Equilibrium Fluctuations for Lattice Gases.

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Abstract: The authors in a previous paper proved the hydrodynamic incompressible limit in $d \geq 3$ for a thermal lattice gas, namely a law of large numbers for the density, velocity field and energy. In this paper the equilibrium fluctuations for this model are studied and a central limit theorem is proved for a suitable modification of the vector fluctuation field $\zeta(t)$, whose components are the density, velocity and energy fluctuations fields. We consider a modified fluctuation field $\xi^\varepsilon(t) = \exp\{-\varepsilon^{-1}tE\}\zeta^\varepsilon$, where E is the linearized Euler operator around the equilibrium and prove that $\xi^\varepsilon(t)$ converges to a vector generalized Ornstein-Uhlenbeck process $\xi(t)$, which is formally solution of the stochastic differential equation $d\xi(t) = N\xi(t)dt + BdW_t$, with $BB^* = -2NC$, where C is the compressibility matrix, N is a matrix whose entries are second order differential operators and B is a mean zero Gaussian field. The relation $-2NC = BB^*$ is the fluctuation-dissipation relation.

Key Words: Fluctuations, Stochastic Cellular Automata, Navier-Stokes Equation.

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1. Introduction.

In this paper we study the equilibrium fluctuations for the stochastic lattice gas introduced in [BEM]. It is a model of particles with discrete velocities jumping on the lattice: a particle with a given velocity moves on the 3-d lattice as the asymmetric simple exclusion process with the jump intensity chosen so to have a drift equal to its velocity. In each site particles collide exchanging velocities in such a way to conserve the number of particles, the momentum in each direction and the energy. This model generalizes the one in [EMY2] to include the case of particles with different kinetic energy. In [BEM] it has been proved the law of large number for this model in the following form. We choose as initial state a Bernoulli measure with density, momentum and energy small perturbation (of order ε) of constant profiles. Then the empirical fields $\nu^\varepsilon_\beta(x,t) = \nu_\beta(\varepsilon^{-1}x, \varepsilon^{-2}t), \beta = 0, \dots, 4$ of density, momentum and energy converge weakly in probability as ε goes to 0 to the solution of the hydrodynamic equations for this model, which are the incompressible Navier-Stokes equation for the velocity field and a diffusive equation (including the transport along the velocity field) for the energy. The dissipative terms in these equations are given in terms of a diffusion tensor $D^{\beta,\nu}_{\alpha,\gamma}, \beta, \nu = 0, \dots, 4,\ \alpha, \gamma = 1, \dots, 3$, which is expressed by the Green-Kubo formulas. The next natural step is to prove the space-time central limit theorem,

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namely that the fluctuations fields starting from the equilibrium state converge to a stationary multi-dimensional Gaussian process with a given space-time covariance. Since the macroscopic behavior of this model is very close to the real hydrodynamics we face the main difficulty of the hydrodynamic fluctuations: the Euler terms and the Navier-Stokes terms live on different time scales. The same feature is responsible for the impossibility of obtaining the compressible dissipative hydrodynamic equations as scaling limit. In fact, the previous result on the law of large number is true for an initial condition which is a small perturbation of the global equilibrium. This perturbation remains small at later times of order ε^{-2} and evolves macroscopically according to the incompressible hydrodynamics. The case of the fluctuations is different because a small perturbation of the equilibrium may become very large and be of order ε^{-1} on times of order ε^{-2}. We go now in some details to explain better this point. The fluctuation fields under diffusive scaling are defined by

$$\zeta_\beta^\varepsilon(t,G) = \varepsilon^{\frac{3}{2}} \sum_x G_\beta(\varepsilon x) \left[I_\beta(\eta_{\varepsilon^{-2}t}(x) - \mathbb{E}[I_\beta] \right],$$

$\beta = 0, \cdots, 4$, where G_β are suitable test functions, $\eta_t(x)$ is the configuration in x at time t and $\mathbb{E}$ is the equilibrium expectation. I_β are the quantities conserved by the dynamics, total number of particles, total momentum and total energy in x.

At time zero the limiting fluctuation fields

$$\lim_{\varepsilon \to 0} \zeta_\beta^\varepsilon(0,G) = \zeta_\beta(0,G)$$

are jointly Gaussian with covariance

$$\mathbb{E}[\zeta_\beta(0,G)\zeta_\nu(0,H)] = C_{\beta,\nu} \int d^3x \ \ G(x)H(x).$$

The matrix $C = (C_{\beta,\nu})$ is called the compressibility matrix. The limit is in the in sense of weak convergence of path measures.

It is not hard to show (it is indeed a by-product of the results and estimates in this paper) that the equilibrium fluctuations under Euler time scale are trivial in the sense that they satisfy in the limit a deterministic equation. This is a general feature first showed in [GP], [FF]. More precisely, the limiting field $\zeta^E = (\zeta_\beta^E), \beta = 0, \cdots, 4$

$$\zeta^E(\tau) = \lim_{\varepsilon \to 0} \zeta_\varepsilon^E(\tau, G) = \lim_{\varepsilon \to 0} \zeta^\varepsilon \left(\varepsilon\tau, G\right)$$

is solution of the deterministic equation

$$d\zeta^E(\tau) = E\zeta^E(\tau)d\tau, \tag{1.1}$$

where E is the linearized Euler operator around the global equilibrium. Equations (1.1) are a system of linear hyperbolic equations. The stochastic noise should appear as a correction of order ε as

$$d\zeta_\varepsilon^E(\tau) = (A + \varepsilon\mathcal{D})\zeta_\varepsilon^E(\tau)d\tau + \sqrt{\varepsilon}BdW_\tau + O(\varepsilon^2),$$

where $\mathcal{D}$ is the linearized Navier-Stokes operator around the global equilibrium and

$$BB^* = -2\mathcal{D}C$$

is the fluctuation-dissipation relation. Hence, to see a finite noise one has to look at longer times $\tau = \varepsilon^{-1}t$. Formally, since $\zeta^\varepsilon(t) = \zeta^E_\varepsilon(\varepsilon^{-1}t)$, we get

$$d\zeta^\varepsilon(t) = (\varepsilon^{-1}E + \mathcal{D})\zeta^\varepsilon(t)dt + BdW_t + O(\varepsilon).$$

Then the limit $\lim_{\varepsilon\to 0}\zeta^\varepsilon(t)$ does not exist because the Euler modes are too big on this time scale. A similar difficulty is present also in the case of ASEP but the analogous of E is simply an operator of the form $v \cdot \nabla_x$ with $v_i = (p_i - q_i)(1-2\alpha)$, p_i, q_i the rates of jumping to the left and right respectively and $\alpha = \mathbb{E}[\eta]$. Therefore, a Galilean shift is sufficient to remove the divergence and in fact in [CLO] the central limit theorem is proved for a fluctuation field of the form

$$Y^\varepsilon(t,G) = \varepsilon^{\frac{d}{2}} \sum_x G(\varepsilon x - \varepsilon^{-1}vt)\left[(\eta_{\varepsilon^{-2}t}(x) - \mathbb{E}[\eta]\right].$$

In our case a possible way to subtract the Euler modes is to consider a modified fluctuation field which moves together with the waves solutions of (1.1), traveling with velocity of order ε^{-1}. Denoting by E^* the adjoint operator of E, we define the fluctuation field as

$$\xi^\varepsilon(t,G) = \zeta^\varepsilon(t, e^{-\frac{t}{\varepsilon}E^*}G).$$

We prove that the limit $\varepsilon \to 0$ exists and satisfies a suitable stochastic differential equation. Before writing the equation, we consider the same problem in a very simple case: let A and M be $K \times K$ matrix with complex entries such that $A = -A^*$ where the adjoint is relative to the scalar product in $\mathbb{R}^K$. Consider the linear ODE system

$$\dot{x}_\varepsilon = (\varepsilon^{-1}A + M)x_\varepsilon, \quad x_\varepsilon(0) = \bar{x}.$$

Then, $y_\varepsilon = e^{-\frac{t}{\varepsilon}A}x_\varepsilon$ is solution of

$$\dot{y}_\varepsilon = e^{-\frac{t}{\varepsilon}A}Me^{\frac{t}{\varepsilon}A}y_\varepsilon, \quad y_\varepsilon(0) = \bar{x}.$$

Consider the limit

$$U := \lim_{\varepsilon\to 0}\int_0^1 ds\, e^{-\frac{s}{\varepsilon}A}Me^{\frac{s}{\varepsilon}A} = \lim_{\varepsilon\to 0}\varepsilon\int_0^{\frac{1}{\varepsilon}} ds\, e^{-sA}Me^{sA}$$

An asymptotic average theorem [EP] states that for any $\delta > 0$ and $T > 0$ there exists $\varepsilon_0 > 0$ such that the solution z of

$$\dot{z} = Uz, \quad z(0) = \bar{x}$$

satisfy

$$\sup_{0<t<T}|y_\varepsilon - z| < \delta, \quad 0 < \varepsilon < \varepsilon_0$$

Therefore, $y = \lim_{\varepsilon\to 0} y_\varepsilon$ is solution of

$$\dot{y} = Uy, \quad y(0) = \bar{x}.$$

The limit U can be characterized in the following way: Let $\mathcal{N}$ be the space of the $K \times K$ matrices with complex entries. $\mathcal{N}$ is a Hilbert space under the inner product

$$(A, B) = \sum_{1\le i,j\le K} A^*_{ij} B_{ij}.$$

For $A \in \mathcal{N}$ define Π_A as the orthogonal projection onto the subspace of the matrices which commute with A

$$\{B \in \mathcal{N} : [B, A] = 0\}.$$

Since the spectrum of A is imaginary it is easy to see that

$$U = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^{\frac{1}{\varepsilon}} ds\, e^{-sA} M e^{sA}$$

is the projector $\Pi_A M$.

Applying this kind of considerations to our problem, by Fourier analysis, we prove that $\xi(t, G) = \lim_{\varepsilon\to 0} \xi^\varepsilon(t, G)$ exists and is a stationary generalized Ornstein-Uhlenbeck process characterized formally by the stochastic differential equation

$$d\xi(t) = N\xi(t)dt + BdW_t,$$

with $BB^* = -2NC$, where C is the compressibility matrix, N is a second order differential operator and BW_t is a mean zero Gaussian field. In particular, this proves the fluctuation-dissipation relation $-2NC = BB^*$ for this model. Denoting by $\hat{M}$ the Fourier transform of a 5×5 matrix whose entries are differential operators, we can characterize N as follows:

$$\hat{N} = \Pi_{\hat{E}} \hat{\mathcal{D}},$$

Π_A the projection on the space of the operators commuting with A. To conclude, we want to stress that this procedure of subtracting the Euler modes works in this case because the equations for the equilibrium fluctuations are linear.

The central limit theorem for equilibrium fluctuations is a well investigated topics [S], [KL]. A standard procedure is to establish first the tightness of the sequence of fluctuation field. Then, the study of the martingale problem allows to identify the unique weak limit as a generalized Ornstein-Uhlenbeck process by the use of Holley-Stroock theory. It is crucial to evaluate some expression in the martingale problem in terms of the fluctuations field. This step, called Boltzmann-Gibbs principle, was first achieved in [BR] for symmetric zero range process. The alternative method by [CY] and [C1] was extended to non gradient systems by [L] and [C2]. The extension is based on a suitable modification of the fluctuation field by adding lower order terms, determined by identifying the diffusion coefficient in the hydrodynamic equations. In [CLO] this approach has been extended to a non-symmetric case by proving a stronger tightness result and as consequence a stronger Boltzmann-Gibbs theorem. We extend the results on tightness and Boltzmann-Gibbs theorem in [CLO] to the present model. Moreover, we prove the convergence of the time averages of the form appearing in the martingale problem, by using and adapting some results in [EP] which studied the convergence of solutions to the linearized Navier-Stokes equations of solutions to the linearized Boltzmann equation. The paper is organized as follows. In Section 2 we define the model and recall the previous results on the hydrodynamic limit that we will need in the sequel. In Section 3 we define the fluctuation field and state the results. In Section 4 we identify the limiting distribution of Q^ε by using Holley-Stroock characterization of Ornstein-Uhlenbeck processes with martingales. The Boltzmann-Gibbs principle is proved in Section 5 together with the tightness of the process. The theorems stating the existence of the time averages are in the Appendix.

2. Model and hydrodynamic limit.

We consider the following model introduced in [BEM], which is a generalization of the model in [EMY2]: given a finite set of velocities $\mathcal{V} \subset \mathbb{R}^3$, particles with velocity $v \in \mathcal{V}$ evolve on the sub-lattice $\Lambda_L = \{-L, \dots, L\}^3$, with periodic boundary conditions, according to an exclusion process. Collisions between two particles can also occur provided that the momentum and the kinetic energy are conserved. The set $\mathcal{V}$ is chosen in the following way:

$$\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2,$$

where $\mathcal{V}_1$ is made of 8 velocities given by

$$\mathcal{V} = \{(\pm 1, \pm 1, \pm 1)\}$$

and $\mathcal{V}_2$ contains 24 velocities, given up to permutation by

$$(\pm \varpi, \pm 1, \pm 1), \tag{2.1}$$

where ϖ is some irrational number suitably chosen.

Formally, if we denote by $\eta(x, v) \in \{0, 1\}$ the number of particles on site $x \in \Lambda_L$ with velocity $v \in \mathcal{V}$, then the infinitesimal generator of the dynamics on the space $\Omega_L = \{\eta = (\eta(x, v),\, x \in \Lambda_L,\, v \in \mathcal{V})\}$ is defined as

$$\mathcal{L} = \mathcal{L}^{\mathrm{ex}} + \mathcal{L}^{\mathrm{c}},$$

where $\mathcal{L}^{\mathrm{ex}}$ is the generator of the nearest neighbor exclusion process with different colors (velocities) and $\mathcal{L}^{\mathrm{c}}$ the generator of the collision process. For a local function f on Ω_L, $\mathcal{L}^{\mathrm{ex}}$ is given by

$$\mathcal{L}^{\mathrm{ex}} f(\eta) = \sum_{v \in \mathcal{V}} \sum_{|e|=1} \sum_{x \in \Lambda_L} (\chi + \frac{1}{2} e \cdot v) \eta(x, v) \left[f\left(\eta^{x, x+e, v}\right) - f\left(\eta\right) \right],$$

where e is a unitary vector of $\mathbb{Z}^3$ (e_α, $\alpha = 1, 2, 3$, will denote the unitary vectors with positive coordinates), $\eta^{x,x+e,v}$ is the configuration obtained after exchanging the values of $\eta(x, v)$ and $\eta(x + e, v)$ and χ is some positive constant large enough such that the jump rate is positive. Note that it is chosen so that the drift of the particles with velocity v is exactly v.

The collisions generator $\mathcal{L}^c$ is given by

$$\mathcal{L}^c f(\eta) = \sum_{x\in\Lambda_L}\sum_{q\in\mathcal{Q}} \left[f\left(\eta^{x,q}\right) - f\left(\eta\right)\right],$$

where $\mathcal{Q}$ is the set of admissible collisions, namely the set of velocity quadruples $q = (v, w, v', w') \in \mathcal{V}^4$ such that $v + w = v' + w'$ and $|v|^2 + |w|^2 = |v'|^2 + |w'|^2$, and $\eta^{x,q}$ is the configuration obtained after the collision on site x between two particles with incoming velocities v, w and outgoing velocities v', w'. Notice that in order to preserve the exclusion rule, we take $\eta^{x,q}$ unchanged with respect to η if one of the conditions $\eta(x, v) = 0$, $\eta(x, w) = 0$, $\eta(x, v') = 1$ or $\eta(x, w') = 1$ is fulfilled.

We denote by $\eta_x = \{\eta(x, v),\ v \in \mathcal{V}\}$ the particle configuration in $x \in \Lambda_L$. For a configuration η, the mass, momentum and kinetic energy in site x are

$$\begin{aligned}
I_0(\eta_x) &= \sum_{v\in\mathcal{V}} \eta(x, v),\\
I_\alpha(\eta_x) &= \sum_{v\in\mathcal{V}} (v\cdot e_\alpha)\eta(x, v)\ ,\ \ \alpha = 1, 2, 3,\\
I_4(\eta_x) &= \sum_{v\in\mathcal{V}} \frac{1}{2}|v|^2\eta(x, v).
\end{aligned}$$

It is easy to check that the quantities $N_\beta(\eta) = \sum_x I_\beta(\eta_x)$, $\beta = 0, \dots, 4$, are conserved by the full dynamics. It is shown in [BEM] that, by choosing suitably the parameter ϖ in (2.1), they are the only conserved quantities, in other words this model has the property of local ergodicity.

As a consequence, the grand canonical measures below are invariant for $\mathcal{L}$

$$\mu_{L,n}(\eta) = Z_{L,n}^{-1} \prod_{x\in\Lambda_L} \exp\Big\{\sum_{\beta=0}^{d+1} n_\beta I_\beta(\eta_x)\Big\}, \tag{2.2}$$

where $n = (n_0, \dots, n_4) \in \mathbb{R}^5$ are the chemical potentials and $Z_{L,n}$ is a normalization constant. All these product measures are absolutely continuous with respect to the reference measure μ obtained by taking n as $n_0 := (r, 0, 0, 0, \theta)$. We set $m_\beta = \mathbb{E}^\mu[I_\beta(\eta_0)]$ for $\beta = 0, \dots, 4$ (notice that $m_\beta = 0$ if $\beta = 1, 2, 3$) and $\tilde{I}_\beta = I_\beta - m_\beta$.

In the sequel we call $\varepsilon = L^{-1}$. The law of the process $(\eta_t(x, v))$ with generator $\varepsilon^{-2}\mathcal{L}$ starting from μ is denoted by $\mathbb{P}^\mu_\varepsilon$ and the corresponding expectation by $\mathbb{E}^\mu_\varepsilon$. We also call $f_0(v) = \mathbb{E}^\mu[\eta(x, v)]$ the density of particles with velocity $v \in \mathcal{V}$ with respect to the reference measure μ. For any function h on $\mathcal{V}$, we put $\langle h\rangle = \sum_{v\in\mathcal{V}} h(v)$.

The currents $w^\beta_{x,\alpha}$ of the conserved quantities I_β, $\beta = 0, \dots, 4$, at site x in direction e_α, $\alpha = 1, 2, 3$, are defined by

$$\mathcal{L}I_\beta(\eta_x) = \sum_{\alpha=1}^{3} \nabla^-_\alpha w^\beta_{x,\alpha},$$

where, if g is a function on Λ_L,

$$\nabla_\alpha^- g(x) = (\nabla_\alpha g)(x - e_\alpha) \quad \text{and} \quad \nabla_\alpha g(x) = g(x + e_\alpha) - g(x).$$

Since the local quantities $I_\beta(\eta_x)$ are conserved by the collision generator, we have $\mathcal{L} I_\beta(\eta_x) = \mathcal{L}^{\mathrm{ex}} I_\beta(\eta_x)$ and the currents can be written as the sum of a symmetric and an antisymmetric parts

$$w_{x,\alpha}^\beta = \chi \nabla_\alpha I_\beta(\eta_x) + w_{x,\alpha}^{(a),\beta}$$

and

$$w_{x,\alpha}^{(a),0} = \big\langle v_\alpha b_{x,\alpha}(v) \big\rangle, \quad w_{x,\alpha}^{(a),\beta} = \big\langle v_\alpha v_\beta b_{x,\alpha}(v) \big\rangle, \ \beta = 1,2,3$$
$$w_{x,\alpha}^{(a),4} = \frac{1}{2} \big\langle v_\alpha |v|^2 b_{x,\alpha}(v) \big\rangle,$$

with

$$b_{x,\alpha}(v) = \eta(x + e_\alpha, v)\eta(x,v) - \frac{1}{2}\big(\eta(x + e_\alpha) + \eta(x,v)\big).$$

Let $\mathcal{G}$ be the space of local functions h on Λ_L such that

$$\mathbb{E}^\mu[h] = 0 \quad \text{and} \quad \frac{\partial \mathbb{E}^{\mu_{L,n}}[h]}{\partial m_\beta(n)}\Big|_{n=n_0} = 0, \quad \beta = 0, \dots, 4, \tag{2.3}$$

where $m_\beta(n) = \mathbb{E}^{\mu_{L,n}}[I_\beta]$. In view of the application of the Boltzmann-Gibbs principle, it is important to modify the currents $w_{x,\alpha}^{(a),\beta}$ so that they are in the space $\mathcal{G}$. It is enough to subtract suitable combinations of the conserved quantities and we now get their explicit expressions.

Let n be the chemical potential $n = n_0 + \delta n = (r + \delta n_0, \delta n_1, \delta n_2, \delta n_3, \theta + \delta n_4)$, then

$$\mathbb{E}^{\mu_{L,n}}\big[w_{0,\alpha}^{(a),0}\big] - c_\alpha^0 = \frac{1}{3}\big\langle |v|^2 h_1 \big\rangle \delta n_\alpha + o(\delta n),$$
$$\mathbb{E}^{\mu_{L,n}}\big[w_{0,\alpha}^{(a),\beta}\big] - c_\alpha^\beta = \delta_{\alpha,\beta}\Big[\frac{1}{3}\big\langle |v|^2 h_1 \big\rangle \delta n_0 + \frac{1}{6}\big\langle |v|^4 \big\rangle h_1 \delta n_0\Big] \delta n_\alpha + o(\delta n), \qquad \beta = 1,2,3,$$
$$\mathbb{E}^{\mu_{L,n}}\big[w_{0,\alpha}^{(a),4}\big] - c_\alpha^4 = \frac{1}{6}\big\langle |v|^4 h_1 \big\rangle \delta n_\alpha + o(\delta n),$$

where $h_0 = f_0(1 - f_0)$, $h_1 = h_0(1 - 2f_0)$ and

$$c_\alpha^\beta = \mathbb{E}^\mu\big[w_{0,\alpha}^{(a),\beta}\big] \tag{2.4}$$

If we denote by $\delta m_\beta = \mathbb{E}^{\mu_{L,n}}[I_\beta(\eta_0)] - m_\beta$, we get

$$\delta n_0 = \frac{1}{\Phi}\big(\big\langle |v|^4 h_0 \big\rangle \delta m_0 - 2\big\langle |v|^2 h_0 \big\rangle \delta m_4\big),$$
$$\delta n_\alpha = \frac{3}{\big\langle |v|^2 h_0 \big\rangle} \delta m_\alpha,$$
$$\delta n_4 = \frac{2}{\Phi}\Big(2\big\langle h_0 \big\rangle \delta m_4 - \big\langle |v|^2 h_0 \big\rangle \delta m_0\Big),$$

where

$$\Phi = \big\langle |v|^4 h_0 \big\rangle \big\langle h_0 \big\rangle - \big\langle |v|^2 h_0 \big\rangle^2 > 0.$$

So, defining for $\beta, \nu = 0, \dots, 4$ and $\alpha = 1, 2, 3$,

$$d_\alpha^{\beta,\nu} = \frac{\partial \mathbb{E}^{\mu_{L,n}}\big[w_{x,\alpha}^{(a),\beta}\big]}{\partial m_\nu(n)}\Big|_{n=n_0},$$

we obtain

$$d_\alpha^{\beta,\nu} = b_0 \delta_{\beta,0}\delta_{\alpha,\nu} + b_4 \delta_{\beta,4}\delta_{\alpha,\nu} + \mathbb{1}_{\{1,2,3\}}(\beta)\delta_{\alpha,\beta}\big[a_0\delta_{\nu,0} + a_4\delta_{\nu,4}\big], \tag{2.5}$$

with

$$b_0 = \frac{\Phi_2}{3\Phi}, \qquad b_4 = 2\frac{\Phi_1}{3\Phi}, \qquad a_0 = \frac{\big\langle |v|^2 h_1 \big\rangle}{\big\langle |v|^2 h_0 \big\rangle}, \qquad a_4 = \frac{\big\langle |v|^4 h_1 \big\rangle}{\big\langle |v|^2 h_0 \big\rangle}, \tag{2.6}$$

$$\Phi_1 = \big\langle h_1 |v|^4 \big\rangle \big\langle h_0 \big\rangle - \big\langle h_1 |v|^2 \big\rangle \big\langle h_0 |v|^2 \big\rangle,$$
$$\Phi_2 = \big\langle h_0 |v|^4 \big\rangle \big\langle h_1 |v|^2 \big\rangle - \big\langle h_1 |v|^4 \big\rangle \big\langle h_0 |v|^2 \big\rangle.$$

Therefore the local function

$$g_\alpha^\beta(\eta) = w_{0,\alpha}^{(a),\beta} - c_\alpha^\beta - \frac{1}{2}\sum_{\nu=0}^{4} d_\alpha^{\beta,\nu}\big(\tilde{I}_\nu(\eta(0)) + \tilde{I}_\nu(\eta(e_\alpha))\big) \tag{2.7}$$

belongs to $\mathcal{G}$.

Slow-fast modes decomposition of the currents

We denote by $\overline{I}_\ell^+ = (\overline{I}_{0,\ell}, \dots, \overline{I}_{4,\ell})$ the empirical averages of the conserved quantities over the block Λ_ℓ of length ℓ:

$$\overline{I}_{\beta,\ell} = \frac{1}{(2\ell+1)^3}\sum_{|y|\le \ell} I_\beta(\eta_y), \quad \beta = 0, \dots, 4.$$

The measure $\hat{\mu}_{\ell,m}$, $m \in \mathbb{R}^5$ is defined as the canonical Gibbs state of $(2\ell+1)^3$ sites with parameters such that $\overline{I}_\ell^+ = m$. It is the uniform probability on the set $\Omega_{\ell,m}$ of configurations on the block Λ_ℓ such that $\overline{I}_\ell^+ = m$. We denote by $\alpha_\ell(g)$ the conditional expectation of g given the averages $\overline{I}_\ell^+$

$$\alpha_\ell(g) = \mathbb{E}^\mu[g|\overline{I}_\ell^+].$$

We call $\mathcal{L}_{s,\ell}$ the symmetric part of the generator $\mathcal{L}$ restricted to the block Λ_ℓ. Since the measures $\hat{\mu}_{\ell,m}$ are the only extremal invariant measures for $\mathcal{L}_{s,\ell}$, we can define $\mathcal{L}_{s,\ell}^{-1} g$ for any function g such that $\alpha_\ell(g) = 0$. Given any local function g on Ω_ℓ, the finite volume "variance" $V_\ell(g,n)$ is

$$V_\ell(g,n) = \frac{1}{(2\ell_1+1)^3}\mathbb{E}^{\mu_{\ell,n}}\Big[\Big(\sum_{|x|\le \ell_1}(\tau_x g - \alpha_\ell(g))\Big)(-\mathcal{L}_{s,\ell})^{-1}\Big(\sum_{|x|\le \ell_1}(\tau_x g - \alpha_\ell(g))\Big)\Big],$$

where $\tau.$ is the translation operator on Ω_L, $\tau_x g(\eta) = g(\tau_x \eta)$, $\ell_1 = \ell - \ell^{1/9}$, ℓ large enough. The "variance" $V(G,n)$ of G is given by

$$V(G,n) = \limsup_{\ell\to\infty} V_\ell(G,n). \tag{2.8}$$

With an abuse of notation, we denote $V_\ell(G,n)$ by $V_\ell(G,r,\theta)$ when n is the chemical potential $n_0 = (r,0,0,0,\theta)$.

We state here the results in [BEM]

Theorem 2.1 . *There exists a rank 2 tensor $\bar D = \big(\bar D^{\beta,\nu}_{\alpha,\gamma}\big)$ ($\bar D^{\beta,\nu}$ positive definite matrix) and a sequence of local functions $h^{(q)} = (h^{(q),\beta}_\alpha$, $\alpha = 1,2,3$, $\beta = 0,\dots,4)$ in $\mathcal{G}$ such that, setting*

$$u^{(q),\beta}_\alpha(\eta) = g^\beta_\alpha(\eta) - \sum_{\gamma=1}^{3}\sum_{\nu=0}^{4}\bar D^{\beta,\nu}_{\alpha,\gamma}\nabla_\gamma \tilde I_\nu(\eta(0)) - \mathcal{L}h^{(q),\beta}_\alpha(\eta), \tag{2.9}$$

where $g^\beta_\alpha(\eta)$ is defined in (2.7), it results

$$\limsup_{q\to\infty}\sum_{\alpha=1}^{3}\sum_{\beta=0}^{4} V(u^{(q),\beta}_\alpha, r,\theta) = 0.$$

Above Theorem actually holds for any function in $\mathcal{G}$.

Lemma 2.2 . *The tensor $\bar D$ satisfies*

$$a\cdot(\bar D C)a = \lim_{q\to\infty}\mathbb{E}^\mu\big[\Gamma(a\cdot h^{(q)})(-\mathcal{L}_s)(a\cdot h^{(q)})\big]. \tag{2.10}$$

In this formula, $a\cdot b = \sum_{\alpha=1}^3\sum_{\beta=0}^4 a^\beta_\alpha b^\beta_\alpha$, $\Gamma(g) = \sum_x \tau_x g$, $\mathcal{L}_s$ is the symmetric part of $\mathcal{L}$ in $L^2(\mu)$, C is the 5×5 compressibility matrix (see (3.3) below for an explicit expression) and $\bar D C$ is the tensor $(\bar D C)^{\beta,\nu}_{\alpha,\gamma} = (\bar D_{\alpha,\gamma} C)^{\beta,\nu}$. We define $D = \bar D + \chi\mathbb{I}$ where $\mathbb{I}^{\beta,\nu}_{\alpha,\gamma} = \delta_{\alpha,\gamma}\delta_{\beta,\nu}$.

Hydrodynamic limit

Given functions $n_\beta(x)$, $\beta = 0,\dots,4$, we consider the Gibbs states with chemical potential $n(x) = (n_0(x),\dots,n_4(x))$

$$\mu_{L,n}(\eta) = Z^{-1}_{L,n}\prod_{x\in\Lambda_L}\exp\Big\{\sum_{\beta=0}^{4} n_\beta(x) I_\beta(\eta_x)\Big\}.$$

Now, assume that the initial distribution of the particles is $\mu_{L,n}$ with $n = (n_\beta)$ the slowly varying chemical potentials given by

$$n_\beta(x) = \lambda_\beta^{(0)} + \varepsilon\lambda_\beta^{(1)}(\varepsilon x) + \varepsilon^2\lambda_\beta^{(2)}(\varepsilon x), \tag{2.11}$$

where $\lambda^{(0)} = (\lambda_\beta^{(0)}) = n_0$ and λ_β^i are smooth functions on the 3-d torus $\mathbb{T}_3$. We define the local equilibrium measure as the Gibbs states $\mu_{L,n(\cdot,t)}$ with $n(\cdot,t)$ the chemical potential given by

$$n_\beta(x,t) = \lambda_\beta^{(0)} + \varepsilon\lambda_\beta^{(1)}(\varepsilon x,t) + \varepsilon^2\lambda_\beta^{(2)}(\varepsilon x,t). \tag{2.12}$$

Furthermore, we assume

$$\operatorname{div} \underline{\lambda}^{(1)} = 0, \quad < h_1 v^2 > \lambda_0^{(1)} + \frac{1}{2} < h_1 v^4 > \lambda_4^{(1)} = 0.$$

Then in [BEM] (see also [EMY2]) it has been proved that the the law of the process at time $t > 0$ is well approximated by the local equilibrium in the sense that the relative entropy per unit volume of the non-equilibrium measure with respect the local equilibrium times ε^{-2} vanishes in the limit $\varepsilon \to 0$.

We can now state the result proved in [BEM] on the hydrodynamic limit. Let $u(z,t)$, $z \in \mathbb{T}_3$, $t \in [0,t_0]$, $t_0 > 0$, be the classical smooth solutions of the following Navier-Stokes equation

$$\begin{aligned} &\operatorname{div} u = 0, \\ &\partial_t u_\beta + \partial_\beta p + K u \cdot \nabla u_\beta = \sum_{\alpha=1}^{3} D_{\alpha,\beta}\partial_\alpha^2 u_\beta, \quad \beta = 1,2,3, \end{aligned} \tag{2.13}$$

with initial condition $u_\alpha(z) = \mathbb{E}^{\mu_{L,n(\varepsilon^{-1}z)}}[I_\alpha(\eta_0)]$ and let $\mathcal{E}(z,t)$ be the solution of the energy equation

$$\frac{\partial}{\partial t}\mathcal{E} + H u \cdot \nabla\mathcal{E} = \sum_{\alpha=1}^{3} \mathcal{K}_\alpha(\partial_\alpha^2\mathcal{E}), \tag{2.14}$$

with initial condition $\mathcal{E}(z) = \mathbb{E}^{\mu_{L,n(\varepsilon^{-1}z)}}[I_4(\eta_0)]$. The constants appearing in (2.13) and (2.14) are given by

$$K = 18\frac{\langle v_1^2 v_2^2 h_2\rangle}{\langle h_0|v|^2\rangle^2},$$

with $h_2 = \frac{1}{2}h_1(1-6f_0(1-f_0))$ and

$$H = \frac{1}{\langle h_0|v|^2\rangle}\frac{\Psi_1 - 2C\Psi_2}{\Phi_2 + C\Phi_1}, \quad C = \frac{1}{2}\frac{\langle h_1|v|^4\rangle}{\langle h_1|v|^2\rangle},$$

where

$$\begin{aligned} \Psi_1 &= \langle h_2|v|^6\rangle\langle h_1|v|^2\rangle - \langle h_2|v|^4\rangle\langle h_1|v|^4\rangle, \\ \Psi_2 &= \langle h_2|v|^4\rangle\langle h_1|v|^2\rangle - \langle h_2|v|^2\rangle\langle h_1|v|^4\rangle. \end{aligned}$$

Let $\mathbb{P}_\varepsilon^{\mu_{L,n}}$ be the law of the process $\eta_t(x,v)$ with generator $\varepsilon^{-2}\mathcal{L}$ starting from the measure $\mu_{L,n}$ defined in (2.11), with chemical potentials $n_\alpha(x)$ of the form (2.12). The density $(\nu_0^\varepsilon(t,z))$, the momentum $((\nu_\beta^\varepsilon(t,z))_{\beta=1,2,3})$ and energy $(\nu_4^\varepsilon(t,z))$ empirical fields are defined as

$$\nu_\beta^\varepsilon(z,t) = \varepsilon^2 \sum_{x\in\Lambda_L} \delta(z-\varepsilon x)\tilde{I}_\beta(\eta_t(x)),$$

where $\tilde{I}_\beta(\eta_x) = I_\beta(\eta_x) - m_\beta$, $m_\beta = \mathbb{E}^\mu[I_\beta(\eta_0)]$ and $\eta_t(x) = \{\eta_t(x,v),\, v\in\mathcal{V}\}$.

Theorem 2.3 *The density, momentum and energy empirical fields converge, for $t\le t_0$, weakly (in space) in $\mathbb{P}_\varepsilon^{\mu_{L,n}}$probability, to $\rho(z,t)dz$, $u(z,t)dz$ and $\mathcal{E}(z,t)dz$, where $a\rho+b\mathcal{E}=c$ for suitable a,b,c.*

Note that the transport coefficients $D_{\alpha,\beta}$ and $\mathcal{K}_\alpha$ are suitable combinations of the diffusion coefficients $D_{\alpha,\gamma}^{\beta,\nu}$ in Theorem 2.1. The explicit expressions are given in [BEM], but we omit them because they do not play any role in this paper.

3. Fluctuation field and results.

In this paper, we are interested in the equilibrium fluctuations of the mass, momentum and energy fields. The initial fluctuations, distributed in terms of the measure μ, are finite but they may become infinite at later very long times because of the effect of waves moving with velocity ε^{-1}, which are the solutions of the linearized (around the equilibrium) Euler equations (linear hyperbolic equations) for this model. To remove the diverging terms we have to modify the usual definition of fluctuation fields not simply by a shift but considering fluctuations which move together with the traveling waves

We denote by U_t^ε the operator $\exp(-\frac{t}{\varepsilon}E^*)$ where E is the linearized Euler operator, a 5×5 matrix whose entries are first order differential operators with constant coefficients,

$$E = \begin{pmatrix} 0 & -a_0\underline{\partial} & 0 \\ -b_0\underline{\partial} & 0 & -b_4\underline{\partial} \\ 0 & -a_4\underline{\partial} & 0 \end{pmatrix}$$

and $*$ is the adjoint with respect the usual scalar product in $L^2(\mathbb{T}_3,\mathbb{R}^5)$ (the constants a_i and b_i are defined in (2.6) and $\underline{\partial} = (\partial_1,\partial_2,\partial_3)$ is the gradient operator).

For any smooth function $G=(G_\beta)_{\beta=0,\dots,4}:\mathbb{T}_3\to\mathbb{R}^5$ consider the (scalar) fluctuation field ξ^ε on the state space $(\mathbb{T}_3)^{\otimes 5}$

$$\xi^\varepsilon(t,G) = \varepsilon^{3/2}\sum_{\beta=0}^{4}\sum_x (U_t^\varepsilon G)_\beta(\varepsilon x)\tilde{I}_\beta(\eta_t(x)). \tag{3.1}$$

It is equivalent to consider the vector fluctuation field $(\xi_\beta^\varepsilon)_{\beta=0,\dots 5}$ on $\mathbb{T}^3$ whose components ξ_0^ε, $(\xi_\beta^\varepsilon)_{\beta=1,\dots 3}$ and ξ_4^ε are respectively the density, momentum and energy fluctuation fields, defined as

$$\xi_\beta^\varepsilon(t,\varphi) = \xi^\varepsilon(t,G^{(\beta)}),\quad \beta=0,\dots,4,$$

where $G^{(\beta)}$ is the vector function with only the β component non vanishing and $G^{(\beta)}_\beta = \varphi$.

We want to study the evolution of the fluctuation fields in the limit $\varepsilon \to 0$ when the fields are initially distributed with the equilibrium measure μ, given by (2.2). We notice that the initial covariance of the limiting fields $\lim_{\varepsilon\to 0} \xi^\varepsilon_\beta(0,\varphi) = \xi_\beta(0,\varphi)$ is

$$\mathbb{E}^\mu_\varepsilon \Big[\xi_\beta(0,\varphi)\xi_\nu(0,\psi)\Big] = C_{\beta,\nu} \int_{\mathbb{T}_3} dx\, \varphi(x)\psi(x), \tag{3.2}$$

where C is the compressibility matrix (5×5)

$$C = \begin{pmatrix} \langle h_0 \rangle & 0 & \langle h_0 \frac{|v|^2}{2}\rangle \\ 0 & \frac{1}{3}\langle |v|^2 h_0\rangle \mathbb{I}_3 & 0 \\ \langle h_0 \frac{|v|^2}{2}\rangle & 0 & \langle h_0 \frac{|v|^4}{4}\rangle \end{pmatrix}, \tag{3.3}$$

with $\mathbb{I}_3$ the 3×3 identity matrix, h_0 defined in the paragraph before (2.4) and $\langle\,\cdot\,\rangle$ in the paragraph after (2.2).

Remark that E is not anti-hermitian in $L^2(\mathbb{T}_3,\mathbb{R}^5)$, since $a_0 \neq b_0$ and $a_4 \neq b_4$. However a straightforward computation shows that EC satisfies $EC + CE^* = 0$.

We want to show that the fluctuation field converges to a stationary Gaussian vector process with a given covariance. The equal time covariance is exactly (3.2) because of the stationarity of the limiting process.

To state the results we need some extra notation. We introduce the Hilbert spaces $\mathcal{H}_k, k\in\mathbb{Z}$ defined by the scalar product

$$\langle G,H\rangle_k = \langle G, L^k H\rangle_0,$$

where $L = I - \Delta$, Δ the Laplacian operator and $\langle .,.\rangle_0$ is the usual inner product of $L^2(\mathbb{T}_3,\mathbb{R}^5)$:

$$\langle G,H\rangle_0 = \sum_{\beta=0}^{4} \int_{\mathbb{T}_3} dx\, G_\beta(x)H_\beta(x). \tag{3.4}$$

Denote by $\|\cdot\|_k$ the norm of $\mathcal{H}_k$ and by $\mathcal{H}_{-k}$ the dual of $\mathcal{H}_k$ with respect to the inner product of $L^2(\mathbb{T}_3,\mathbb{R}^5)$. The fluctuation field $(\xi^\varepsilon(t))_{t\geq 0}$ is a distribution valued stochastic process taking values in the Sobolev space $\mathcal{H}_{-k_0}$ for some suitable k_0. Its path space is $D([0,T],\mathcal{H}_{-k_0})$ $(T>0)$, the space of functions with values in $\mathcal{H}_{-k_0}$, right continuous with left limits, endowed with the uniform (in time) weak (in space) topology. We call Q^ε the law of $(\xi^\varepsilon(t))_{t\geq 0}$ when the process is initially distributed according to the equilibrium measure μ. It is therefore a probability measure on the space $D([0,T],\mathcal{H}_{-k_0})$.

By analogy with (3.4), we define for local functions $g = (g_0,\dots,g_4)$ on Ω_L and smooth functions $G = (G_0,\dots,G_4)$ on $\mathbb{T}_3$

$$\langle G,g\rangle_{0,L} = \varepsilon^{3/2} \sum_{\beta=0}^{4} \sum_{x\in\Lambda_L} G_\beta(\varepsilon x)\tau_x g_\beta. \tag{3.5}$$

So the fluctuation field (3.1) can be rewritten as

$$\xi^{\varepsilon}(t,G) = \big\langle U_t^{\varepsilon} G, \tilde{I}(\eta_t(0)) \big\rangle_{0,L}.$$

We introduce the linearized Navier-Stokes operator $\mathcal{D}$ as

$$\mathcal{D}G = \sum_{\alpha,\gamma=1}^{3} D_{\alpha,\gamma} \partial_\alpha \partial_\gamma G. \tag{3.6}$$

Then $N = \pi_E(\mathcal{D})$ is the operator defined as the limit

$$\lim_{\varepsilon \to 0} \frac{1}{T} \int_0^T dt \, \exp\big(-\frac{t}{\varepsilon} E\big) \mathcal{D} \exp\big(\frac{t}{\varepsilon} E\big) G = \pi_E(\mathcal{D}) G$$

and $\pi_E(\mathcal{D})^*$ the adjoint of N with respect to the inner product $\langle \cdot, \cdot \rangle_0$ in $L^2(\mathbb{T}_3, \mathbb{R}^5)$. The main result of this paper is

Theorem 3.1 *The probability measures (Q^{ε}) converge weakly in $D([0,T], \mathcal{H}_{-k_0})$ to the law Q of the stationary generalized Ornstein-Uhlenbeck process ξ with mean 0 and covariance*

$$\mathbb{E}^Q \Big[\xi_\beta(s,\phi) \xi_\nu(t,\psi) \Big] = \int_{\mathbb{T}_3} dx \, \big((C S_{|t-s|})_{\beta,\nu} \phi \big)(x) \psi(x),$$

where $(S_t)_{t \geq 0}$ is the semi-group in $L^2(\mathbb{T}_3, \mathbb{R}^5)$ associated to $\pi_E(\mathcal{D})^*$ and C the compressibility matrix. It is formally characterized by the SDE

$$\begin{aligned} d\xi(t) &= N\xi(t) dt + B dW_t, \\ BB^* &= -2NC. \end{aligned}$$

One of the main ingredient needed while studying the equilibrium fluctuations is the so-called Boltzmann-Gibbs principle which states that the non conserved quantities arising in the conservation laws may be replaced by linear combinations of the conserved ones. In the context of a non gradient system, the usual statement is not valid and some corrections to the fluctuation field have to be introduced (see [C], [Lu]). The situation in the case of an asymmetric system is more delicate since the usual Boltzmann-Gibbs estimate is not sharp enough and one has to prove a stronger result ([CLO]). We need to generalize such a result to the present setup. Indeed we prove the following

Theorem 3.2 **(Boltzmann-Gibbs principle)** *Assume that $h \in \mathcal{G}$ (see (2.3)). Then, for any smooth function $G : \mathbb{R}_+ \times \mathbb{T}_3 \to \mathbb{R}$, the following estimate holds*

$$\limsup_{\varepsilon \to 0} \mathbb{E}_\varepsilon^\mu \left[\sup_{0 \le t \le T} \left(\varepsilon^{3/2-1} \int_0^t \sum_x G(s, \varepsilon x) \tau_x h(\eta, s)\, ds \right)^2 \right] \le cT \|G\|_0^2 V(h; r, \theta),$$

where V is the infinite volume variance defined in (2.8).

4. Limiting distribution of the fluctuation field.

The theory of Holley-Stroock [HS] characterizes the law Q of the Ornstein-Uhlenbeck process ξ described in Theorem 3.1 by the following martingale problem:

$$\begin{aligned} M_1(t, G) &= \xi(t, G) - \xi(0, G) - \int_0^t ds\, \xi(s, \pi_E(\mathcal{D})^* G), \\ M_2(t, G) &= \big(M_1(t, G)\big)^2 + 2t \left\langle G, C\, \pi_E(\mathcal{D})^* G \right\rangle_0 \end{aligned} \tag{4.1}$$

are martingales under Q. In this section, we will prove that any limit law $\bar{Q}$ of Q^ε satisfies (4.1). Therefore from the tightness of (Q^ε) (see Theorem 5.5 of section 5), it has to converge to Q and Theorem 3.1 follows.

The processes analogous to (4.1) for Q^ε are

$$\begin{aligned} M_1^\varepsilon(t, G) &= \xi^\varepsilon(t, G) - \xi^\varepsilon(0, G) - \int_0^t ds\, \xi^\varepsilon(s, \pi_E(\mathcal{D})^* G), \\ M_2^\varepsilon(t, G) &= \big(M_1^\varepsilon(t, G)\big)^2 + 2t \left\langle G, C\, \pi_E(\mathcal{D})^* G \right\rangle_0 \end{aligned}$$

and we want to show that these processes are martingales up to some error terms which vanish as ε goes to 0. Given local functions $h = (h_\alpha)_{\alpha=1,2,3} = (h_\alpha^\beta)_{\alpha=1,2,3;\beta=0,\dots,4} \in \mathcal{G}$, we introduce the modified fluctuation field

$$\zeta^\varepsilon(t, G, h) = \xi^\varepsilon(t, G) - \varepsilon \sum_{\alpha=1}^3 \left\langle \partial_\alpha (U_t^\varepsilon G), h_\alpha \right\rangle_{0,L},$$

where $\left\langle \cdot, \cdot \right\rangle_{0,L}$ was defined in (3.5). Actually we will choose for h the terms of the sequence $h^{(q)}$ defined in Theorem 2.1, but we will omit the label q for sake of shortness. It is clear that the difference between $\zeta^\varepsilon(t, G, h)$ and $\xi^\varepsilon(t, G)$ vanishes in $L^2(\mathbb{P}_\varepsilon^\mu)$ with ε. Moreover, it is well known that the following processes are martingales with respect to the usual filtration related to the process $(\eta_t(x, v))$

$$\begin{aligned} M_1^\varepsilon(t, G, h) &= \zeta^\varepsilon(t, G, h) - \zeta^\varepsilon(0, G, h) - \int_0^t \gamma_1^\varepsilon(s, G, h)\, ds, \\ M_2^\varepsilon(t, G, h) &= \big(M_1^\varepsilon(t, G, h)\big)^2 - \int_0^t \gamma_2^\varepsilon(s, G, h)\, ds, \end{aligned} \tag{4.2}$$

with

$$\gamma_1^\varepsilon(t,G,h) = \big(\partial_t + \varepsilon^{-2}\mathcal{L}\big)\zeta^\varepsilon(t,h,G),$$
$$\gamma_2^\varepsilon(t,G,h) = \big(\partial_t + \varepsilon^{-2}\mathcal{L}\big)\big(\zeta^\varepsilon(t,h,G)^2\big) - 2\big(\zeta^\varepsilon(t,h),G\big)\big(\partial_t + \varepsilon^{-2}\mathcal{L}\big)\big(\zeta^\varepsilon(t,h),G\big).$$

We first compute the compensator γ_1^ε. Let $w_{x,\alpha}^{(a)} = (w_{x,\alpha}^{(a),\beta})_{\beta=0,\dots,4}$. Then

$$\begin{aligned}\gamma_1^\varepsilon(t,G,h) =& \big\langle \partial_t(U_t^\varepsilon G), \tilde I(\eta_t(0))\big\rangle_{0,L} - \varepsilon^{-1}\sum_{\alpha=1}^3 \big\langle \partial_\alpha(U_t^\varepsilon G), \mathcal{L}h_\alpha(\eta_t)\big\rangle_{0,L} \\ &- \varepsilon^{-2}\sum_{\alpha=1}^3 \big\langle \nabla_\alpha(U_t^\varepsilon G), \chi\nabla_\alpha \tilde I(\eta_t(0)) + w_{0,\alpha}^{(a)}(\eta_t)\big\rangle_{0,L} + R_5(t,G,h),\end{aligned} \tag{4.3}$$

where, remembering that $\partial_t U_t^\varepsilon = -\varepsilon^{-1}E^* U_t^\varepsilon$,

$$R_5(t,G,h) = \sum_{\alpha=1}^3 \big\langle \partial_\alpha(-E^*U_t^\varepsilon)G, h_\alpha(\eta_t)\big\rangle_{0,L}.$$

Now, given $\bar D_{\alpha,\gamma} = (\bar D_{\alpha,\gamma}^{\beta,\nu})_{\beta,\nu=0,\dots,4}$, $\alpha,\gamma = 1,2,3$, we add and subtract the term

$$\sum_{\alpha,\gamma=1}^3 \big\langle \bar D_{\alpha,\gamma}\partial_\alpha(U_t^\varepsilon G), \nabla_\gamma \tilde I(\eta_t(0))\big\rangle_{0,L}$$

in (4.3). Then $\gamma_1^\varepsilon(t,G,h)$ is equal to

$$\begin{aligned}&\big\langle \partial_t(U_t^\varepsilon G), \tilde I(\eta_t(0))\big\rangle_{0,L} + \big\langle \mathcal{D}^*(U_t^\varepsilon G), \tilde I(\eta_t(0))\big\rangle_{0,L} \\ &- \varepsilon^{-1}\sum_{\alpha=1}^3 \Big\langle \partial_\alpha(U_t^\varepsilon G), w_{0,\alpha}^{(a)}(\eta_t) - c_\alpha - \sum_{\gamma=1}^3 \bar D_{\alpha,\gamma}\nabla_\gamma \tilde I(\eta_t(0)) - \mathcal{L}h_\alpha(\eta_t)\Big\rangle_{0,L} \\ &+ R_5(t,G,h) + R_1(t,G) + R_2(t,G) + R(t,G),\end{aligned} \tag{4.4}$$

with $\mathcal{D}^*$ the adjoint in $L^2(\mathbb{T}_3,\mathbb{R}^5)$ of the differential operator $\mathcal{D}$ in (3.6), $c_\alpha = (c_\alpha^\beta)_{\beta=0,\dots,4}$ the equilibrium value of the current $w_\alpha^{(a),\beta}$ (see (2.4)) and

$$R_1(t,G) = \varepsilon^{-1}\sum_{\alpha=1}^3 \big\langle (\partial_\alpha - \varepsilon^{-1}\nabla_\alpha)(U_t^\varepsilon G), \chi\nabla_\alpha \tilde I(\eta_t(0))\big\rangle_{0,L},$$

$$R_2(t,G) = \sum_{\alpha,\gamma=1}^3 \big\langle (\varepsilon^{-1}\nabla_\gamma^- - \partial_\gamma)\partial_\alpha D_{\alpha,\gamma}(U_t^\varepsilon G), \tilde I_\nu(\eta_t(0))\big\rangle_{0,L},$$

$$R(t,G) = \varepsilon^{-1} \sum_{\alpha=1}^{3} \big\langle \big(\partial_\alpha - \varepsilon^{-1}\nabla_\alpha\big)\,(U_t^\varepsilon G), w_{0,\alpha}^{(a)} - c_\alpha \big\rangle_{0,L}.$$

¿From the definition of $U_t^\varepsilon G$, the first term of the sum (4.4) can be written as

$$\varepsilon^{-1} \sum_{\alpha=1}^{3} \langle (-E^* U_t^\varepsilon) G, \tilde{I}(\eta_t(0)) \rangle_{0,L} = \varepsilon^{-1} \sum_{\alpha=1}^{3} \langle \partial_\alpha (U_t^\varepsilon G), d_\alpha \tilde{I}(\eta_t(0)) \rangle_{0,L}$$

where the coefficients of the matrix $d_\alpha = (d_\alpha^{\beta,\nu})_{\beta,\nu=0\ldots,4}$ were defined in (2.5). Recalling the definition of the local functions $g_\alpha = (g_\alpha^\beta)_{\beta=0\ldots,4}$ and $u_\alpha = (u_\alpha^\beta)_{\beta=0\ldots,4}$ in (2.7), (2.9) (we omit the label q), we obtain

$$\gamma_1^\varepsilon(t,G,h) = \langle \mathcal{D}^*(U_t^\varepsilon G), \tilde{I}_\nu(\eta_t(0)) \rangle_{0,L} + \sum_{i=1}^{4} R_i(t,G) + \sum_{i=5}^{6} R_i(t,G,h),$$

where

$$R_3(t,G) = \varepsilon^{-1} \sum_{\alpha=1}^{3} \Big\langle \Big(\partial_\alpha - \varepsilon^{-1}\nabla_\alpha + \frac{1}{2}\nabla_{e_\alpha}^{-}\partial_\alpha \Big)(U_t^\varepsilon G), d_\alpha \tilde{I}(\eta_t(0)) \Big\rangle_{0,L},$$

$$R_4(t,G) = \varepsilon^{-1} \sum_{\alpha=1}^{3} \langle (\partial_\alpha - \varepsilon^{-1}\nabla_\alpha)(U_t^\varepsilon G), g_\alpha(\eta_t) \rangle_{0,L}$$

and

$$R_6(t,G,h) = -\varepsilon^{-1} \sum_{\alpha=1}^{3} \langle \partial_\alpha (U_t^\varepsilon G), u_\alpha(\eta_t) \rangle_{0,L}.$$

To prove that the compensator $\int_0^t \gamma_1^\varepsilon(t,G,h)ds$ is converging, we have to control the remainder terms.

The remainders for $i = 1,2,3$ are easily controlled by the following

Lemma 4.1 *Let h be a mean zero local function and $G : \mathbb{R}_+ \times \mathbb{T}^3 \to \mathbb{R}$ a continuous function. Then there exist a constant c depending only on h such that, for all $t \geq 0$ and all $\varepsilon > 0$*

$$\mathbb{E}_\varepsilon^\mu \left[\sup_{0 \le t \le T} \left(\int_0^t ds\, \varepsilon^{3/2} \sum_x G(s,\varepsilon x)\tau_x h(\eta_s) \right)^2 \right] \leq cT^2 \|G\|_\infty^2.$$

The proof is an easy consequence of the Schwartz inequality and the stationarity of $\mathbb{P}_\varepsilon^\mu$. We refer to Lemma 4.1 in [CLO] for details.

By Taylor expanding and using Lemma 4.1 we immediately obtain

$$\lim_{\varepsilon\to 0} \mathbb{E}_\varepsilon^\mu \left[\left(\int_0^t ds\, R_i(s,G) \right)^2 \right] = 0.$$

for $i = 1, 2, 3$.

The other terms are estimated by using the refined Boltzmann-Gibbs principle (Theorem 3.2) because the functions g_α^β and u_α^β are in $\mathcal{G}$ ($h_\alpha^\beta \in \mathcal{G}$ by hypothesis). We get

$$\lim_{\varepsilon\to 0} \mathbb{E}_\varepsilon^\mu \left[\left(\int_0^t ds\, R_4(s, G) \right)^2 \right] = \lim_{\varepsilon\to 0} \mathbb{E}_\varepsilon^\mu \left[\left(\int_0^t ds\, R_5(s, G, h) \right)^2 \right] = 0$$

and

$$\limsup_{\varepsilon\to 0} \mathbb{E}_\varepsilon^\mu \left[\left(\int_0^t ds\, R_6(s, G, h) \right)^2 \right] \le c\, t \max_{\alpha=1,2,3} \|\partial_\alpha (U_\cdot^\varepsilon G)\|_0^2 \sum_{\alpha=1}^{3} \sum_{\beta=0}^{4} V(u_\alpha^\beta; r, \theta).$$

¿From the definition of the semi-group (U_t^ε), it is clear that $\|\partial_\alpha (U_\cdot^\varepsilon G)\|_0^2$ is uniformly bounded in ε. Moreover the diffusion coefficients $\bar{D}_{\alpha,\gamma}^{\beta,\nu}$ are chosen in such a way that, since we take for h_α^β the terms of the sequence $(h_\alpha^{(q),\beta})$ given in Theorem 2.1, we have

$$\lim_{q\to\infty} V(u_\alpha^{(q),\beta}; r, \theta) = 0.$$

We have shown so far that there exists a random variable R_ε^q which converges to 0 in $L^2(\mathbb{P}_\varepsilon^\mu)$ as $\varepsilon \to 0$ and then $q \to \infty$ such that

$$M_1^\varepsilon(t, G, h) = \xi^\varepsilon\big(t, G\big) - \xi^\varepsilon\big(0, G\big) - \int_0^t ds\, \big\langle \mathcal{D}^*(U_s^\varepsilon G), \tilde{I}(\eta_s(0)) \big\rangle_{0,L} + R_\varepsilon^q. \tag{4.5}$$

We would like to have instead of the third term in (4.5) a term of the form

$$\big\langle U_s^\varepsilon(HG), \tilde{I}(\eta_s(0)) \big\rangle_{0,L}$$

for some suitable operator H that we could then rewrite as $\xi^\varepsilon(s, HG)$, so to identify the limiting martingale problem. We proceed in the following way:

$$\big\langle \mathcal{D}^*(U_s^\varepsilon G), \tilde{I}(\eta_s(0)) \big\rangle_{0,L} = \big\langle U_s^\varepsilon (U_s^\varepsilon)^{-1} \mathcal{D}^* U_s^\varepsilon G, \tilde{I}(\eta_s(0)) \big\rangle_{0,L}.$$

By Lemma A.2 and Lemma A.3

$$\lim_{\varepsilon\to 0} \mathbb{E}_\varepsilon^\mu \left[\left(\int_0^t ds \left[\xi^\varepsilon\big(s, (U_s^\varepsilon)^{-1} \mathcal{D}^* U_s^\varepsilon G\big) - \xi^\varepsilon\big(s, \pi_{E^*}(\mathcal{D}^*) G\big) \right] \right)^2 \right] = 0.$$

Hence, noticing that $\pi_{E^*}(\mathcal{D}^*) = \pi_E(\mathcal{D})^*$, we have proved that there exists a random variable C_ε^q which converges to 0 in $L^2(\mathbb{P}_\varepsilon^\mu)$ when $\varepsilon \to 0$ and then $q \to \infty$ such that

$$M_1^\varepsilon(t, G, h) = \xi^\varepsilon(t, G) - \xi^\varepsilon(0, G) - \int_0^t ds\, \xi^\varepsilon\big(s, \pi_E(\mathcal{D})^* G\big) + C_\varepsilon^q.$$

We now compute the compensator γ_2^ε in (4.2). We first remark that

$$\gamma_2^\varepsilon(t,G,h) = \big(\varepsilon^{-2}\mathcal{L}\big)\big(\zeta^\varepsilon(t,G,h)^2\big) - 2\zeta^\varepsilon(t,G,h)\big(\varepsilon^{-2}\mathcal{L}\big)\zeta^\varepsilon(t,G,h).$$

We introduce the operator $\mathcal{L}^{(2)} = \mathcal{L}^{\mathrm{ex},(2)} + \mathcal{L}^{\mathrm{c},(2)}$ for local functions f and g as

$$\begin{aligned}\mathcal{L}^{\mathrm{ex},(2)}(f;g) &= \mathcal{L}^{\mathrm{ex}}(fg) - f\mathcal{L}^{\mathrm{ex}}g - g\mathcal{L}^{\mathrm{ex}}f,\\ \mathcal{L}^{\mathrm{c},(2)}(f;g) &= \mathcal{L}^{\mathrm{c}}(fg) - f\mathcal{L}^{\mathrm{c}}g - g\mathcal{L}^{\mathrm{c}}f.\end{aligned}$$

Then we obtain

$$\gamma_2^\varepsilon(t,G,h) = Y_1^\varepsilon(t,G) + Y_2^\varepsilon(t,G,h) + Y_3^\varepsilon(t,G,h),$$

where

$$Y_1^\varepsilon(t,G) = \varepsilon\sum_{\beta,\nu=0}^{4}\sum_{x,y}(U_t^\varepsilon G)_\beta(\varepsilon x)(U_t^\varepsilon G)_\nu(\varepsilon y)\mathcal{L}^{\mathrm{ex},(2)}\big(\tilde I_\beta(\eta_x);\tilde I_\nu(\eta_y)\big),$$

$$Y_2^\varepsilon(t,G,h) = -2\varepsilon^2\sum_{\alpha=1}^{3}\sum_{\beta,\nu=0}^{4}\sum_{x,y}(U_t^\varepsilon G)_\beta(\varepsilon x)\partial_\alpha(U_t^\varepsilon G)_\nu(\varepsilon y)\mathcal{L}^{\mathrm{ex},(2)}\big(\tilde I_\beta(\eta_x);\tau_y h_\alpha^\nu)\big),$$

$$Y_3^\varepsilon(t,G,h) = \varepsilon^3\sum_{\alpha,\gamma=1}^{3}\sum_{\beta,\nu=0}^{4}\sum_{x,y}\partial_\alpha(U_t^\varepsilon G)_\beta(\varepsilon x)\partial_\gamma(U_t^\varepsilon G)_\nu(\varepsilon y)\mathcal{L}^{(2)}\big(\tau_x h_\alpha^\beta;\tau_y h_\gamma^\nu)\big).$$

¿From the explicit formulas (4.7), (4.8) and (4.9) that we will get below for $Y_i(t,G)$, $i=1,2,3$ and the use of Lemma 4.1 it is easy to see that

$$\int_0^t ds\,\Big(\gamma_2^\varepsilon(s,h,G) - \mathbb{E}_\varepsilon^\mu\left[\gamma_2^\varepsilon(s,h,G)\right]\Big)$$

is converging to 0 in $L^2(\mathbb{P}_\varepsilon^\mu)$. So, all we need to compute is $\mathbb{E}_\varepsilon^\mu\big[\gamma_2^\varepsilon(t,h,G)\big]$.

Notice that

$$\begin{aligned}\mathcal{L}^{\mathrm{ex},(2)}(f;g) &= \sum_{x,e,v} b(x,x+e,v)\nabla_{x,x+e,v}f\,\nabla_{x,x+e,v}g,\\ \mathcal{L}^{\mathrm{c},(2)}(f;g) &= \sum_{x,q}\nabla_{x,q}f\,\nabla_{x,q}g,\end{aligned} \tag{4.6}$$

with $\nabla_{x,x+e,v}f = f(\eta^{x,x+e,v}) - f(\eta)$, $\nabla_{x,q}f = f(\eta^{x,q}) - f(\eta)$ and

$$b(x,y,v) = \Big(\chi + \frac{1}{2}v\cdot(y-x)\Big)\eta(x,v)(1-\eta(y,v)).$$

So, if we let $\phi_0(v) = 1$, $\phi_\beta(v) = v_\beta$ for $\beta = 1, 2, 3$ and $\phi_4(v) = \frac{1}{2}|v|^2$, a straightforward computation leads to the following

$$\begin{aligned} Y_1^\varepsilon(t, G) =& \varepsilon^3 \sum_{\alpha=1}^{3} \sum_{\beta,\nu=0}^{4} \sum_x \Big(\partial_\alpha (U_t^\varepsilon G)_\beta \partial_\alpha (U_t^\varepsilon G)_\nu \Big)(\varepsilon x) \times \\ & \sum_v \Big[b(x, x+e_\alpha, v) + b(x+e_\alpha, x, v) \Big] \phi_\beta(v) \phi_\nu(v) + O(\varepsilon). \end{aligned} \tag{4.7}$$

Therefore

$$\begin{aligned} \mathbb{E}^\mu \Big[Y_1^\varepsilon(t, G) \Big] &= \varepsilon^3 2\chi \sum_{\alpha=1}^{3} \sum_{\beta,\nu=0}^{4} \sum_x C_{\beta,\nu} \Big(\partial_\alpha (U_t^\varepsilon G)_\beta \partial_\alpha (U_t^\varepsilon G)_\nu \Big)(\varepsilon x) + O(\varepsilon) \\ &= -2\chi \big\langle (U_t^\varepsilon G), \Delta C (U_t^\varepsilon G) \big\rangle_0 + O(\varepsilon), \end{aligned}$$

where Δ is the vectorial Laplacian operator defined as $(\Delta G)_\beta = \Delta G_\beta$. Observe that

$$\big\langle (U_t^\varepsilon G), \Delta C (U_t^\varepsilon G) \big\rangle_0 = \big\langle G, e^{-\frac{t}{\varepsilon} E} C e^{-\frac{t}{\varepsilon} E^*} \Delta G \big\rangle_0 = \big\langle G, C e^{\frac{t}{\varepsilon} E^*} e^{-\frac{t}{\varepsilon} E^*} \Delta G \big\rangle_0,$$

where we have used that $EC = -CE^*$. In conclusion, $Y_1^\varepsilon(t, G)$ converges in $L^2(\mathbb{P}_\varepsilon^\mu)$ to $-2\chi \big\langle G, C\Delta G \big\rangle_0$.

We get in the same way

$$\begin{aligned} Y_2^\varepsilon(t, G, h) =& 2\varepsilon^3 \sum_{\alpha,\gamma=1}^{3} \sum_{\beta,\nu=0}^{4} \sum_x \Big(\partial_\alpha (U_t^\varepsilon G)_\beta \partial_\gamma (U_t^\varepsilon G)_\nu \Big)(\varepsilon x) \times \\ & \sum_v \Big[b(x, x+e_\alpha, v) - b(x+e_\alpha, x, v) \Big] \phi_\beta(v) \nabla_{x, x+e_\alpha, v} \Gamma(h_\alpha^\nu) + O(\varepsilon), \end{aligned} \tag{4.8}$$

where $\Gamma(h_\alpha^\nu) = \sum_x \tau_x h_\alpha^\nu$. Since μ is invariant for the jump generator of particles with a given velocity, it is easy to check that

$$\mathbb{E}^\mu \Big[\eta(x, v)(1 - \eta(x+e_\alpha, v)) \nabla_{x, x+e_\alpha, v} \Gamma(h_\alpha^\nu) \Big] = 0,$$

which implies that the time integral of $Y_2^\varepsilon(t, G, h)$ converges to 0 in $L^2(\mathbb{P}_\varepsilon^\mu)$ by Lemma 4.1.

The last term $Y_3^\varepsilon(t, G, h)$ is given by

$$\begin{aligned} & \varepsilon^3 \sum_{\alpha,\gamma=1}^{3} \sum_{\beta,\nu=0}^{4} \sum_x \Big(\partial_\alpha (U_t^\varepsilon G)_\delta \partial_\gamma (U_t^\varepsilon G)_\nu \Big)(\varepsilon x) \times \\ & \Bigg[\sum_{v; |e|=1} b(x, x+e, v) \nabla_{x, x+e, v} \Gamma(h_\alpha^\beta) \nabla_{x, x+e, v} \Gamma(h_\gamma^\nu) + \sum_q \nabla_{x,q} \Gamma(h_\alpha^\beta) \nabla_{x,q} \Gamma(h_\gamma^\nu) \Bigg]. \end{aligned} \tag{4.9}$$

By using again Lemma 4.1 it is immediate to show that the time integral of Y_3^ε converges in $L^2(\mathbb{P}_\varepsilon^\mu)$ to its average that we are going to compute.

Let $\mathcal{L}_s^{\mathrm{ex}}$ be respectively the symmetric part of $\mathcal{L}^{\mathrm{ex}}$ in $L^2(\mu)$. It is easy to check that for any local function f and g

$$\sum_{v;|e|=1} \mathbb{E}^\mu \Big[\eta(0,v)(1-\eta(e,v))\nabla_{0,e,v}\Gamma(f)\nabla_{0,e,v}\Gamma(g)\Big] = 2\mathbb{E}^\mu\Big[\Gamma(f)(-\mathcal{L}_s^{\mathrm{ex}})g\Big]$$

and

$$\sum_q \mathbb{E}^\mu\Big[\nabla_{0,q}\Gamma(f)\nabla_{0,q}\Gamma(g)\Big] = 2\mathbb{E}^\mu\Big[\Gamma(f)(-\mathcal{L}^{\mathrm{c}})g\Big].$$

Therefore

$$\mathbb{E}^\mu\left[Y_3^\varepsilon(t,G)\right] = 2\varepsilon^3 \sum_x \mathbb{E}^\mu\Big[\Gamma\big(\partial(U_t^\varepsilon G)(\varepsilon x)\cdot h\big)(-\mathcal{L}_s)(\partial(U_t^\varepsilon G)(\varepsilon x)\cdot h)\Big],$$

where $\mathcal{L}_s$ and $a\cdot b$ were defined after (2.10). Remember that the functions $h=(h_\alpha^\beta)$ are chosen as the terms of the sequence $(h_\alpha^{(q),\beta})$ in Theorem 2.1. Lemma 2.2 asserts that

$$\lim_{q\to\infty} \mathbb{E}^\mu\Big[\Gamma\big(a\cdot h^{(q)}\big)(-\mathcal{L}_s)(a\cdot h^{(q)})\Big] = 2a\cdot(\bar{D}C)a.$$

Hence,

$$\begin{aligned}\mathbb{E}^\mu\left[Y_3^\varepsilon(t,G,h)\right] &= 2\varepsilon^3\sum_x \underline{\partial}(U_t^\varepsilon G)\cdot(\bar{D}C)\underline{\partial}(U_t^\varepsilon G) + o_q(1)\\ &= -2\big\langle U_t^\varepsilon G, (\bar{\mathcal{D}}C)(U_t^\varepsilon G)\big\rangle_0 + o_q(1) + O(\varepsilon),\end{aligned}$$

where, denoting by $\bar{D}_{\alpha,\gamma}$ the matrix $(\bar{D}_{\alpha,\gamma}^{\beta,\nu})_{\beta,\nu=0\dots,4}$,

$$\bar{\mathcal{D}}G = \sum_{\alpha,\gamma=1}^{3} \bar{D}_{\alpha,\gamma}\partial_\alpha\partial_\gamma G.$$

With the property $\bar{\mathcal{D}}C = C\bar{\mathcal{D}}^*$, we get

$$\big\langle U_t^\varepsilon G, (\bar{\mathcal{D}}C)(U_t^\varepsilon G)\big\rangle_0 = \big\langle e^{-\frac{t}{\varepsilon}E^*}G, (C\bar{\mathcal{D}}^*)e^{-\frac{t}{\varepsilon}E^*}G\big\rangle_0 = \big\langle G, Ce^{-\frac{t}{\varepsilon}E^*}\bar{\mathcal{D}}^*e^{-\frac{t}{\varepsilon}E^*}G\big\rangle_0$$

and by Lemma A.2

$$\lim_{\varepsilon\to 0}\int_0^t ds\ \exp\big(\frac{s}{\varepsilon}E^*\big)\bar{\mathcal{D}}^*\exp\big(-\frac{s}{\varepsilon}E^*\big) = t\,\pi_{E^*}(\bar{\mathcal{D}}^*) = t\,\pi_E(\bar{\mathcal{D}})^*,$$

so

$$\lim_{\varepsilon\to 0}\mathbb{E}^\mu\Big[\int_0^t ds\, Y_3^\varepsilon(s,G,h)\Big] = -2t\,\big\langle G, C\,\pi_E(\bar{\mathcal{D}})^*G\big\rangle_0.$$

To summarize, we have proved that there exists a random variable R^q_ε vanishing in $L^2(\mathbb{P}^\mu_\varepsilon)$ in the limits $\varepsilon \to 0$ and then $q \to \infty$ such that

$$\begin{aligned} M^\varepsilon_2(t,G,h) &= \big(M^\varepsilon_1(t,G,h)\big)^2 + 2t\big\langle G, C\,\pi_E(\bar{\mathcal{D}})^* G\big\rangle_0 + 2t\big\langle G, C\,\Delta G\big\rangle_0 + R^q_\varepsilon \\ &= \big(M^\varepsilon_1(t,G,h)\big)^2 + 2t\big\langle G, C\,\pi_E(\mathcal{D})^* G\big\rangle_0 + R^q_\varepsilon. \end{aligned}$$

This completes the proof of Theorem 3.1, once the Boltzmann-Gibbs principle and Lemmas A.2 and A.3 are proved.

5. The Boltzmann-Gibbs principle.

Since we closely follow the strategy proposed in [CLO] to prove Theorem 3.2, we will only focus our attention to the points where non trivial changes are necessary.

One of the ingredients in the proof is the equivalence of ensembles, which is classical for Bernoulli product measures but, as far as we know, is not in our case. We state below a weaker statement which will suffice to our purpose.

For a given chemical potential $n \in \mathbb{R}^5$, let $M(n) = (M_0(n), \dots, M_4(n))$ be defined as $M_\beta(n) = \mathbb{E}^{\mu_{L,n}}[I_\beta(\eta_0)]$. If we put $A = M(\mathbb{R}^5)$, it is easy to verify that $n \mapsto M(n)$ is a C^1 diffeomorphism from $\mathbb{R}^5$ onto A, in particular the inverse function $M \mapsto n(M)$ is continuous on A. Given $a > 0$, we introduce the set A^a of $M \in A$ such that, $|n(M) - n_0| \le a$, with $n_0 = (r, 0, 0, 0, \theta)$ the equilibrium chemical potentials. We denote by $\bar\mu_{L,M}$ the grand canonical measure $\mu_{L,n(M)}$ which satisfies therefore $E^{\bar\mu_{L,M}}[I_\beta(\eta_0)] = M_\beta$ for $\beta = 0, \dots, 4$.

Recall that $\bar I^+_L(\eta) = \big(\bar I^+_{0,L}(\eta), \dots, \bar I^+_{4,L}(\eta)\big)$ are the empirical averages of the conserved quantities in Λ_L. For any particle configuration η in Ω_L, we call $\bar N^v_L(\eta)$, $v \in \mathcal{V}$, the average number of particles with velocity v in Λ_L.

Also recall the definition of $\phi_\beta(v)$ before (4.6). Given $k = (k_v)_{v\in\mathcal{V}}$, we set $I_\beta(k) = \sum_v \phi_\beta(v) k_v$ and $I^+(k) = \big(I_0(k), \dots, I_4(k)\big)$.

Lemma 5.1 (Equivalence of ensembles) *Let h be a local function. Then there exists a constant $c = c(h,a)$ such that*

$$\Big|\mathbb{E}^\mu\big[h\big|\bar I^+_L = M\big] - E^{\bar\mu_{L,M}}[h]\Big| \le c\,\varepsilon^3$$

uniformly in $M \in A^a$.

Proof. Let ℓ be the number of velocities in $\mathcal{V}$ and denote by ν_α, $\alpha = (\alpha_v)_{v\in\mathcal{V}}$, the product measure on Ω_L of Bernoulli measures with parameters $\alpha = (\alpha_v)_v$, i.e. $E^{\nu_\alpha}[\eta(x,v)] = \alpha_v$. A straightforward extension of the classical strong equivalence of ensembles asserts that for any local function g,

$$\Big|E^{\nu_\alpha}\big[g\big|\bar N^v_L = k_v, v \in \mathcal{V}\big] - E^{\nu_k}[h]\Big| \le C(h)\varepsilon^3 \tag{5.1}$$

uniformly in $k = (k_v)_{v\in\mathcal{V}} \in B_L = \{0, L^{-3}, \dots, 1\}^\ell$.

We first compute the term $\mathbb{E}^{\mu}\left[h\middle|\bar{I}_L^+ = M\right]$. Since this expectation does not depend on the chemical potential (here n_0), it is equal to $E^{\nu_{1/2}}\left[h\middle|\bar{I}_L^+ = M\right]$ with the obvious abuse of notation $1/2 = (1/2, \ldots, 1/2)$. Therefore, from (5.1),

$$\mathbb{E}^{\mu}\left[h\middle|\bar{I}_L^+ = M\right] = \frac{\sum_{k\in B_L, \bar{I}_L^+(k)=M} \nu_{1/2}\big(\bar{N}_L^v = k_v, v\in\mathcal{V}\big)E^{\nu_k}[h]}{\sum_{k\in B_L, \bar{I}_L^+(k)=M} \nu_{1/2}\big(\bar{N}_L^v = k_v, v\in\mathcal{V}\big)} + O(\varepsilon^3). \tag{5.2}$$

Since the particles with different velocities are independent

$$\nu_{1/2}\big(\bar{N}_L^v = k_v, v\in\mathcal{V}\big) = \prod_{v\in\mathcal{V}} \nu_{1/2}\big(\bar{N}_L^v = k_v\big)$$

and the asymptotics of a single term in the product above is given by the Stirling formula

$$\nu_{1/2}\big(\bar{N}_L^v = k\big) = \frac{1}{\sqrt{2\pi\varepsilon^{-3}k(1-k)}}\exp\Big[-\varepsilon^{-3}(s(k)+\log 2)\Big]\Big(1+O\Big(\frac{\varepsilon^3}{k(1-k)}\Big)\Big), \tag{5.3}$$

where $s(k) = k\log k + (1-k)\log(1-k)$ is the entropy. In particular, if $(k_v)_v$ belongs to $B_L^\delta := B_L \cap [\delta, 1-\delta]^\ell$ for some small $\delta > 0$, then

$$\nu_{1/2}\big(\bar{N}_L^v = k_v, v\in\mathcal{V}\big) = \frac{1}{\sqrt{(2\pi\varepsilon^{-3})^\ell \prod_v k_v(1-k_v)}}\exp\big[-\varepsilon^{-3}\sum_v(s(k_v)+\log 2)\big]\big(1+O(\varepsilon^3)\big).$$

The fact that the entropy is convex suggests to use the Laplace method to derive the asymptotics of both terms in the ratio (5.2). This is the aim of Lemma 5.2 below which is stated in the $\ell = 1$ case without any constraint on k, nevertheless the generalization to higher dimension with constrains is easy because, up to a linear change of variables $k \mapsto k'$, the sums over k in (5.2) with constraints can be written as a sum without constraint over k' in a cube of dimension $\ell - 5$ (5 is the number of linear conditions $I^+(k) = M$). Therefore, we have

$$\sum_{k\in B_L^\delta, \bar{I}^+(k)=M} \nu_{1/2}\big(\bar{N}_L^v = k_v, v\in\mathcal{V}\big)E^{\nu_k}[h]$$
$$= \frac{T_\varepsilon}{\sqrt{(2\pi\varepsilon^{-3})^\ell\prod_v k_v^*(1-k_v^*)}}\exp\Big[-\varepsilon^{-3}\sum_v(s(k_v^*)+\log 2)\Big]E^{\nu_{k^*}}[h]\big(1+O(\varepsilon^3)\big),$$

where k^* is the minimizer of $\sum_v(s(k_v)+\log 2)$ under the constraints $k\in[\delta, 1-\delta]^n$ and $I^+(k) = M$,

$$T_\varepsilon = \sum_{\substack{k\in B_L^\delta, \bar{I}^+(k)=M\\ |k-k^*|\le\varepsilon^{3\alpha}}} \exp\Big[-\frac{\sum_v s''(k_v^*)}{2}\varepsilon^{-3}(k-k^*)^2\Big].$$

with $0 < \alpha < 1/2$. Notice that this result holds provided that the minimizer k^* satisfies $k^* \in]\delta, 1-\delta[^\ell$, that will be shown below. As a consequence, the ratio (5.2) is equal to

$$E^{\nu_{k^*}}[h]\big(1+O(\varepsilon^3)\big),$$

provided that the contributions from "bad" configurations are negligible.

Let κ be the minimizer of $\sum_v (s(k_v)+\log 2)$ under the constraints $k \in [0,1]^\ell$, $I^+(k) = M$. ¿From Lagrange optimization theorem, κ has to minimize the function

$$\sum_v (s(k_v) + \log 2) + \sum_{\beta=0}^{4} \gamma_\beta \sum_v \big(\phi_\beta(v) k_v - M_\beta\big),$$

where ϕ_β have been defined in the line before (4.7) and (γ_β) are Lagrangian multipliers. So the minimizer satisfies

$$s'(\kappa_v) = \sum_{\beta=0}^{4} \gamma_\beta \phi_\beta(v), \quad v \in \mathcal{V}.$$

Since the derivative of the entropy $s'(\alpha)$ is equal to the associated chemical potential $\lambda = \log \frac{\alpha}{1-\alpha}$, we have $\nu_\kappa = \mu_{L,\gamma}$, $\gamma = (\gamma_0, \dots, \gamma_4)$ but the constraint $I^+(\kappa) = M$ implies that $\gamma = n(M)$ that is to say $\nu_\kappa = \bar\mu_{L,M}$ and in particular $E^{\nu_\kappa}[h] = E^{\bar\mu_{L,M}}[h]$. Moreover, if $\lambda_v = \log \frac{\kappa_v}{1-\kappa_v}$ is the chemical potential related to κ_v, then we have

$$\lambda_v = \sum_{\beta=0}^{4} \phi_\beta(v) n_\beta(M).$$

¿From the assumption $M \in A^a$, the previous equality implies that we can choose $\delta > 0$ small enough such that $\kappa \in]2\delta, 1-2\delta[^\ell$ uniformly in $M \in A^a$. Such a choice of δ implies that $k^* = \kappa$.

So the lemma will be proved if we finally show that the contribution of the "bad" k ($k \in B_L \setminus [\delta, 1-\delta]^\ell$) inside the sums in the numerator and denominator of the ration (5.2) is irrelevant with respect to the leading term. From Stirling formula (5.3), there is $c > 0$ such that

$$\sum_{k \notin B_L^\delta, \bar I^+(k) = M} \nu_{1/2}\big(\bar N_L^v = k_v, v \in \mathcal{V}\big) \le \exp\big[-\varepsilon^{-3} \sum_v (s(k_v) + \log 2) - c \log \varepsilon\big].$$

¿From the discussion above, there exists $b > 0$ such that $\sum_v s(k_v) \ge \sum_v s(\kappa_v) + b$, therefore

$$\sum_{k \notin B_L^\delta, \bar I^+(k) = M} \nu_{1/2}\big(\bar N_L^v = k_v, v \in \mathcal{V}\big) \le c \exp\big[-\varepsilon^{-3} \sum_v (s(\kappa_v) + \log 2)\big] \exp[-b\varepsilon^{-3}/2].$$

□

Lemma 5.2 *Let ψ and ϕ be smooth functions on $[0,1]$, ψ concave, ϕ non negative. Assume that the maximizer θ of ψ is in $]0,1[$, then*

$$\sum_{i=0}^{N} \phi\big(\frac{i}{N}\big) \exp\big[N\psi\big(\frac{i}{N}\big)\big] = S_N\big(\alpha, \frac{\psi''(\theta)}{2}\big) \phi(\theta) \exp[N\psi(\theta)] \Big(1 + O\big(\frac{1}{N}\big)\Big),$$

with θ the maximizer of ψ and

$$S_N(\alpha, a) = \sum_{|i-N\theta|\le N^{1-\alpha}} \exp\Big[a\frac{(i-N\theta)^2}{N}\Big], \ 0<\alpha<\frac{1}{2}.$$

Proof. We start by factorizing the leading term $\exp[N\psi(\theta)]$ in the sum. For simplicity call

$$U_N(i) = \phi\big(\frac{i}{N}\big)\exp N\big[\psi\big(\frac{i}{N}\big) - \psi(\theta)\big].$$

¿From the assumption on θ, if $\delta > 0$ there exists a constant $c(\delta) > 0$ such that

$$\sum_{|i-N\theta|>N\delta} U_N(i) \le \exp[-Nc(\delta)].$$

Moreover, choosing δ small enough ensures that $\psi(x) - \psi(\theta) \le -c(x-\theta)^2$ provided that $|x-\theta| \le \delta$, where $c > 0$ is a constant which will change from line to line. Then, given $0 < \alpha < \frac{1}{2}$,

$$\sum_{N^{1-\alpha}<|i-N\theta|\le N\delta} U_N(i) \le \exp[-N^{1-2\alpha}c(\delta)]. \tag{5.4}$$

So the main contribution is coming from $\sum_{|i-N\theta|\le N^{1-\alpha}} U_N(i)$. Using Taylor expansion, we see that in this range of i's,

$$U_N(i) = \exp\Big[b_0\frac{j^2}{N}\Big]\left(a_0 + a_1\Big(\frac{j}{N}\Big) + a_0b_1N\Big(\frac{j}{N}\Big)^3 + O\Big(\Big(\frac{j}{N}\Big)^2\Big) + O\Big(N\Big(\frac{j}{N}\Big)^4\Big)\right),$$

where $j = \frac{i}{N} - \theta$, $a_0 = \phi(\theta)$, $a_1 = \phi'(\theta)$, $b_0 = \frac{\psi''(\theta)}{2} < 0$ and $b_1 = \frac{\psi^{(3)}(\theta)}{6}$. By the "almost oddness" of j,

$$\sum_{|i-N\theta|\le N^{1-\alpha}} \exp\Big[b_0\frac{j^2}{N}\Big]\left(a_1\Big(\frac{j}{N}\Big) + a_0b_1N\Big(\frac{j}{N}\Big)^3\right) = O\big(\frac{1}{N}\big)S_N(\alpha, b_0). \tag{5.5}$$

We also remark

$$\sum_{|i-N\theta|\le N^{1-\alpha}} \Big(\frac{j}{N}\Big)^2 \exp\Big[b_0\frac{j^2}{N}\Big] \le \frac{c}{N}\sum_{|i-N\theta|\le N^{1-\alpha}} \exp\Big[b_0\frac{j^2}{2N}\Big],$$

so that

$$\sum_{|i-N\theta|\le N^{1-\alpha}} \Big(\frac{j}{N}\Big)^2 \exp\Big[b_0\frac{j^2}{N}\Big] = O\big(\frac{1}{N}\big)S_N\big(\alpha, \frac{b_0}{2}\big), \tag{5.6}$$

We get in the same way

$$\sum_{|i-N\theta|\leq N^{1-\alpha}} N\Big(\frac{j}{N}\Big)^4 \exp\Big[b_0\frac{j^2}{N}\Big] = O\big(\frac{1}{N}\big)S_N\big(\alpha,\frac{b_0}{2}\big). \tag{5.7}$$

Finally, comparing $N^{-1/2}S_N(\alpha,a)$ $(a<0)$ with the integral of a Gaussian, it is easy to check that $S_N(\alpha,a)=c(a)\sqrt{N}+O(1)$, therefore

$$S_N\big(\alpha,\frac{b_0}{2}\big) = O(1)S_N(\alpha,b_0). \tag{5.8}$$

Putting together formulas (5.4) to (5.8), the Lemma is proved.
□

Even if the equivalence of ensembles that we stated in Lemma 5.1 is weaker than the classical one, it is enough to prove the following result which is actually the only estimate needed in the proof of Boltzmann-Gibbs principle.

Corollary 5.3 *If $h\in\mathcal{G}$ is a local function then*

$$\mathbb{E}^\mu\Big[\big(\mathbb{E}^\mu\big[h\big|\bar{I}_L^+=M\big]\big)^2\Big]\leq c\varepsilon^6.$$

Proof. Let $\hat{h}=\mathbb{E}^\mu\big[h\big|\bar{I}_L^+\big]$, $\tilde{h}(M)=E^{\bar{\mu}_{L,M}}[h]$ and consider the decomposition

$$\mathbb{E}^\mu\big[\hat{h}^2\big]\leq 2\mathbb{E}^\mu\Big[\big(\hat{h}-\tilde{h}(\bar{I}_L^+)\big)^2\Big]+2\mathbb{E}^\mu\Big[\big(\tilde{h}(\bar{I}_L^+)\big)^2\Big].$$

Since h is in $\mathcal{G}$, we have

$$\tilde{h}(m)=0 \quad\text{and}\quad \frac{\partial\tilde{h}}{\partial M_\beta}\Big|_{M=m}=0,$$

with m_β the equilibrium values of $I_\beta(\eta_0)$. Therefore

$$\big|\tilde{h}(\bar{I}_L^+)\big|\leq c\sum_{\beta,\nu=0}^{4}\big|\big(\bar{I}_{\beta,L}(\eta)-m_\beta\big)\big(\bar{I}_{\nu,L}(\eta)-m_\nu\big)\big|.$$

Hence $\mathbb{E}_\varepsilon^\mu\big[(\tilde{h}(\bar{I}_L^+))^2\big]\leq c\varepsilon^6$.

On the other hand, it results from Lemma 5.1 that for any $a>0$

$$\mathbb{E}^\mu\Big[\big(\hat{h}-\tilde{h}(\bar{I}_L^+)\big)^2\Big]\leq c(a)\varepsilon^6+cP^\mu\big[I_L^+\notin A^a\big].$$

¿From the continuity of the function $M \mapsto n(M)$, there exists $b > 0$ such that

$$P^{\mu}\left[I_L^+ \notin A^a\right] \leq P^{\mu}\left[|I_L^+ - m| > b\right].$$

Finally, since $I_L^+ = \frac{1}{|\Lambda_L|}\sum_x I^+(\eta_x)$ with $I^+(\eta_x)$ i.i.d. random vectors with finite exponential moments and expectation m under μ, a large deviation estimate provides

$$P^{\mu}\left[|I_L^+ - m| > b\right] \leq \exp(-c\varepsilon^3).$$

□

The first result used in [CLO] (Lemma 4.3) before establishing Boltzmann-Gibbs is a general estimate bounding the equilibrium expectation of the squared time integral of zero mean functions of Markov processes by their $\|\cdot\|_{-1}$ norm. More precisely, if X is a Markov process on the finite state space $\mathcal{E}$ with generator L and ergodic invariant measure π, then there exists a (universal) constant $c > 0$ such that for any function $f : [0,T] \times \mathcal{E} \to \mathbb{R}$ satisfying $\mathbb{E}^{\pi}[f(t, X_t)] = 0$ for any $t \in [0,T]$, we have

$$\mathbb{E}^{\pi}\left[\sup_{0\leq t\leq T}\left(\int_0^t ds\, f(s, X_s)\right)^2\right] \leq c\int_0^T ds\, \|f(s,\cdot)\|_{-1}^2, \tag{5.9}$$

where

$$\|f\|_{-1}^2 = \sup_g\{\langle f, g\rangle + \langle f, L^s g\rangle\}$$

and L^s is the symmetric part of L in $L^2(\pi)$.

The next lemma (Lemma 4.4 in [CLO]) is needed to control remainder terms in the proof of the Boltzmann-Gibbs principle.

Lemma 5.4 *For any local function $h \in \mathcal{G}$, there exists a constant $c(h) > 0$ such that for any subset B of Λ_L, any smooth function $G : [0,T] \times \mathbb{T}_3 \to \mathbb{R}$ and ε small enough,*

$$\mathbb{E}_{\varepsilon}^{\mu}\left[\sup_{0\leq t\leq T}\left(\int_0^t ds\, \varepsilon^{3/2-1}\sum_{x\in B} G(s,\varepsilon x))\tau_x h(\eta_s)\, ds\right)^2\right] \leq c(h)(1+T)\int_0^T ds\, \varepsilon^3 \sum_{x\in B} G^2(s,\varepsilon x). \tag{5.10}$$

Proof. Following [EMY2] (section 4), we introduce an alternative representation for the particle configuration $\eta_x = \big(\eta(x,v)\big)_{v\in\mathcal{V}}$ at site x: one can find $n(n-5)$ numbers $c_\beta(v)$, $\beta = -n+5, \ldots, -1$ and $v \in \mathcal{V}$, such that if we put

$$I_\beta(\eta_x) = \sum_{v\in\mathcal{V}} c_\beta(v)\eta(x,v),$$

then the map $\eta_x \mapsto (I_\beta(\eta_x))_{\beta=-n+5,\ldots,4}$ is one to one. Moreover the coefficients $c_\beta(v)$ can be chosen in such a way that the covariances $\mathbb{E}^{\mu}[I_\beta(\eta_x); I_\nu(\eta_x)]$, $\beta \neq \nu$, vanish (except $\beta, \nu \in \{0,4\}$). We also introduce the variables $\tilde{I}_\beta(\eta_x) = I_\beta(\eta_x) - \mathbb{E}^{\mu}[I_\beta(\eta_x)]$.

Let $\mathcal{G}^{ex}$ be the space of functions h such that $\mathbb{E}^\mu[h] = 0$ and $\sum_x \mathbb{E}^\mu[h; I_\beta(\eta_x)] = 0$ for any $\beta = -n+5, \dots, 4$. The integration by parts lemma valid for ASEP (Lemma 6.1 in [EMY1]) easily generalizes to a superposition of ASEP.

We now turn to the proof. Fix $h \in \mathcal{G}$, we can find coefficients $(a_\beta)_{\beta<0}$ such that $h - \sum_{\beta<0} a_\beta \bar{I}_{\beta,L}$ is in $\mathcal{G}^{ex}$, where $\bar{I}_{\beta,L}(\eta) = |\Lambda_L|^{-1} \sum_x I_\beta(\eta_x)$. Therefore, it is enough to prove the lemma in the case where $h \in \mathcal{G}^{ex}$ and in the case where $h = \bar{I}_{\beta,L}(\eta)$. The first case is a straightforward generalization of Lemma 4.4 in [CLO] since the integration by parts formula is valid in $\mathcal{G}^{ex}$. In the second case, denote by $\hat{I}_\beta(\eta_x)$ ($\beta < 0$ fixed) the conditional expectation of $I_\beta(\eta_x)$ with respect to the empirical averages of the conserved quantities $\bar{I}_L^+$. Then the left hand side of (5.10) is bounded above (up to a factor 2) by the sum of the two terms

$$\mathbb{E}_\varepsilon^\mu \left[\sup_{0\le t\le T} \left(\int_0^t ds\, \varepsilon^{3/2-1} \bar{G}_s^B \sum_x \big(I_\beta(\eta_x) - \hat{I}_\beta(\eta_x) \big)\, ds \right)^2 \right], \tag{5.11}$$

where $\bar{G}_s^B := \varepsilon^3 \sum_{x\in B} G(s, \varepsilon x)$, and

$$\mathbb{E}_\varepsilon^\mu \left[\sup_{0\le t\le T} \left(\int_0^t ds\, \varepsilon^{3/2-1} \sum_{x\in B} G(s,\varepsilon x) \hat{I}_\beta(\eta_0)\, ds \right)^2 \right], \tag{5.12}$$

¿From the inequality (5.9), (5.11) is less or equal to

$$c\, V_L\big(\tilde{I}_\beta(\eta_0), r, \theta\big) \int_0^T ds\, \big(\bar{G}_s^B\big)^2$$

and by corollary 4.6 of [EMY2], $V\big(\tilde{I}_\beta(\eta_0), r, \theta\big) = \limsup_L V_L\big(\tilde{I}_\beta(\eta_0), r, \theta\big) < +\infty$. So (5.11) is bounded above by

$$c \int_0^T ds\, \varepsilon^3 \sum_{x\in B} G^2(s, \varepsilon x)$$

(c a positive constant). Finally, by stationarity of μ, the term (5.12) is less than

$$\varepsilon^{-5} \mathbb{E}^\mu\big[(\hat{I}_\beta(\eta_0))^2\big] T \int_0^T ds\, \big(\bar{G}_s^B\big)^2.$$

¿From Corollary 5.3, $\mathbb{E}^\mu\big[(\hat{I}_\beta(\eta_0))^2\big] \le c\,\varepsilon^6$ and (5.12) is going to zero as $\varepsilon \to 0$.
□

Finally, Corollary 5.3 and Lemma 5.4 allows to extend straightforwardly the proof of the Boltzmann-Gibbs principle given in section 4 of [CLO] and then to obtain Theorem 3.2.

We conclude this section by pointing out that the arguments for the proof of tightness (section 5 of [CLO]) can be easily adapted to our case. Notice that, up to now, we did not need to have the supremum over time inside the expectation in the Boltzmann-Gibbs statement, however it is used in this part to control some terms arising in martingale compensators. So, we can state

Theorem 5.5 *The family of probability $(Q^\varepsilon)_{\varepsilon>0}$ on $D([0,T],\mathcal{H}_{-k_0})$ is tight since*

$$\lim_{M\to\infty}\lim_{\varepsilon\to 0}\mathbb{P}^\mu_\varepsilon\Big(\sup_{0\le t\le T}\|\xi^\varepsilon_t\|_{-k_0}>M\Big)=0 \tag{5.13}$$

and for any $a>0$

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\mathbb{P}^\mu_\varepsilon\Big(\sup_{\substack{|s-t|<\delta\\ 0\le s,t\le T}}\|\xi^\varepsilon_t-\xi^\varepsilon_s\|_{-k_0}>a\Big)=0. \tag{5.14}$$

Appendix

Let $\mathcal{A}$ be the space of $n\times n$ matrices with complex entries. $\mathcal{A}$ is a Hilbert space under the scalar product

$$(X,Y)=\sum_{1\le k,\ell\le n}\bar{X}_{k,\ell}Y_{k,\ell}.$$

Given a matrix A in $\mathcal{A}$ the projector Π_A is defined as the orthogonal projection onto $\mathcal{C}(A)$, the commutator space of A

$$\mathcal{C}(A)=\{M\in A:[M,A]=0\},\quad [M,A]:=MA-AM.$$

Lemma A.1 *Let A be a diagonalizable matrix,* $\mathrm{Sp}(A)\in i\mathbb{R}$. *Then, for any matrix M and $t>0$*

$$\lim_{\varepsilon\to 0}\frac{1}{t}\int_0^t ds\,\exp\big(\frac{s}{\varepsilon}A\big)M\exp\big(-\frac{s}{\varepsilon}A\big)=\Pi_A(M).$$

Proof. We follow the proof in [EP]. Let P be a non-singular matrix and R a real diagonal matrix such that $A=P^{-1}iRP$. Let $\{S_j, j=1,\dots,m\}$ be a partition of the integers $\{j=1,\dots,n\}$ such that

$$\begin{aligned}&R_k=R_\ell\quad\text{if}\quad k,\ell\in S_j\quad\text{for some } j,\\ &R_k\neq R_\ell\quad\text{otherwise,}\end{aligned}$$

where $R_j, j=1,\cdots,n$ are the eigenvalues of R. We define the bar operation in the following way: let $K=(K_{k,\ell})\in\mathcal{A}$ be

$$K_{k,\ell}=\begin{cases}1 & \text{if}\quad k,\ell\in S_j\quad\text{for some } j,\\ 0 & \text{otherwise.}\end{cases}$$

Then $\overline{M}$, $M \in \mathcal{A}$, is defined as

$$\overline{M}_{k,\ell} = K_{k,\ell} M_{k,\ell}.$$

Observe that $\overline{M}$ is the diagonal part of M in the simple case of R with distinct eigenvalues.

We have that

$$\exp\big(\frac{s}{\varepsilon}A\big) M \exp\big(-\frac{s}{\varepsilon}A\big) = P^{-1}\exp\big(\frac{is}{\varepsilon}R\big) P M P^{-1} \exp\big(-\frac{is}{\varepsilon}R\big)P.$$

It is proved in [EP] that

$$P^{-1}\overline{PMP^{-1}}P$$

is a projection onto $\mathcal{C}(A)$. Hence $\overline{M} = \Pi_R(M)$ because R is diagonal. Moreover, $\Pi_R(M) = \Pi_A(M)$ because R is diagonal. So it is enough to prove that for any M

$$\lim_{\varepsilon\to 0} \frac{1}{t}\int_0^t ds\ \exp\big(\frac{is}{\varepsilon}R\big) M \exp\big(-\frac{is}{\varepsilon}R\big) = \overline{M}.$$

In [EP] it is also shown that for any matrix M there exists a matrix S such that M can be decomposed as

$$M = \overline{M} + [S, R].$$

Since $\overline{M}$ commutes with R

$$\exp\big(\frac{is}{\varepsilon}R\big) M \exp\big(-\frac{is}{\varepsilon}R\big) = \overline{M} + \exp\big(\frac{is}{\varepsilon}R\big)[S,R]\exp\big(-\frac{is}{\varepsilon}R\big).$$

The second term on the r.h.s gives

$$\Big(\exp\big(\frac{is}{\varepsilon}R\big)[S,R]\exp\big(-\frac{is}{\varepsilon}R\big)\Big)_{k,\ell} = S_{k,\ell}(R_\ell - R_k)\exp\big(\frac{is}{\varepsilon}(R_k - R_\ell)\big),$$

where $S = (S_{k,\ell})$ and $R = (R_{k,\ell}) = (R_k \delta_{k,\ell})$. As a consequence,

$$\lim_{\varepsilon\to 0} \frac{1}{t}\int_0^t ds \exp\big(\frac{is}{\varepsilon}R\big)[S,R]\exp\big(-\frac{is}{\varepsilon}R\big) = 0.$$

Lemma A.2 *Let $\mathbb{E}$ be a first order differential operator such that its Fourier transform $\hat{\mathbb{E}}(k)$ satisfies $\mathrm{Sp}(\hat{\mathbb{E}}(k)) \in i\mathbb{R}$ for any k and let $\mathbb{D} = \sum_{\alpha,\gamma=1}^3 D_{\alpha,\gamma}\partial_\alpha\partial_\gamma$ be a second order differential operator, where $D = (D_{\alpha,\gamma}) = (D_{\alpha,\gamma}^{\beta,\nu})$ is a definite positive rank 2 tensor. Then there exists a definite positive second order differential operator $\pi_{\mathbb{E}}(\mathbb{D})$ such that for any G smooth*

$$\lim_{\varepsilon\to 0}\Big\| \int_0^t ds\ \Big[\exp\big(\frac{s}{\varepsilon}\mathbb{E}\big)\mathbb{D}\exp\big(-\frac{s}{\varepsilon}\mathbb{E}\big) - \pi_{\mathbb{E}}(\mathbb{D})\Big] G\Big\|_0 = 0.$$

Proof. Let $\hat{\mathbb{D}}(k)$ be the Fourier transform of $\mathbb{D}$

$$\hat{\mathbb{D}}(k) = -\sum_{\alpha,\gamma=1}^{3} D_{\alpha,\gamma}^{\beta,\nu} k_\alpha k_\gamma \hat{G}(k)$$

It is enough to prove that for any $t > 0$ and for any G smooth there exist a matrix $\hat{\pi}_E(\mathbb{D})$ such that

$$\lim_{\varepsilon\to 0}\Big\| \int_0^t ds \left[\exp\big(\frac{s}{\varepsilon}\hat{\mathbb{E}}\big)\hat{\mathbb{D}}\exp\big(-\frac{s}{\varepsilon}\hat{\mathbb{E}}\big) - \hat{\pi}_{\mathbb{E}}(\mathbb{D})\right] G\Big\|_0 = 0$$

where $\|\cdot\|_0$ is the usual norm in $L^2(\mathbb{T}_3, \mathbb{R}^5)$. Choosing $\hat{\pi}_{\mathbb{E}}(\mathbb{D}) = \pi_{\hat{\mathbb{E}}}(\hat{\mathbb{D}})$, that is an easy consequence of Lemma A.1 via dominated convergence theorem since, by assumption, $\hat{\mathbb{E}}$ is diagonalizable with pure complex eigenvalues which implies $\big\|\exp\big(\frac{s}{\varepsilon}\mathbb{E}\big)\big\|_0 \leq$ const. Finally, since $\hat{\mathbb{D}}$ is definite positive, the same is true for $\pi_{\hat{\mathbb{E}}}(\hat{\mathbb{D}})$.
□

Notice that Lemma A.2 implies that for any $0 \leq s \leq t$,

$$\lim_{\varepsilon\to 0}\Big\| \int_s^t du \left[\exp\big(\frac{u}{\varepsilon}\mathbb{E}\big)\mathbb{D}\exp\big(-\frac{u}{\varepsilon}\mathbb{E}\big) - \pi_{\mathbb{E}}(\mathbb{D})\right] G\Big\|_0 = 0$$

Lemma A.3 *Let $A^\varepsilon(s)$, A be linear operators from $\mathcal{H}_{k_0+2}$ to $\mathcal{H}_{k_0}$ such that*

$$\sup_{\varepsilon, 0\leq s\leq t} \big\|A^\varepsilon(s)\big\|_{k_0+2\to k_0} < \infty$$

and for any $G \in \mathcal{H}_{k+2}$ and $0 \leq s < t$

$$\lim_{\varepsilon\to 0}\left\| \int_s^t du\, [A^\varepsilon(u) - A] G\right\|_{k_0} = 0.$$

Then, for any $G \in \mathcal{H}_{k_0+2}$

$$\lim_{\varepsilon\to 0} E_\varepsilon^\mu\left[\Big(\int_0^t ds\, \xi^\varepsilon\big(s, [A^\varepsilon(s) - A]G\big)\Big)^2\right] = 0,$$

where ξ_s^ε is the fluctuation field.

Proof. We set $\langle \xi_t^\varepsilon, G\rangle = \xi^\varepsilon(t, G)$. Let $0 = t_0 < t_1 < \ldots < t_\ell = t$ be a subdivision of the interval $[0, t]$ of size $\delta > 0$. Then

$$\int_0^t ds\, \xi^\varepsilon(s, A^\varepsilon(s)G) = \sum_{i=0}^{\ell-1}\Big\langle \xi_{t_i}^\varepsilon, \int_{t_i}^{t_i+1} ds\, A^\varepsilon(s)G\Big\rangle + R_1^\varepsilon,$$

with

$$R_1^\varepsilon = \sum_{i=0}^{\ell-1} \int_{t_i}^{t_i+1} ds \, \langle \xi_s^\varepsilon - \xi_{t_i}^\varepsilon, A^\varepsilon(s) G \rangle.$$

Since

$$|R_1^\varepsilon| \le t \sup_{\substack{|s_1-s_2|\le\delta \\ 0\le s_1,s_2\le t}} \|\xi_{s_1}^\varepsilon - \zeta_{s_2}^\varepsilon\|_{-k_0} \sup_{0\le s\le t} \|A^\varepsilon(s)\|_{k_0+2\to k_0} \|G\|_{k_0+2},$$

it results from tightness (5.14) that for any $\delta > 0$,

$$\lim_{\delta\to 0} \lim_{\varepsilon\to 0} P_\mu^\varepsilon\big(|R_1^\varepsilon| > \delta\big) = 0.$$

Moreover

$$\int_0^t ds \, \langle \xi_s^\varepsilon, A^\varepsilon(s) G \rangle = \sum_{i=0}^{\ell-1} \langle \xi_{t_i}^\varepsilon, AG \rangle (t_{i+1} - t_i) + R_1^\varepsilon + R_2^\varepsilon,$$

with

$$|R_2^\varepsilon| = \left| \sum_{i=0}^{\ell-1} \int_{t_i}^{t_{i+1}} ds \, \langle \xi_{t_i}^\varepsilon (A^\varepsilon(s) - A) G \rangle \right|$$

$$\le t \sup_{0\le s\le t} \|\zeta_s^\varepsilon\|_{-k_0} \max_i \left\| \int_{t_i}^{t_{i+1}} ds [A^\varepsilon(s) - A] G \right\|_{k_0}.$$

¿From assumption

$$\lim_{\varepsilon\to 0} \max_i \left\| \int_{t_i}^{t_{i+1}} ds \, [A^\varepsilon(s) - A] G \right\|_{k_0} = 0.$$

So, using tightness (5.13), we get for $M > 0$ and ε small enough

$$P_\mu^\varepsilon\big(|R_2^\varepsilon| > \delta\big) \le P_\mu^\varepsilon\Big(\sup_{0\le s\le t} \|\xi_s^\varepsilon\|_{-k_0} > \frac{\delta}{M} \Big)$$

which vanishes in the limit $M \to 0$ after $\varepsilon \to 0$. With the same kind of arguments (using tightness again), we get

$$\sum_{i=0}^{\ell-1} \langle \xi_{t_i}^\varepsilon, AG \rangle (t_{i+1} - t_i) = \int_0^t ds \, \langle \xi_s^\varepsilon, AG \rangle + R_3^\varepsilon,$$

where

$$\lim_{\delta\to 0} \lim_{\varepsilon\to 0} P_\mu^\varepsilon\big(|R_4^\varepsilon| > \delta\big) = 0.$$

We have proved so far that $\int_0^t ds\big(\xi^\varepsilon(s, A^\varepsilon(s)G) - \xi^\varepsilon(s, AG)\big)$ converges to 0 in $\mathbb{P}_\varepsilon^\mu$ probability. To assert that the convergence occurs in $L^2(P^\mu)$ it suffices e.g. to check that

$$\sup_\varepsilon \mathbb{E}_\varepsilon^\mu \left[\left(\int_0^t ds \, \langle \xi_s^\varepsilon, (A^\varepsilon(s) - A) G \rangle \right)^4 \right] < \infty,$$

which is clear from the assumptions on the operator $A^\varepsilon(s)$ and A.

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