The full eigenvector decomposition then has a term, generically nonzero, that grows in the same way for other points $x_i$, and so the same divergence occurs at points throughout the support interval. The regions where $a_0$ or $a_5$ is greater than 1, leading to this divergence in the cascade iteration, are shown by shading in Figure 4.

The coefficients here are not restricted, as they are in [Wan00], to $a_i \leq 1$, so we do not have cycles in the cascade iteration: the terms grow indefinitely. As a result, the condition $a_5 > 1$ or $a_0 > 1$ is sufficient, but not necessary, for divergence.

For the example with $g = 3$ specified in (6.50) above, the formulas for the coefficients $a_0, a_1, \dots, a_5$ of the polynomial $m_0^{(A)}(z)$ in (8.11) may be derived as follows. From (6.43) we have

$$A(z) = V(Q_\theta^\perp + zQ_\theta)(Q_\rho^\perp + zQ_\rho) \tag{A.9}$$

with

$$V = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{A.10}$$

(A.11)
$$Q_\theta = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}\right),$$

and

$$Q_\theta^\perp = Q_{\theta+(\pi/2)} \tag{A.12}$$

Then the coefficients $a_0, a_1, \dots, a_5$ are:

(A.13)
$$\begin{aligned}
a_0 &= \frac{1}{4}(1 - \cos 2\theta - \sin 2\theta - \cos 2\rho - \sin 2\rho + \cos(2\theta - 2\rho) + \sin(2\theta - 2\rho)),\\
a_1 &= \frac{1}{4}(1 + \cos 2\theta - \sin 2\theta + \cos 2\rho - \sin 2\rho + \cos(2\theta - 2\rho) - \sin(2\theta - 2\rho)),\\
a_2 &= \frac{1}{2}(1 - \cos(2\theta - 2\rho) - \sin(2\theta - 2\rho)),\\
a_3 &= \frac{1}{2}(1 - \cos(2\theta - 2\rho) + \sin(2\theta - 2\rho)),\\
a_4 &= \frac{1}{4}(1 + \cos 2\theta + \sin 2\theta + \cos 2\rho + \sin 2\rho + \cos(2\theta - 2\rho) + \sin(2\theta - 2\rho)),\\
a_5 &= \frac{1}{4}(1 - \cos 2\theta + \sin 2\theta - \cos 2\rho + \sin 2\rho + \cos(2\theta - 2\rho) - \sin(2\theta - 2\rho)).
\end{aligned}$$

These can be seen to meet the conditions (1.5)–(1.6) for the coefficients of a scaling function, The even- and odd-indexed coefficients also sum to a constant separately:

$$\sum_{i=0}^{2} a_{2i} = \sum_{i=0}^{2} a_{2i+1} = 1 \tag{A.14}$$

(see [ReWe98, eq. (9.12)]), which is what makes the constant vector $\begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}$ an eigenvector or the $a_i$-matrices above. Of course, the coefficients $a_i$, as functions of $\theta$ and $\rho$, have the periodicity (with period $\pi$ in both angles $\theta$ and $\rho$) of the projections $Q_\theta$ and $Q_\rho$ they were derived from:

$$a_i(\theta,\rho) = a_i(\theta + m\pi, \rho + n\pi), \qquad m,n \in \mathbb{Z}. \tag{A.15}$$

In addition, they are related in pairs by the reflection relation

$$a_i(\theta,\rho) = a_{5-i}(-\theta,-\rho), \qquad i = 0,\dots,5. \tag{A.16}$$

These relations carry through the successive stages of the local cascade computation (A.4) as the periodicity

$$\varphi_{\theta,\rho}^{(n)}(x) = \varphi_{\theta+m\pi,\rho+n\pi}^{(n)}(x) \tag{A.17}$$

and the reflection symmetry

$$\varphi_{\theta,\rho}^{(n)}(x) = \varphi_{\theta-\pi,\rho-\pi}^{(n)}(x) = \varphi_{\pi-\theta,\pi-\rho}^{(n)}\left(x_f^{(n)} - x\right), \tag{A.18}$$

where $x_f$ is the last point to which a value is assigned by the $n$th stage. By (A.7) we have

$$x_f^{(n)} = \left(q^{(n)} - 1\right) \cdot 2^{-n} = 5 - 5 \cdot 2^{-n}. \tag{A.19}$$

There are pairwise relations involving a translation in the $(\theta,\rho)$ plane by half the period:

$$a_{2i+j}(\theta,\rho) = a_{2(2-i)+j}\left(\theta - \frac{\pi}{2}, \rho - \frac{\pi}{2}\right), \qquad i = 0,1,2, \qquad j = 0,1. \tag{A.20}$$

There are also twofold and threefold affine symmetries, such as the invariance of $a_2$ under the twofold transformation

$$\theta \longmapsto -\theta, \qquad \rho \longmapsto -\rho + \frac{\pi}{4}, \tag{A.21}$$

or the invariance of $a_0$ under the threefold transformation

$$\theta \longmapsto -\rho + \frac{\pi}{4}, \qquad \rho \longmapsto \theta - \rho + \frac{\pi}{2}, \tag{A.22}$$

which has $a_0$'s three local extrema as its fixed points in the $\pi$-periodic context.

Contour plots of $a_0$ and $a_2$ in the $\theta,\rho$-plane are shown in Figure 3 below; the other $a_i$'s can be derived from these by the translation in (A.20) or the rotation around the origin in (A.16), or around the point $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$ when this is combined with translation by $\pi$ in both angles.

Discussion of what variation $\varphi^{(n)}$ has between $x_i$ and $x_{i+1}$ is somewhat metaphysical: it only matters that $\varphi^{(n)}(x_i)$ is associated with the point $x_i = i \cdot 2^{-n}$. For example, if we said $\varphi^{(n)}(x) = \varphi^{(n)}(i \cdot 2^{-n})$ for $i \cdot 2^{-n} \le x < \left(i + \frac{1}{2}\right) \cdot 2^{-n}$ and $\varphi^{(n)}(x) = 0$ for $\left(i + \frac{1}{2}\right) \cdot 2^{-n} \le x < (i+1) \cdot 2^{-n}$, then the same "shape" would occur in every bottom-level interval of every cascade stage, preventing continuity from appearing in the $n \to \infty$ limit even if it would otherwise have appeared. Other variations within the shortest dyadic intervals at a given stage have a similar arbitrariness; $\varphi^{(n)}$ is really only defined pointwise for finite $n$. When $n$ goes to infinity,

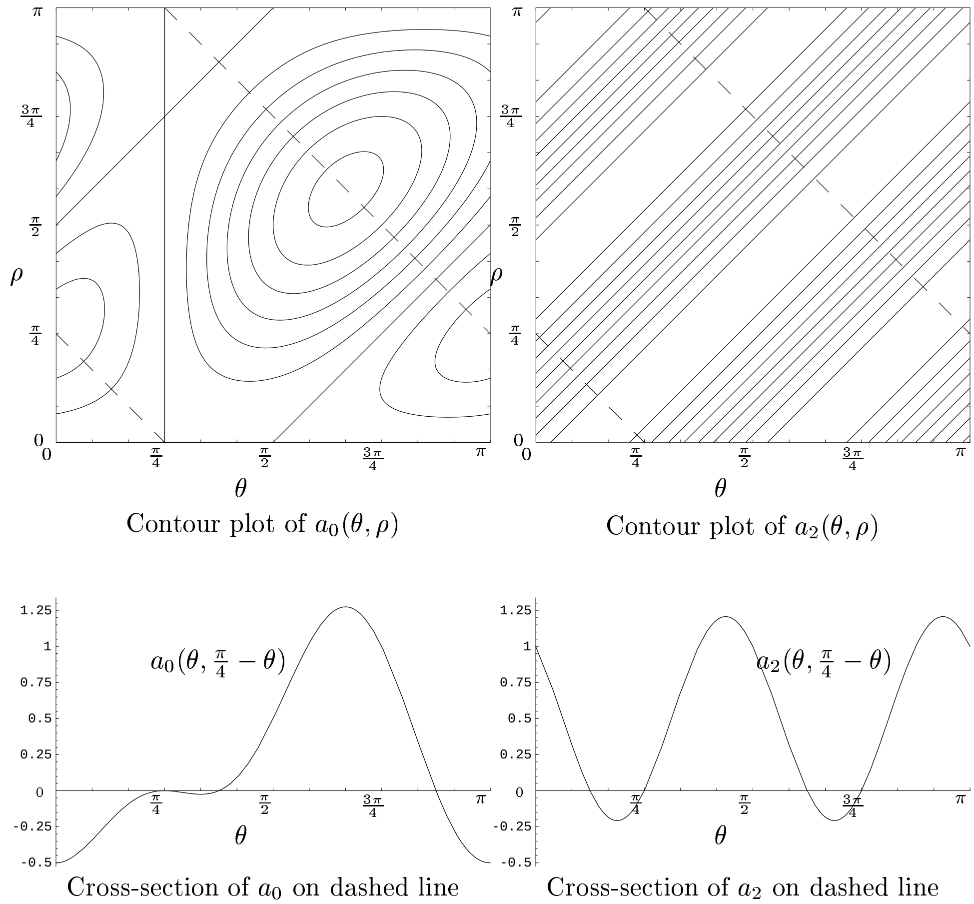



FIGURE 3. Contour plots of $a_0$ and $a_2$. For $a_0$, the three lines $\theta = \pi/4$, $\rho = 0$ (or $\pi$), and $\theta - \rho = \pi/2$ are contours of $a_0 = 0$, whose intersections are saddle points; all the maxima and minima of $a_0$ are located on the dashed line $\theta + \rho = \pi/4$ (or $5\pi/4$), displayed in the lower plot. For $a_2$, contours of constant $a_2$ are diagonal lines of the form $\theta - \rho =$ constant.

continuity arises in some cases. On the other hand, for a given $x$ and finite $n$, $\varphi_{\theta,\rho}^{(n)}(x)$ as a function of $\theta$ and $\rho$ is continuous, since it is a polynomial (of order $n$) in the $a_i$'s; when the order $n$ goes to infinity, singularities arise.

To decide when the corresponding scaling function $\varphi_{\theta,\rho}(x)$ generates a wavelet in the *strict sense* or merely a *tight frame*, as discussed in Section 1 above, we use the method of Cohen [Coh92b, CoRy95] (see also [BEJ00]): We identify cycles on $\mathbb{T}$ for the doubling map $z \mapsto z^2$, i.e., a finite cyclic subset unequal to $\{1\}$ and invariant under $z \mapsto z^2$. The result is that $\varphi(x)$ generates a "strict" wavelet if and only if

$$\left\{ z \in \mathbb{T} \,;\, m_0^{(\varphi)}(-z) = 0 \right\} \tag{A.23}$$

does not contain a nontrivial cycle.

The cycles on $\mathbb{T}$ are not subgroups of $\mathbb{T}$ but rather cyclic orbits on $\mathbb{T}$ under the $z \mapsto z^2$ action of one of the cyclic groups $\mathbb{Z}_k$, $k = 1, 2, \ldots$. Such a cyclic orbit $C_k$ with $k$ distinct points $z_1, \ldots, z_k$ must be of the form $z_1 \to z_2 \to \cdots \to z_k \to z_1$, where $z_{i+1} = z_i^2$ if $i = 1, 2, \ldots, k-1$, and $z_k^2 = z_1$. Hence points $c$ in an orbit $C_k$ must satisfy $c^{2^k} = c$, and each $c$ must be a $\left(2^k - 1\right)$'th root of 1. Different orbits must be disjoint, and their union will be invariant under $z \mapsto z^2$ acting on $\mathbb{T}$. The converse is not true. Note also that we can have different $\left(2^k - 1\right)$'th roots $c$ of 1 defining different cyclic orbits for the same $k$. If $k = 1$ or $k = 2$, then in each case there is only one orbit, but if $k = 3$, there are two choices. Since $m_0^{(\theta,\rho)}(z)$ is for each $\theta, \rho$ a polynomial of degree at most 5, the cardinality of a cycle contained in (A.23) is at most 4. Thus, if $z$ is contained in such a cycle, we must have one of the possibilities $z^2 = z$, $z^4 = z$, $z^8 = z$. Hence the cycles of length at most 3 are $\{1\}$, $\left\{\omega, \omega^2\right\}$ where $\omega = e^{i2\pi/3}$, $\left\{\zeta, \zeta^2, \zeta^4\right\}$ where $\zeta := e^{i2\pi/7}$, and $\left\{\bar{\zeta}, \bar{\zeta}^2, \bar{\zeta}^4\right\} = \left\{\zeta^6, \zeta^5, \zeta^3\right\}$. But as $m_0(-1) = 0$ always, $(z+1)$ is always a factor of $m_0(z)$, and since the cycle should be different from the trivial cycle $\{1\}$, we are reduced to the case $\left\{\omega, \omega^2\right\}$. The other cycles would make $m_0^{(\theta,\rho)}$ divisible by a polynomial of degree at least 4.

Thus we have the following cases: $m_0^{(\theta,\rho)}(z)$ may be divisible by

$$p_3(z) = \prod_{k=0}^{2} \left(\omega^k + z\right) = 1 + z^3, \tag{A.24}$$

by

$$\begin{aligned} p_4(z) &= (1+z)(\zeta + z)\left(\zeta^2 + z\right)\left(\zeta^4 + z\right) \\ &= 1 + \bar{\beta} z - z^2 + \beta z^3 + z^4, \end{aligned} \tag{A.25}$$

or by

$$\begin{aligned} p_4^{(\#)}(z) = \overline{p_4(\bar{z})} &= (1+z)\left(\zeta^3 + z\right)\left(\zeta^5 + z\right)\left(\zeta^6 + z\right) \\ &= 1 + \beta z - z^2 + \bar{\beta} z^3 + z^4, \end{aligned} \tag{A.26}$$

where $\zeta$ (as above) and $\beta$ are defined as

$$\zeta := e^{i\frac{2\pi}{7}}, \qquad \beta := 1 + \zeta + \zeta^2 + \zeta^4 = \frac{1}{2} + i\frac{\sqrt{7}}{2}. \tag{A.27}$$

**Proposition A.1.** *There are only four cases of QMF-functions*

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5, \qquad a_k \in \mathbb{R}, \tag{A.28}$$

*which give tight frames that are not strict wavelets. In addition to the* $\mathcal{P}_3(\mathbb{T}, \mathrm{U}_2(\mathbb{C}))$*-conditions, they satisfy*

$$\sum_{k=0}^{5} a_k = 2. \tag{A.29}$$

*The four correspond to the three loops*

$$\frac{1}{\sqrt{2}} \begin{pmatrix} z & z^2 \\ 1 & -z \end{pmatrix}, \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & z \\ z & -z^2 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} z^2 & 1 \\ z^2 & -1 \end{pmatrix}. \tag{A.30}$$

*and the loop*

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & z^2 \\ 1 & -z^2 \end{pmatrix}. \tag{A.31}$$

*The wavelet representation $T^{(A)}$ is irreducible for the first two of the four, and reducible for the last two. The values of $\lambda_0(A)$ are as follows: $1/2$, $1/2$, $0$, and $1$, respectively. The first three have cycles of order 2 and the last one a cycle of order 4. The corresponding system of coefficients is as follows:*

$$\begin{array}{cccccc} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ \hline 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array} \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \textit{two-cycle} \\ \quad \textit{four-cycle} \end{array} \tag{A.32}$$

*and so all four cases are Haar wavelets. The scaling functions $\varphi(x)$ may be taken as in Table* 2.

*Proof.* If $m_0(z)$ is divisible by $1+z^3$, then its six coefficients $a_0, a_1, \dots, a_5$ must be of the form $c_0, c_1, c_2, c_0, c_1, c_2$, and the associated loop $\mathbb{T} \to \mathrm{U}_2(\mathbb{C})$,

$$\begin{pmatrix} c_0 + c_2 z + c_1 z^2 & c_1 + c_0 z + c_2 z^2 \\ \bar{c}_2 + \bar{c}_0 z + \bar{c}_1 z^2 & -\left(\bar{c}_1 + \bar{c}_2 z + \bar{c}_0 z^2\right) \end{pmatrix}. \tag{A.33}$$

The corresponding $\mathrm{U}_2(\mathbb{C})$-conditions then yield:

$$2\bar{c}_0 c_2 + \bar{c}_1 c_0 + \bar{c}_2 c_1 = 0, \qquad \bar{c}_0 c_1 + \bar{c}_1 c_2 = 0. \tag{A.34}$$

Substitution of the second into the first yields $\bar{c}_0 c_2 = 0$. Hence, of the three numbers $c_0, c_1, c_2$, at most one, and therefore precisely one, can be nonzero. But each of the three cases is determined up to scale, and condition (A.29) decides the scale. We are therefore led to the three loops in (A.30), and the rule (8.11) then gives the three scaling functions $\varphi(x)$ which are listed in (A.32) and Table 2.

The cycle of the last line in Table 2 is of order 4. Let $\mu := e^{i2\pi/5} = \lambda^3$ ($\lambda := e^{i2\pi/15}$). Then the cycle is $\left\{\mu, \mu^2, \mu^4, \mu^3\right\}$, and $\prod_{k=0}^{4}\left(\mu^k + z\right) = z^5 + 1$, which is the $m_0(z)$ for the last line of Table 2. (It is from a root of 1 of order $2^l - 1$ $(= 15)$ for $l = 4$.)

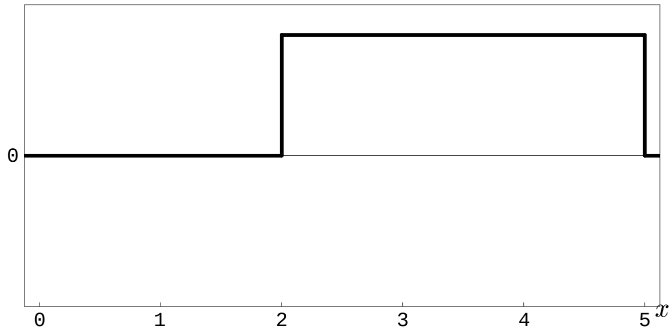
The other length-3 loops which would be possible are, as noted, $\left\{\zeta, \zeta^2, \zeta^4\right\}$ and $\left\{\bar{\zeta}, \bar{\zeta}^2, \bar{\zeta}^4\right\}$, with $\zeta = e^{i2\pi/7}$. We claim that they do *not* in fact occur.

If one of them did occur, then the corresponding $m_0(z)$ would be divisible by either $p_4(z)$, or by $p_4^{(\#)}(z)$. But $p_4(1) = p_4^{(\#)}(1) = 2$, so the factorization would be $m_0(z) = p_4(z)\, l(z)$ where $l(z) = a + (1-a)z$. (We have picked the normalization of $m_0(z)$ given by $m_0(1) = 2$ for convenience.) From the formulas (A.10) and (A.13) we note that the coefficients $a_0, a_1, \dots, a_5$ are real. Divisibility by $p_4(z)$ means that $-1$, $-\zeta$, $-\zeta^2$, and $-\zeta^4$ are roots of $m_0(z)$. So the complex conjugates $-\bar{\zeta}$, $-\bar{\zeta}^2$, $-\bar{\zeta}^4$ are also roots. But that would give us all seven points, $-1, -\zeta, -\zeta^2, -\zeta^3, -\zeta^4, -\zeta^5, -\zeta^6$, as distinct roots of $m_0(z)$, which is impossible since $m_0$ is of degree at most 5. □

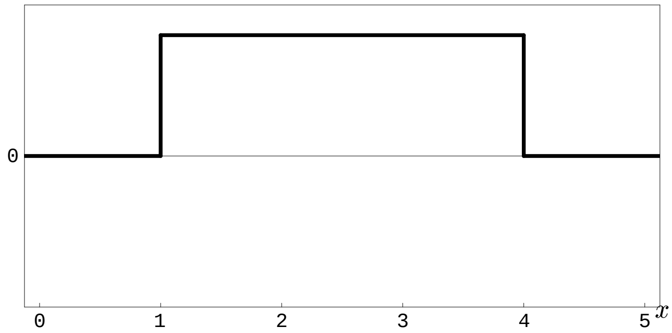
In conclusion, when $g = 3$, the variety of the wavelets which are only tight frames sits on a finite subset of the full variety of all $\mathcal{P}_3(\mathbb{T}, \mathrm{U}_2(\mathbb{C}))$ examples.

On the following pages are plots, for various values of the angles $\theta$ and $\rho$, of the wavelet scaling functions $\varphi_{\theta,\rho}(x)$ at the 8th cascade level, computed by the local algorithm described in (A.4) above. The layout of the plots is shown in the chart in Table 3. The plots for $\theta = \pi$ or $\rho = \pi$ beyond the top and right of this chart are the same as those for $\theta = 0$ or $\rho = 0$, because of the periodicity (A.17).

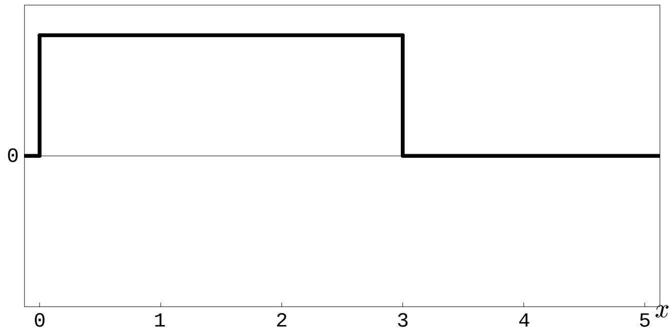
Table 2. $\varphi(x)$ in the four cases: Tight frames corresponding to cycles of length two, and a four-cycle.



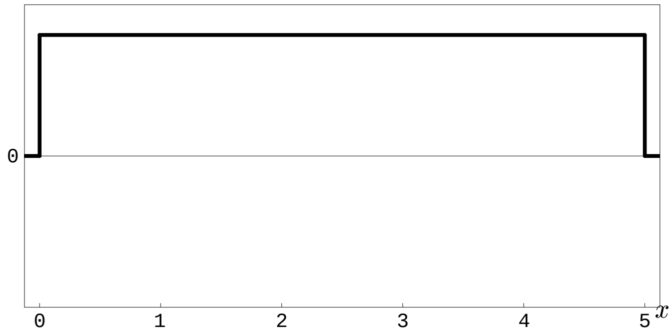

$\theta = \pi/4$,
$\rho = \pi/2$;




$\theta = 0\ (or\ \pi)$,
$\rho = 0\ (or\ \pi)$;




$\theta = 3\pi/4$,
$\rho = \pi/2$;




$\theta = \rho = \pi/2$.

TABLE 3. Layout of scaling function plots. *Round solid points* (●): approximate locations of "ultra-smooth" wavelet scaling functions in relation to scaling functions plotted. *Shading*: divergence due to $a_0 > 1$ or $a_5 > 1$. *Boxes*: marginal divergence due to $a_0 = 1$ or $a_5 = 1$.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho = 11\pi/12$ | al | bl | cl | dl | el | fl | gl | hl | il | jl | kl | ll |
| $5\pi/6$ | ak | bk | ck | dk | ek | fk | gk | hk | ik | jk | kk | lk |
| $3\pi/4$ | aj | bj | cj | dj | ej | fj | gj | hj | ij | $\boxed{\text{jj}}$ | kj | lj |
| $2\pi/3$ | ai | bi | ci | di | ei | fi | gi | hi | ii | ji | ki | li |
| | | p. 64 | | | p. 65 | | | p. 66 | | | p. 67 | |
| $7\pi/12$ | ah | bh | ch | dh | eh | fh | gh | hh | ih | jh | kh | lh |
| $\pi/2$ | ag | bg | cg | $\boxed{\text{dg}}$ | eg | fg | $\boxed{\text{gg}}$ | hg | ig | $\boxed{\text{jg}}$ | kg | lg |
| $5\pi/12$ | af | bf | cf | df | ef | ff | gf | hf | if | jf | kf | lf |
| $\pi/3$ | ae | be | ce | de | ee | fe | ge | he | ie | je | ke | le |
| | | p. 60 | | | p. 61 | | | p. 62 | | | p. 63 | |
| $\pi/4$ | ad | bd | cd | $\boxed{\text{dd}}$ | ed | fd | gd | hd | id | jd | kd | ld |
| $\pi/6$ | ac | bc | cc | dc | ec | fc | gc | hc | ic | jc | kc | lc |
| $\pi/12$ | ab | bb | cb | db | eb | ec | gb | hb | ib | jb | kb | lb |
| $\rho = 0$ | aa | ba | ca | da | ea | fa | ga | ha | ia | ja | ka | la |
| | | p. 56 | | | p. 57 | | | p. 58 | | | p. 59 | |
| | $\theta = 0$ | $\frac{\pi}{12}$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{5\pi}{12}$ | $\frac{\pi}{2}$ | $\frac{7\pi}{12}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\frac{11\pi}{12}$ |

TABLE 4. Embedding of the $g = 2$ family in the $g = 3$ family.

| **Support interval:** | $(\theta,\rho)$ **values:** | **Continuous** $\varphi_{\theta,\rho}(x)$ **at:** |
|---|---|---|
| $x \in [0,3]$ | $\{(\theta,\rho) : \theta = 3\pi/4\}$ | $\theta = 3\pi/4,\ \rho \in (0,\pi/4) \cup (3\pi/4,\pi)$ |
| $x \in [1,4]$ | $\{(\theta,\rho) : \rho = 0\}$ | $\rho = 0,\ \theta \in (\pi/4,\pi/2) \cup (\pi/2,3\pi/4)$ |
| $x \in [2,5]$ | $\{(\theta,\rho) : \theta = \pi/4\}$ | $\theta = \pi/4,\ \rho \in (0,\pi/4) \cup (3\pi/4,\pi)$ |

The "ultra-smooth" scaling function with $m_0^{(\theta,\rho)}(z)$ divisible by $(1+z)^3$, shown in Figure 2 above, and its counterpart under the symmetry (A.18), lie at the positions shown by a round solid point (●) in both Table 3 and Figure 4.

The scaling functions from the $g = 2$ family in [BrJo99b] appear as subsets of the $g = 3$ family here, supported on various subintervals of $[0,5]$ of length 3. The correspondence results, for particular values of $(\theta,\rho)$, from the vanishing of two of the $a_i$ coefficients, and the equality of the other four $a_i$'s to the four coefficients of the $g = 2$ family. The values of $(\theta,\rho)$ corresponding to continuous scaling functions in the $g = 2$ family [CoHe92, CoHe94, Wan95, Wan96, DauL92] (see [BrJo99b, Remark 3.1]) are indicated in Table 4. The values of $(\theta,\rho)$ that give these known continuous scaling functions are indicated graphically in Figure 4, along with the vanishing-moment points where the polynomial $m_0^{(\theta,\rho)}(z)$ is divisible by $(1+z)^2$ and by $(1+z)^3$ (see Proposition 8.12), and the tight-frame cases (see Proposition A.1). Some regions of the $(\theta,\rho)$ plane where the cascade approximants do not converge to a continuous scaling function are also indicated in the same figure.

Putting all the 144 pictures together as illustrated in Table 3, we get graphic support for the observation that the two spin-vectors in the factorization (A.9) produce more smoothness of $x \mapsto \varphi_{\theta,\rho}(x)$ when they are not aligned, i.e., off the

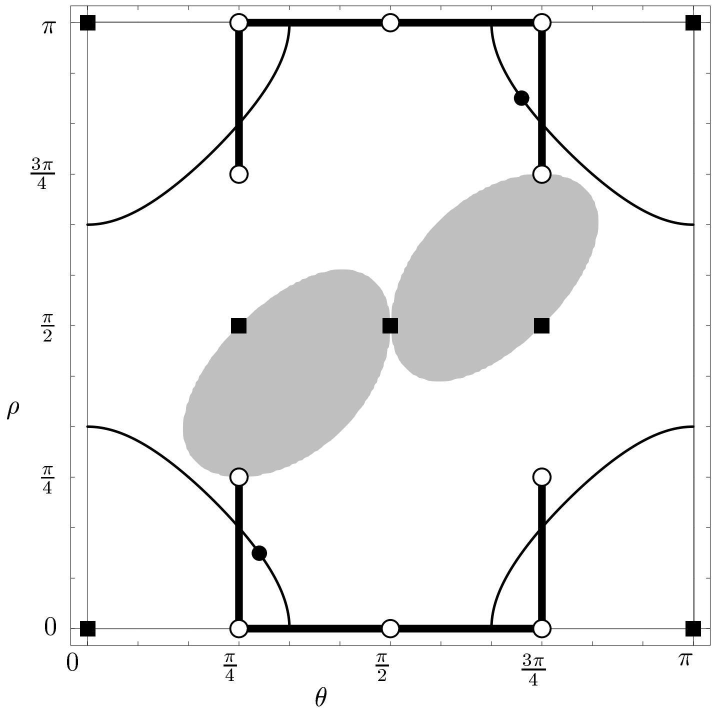
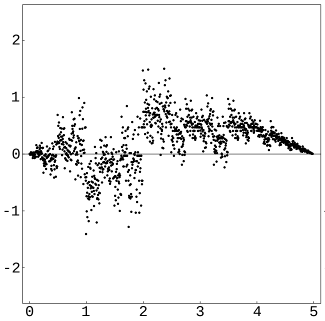


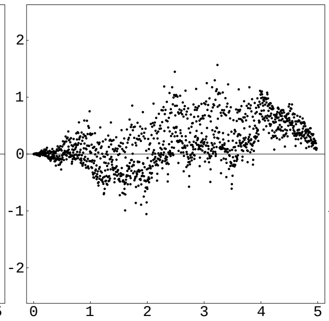

FIGURE 4. *Thin curved lines* (in the four corners): vanishing first moment of $\psi$ (Proposition 8.12(a)). *Round solid points* (●): vanishing second moment of $\psi$; "ultra-smooth" wavelet scaling function (Proposition 8.12(b), Figure 2). *Thick straight lines*: embedding of (continuous portion of) $g = 2$ family in $g = 3$ family (Table 4). *Round open points* (○): translated Haar functions within $g = 2$ family (plots "da", "dd", "dj", "ga", "ja", "jd", "jj"). *Square points*: tight frames (Proposition A.1, plots "aa", "dg", "gg", "jg"). *Shading*: divergence due to $a_0 > 1$ or $a_5 > 1$.

diagonal $\theta = \rho$. It also shows that, close to one of the true Haar wavelets, i.e., when $\varphi$ is the indicator function of some $[k, k+1)$, there is a continuous $\varphi$, while close to a mock Haar wavelet (i.e., one that is only a tight frame) it appears that the graph of the scaling functions have Hausdorff dimension $> 1$, hence the "fractal" appearance.

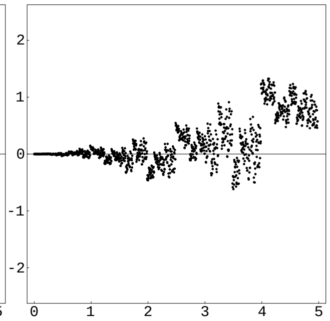
*Acknowledgements.* We are very grateful to Ola Bratteli, Ken Davidson, and David Kribs for enlightening discussions, and to Brian Treadway and Cymie Wehr for expert typesetting, and graphics artwork.



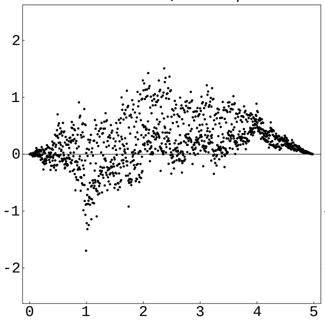

ad: $\theta = 0,\ \rho = \pi/4$



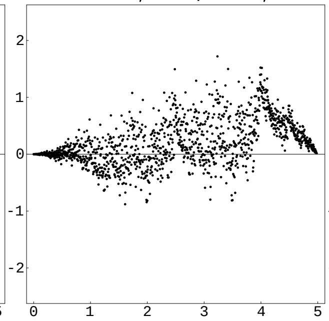

bd: $\theta = \pi/12,\ \rho = \pi/4$



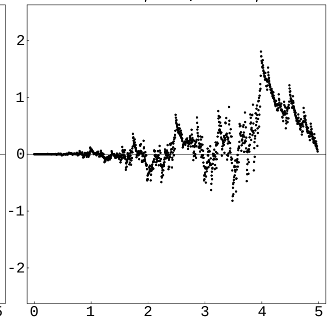

cd: $\theta = \pi/6,\ \rho = \pi/4$



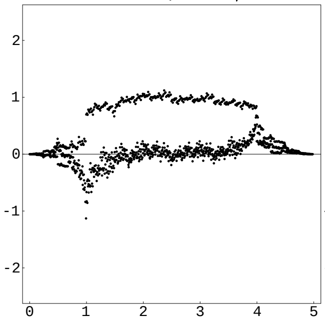

ac: $\theta = 0,\ \rho = \pi/6$



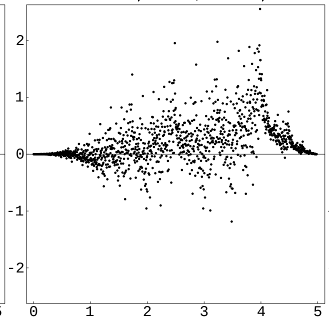

bc: $\theta = \pi/12,\ \rho = \pi/6$



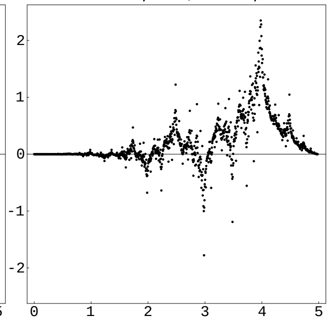

cc: $\theta = \pi/6,\ \rho = \pi/6$



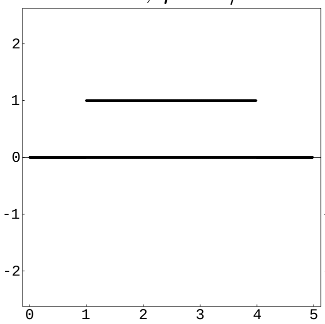

ab: $\theta = 0,\ \rho = \pi/12$



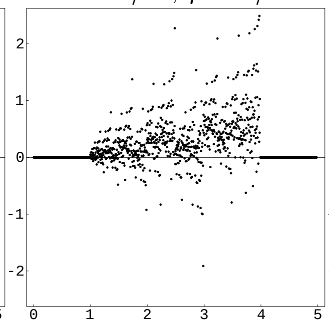

bb: $\theta = \pi/12,\ \rho = \pi/12$



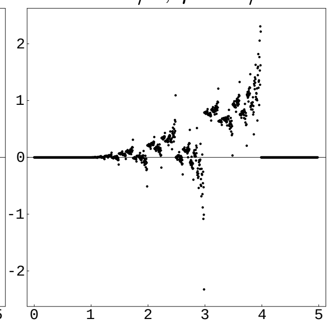

cb: $\theta = \pi/6,\ \rho = \pi/12$



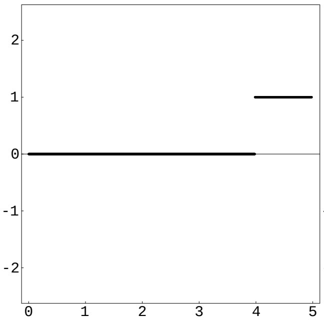

aa: $\theta = 0,\ \rho = 0$



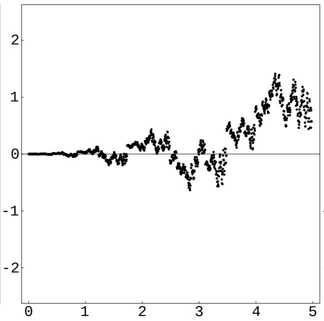

ba: $\theta = \pi/12,\ \rho = 0$



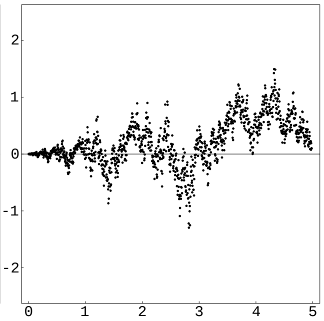

ca: $\theta = \pi/6,\ \rho = 0$



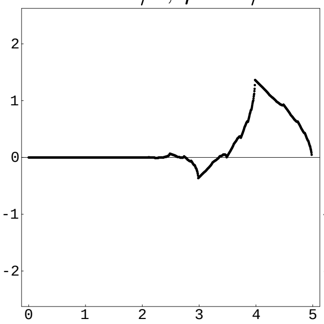
dd: $\theta = \pi/4,\ \rho = \pi/4$



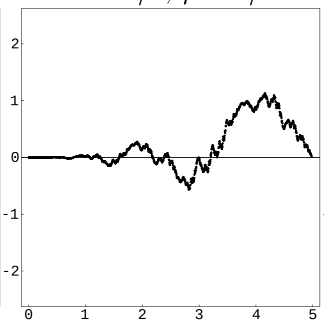
ed: $\theta = \pi/3,\ \rho = \pi/4$



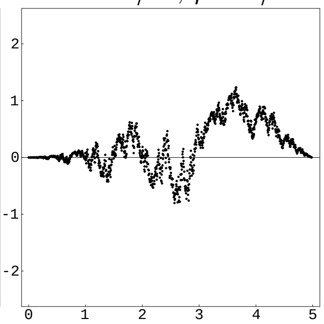
fd: $\theta = 5\pi/12,\ \rho = \pi/4$



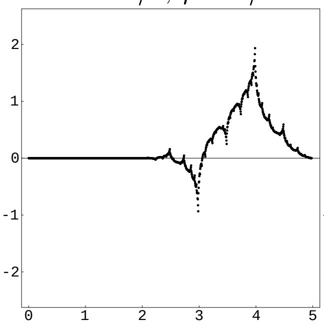
dc: $\theta = \pi/4,\ \rho = \pi/6$



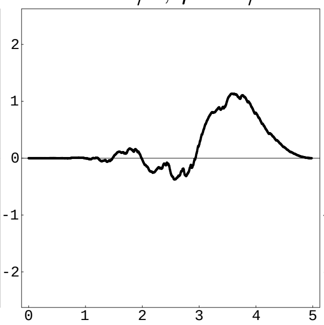
ec: $\theta = \pi/3,\ \rho = \pi/6$



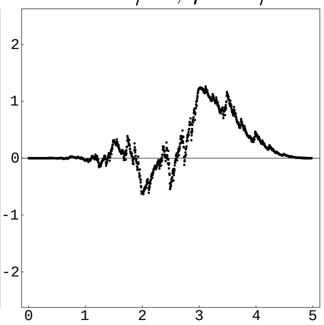
fc: $\theta = 5\pi/12,\ \rho = \pi/6$



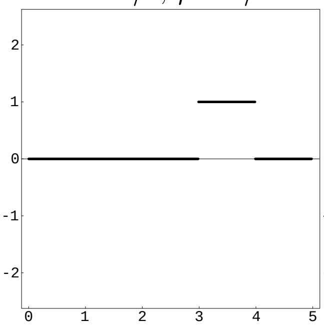
db: $\theta = \pi/4,\ \rho = \pi/12$



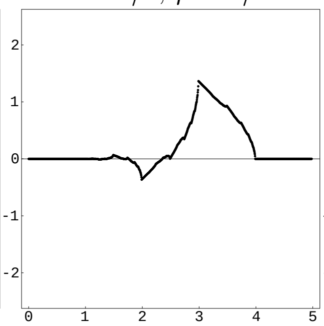
eb: $\theta = \pi/3,\ \rho = \pi/12$



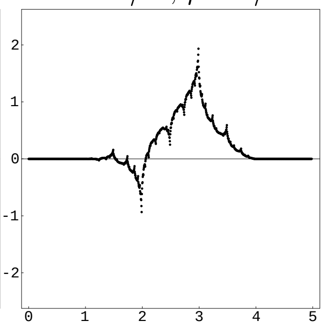
fb: $\theta = 5\pi/12,\ \rho = \pi/12$



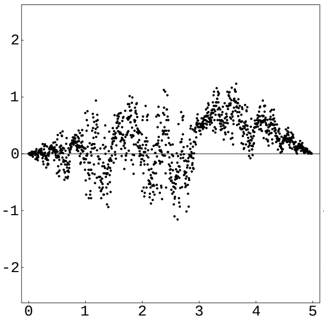
da: $\theta = \pi/4,\ \rho = 0$



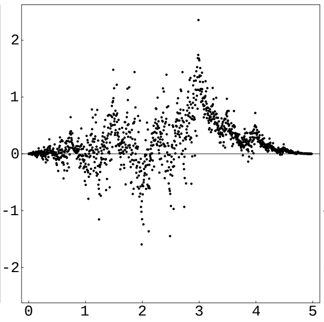
ea: $\theta = \pi/3,\ \rho = 0$



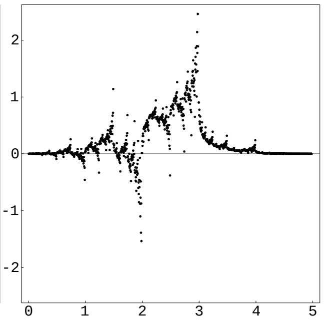
fa: $\theta = 5\pi/12,\ \rho = 0$



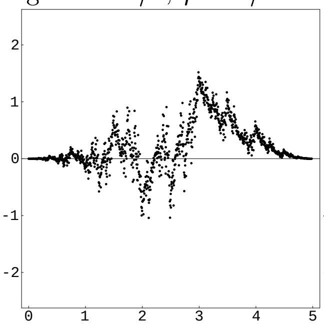
gd: $\theta = \pi/2,\ \rho = \pi/4$



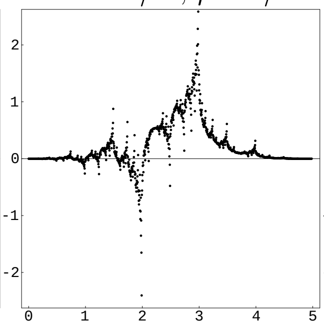
hd: $\theta = 7\pi/12,\ \rho = \pi/4$



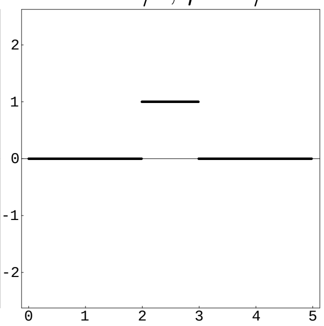
id: $\theta = 2\pi/3,\ \rho = \pi/4$



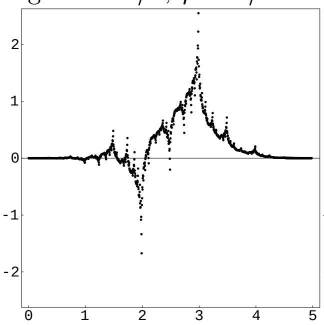
gc: $\theta = \pi/2,\ \rho = \pi/6$



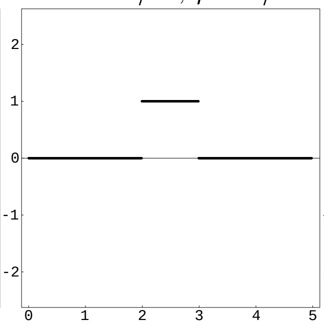
hc: $\theta = 7\pi/12,\ \rho = \pi/6$



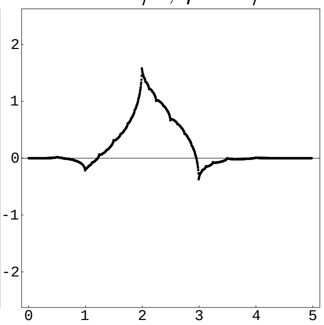
ic: $\theta = 2\pi/3,\ \rho = \pi/6$



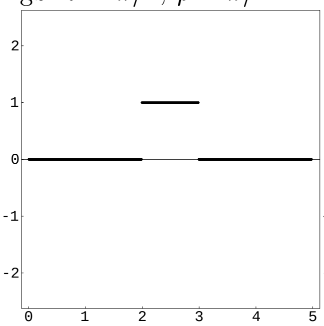
gb: $\theta = \pi/2,\ \rho = \pi/12$



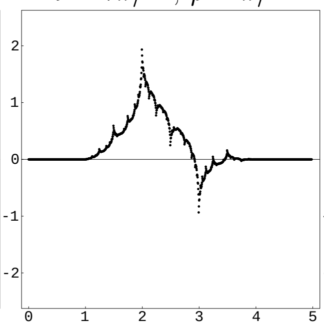
hb: $\theta = 7\pi/12,\ \rho = \pi/12$



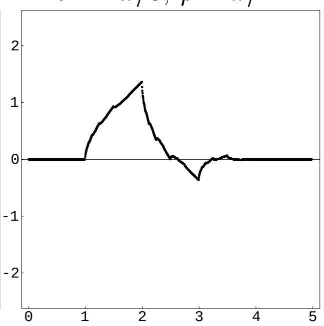
ib: $\theta = 2\pi/3,\ \rho = \pi/12$



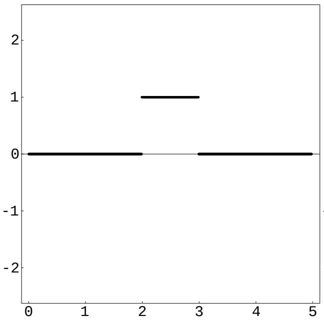
ga: $\theta = \pi/2,\ \rho = 0$



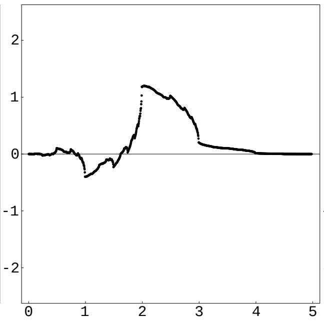
ha: $\theta = 7\pi/12,\ \rho = 0$



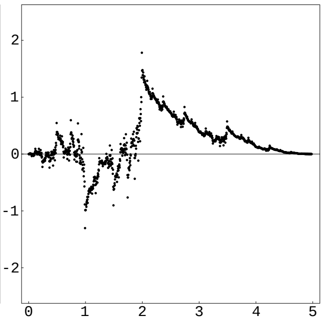
ia: $\theta = 2\pi/3,\ \rho = 0$



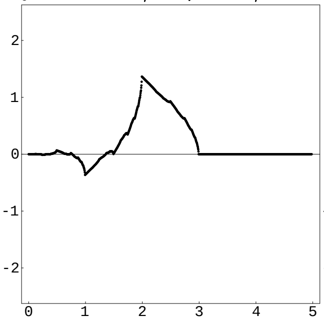
jd: $\theta = 3\pi/4,\ \rho = \pi/4$



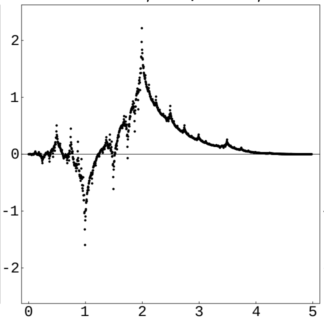
kd: $\theta = 5\pi/6,\ \rho = \pi/4$



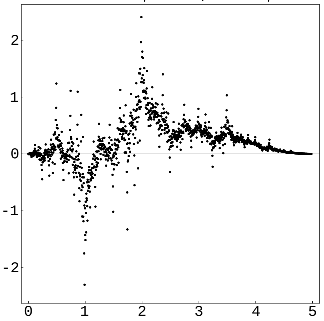
ld: $\theta = 11\pi/12,\ \rho = \pi/4$



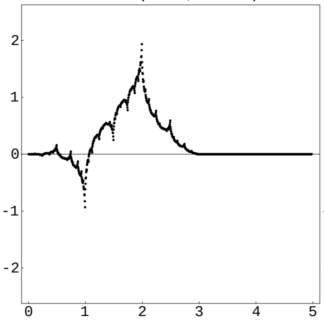
jc: $\theta = 3\pi/4,\ \rho = \pi/6$



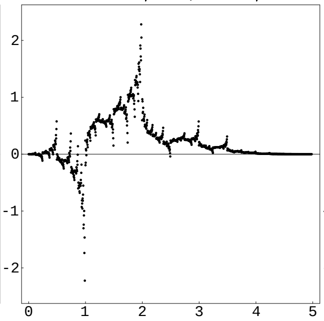
kc: $\theta = 5\pi/6,\ \rho = \pi/6$



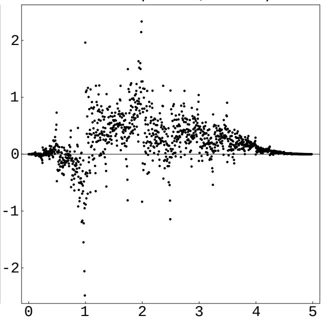
lc: $\theta = 11\pi/12,\ \rho = \pi/6$



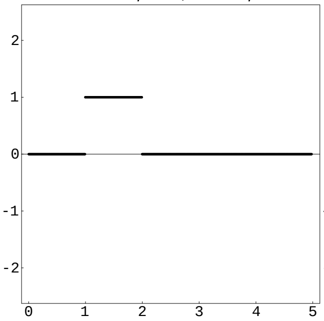
jb: $\theta = 3\pi/4,\ \rho = \pi/12$



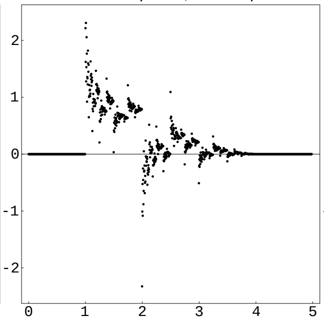
kb: $\theta = 5\pi/6,\ \rho = \pi/12$



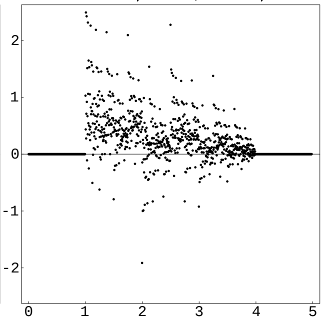
lb: $\theta = 11\pi/12,\ \rho = \pi/12$



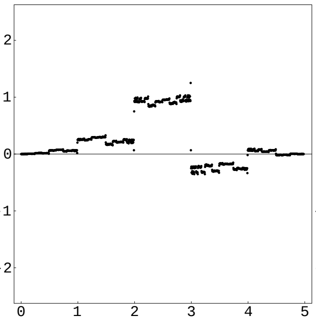
ja: $\theta = 3\pi/4,\ \rho = 0$



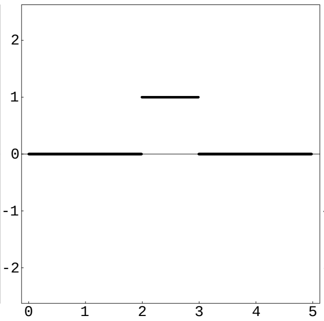
ka: $\theta = 5\pi/6,\ \rho = 0$



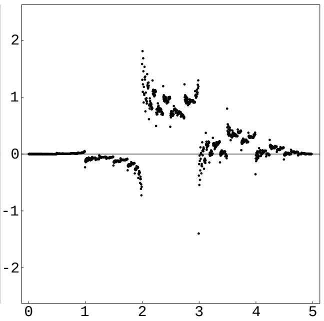
la: $\theta = 11\pi/12,\ \rho = 0$



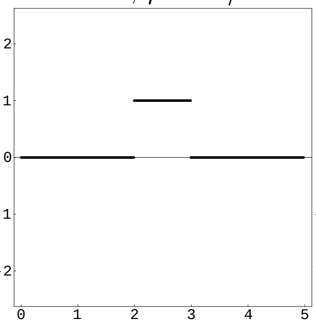

ah: $\theta = 0,\ \rho = 7\pi/12$



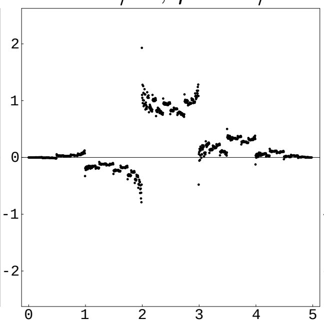

bh: $\theta = \pi/12,\ \rho = 7\pi/12$



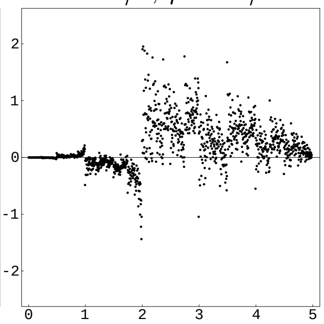

ch: $\theta = \pi/6,\ \rho = 7\pi/12$



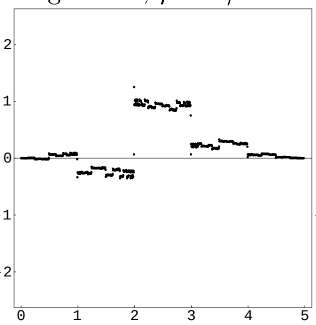

ag: $\theta = 0,\ \rho = \pi/2$



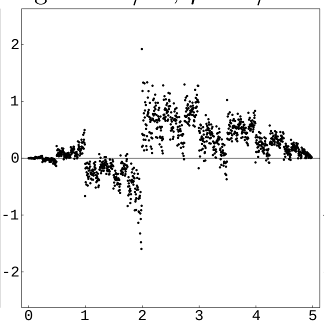

bg: $\theta = \pi/12,\ \rho = \pi/2$



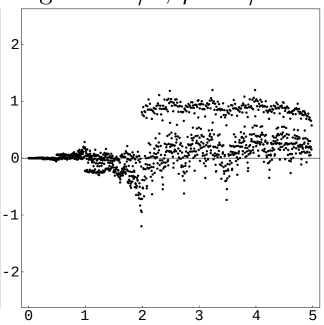

cg: $\theta = \pi/6,\ \rho = \pi/2$



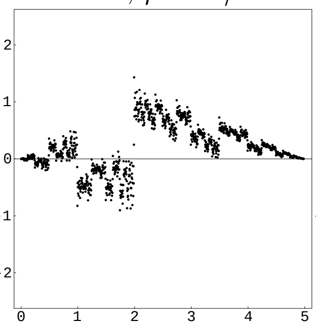

af: $\theta = 0,\ \rho = 5\pi/12$



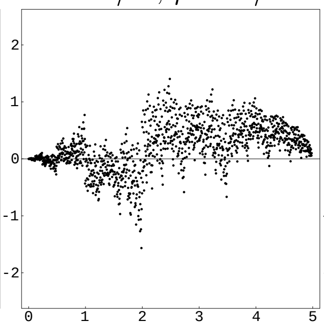

bf: $\theta = \pi/12,\ \rho = 5\pi/12$



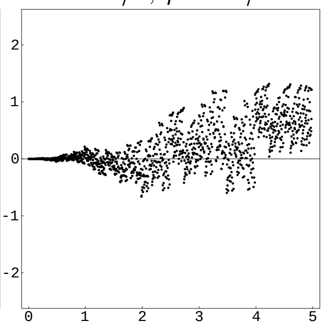

cf: $\theta = \pi/6,\ \rho = 5\pi/12$



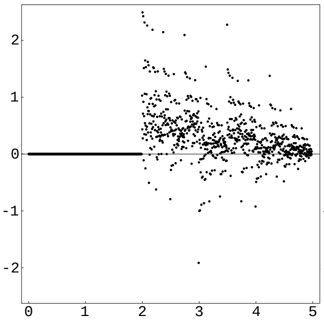

ae: $\theta = 0,\ \rho = \pi/3$



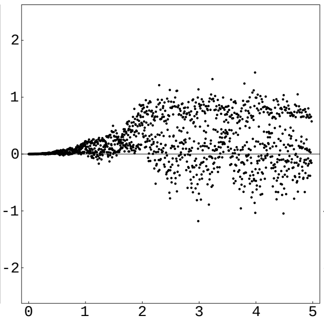

be: $\theta = \pi/12,\ \rho = \pi/3$



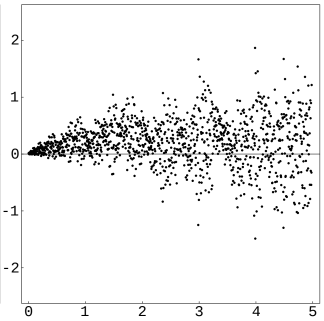

ce: $\theta = \pi/6,\ \rho = \pi/3$



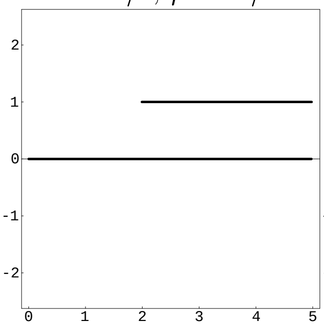
dh: $\theta = \pi/4,\ \rho = 7\pi/12$



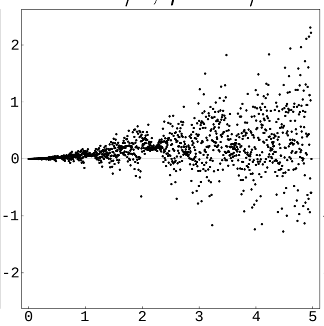
eh: $\theta = \pi/3,\ \rho = 7\pi/12$



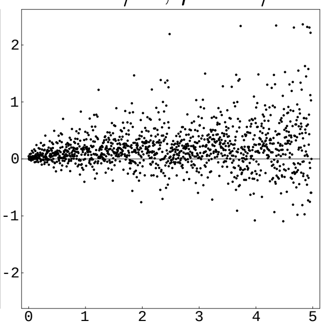
fh: $\theta = 5\pi/12,\ \rho = 7\pi/12$



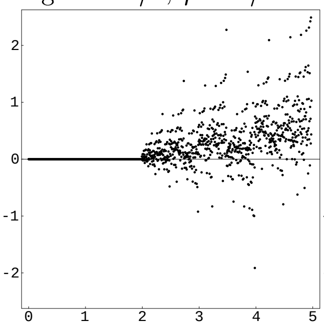
dg: $\theta = \pi/4,\ \rho = \pi/2$



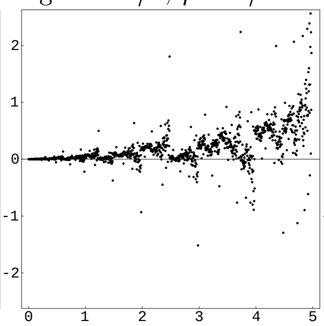
eg: $\theta = \pi/3,\ \rho = \pi/2$



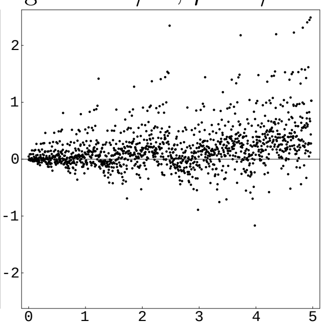
fg: $\theta = 5\pi/12,\ \rho = \pi/2$



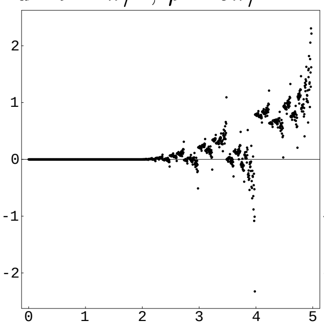
df: $\theta = \pi/4,\ \rho = 5\pi/12$



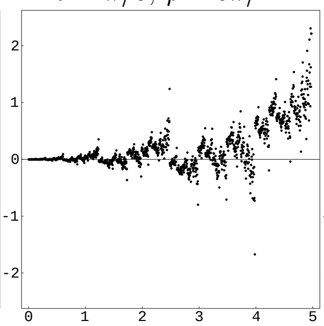
ef: $\theta = \pi/3,\ \rho = 5\pi/12$



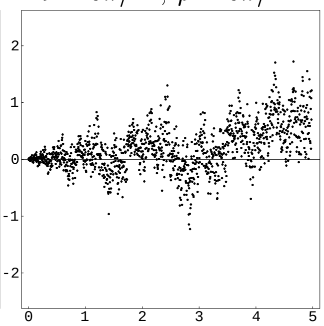
ff: $\theta = 5\pi/12,\ \rho = 5\pi/12$



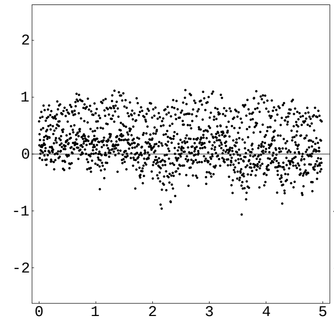
de: $\theta = \pi/4,\ \rho = \pi/3$



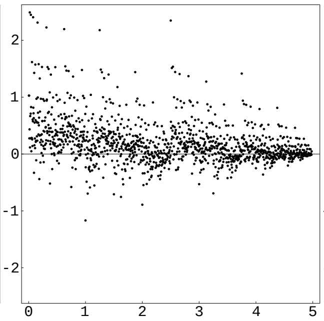
ee: $\theta = \pi/3,\ \rho = \pi/3$



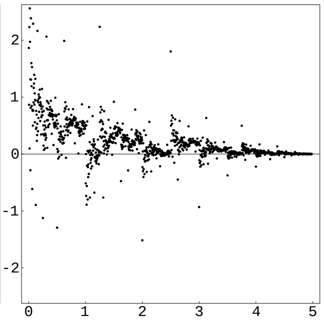
fe: $\theta = 5\pi/12,\ \rho = \pi/3$



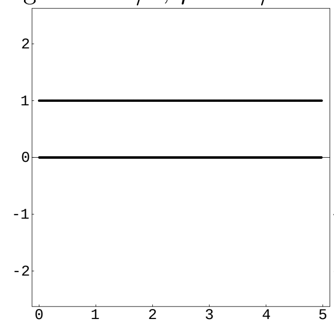
gh: $\theta = \pi/2,\ \rho = 7\pi/12$



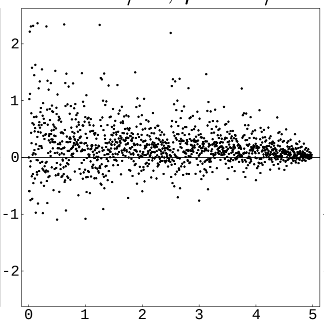
hh: $\theta = 7\pi/12,\ \rho = 7\pi/12$



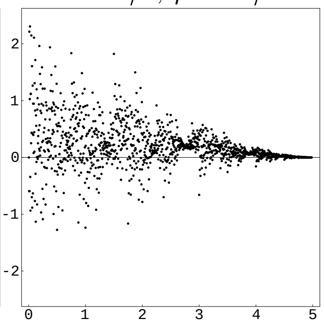
ih: $\theta = 2\pi/3,\ \rho = 7\pi/12$



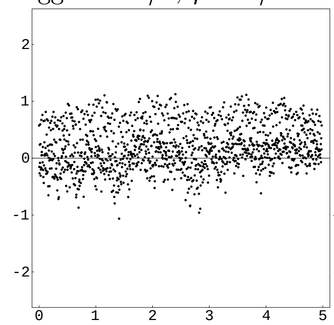
gg: $\theta = \pi/2,\ \rho = \pi/2$



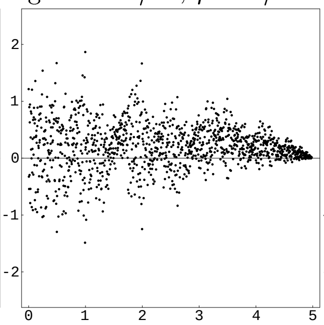
hg: $\theta = 7\pi/12,\ \rho = \pi/2$



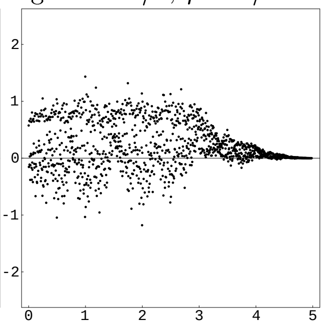
ig: $\theta = 2\pi/3,\ \rho = \pi/2$



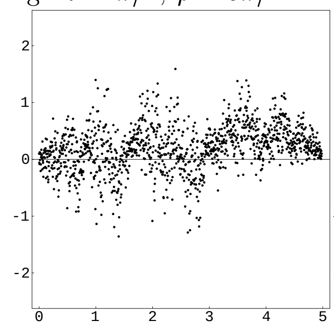
gf: $\theta = \pi/2,\ \rho = 5\pi/12$



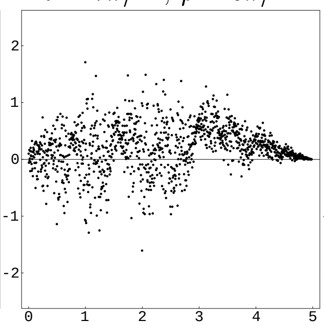
hf: $\theta = 7\pi/12,\ \rho = 5\pi/12$



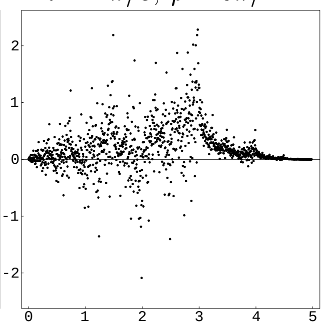
if: $\theta = 2\pi/3,\ \rho = 5\pi/12$



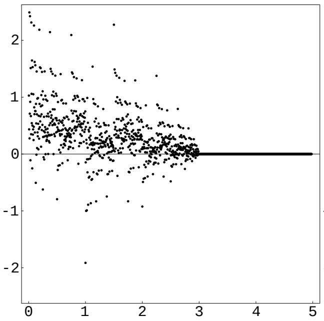
ge: $\theta = \pi/2,\ \rho = \pi/3$



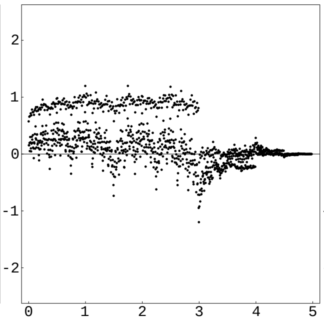
he: $\theta = 7\pi/12,\ \rho = \pi/3$



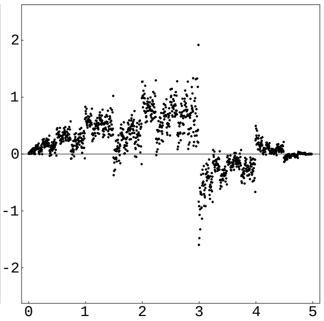
ie: $\theta = 2\pi/3,\ \rho = \pi/3$



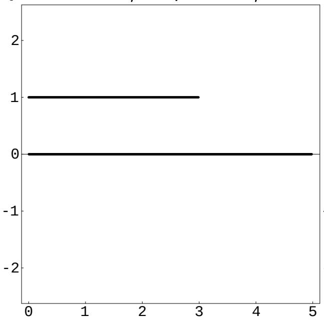

jh: $\theta = 3\pi/4,\ \rho = 7\pi/12$



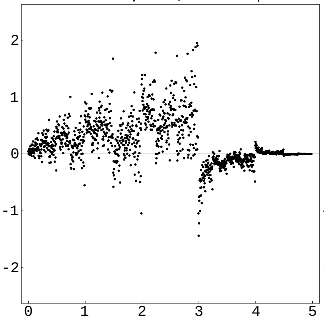

kh: $\theta = 5\pi/6,\ \rho = 7\pi/12$



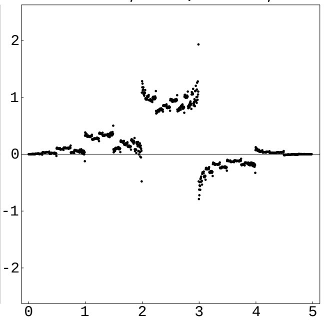

lh: $\theta = 11\pi/12,\ \rho = 7\pi/12$



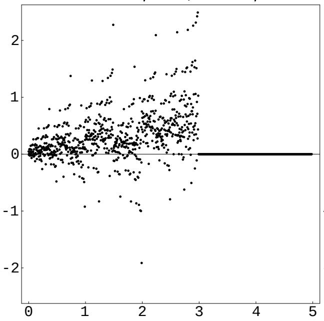

jg: $\theta = 3\pi/4,\ \rho = \pi/2$



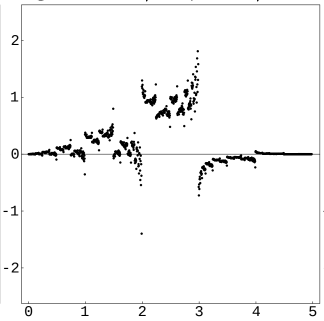

kg: $\theta = 5\pi/6,\ \rho = \pi/2$



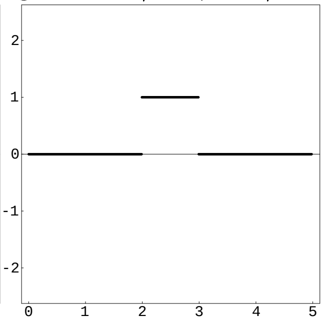

lg: $\theta = 11\pi/12,\ \rho = \pi/2$



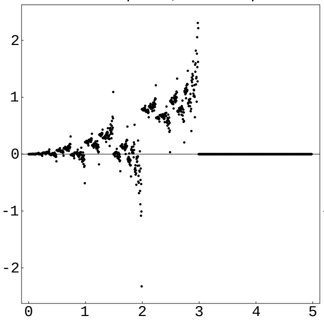

jf: $\theta = 3\pi/4,\ \rho = 5\pi/12$



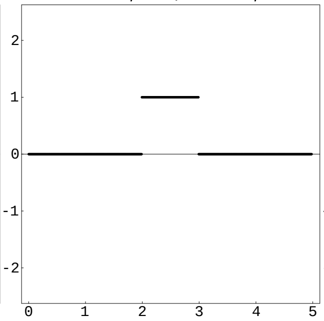

kf: $\theta = 5\pi/6,\ \rho = 5\pi/12$



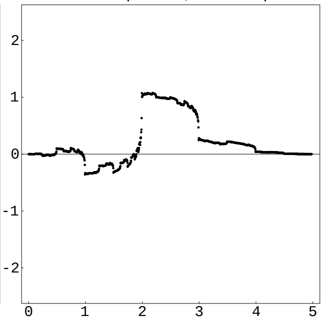

lf: $\theta = 11\pi/12,\ \rho = 5\pi/12$



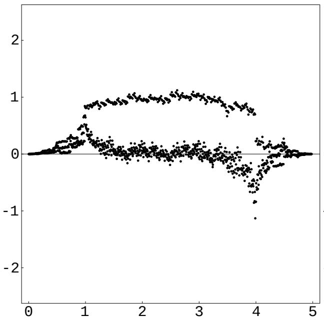

je: $\theta = 3\pi/4,\ \rho = \pi/3$



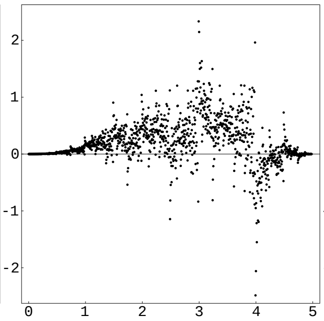

ke: $\theta = 5\pi/6,\ \rho = \pi/3$



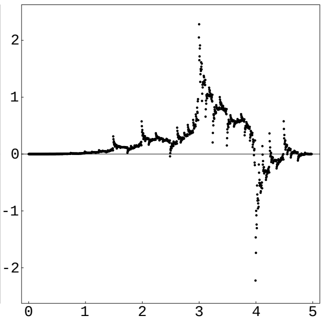

le: $\theta = 11\pi/12,\ \rho = \pi/3$



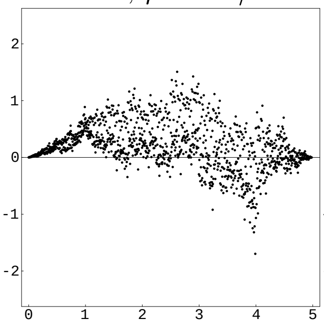
al: $\theta = 0,\ \rho = 11\pi/12$



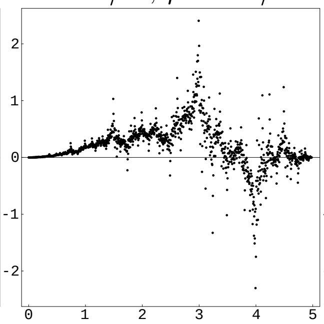
bl: $\theta = \pi/12,\ \rho = 11\pi/12$



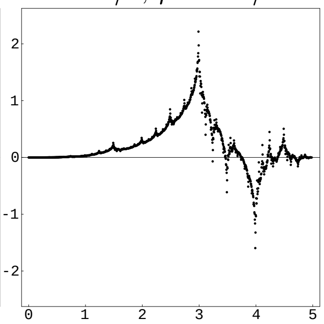
cl: $\theta = \pi/6,\ \rho = 11\pi/12$



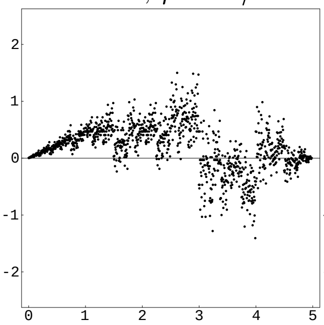
ak: $\theta = 0,\ \rho = 5\pi/6$



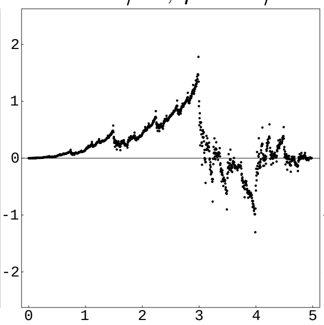
bk: $\theta = \pi/12,\ \rho = 5\pi/6$



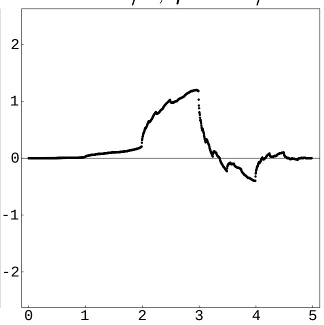
ck: $\theta = \pi/6,\ \rho = 5\pi/6$



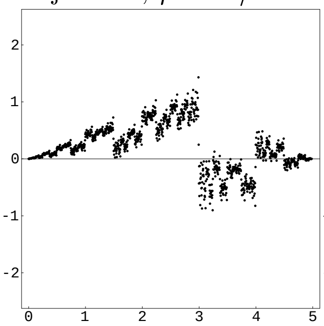
aj: $\theta = 0,\ \rho = 3\pi/4$



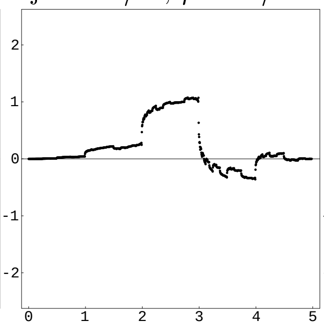
bj: $\theta = \pi/12,\ \rho = 3\pi/4$



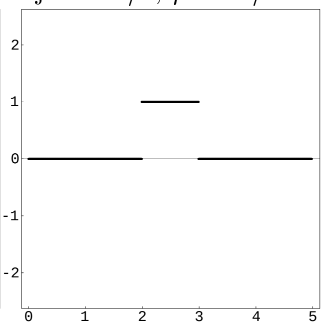
cj: $\theta = \pi/6,\ \rho = 3\pi/4$



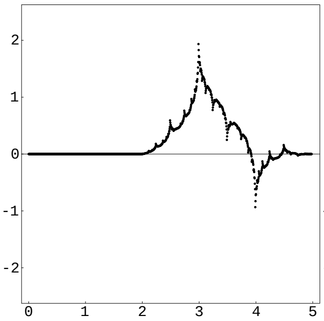
ai: $\theta = 0,\ \rho = 2\pi/3$



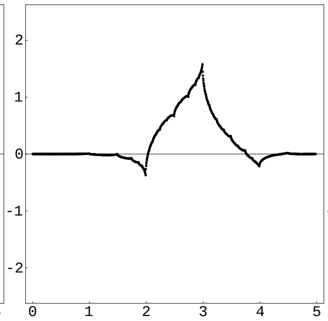
bi: $\theta = \pi/12,\ \rho = 2\pi/3$



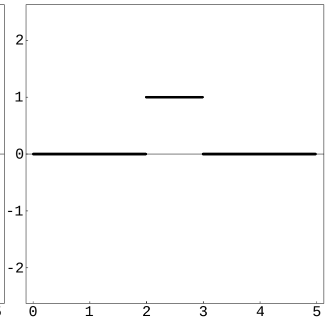
ci: $\theta = \pi/6,\ \rho = 2\pi/3$



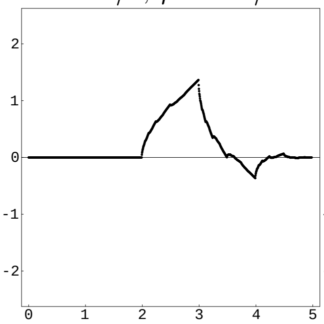
dl: $\theta = \pi/4,\ \rho = 11\pi/12$



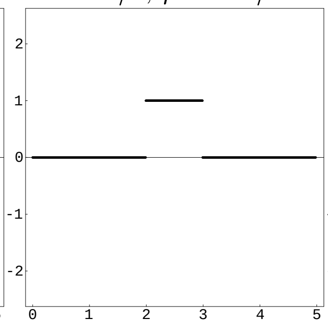
el: $\theta = \pi/3,\ \rho = 11\pi/12$



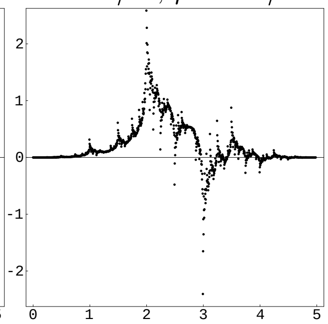
fl: $\theta = 5\pi/12,\ \rho = 11\pi/12$



dk: $\theta = \pi/4,\ \rho = 5\pi/6$



ek: $\theta = \pi/3,\ \rho = 5\pi/6$



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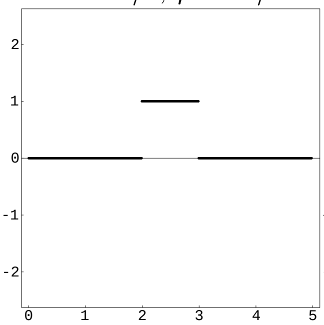

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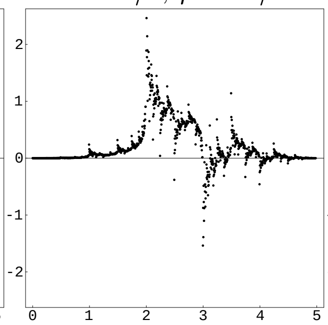

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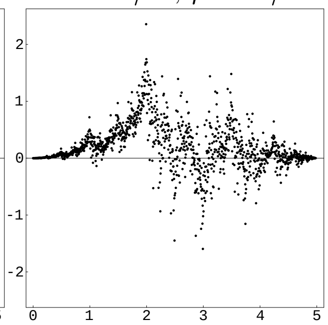

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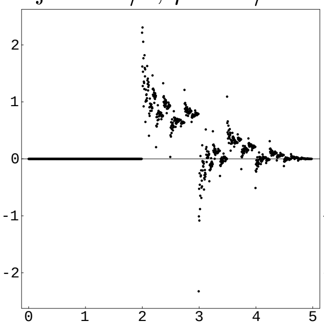
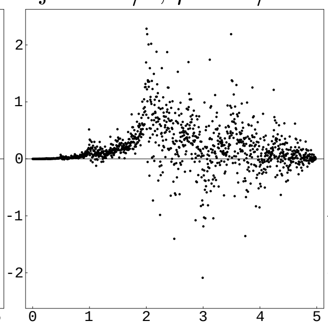


di: $\theta = \pi/4,\ \rho = 2\pi/3$



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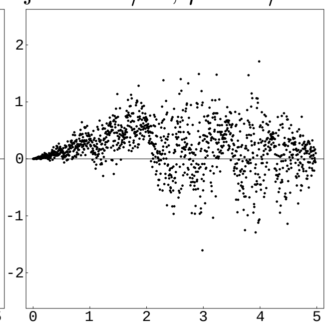

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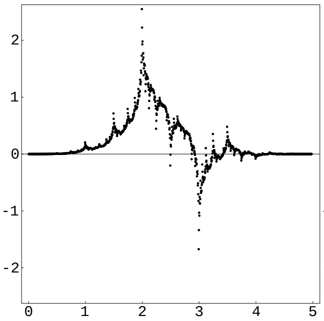
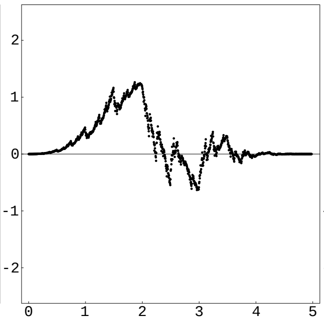


gl: $\theta = \pi/2,\ \rho = 11\pi/12$



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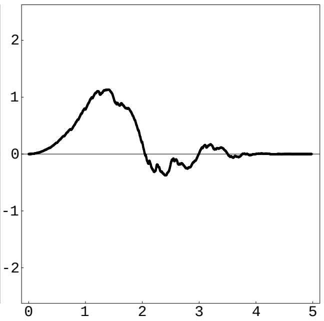
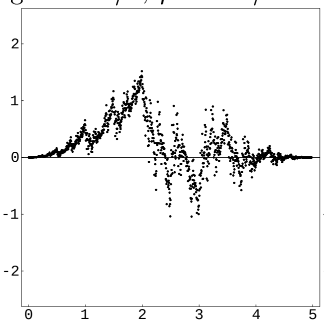
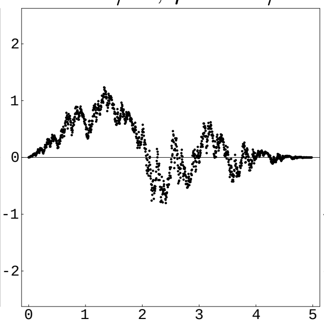


il: $\theta = 2\pi/3,\ \rho = 11\pi/12$



gk: $\theta = \pi/2,\ \rho = 5\pi/6$



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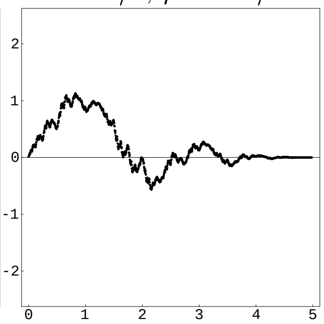

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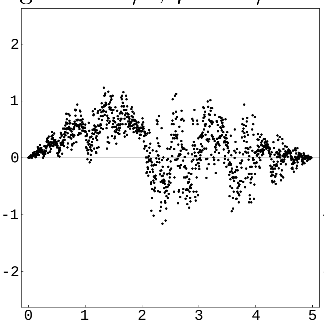
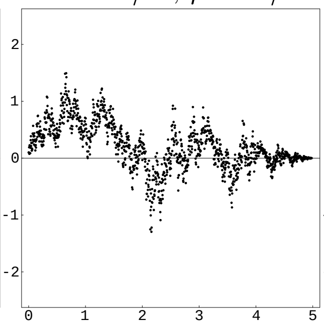


gj: $\theta = \pi/2,\ \rho = 3\pi/4$



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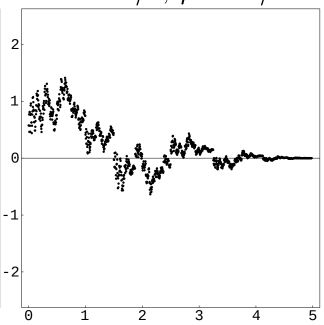
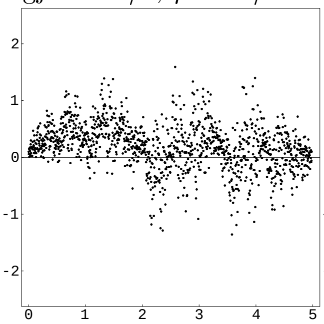
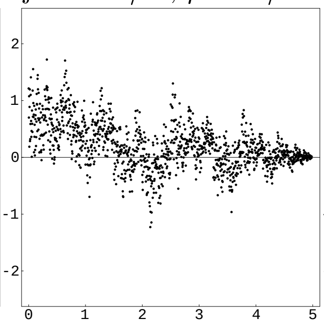


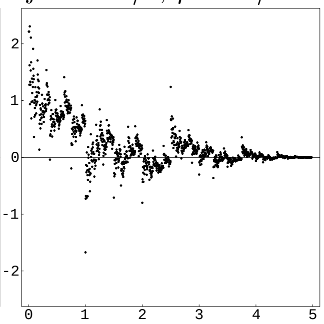
ij: $\theta = 2\pi/3,\ \rho = 3\pi/4$



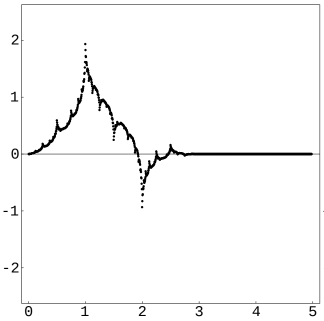
gi: $\theta = \pi/2,\ \rho = 2\pi/3$



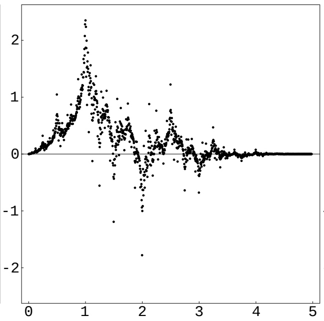
hi: $\theta = 7\pi/12,\ \rho = 2\pi/3$



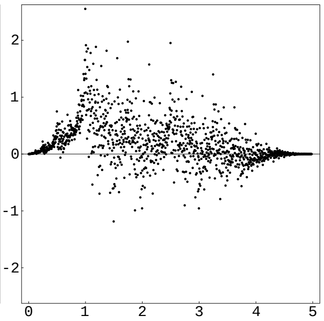
ii: $\theta = 2\pi/3,\ \rho = 2\pi/3$



jl: $\theta = 3\pi/4,\ \rho = 11\pi/12$



kl: $\theta = 5\pi/6,\ \rho = 11\pi/12$



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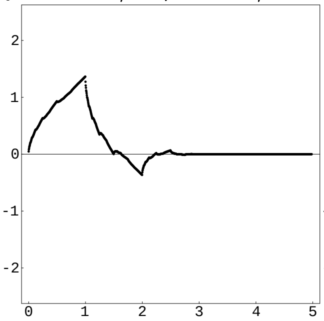
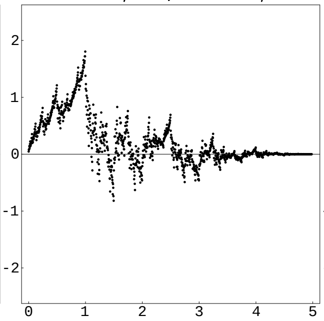
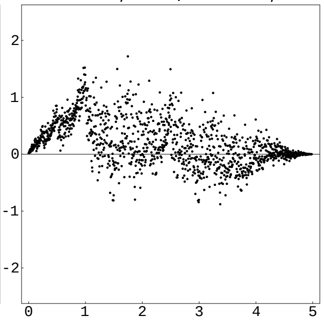


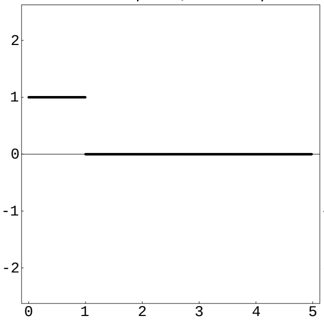
jk: $\theta = 3\pi/4,\ \rho = 5\pi/6$



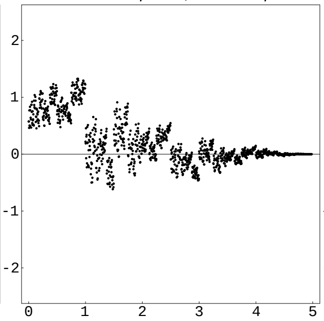
kk: $\theta = 5\pi/6,\ \rho = 5\pi/6$



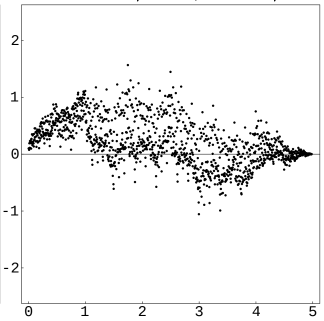
lk: $\theta = 11\pi/12,\ \rho = 5\pi/6$



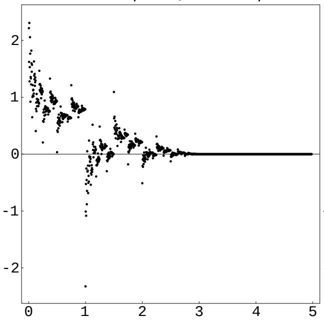
jj: $\theta = 3\pi/4,\ \rho = 3\pi/4$



kj: $\theta = 5\pi/6,\ \rho = 3\pi/4$



lj: $\theta = 11\pi/12,\ \rho = 3\pi/4$



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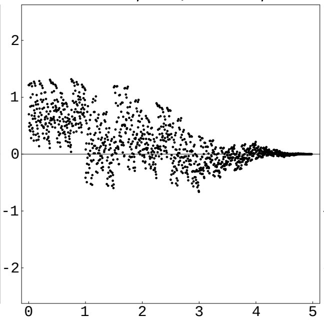

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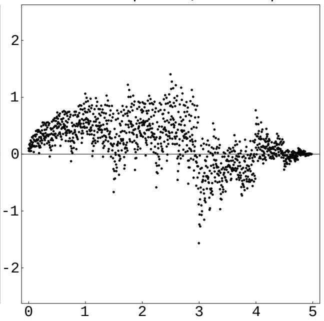

li: $\theta = 11\pi/12,\ \rho = 2\pi/3$

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