MONOTONICITY, $\mathcal{J}$-ALGEBRA OF POTAPOV AND LYAPUNOV EXPONENTS

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January 15, 1999

Abstract. We present a new approach and a generalization of the estimates of Lyapunov exponents developed first in [W2] in the symplectic case. The work of Lewowicz [L], Markarian [M], and our [W1], [W2], [W5], are combined with the $\mathcal{J}$–algebra of Potapov, [P1],[P2],[P3]. We obtain a general theory which we then specify to the symplectic case. The appendix contains a simple application to the gas of hard spheres.

§0. Introduction.

We will discuss a general method for establishing hyperbolic properties in concrete dynamical systems.

Hyperbolicity is defined in the language of the linearized system (the derivative in the case of discrete time). It amounts to exponential growth or decay of solutions of the linearized system. To capture such a behavior we use indefinite quadratic forms. If an indefinite quadratic form (more precisely a field of such forms on the tangent bundle of the phase space) evaluated on any nonzero solution grows, then it provides us with a distinguished cone (sector) of positive tangent vectors which is taken into itself by the linearized dynamics. Once a solution enters this cone the quadratic form can be used as a measure of the magnitude of the solution and the growth can be usually strengthened to exponential growth with the help of recurrence. The decay of some solutions is obtained by reversing time and using the absolute value of our quadratic form.

This crucial idea was put forward by Lewowicz [L] and independently in [W1]. It also appeared in the study of exponential dichotomies in linear differential equations, Coppel, [C], Samoilenko and Kulik, [Sa-K].

We will consider a more general situation when solutions of the linearized system stay in the positive cone forever, once they enter it, but the form defining the cone is not necessarily increased.

To illustrate the contents of this paper let us consider the linear system of differential equations (with periodic coefficients) of the form.

$$\begin{aligned} \dot{x} &= \alpha(t)x + B(t)y, \\ \dot{y} &= C(t)x + \delta(t)y, \end{aligned} \tag{0.1}$$

where $x \in \mathbb{R}^m, y \in \mathbb{R}^m$, the functions α and δ have scalar values and B, C are $m \times m$ matrix valued functions.

We introduce the quadratic form $\mathcal{Q}(x,y) = \langle x, y\rangle$ where $\langle\cdot,\cdot\rangle$ is the arithmetic scalar product on $\mathbb{R}^m$. We have

$$\frac{d}{dt}\mathcal{Q} = (\alpha(t) + \delta(t))\,\langle x, y\rangle + \langle B(t)y, y\rangle + \langle C(t)x, x\rangle.$$

Under the assumption that the symmetric parts of B and C are positive definite we obtain the following property

$$\frac{d}{dt}\mathcal{Q} > 0, \quad \text{if} \quad \mathcal{Q} = 0.$$

If this property holds we will say that the linear flow defined by (0.1) is strictly $\mathcal{Q}$-separated. We will explore the consequences of this property.

We will establish that if the system (0.1) is strictly $\mathcal{Q}$-separated then the linear flow has a dominated splitting, i.e., there is an invariant splitting of $\mathbb{R}^m \times \mathbb{R}^m$ into m-dimensional subspaces E^- and E^+ such that the largest Floquet multiplier on E^- is strictly smaller (in absolute value) than the smallest Floquet multiplier on E^+. The quadratic form $\mathcal{Q}$ is negative on E^- and positive on E^+. We will also introduce quantitative description of the spectral gap.

We will use the language of matrix cocycles rather than that of differential equations and we will obtain estimates of Lyapunov exponents. The results can be readily translated into the language of differential equations.

The system (0.1) with $\alpha = \delta \equiv 0, B(t) \equiv I$ and symmetric positive definite $C(t)$ appears in the study of geodesic flows on manifolds of negative sectional curvature. In this situation we obtain the uniform expansion on E^+ and the uniform contraction on E^-, i.e., the Anosov property.

More generally in the symplectic case, if a system is $\mathcal{Q}$-separated then it is Anosov.

The symplectic case was studied earlier in [W2],[W5], where a special case of the estimates of Theorem 3.10 was obtained.

This method of estimating Lyapunov exponents for the Hamiltonian dynamics has two major applications, the Boltzmann-Sinai gas of hard spheres [Ch-S],[W2], [W5],[S-Sz], and the systems of falling balls in 1-dimension, [W3],[W4], [S1],[W6],[W7].

In the Appendix we present an application of the estimates of Lyapunov exponents from Theorem 3.10 to the gas of hard spheres.

The primary motivation for this paper was our study of the geodesic flows on Weyl manifolds, which we call W-flows, [W9],[W8]. The linearization of these flows has (in appropriate coordinates) the form (0.1) with $\delta \equiv 0, B(t) \equiv I$, and $C(t)$ not necessarily symmetric.

§1. The $\mathcal{J}$–algebra of Potapov and contraction/expansion properties.

Let us consider an n-dimensional linear space V over $\mathbb{R}$ with a chosen pseudo Euclidean structure, i.e., a nondegenerate quadratic form $\mathcal{J}$ with the positive index of inertia equal to p and the negative index of inertia equal to q, $p+q=n, p \geq 1, q \geq 1$. Depending on the sign of $\mathcal{J}$ we have positive and negative vectors. We denote by

$$\mathcal{C}_{\pm} = \{v \in V | \pm \mathcal{J}(v) > 0\} \cup \{0\}$$

the open cones of, respectively, positive and negative vectors (with the zero vector included), and by $\mathcal{C}_0$ their common boundary, $\mathcal{C}_0 = \{v \in V | \mathcal{J}(v) = 0\}$.

A linear subspace is called positive (negative), if it is contained in $\mathcal{C}_+$ ($\mathcal{C}_-$). The maximal dimension of a positive subspace is p and of a negative subspace q. Let P be a positive subspace of dimension p (a maximal positive subspace) and let Q be the $\mathcal{J}$–orthogonal complement of P. The dimension of Q is q and it is by necessity a negative subspace. We can introduce an auxiliary scalar product in V by using $\mathcal{J}$ in P, $-\mathcal{J}$ in Q and declaring P and Q to be orthogonal. We choose coordinates in $P = \mathbb{R}^p$ and $Q = \mathbb{R}^q$ in such a way that $\mathcal{J}$ and $-\mathcal{J}$ become the arithmetic scalar products in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. As a result we obtain the following coordinate representation: $V = \mathbb{R}^p \times \mathbb{R}^q$ and for $v = (v_1, v_2), v_1 \in \mathbb{R}^p, v_2 \in \mathbb{R}^q$

$$\mathcal{J}(v) = \langle Jv, v\rangle = v_1^2 - v_2^2, \quad \text{where} \quad J = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$$

I_p, I_q are identity matrices in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. We will use freely this coordinate representation when we find it convenient. It is important though for discussing diffeomorphisms and flows on manifolds that the following theory depends on the choice of the form $\mathcal{J}$ alone.

Definition 1.1. *A linear nonsingular operator $S : V \to V$ is*

(1) *$\mathcal{J}$–separated, if $S\mathcal{C}_+ \subset \mathcal{C}_+$,*
(2) *strictly $\mathcal{J}$–separated, if $S(\mathcal{C}_0 \cup \mathcal{C}_+) \subset \mathcal{C}_+$,*
(3) *$\mathcal{J}$–monotone, if $\mathcal{J}(Sv) \geq \mathcal{J}(v)$ for every $v \in V$,*
(4) *strictly $\mathcal{J}$–monotone, if $\mathcal{J}(Sv) > \mathcal{J}(v)$ for every $v \in V, v \neq 0$,*
(5) *$\mathcal{J}$–isometry, if $\mathcal{J}(Sv) = \mathcal{J}(v)$ for every $v \in V$,*

Note that the five classes of operators are semigroups with respect to composition. The inclusions between them are as follows

$$\begin{array}{ccccc} (1) & \supset & (3) & \supset & (5) \\ \cup & & \cup & & \\ (2) & \supset & (4) & & \end{array}$$

We extend the definitions of properties 1 - 5 to one parameter subgroups of linear operators (elements of the Lie algebra $gl(n)$) in the natural way. An element G of $gl(n)$ has property (i) if the operator e^{tG} has property (i) for all $t > 0$.

Let us further define a $\mathcal{J}$–symmetric operator S as a linear operator on V such that $\langle JS\cdot, \cdot\rangle$ is a symmetric bilinear form on V.

Potapov, [P1],[P2],[P3], obtained fundamental results about the structure and properties of $\mathcal{J}$–monotone operators, which he called $\mathcal{J}$–noncontractive. Potapov developed his theory for a pseudo unitary form $\mathcal{J}$ rather than pseudo Euclidean, but we will concentrate on the real case. We will comment along the way on the complex versions of theorems.

We will formulate a series of propositions about the $\mathcal{J}$-algebra of Potapov. We will prove only the infinitesimal versions. The (harder) proofs in the general case can be found in the papers of Potapov, [P1],[P2],[P3].

1. If S is $\mathcal{J}$–separated then the set of positive numbers r such that $\frac{1}{r}S$ is $\mathcal{J}$–monotone is a closed interval $[r_-, r_+], r_- > 0$, possibly degenerating to a point.

2. If S is strictly $\mathcal{J}$–separated then the set of positive numbers r such that $\frac{1}{r}S$ is strictly $\mathcal{J}$–monotone is an open interval $(r_-, r_+), r_- > 0$.

These properties are the consequence of the theorem of Kühne.

Theorem 1.2. *If a symmetric bilinear form $F(\cdot,\cdot)$ is nonnegative on $\mathcal{C}_0$ then*

$$\inf_{v\in\mathcal{C}_+} \frac{F(v,v)}{\langle Jv,v\rangle} \geq \sup_{u\in\mathcal{C}_-} \frac{F(u,u)}{\langle Ju,u\rangle},$$

and for every number r between the infimum and the supremum $F(v,v) \geq r\langle Jv,v\rangle$, for all v.

Proof. Suppose to the contrary that there is a positive vector $v_0 \in \mathcal{C}_+$ and a negative vector $u_0 \in \mathcal{C}_-$ which violate the inequality, i.e., assuming $\langle Jv_0, v_0\rangle = 1, \langle Ju_0, u_0\rangle = -1$ we have $F(v_0,v_0) + F(u_0,u_0) < 0$. We can find the angle α such that the linear combinations $v_1 = \cos\alpha v_0 + \sin\alpha u_0$ and $v_2 = -\sin\alpha v_0 + \cos\alpha u_0$ are in $\mathcal{C}_0$. Then $F(v_1,v_1) \geq 0,\ F(v_2,v_2) \geq 0$. We obtain the contradiction

$$F(v_1,v_1) + F(v_2,v_2) = F(v_0,v_0) + F(u_0,u_0) \geq 0.$$

□

The proof of the pseudo unitary version of Theorem 1.2 can be found in [P2].

3. Every $\mathcal{J}$–separated operator S has a unique representation $S = RU$, where U is a $\mathcal{J}$–isometry and R is $\mathcal{J}$–symmetric and has only positive eigenvalues. The operator R is equal to $(SJS^*J)^{\frac{1}{2}}$ and it is called the modulus of S. Note that although the adjoint S^* depends on the auxiliary Euclidean structure in V the modulus R depends on the form $\mathcal{J}$ alone.

4. For a strictly $\mathcal{J}$–separated S there is a $\mathcal{J}$–isometry W such that the modulus R of S is diagonalized by W, i.e., $W^{-1}RW$ is the diagonal matrix $\begin{bmatrix} R_+ & 0 \\ 0 & R_- \end{bmatrix}$. Moreover $R_+ \geq r_+ I_p$ and $0 < R_- \leq r_- I_q$, where $r_+ > r_- > 0$ were introduced in 2.

5. For a $\mathcal{J}$–separated S the p largest eigenvalues of the modulus R are not smaller than r_+ and the q smallest eigenvalues do not exceed r_-. Let us denote these eigenvalues by

$$r_p^+ \geq \cdots \geq r_2^+ \geq r_1^+ \geq r_+ \geq r_- \geq r_1^- \geq r_2^- \geq \cdots \geq r_q^- > 0.$$

Moreover we have $r_1^+ = r_+$ and $r_1^- = r_-$.

Clearly a $\mathcal{J}$–separated S is (strictly) $\mathcal{J}$–monotone if and only if $r_1^+ = r_+ \geq 1$ ($>$ 1) and $r_1^- = r_- \leq 1$ (< 1).

The proofs depend on the celebrated theorem of Potapov, that S is $\mathcal{J}$–separated if and only if S^* is $\mathcal{J}$–separated. The infinitesimal version is quite simple. By

Theorem 1.2 $G \in gl(n)$ is $\mathcal{J}$–separated if and only if there is a number t (not necessarily positive) such that $G^*J + JG \geq tJ$. Multiplying that inequality by J on the left and on the right we obtain $JG^* + GJ \geq tJ$.

The infinitesimal versions of the claims 3 – 5 are equally simple to prove. First of all any element G of the Lie algebra $gl(n)$ can be split uniquely into the $\mathcal{J}$–symmetric

$G_s = \frac{1}{2}(G + JG^*J)$ and the $\mathcal{J}$–antisymmetric $G_a = \frac{1}{2}(G - JG^*J)$.

The $\mathcal{J}$–antisymmetric elements of $gl(n)$ are the infinitesimal $\mathcal{J}$–isometries. If G is strictly $\mathcal{J}$–separated then $2JG_s = G^*J + JG > tJ$. It follows that the eigenvalues of G_s are real and they are bigger than $\frac{t}{2}$ if the eigenvector is a positive vector and smaller than $\frac{t}{2}$ if the eigenvector is a negative vector. (To establish that the spectrum of G_s is real we need to use the pseudo unitary $\mathcal{J}$-form in the complexified space. It goes through exactly as in the case of symmetric matrices and we leave it to the reader.)

If G_s is strictly $\mathcal{J}$–separated then it is diagonalizable by a $\mathcal{J}$–isometry. Indeed it must have an invariant subspace P which is maximal positive and an invariant subspace Q which is maximal negative. (It is easier to think about $S = e^{\epsilon G_s}$. $S(S^{-1})$ takes maximal positive(negative) subspaces into maximal positive(negative) subspaces. Hence by some kind of fixed point theorem there must be invariant subspaces.) Similarly to the case of symmetric matrices the subspaces P and Q are $\mathcal{J}$–orthogonal. Hence there is a $\mathcal{J}$–isometry U such that

$$U^{-1}G_sU = \begin{bmatrix} G_+ & 0 \\ 0 & G_- \end{bmatrix}.$$

Moreover $G_\pm$ are symmetric. So we can further diagonalize them. Also $G_+ > \frac{1}{2}tI_p$ and $G_- < \frac{1}{2}tI_q$. Indeed since the defining property for a $\mathcal{J}$-isometry U is $U^{-1} = JU^*J$, we have

$$2J(U^{-1}G_sU) = U^*2JG_sU \geq tU^*JU = tJ.$$

We proceed with the development of further properties of $\mathcal{J}$-separated operators; we will freely use properties 1 – 5. For a d–dimensional positive linear subspace $L \subset \mathcal{C}_+$ the restriction of $\mathcal{J}$ is a scalar product. If S is $\mathcal{J}$–separated then $SL \subset \mathcal{C}_+$ is also a positive subspace with the scalar product furnished by $\mathcal{J}$. Hence the subspaces L and SL are equipped with (unoriented) volume elements, which we call the $\mathcal{J}$–volumes. Let $\alpha_d(L; S)$ be the coefficient of the $\mathcal{J}$–volume expansion for the linear map S restricted to L. Further let $\sigma_d(S)$ be the infimum of $\alpha_d(L; S)$ over all positive d–dimensional subspaces L. The important property of the coefficient σ_d is the supermultiplicativity: for $\mathcal{J}$–separated S_1 and S_2 we have

$$\sigma_d(S_1S_2) \geq \sigma_d(S_1)\sigma_d(S_2). \tag{1.1}$$

Using the notation introduced in 5. we have

Proposition 1.3. $\sigma_d(S) = r_1^+ r_2^+ \dots r_d^+$.

Proof. The operator $C_\epsilon = \begin{bmatrix} (1+\epsilon)I_p & 0 \\ 0 & (1-\epsilon)I_q \end{bmatrix}$ is strictly $\mathcal{J}$–monotone for $0 < \epsilon < 1$. If S is $\mathcal{J}$–separated then $C_\epsilon S$ is strictly $\mathcal{J}$–separated. Hence by the continuity of the modulus R and the coefficient $\alpha_d(L; S)$ as functions of S, it is sufficient to prove the Proposition for a strictly $\mathcal{J}$–separated operator S. For such operators we can diagonalize the modulus as formulated in 4.

Since S differs from its modulus R by a $\mathcal{J}$–isometry U we get that $\alpha_D(L;S)$ is equal to $\alpha_d(UL;R)$. Every d–dimensional positive subspace L can be included in a p–dimensional positive subspace T (maximal positive subspace). The subspace T is the graph of the linear mapping $T:\mathbb{R}^p\to\mathbb{R}^q$ such that $T^*T<I_p$. We will slightly abuse the notation denoting by the same letter T the linear map and its graph. In the coordinate representation T is a $q\times p$ matrix. Using $v_1\in\mathbb{R}^p$ as a coordinate on T we obtain that the scalar product $\mathcal{J}$ on T is equal to $\langle (I_p-T^*T)v_1,v_1\rangle$ and the action of R on T is given by $v_1\mapsto R_+v_1$. The image space RT is the graph of the linear map $R_-TR_+^{-1}$. To estimate the coefficients of $\mathcal{J}$–volume expansion under the action of the restriction of R to the subspace T we need to estimate the singular values of

$$(I_p-R_+^{-1}T^*R_-^2TR_+^{-1})^{\frac12}R_+(I_p-T^*T)^{-\frac12},$$

i.e., we need to estimate the eigenvalues of

$$(I_p-T^*T)^{-\frac12}(R_+^2-T^*R_-^2T)(I_p-T^*T)^{-\frac12}\geq (I_p-T^*T)^{-\frac12}R_+^2(I_p-T^*T)^{-\frac12}.$$

The i-th eigenvalue in ascending order of the last matrix cannot be smaller than $(r_i^+)^2$. It gives us the desired estimate $\alpha_d(L,S)\geq r_1^+r_2^+\dots r_d^+$. To get the equality for the infimum we need to consider the subspace $T=0$. $\square$

Proposition 1.3 allows the interpretation of the eigenvalues of the modulus as pseudo Euclidean singular values. We will call them simply positive and negative singular values of the $\mathcal{J}$–separated operator S (note that both positive and negative singular values are positive numbers). In particular they are not changed if we compose S on the left and/or on the right with $\mathcal{J}$–isometries.

In the pseudo unitary case we consider d-dimensional complex subspaces which have $2d$-dimensional real $\mathcal{J}$-volume. Essentially the same proof will give us $\sigma_d(S)=(r_1^+r_2^+\dots r_d^+)^2$.

Corollary 1.4. *For $\mathcal{J}$-separated S_1 and S_2 we have*

$$r_1^+(S_1S_2)\geq r_1^+(S_1)r_1^+(S_2)\quad \textit{and}\quad r_1^-(S_1S_2)\leq r_1^-(S_1)r_1^-(S_2).$$

Proof. The first part follows immediately from Proposition 1.3.

For a $\mathcal{J}$-separated S the operator S^{-1} is $(-\mathcal{J})$-separated and the positive pseudo Euclidean singular values of S^{-1} (with respect to $-\mathcal{J}$) are equal to $\frac{1}{r_1^-(S)},\dots,\frac{1}{r_q^-(S)}$. The second part follows now from the first applied to $S_2^{-1}S_1^{-1}$ and $-\mathcal{J}$. $\square$

Let us consider the manifold $\mathcal{L}_+$ of all p-dimensional subspaces contained in $\mathcal{C}_+$, It can be identified with the manifold of $q\times p$ matrices T such that $T^*T<I_p$. This manifold carries the Riemannian metric of the symmetric space [Py],

$$dT^2=tr\left((I_p-T^*T)^{-1}dT^*(I_q-TT^*)^{-1}dT\right). \tag{1.2}$$

To get (1.2) we need to observe that the $\mathcal{J}$–isometries act transitively on $\mathcal{L}_+$. The isometry that takes T into the zero subspace $\mathbb{R}^p \times \{0\}$ is given by

$$\begin{bmatrix} (I_p - T^*T)^{-\frac{1}{2}} & (I_p - TT^*)^{-\frac{1}{2}}T^* \\ -(I_q - TT^*)^{-\frac{1}{2}}T & (I_q - TT^*)^{-\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} (I_p - T^*T)^{-\frac{1}{2}} & -T^*(I_q - TT^*)^{-\frac{1}{2}} \\ T(I_p - T^*T)^{-\frac{1}{2}} & (I_q - TT^*)^{-\frac{1}{2}} \end{bmatrix}^{-1}.$$

Hence the metric $tr(dT^*dT)$ at 0 is transported to T as

$$tr((I_p - T^*T)^{-\frac{1}{2}} dT^*(I_q - TT^*)^{-1} dT(I_p - T^*T)^{-\frac{1}{2}}).$$

A $\mathcal{J}$–separated operator S maps naturally $\mathcal{L}_+$ into itself. This mapping is a contraction. More precisely we have

Theorem 1.5. *For a $\mathcal{J}$–separated operator S and $T_1, T_2 \in \mathcal{L}_+$*

$$dist(ST_1, ST_2) \leq \frac{r_1^-}{r_1^+} dist(T_1, T_2),$$

and the contraction coefficient is sharp.

Proof. By continuity it is sufficient to prove the theorem for a strictly $\mathcal{J}$–separated operator A. We take advantage of the polar form of $S = RU$ where R is the modulus and U is the $\mathcal{J}$–isometry. Hence it is enough to show the contraction for an operator of the form described in 4. We will establish the infinitesimal contraction property. The action of this operator on $\mathcal{L}_+$ is given by $T \mapsto R_- T R_+^{-1}$. The derivative of this action is $dT \mapsto R_- dT R_+^{-1}$. Hence we need to compare the spectra of

$$(I_p - T^*T)^{-\frac{1}{2}} dT^*(I_q - TT^*)^{-1} dT(I_p - T^*T)^{-\frac{1}{2}} \quad \text{and}$$
$$(I_p - R_+^{-1}T^*R_-^2TR_+^{-1})^{-\frac{1}{2}} R_+^{-1} dT^* R_-(I_q - R_-TR_+^{-2}T^*R_-)^{-1} R_- dT R_+^{-1}(I_p - R_+^{-1}T^*R_-^2TR_+^{-1})^{-\frac{1}{2}}.$$

We will need the following

Lemma 1.6. *For nonsingular symmetric matrices X, Y and a symmetric matrix Z, if*

$X^2 \leq Y^2$ then the spectrum of YZY is not smaller than the spectrum of XZX, i.e., the i-th eigenvalue (in ascending order) of XZX does not exceed the i-th eigenvalue of YZY.

Note that we do not claim that $XZX \leq YZY$, which is not guaranteed by our assumptions.

Proof. The eigenvalues of XZX and YZY can be obtained as min-max of the Rayleigh quotients

$$\frac{\langle XZXv, v\rangle}{\langle v, v\rangle} = \frac{\langle Zu, u\rangle}{\langle X^{-2}u, u\rangle} \leq \frac{\langle Zu, u\rangle}{\langle Y^{-2}u, u\rangle} = \frac{\langle YZYw, w\rangle}{\langle w, w\rangle},$$

for $u = Xv$ and $u = Yw$. $\square$

The last operator has the same spectrum as the operator

(1.3)
$$R_+^{\frac{1}{2}}(R_+^2 - T^*R_-^2T)^{-\frac{1}{2}}R_+^{-\frac{1}{2}}dT^*(R_-^{-2} - TR_+^{-2}T^*)^{-1}dTR_+^{-\frac{1}{2}}(R_+^2 - T^*R_-^2T)^{-\frac{1}{2}}R_+^{\frac{1}{2}}.$$

Now applying Lemma 1.6 we can "peel off" the outside factors using

$$\begin{gathered} R_+ \geq r_1^+ I_p,\ R_+^2 \geq (r_1^+)^2 I_p, \quad R_- \geq r_1^- I_q,\ R_-^2 \geq (r_1^-)^2 I_q, \\ (R_+^2 - T^*R_-^2T) \geq (r_1^+)^2(I_p - T^*T), \quad (R_-^{-2} - TR_+^{-2}T^* \geq (r_1^-)^{-2}(I_q - TT^*). \end{gathered}$$

We obtain that the spectrum of (1.3) does not exceed the spectrum of

$$\frac{1}{(r_1^+)^2}(I_p - T^*T)^{-\frac{1}{2}}dT^*(R_-^{-2} - TR_+^{-2}T^*)^{-1}dT(I_p - T^*T)^{-\frac{1}{2}} \leq$$
$$\left(\frac{r_1^-}{r_1^+}\right)^2 (I_p - T^*T)^{-\frac{1}{2}}dT^*(I_q - TT^*)^{-1}dT(I_p - T^*T)^{-\frac{1}{2}}.$$

To establish that the coefficient of contraction is sharp it is enough to take $T = 0$ and calculate the infinitesimal coefficient of contraction which is equal to $\frac{r_1^-}{r_1^+}$. $\square$

This theorem extends the result of Bougerol, [B], about the contraction of the space of positive Lagrangian subspaces under the action of monotone symplectic operators.

In the proof we have obtained the estimate of all eigenvalues and not just the trace. It leads to the contraction property for other metrics, for instance the Finsler metric determined by $\|dT^*dT\|$ at zero subspace, in parallel with [L-W].

Theorem 1.5 holds also in the pseudo unitary case and the proof is essentially the same. In such a case the space of complex positive subspaces of dimension p is the classical bounded domain. As shown by Suzuki, [Su], for this domain the Caratheodory and Kobayashi metrics coincide with the Finsler metric determined by $\|dT^*dT\|$ at zero subspace. Hence this metric must be contracted by any holomorphic map of the domain into itself and not only by the Möbius maps as in Theorem 1.5. The Kahler metric of the symmetric space is not contracted by all holomorphic maps (cf. [Ko]), but it is contracted by Möbius maps defined by $\mathcal{J}$-separated operators, which is the content of Theorem 1.5 in the pseudo unitary case.

§2. The symplectic case.

Let us apply these general ideas to the symplectic space and symplectic operators. We have $p = q$, $V = \mathbb{R}^p \times \mathbb{R}^p$ and the symplectic form $\omega(v, w) = \langle v_1, w_2\rangle - \langle v_2, w_1\rangle$. We consider a special type of a $\mathcal{J}$–form, which we call a $\mathcal{Q}$–form. It was defined geometrically in [W5] . A $\mathcal{Q}$– form is defined by a pair of transversal Lagrangian subspaces L_1, L_2. Using the standard splitting $v = v_1 + v_2, v_i \in L_i, i = 1, 2$, we put $\mathcal{Q}(v) = \omega(v_1, v_2)$.

The positive cone $\mathcal{C}_+$ of a $\mathcal{Q}$–form is called a symplectic sector between L_1 and L_2. The boundary of a symplectic sector contains many Lagrangian subspaces, but only two isolated ones, L_1 and L_2. These two Lagrangian subspaces act as the "sides" of the sector, which they actually are only when $p = 1$.

We can choose symplectic coordinates in such a way that $L_1 = \mathbb{R}^p \times \{0\}, L_2 = \{0\} \times \mathbb{R}^p$ and accordingly $\mathcal{Q}(v) = \langle v_1, v_2\rangle$. We will call it the standard $\mathcal{Q}$-form.

The following theorem was published by Potapov in [P2], and independently in [W1]. The proof below is essentially taken from [W1].

Theorem 2.1. *If a symplectic operator is (strictly) $\mathcal{Q}$–separated then it is (strictly) $\mathcal{Q}$–monotone.*

Before providing the proof we develop a useful factorization of $\mathcal{Q}$–separated symplectic operators. We will use the standard symplectic form and the standard $\mathcal{Q}$-form in $V = \mathbb{R}^p \times \mathbb{R}^p$.

Let

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a symplectic operator on $V = \mathbb{R}^p \times \mathbb{R}^p$. Hence A, B, C, D are $p \times p$ matrices such that $C^*A = A^*C, D^*B = B^*D,$ and $D^*A - B^*C = I$.

Lemma 2.2. *If a symplectic S is $\mathcal{Q}$–separated then A and D are invertible.*

Proof. Suppose that there is $\widehat{v}_1 \in \mathbb{R}^p$ such that $A\widehat{v}_1 = 0$. It follows that $\widehat{v}_1 = (D^*A - B^*C)\widehat{v}_1 = -B^*C\widehat{v}_1$.

For $v = (\widehat{v}_1, v_2) \in \mathcal{C}_+$ we must have $Sv = (Bv_2, C\widehat{v}_1 + Dv_2) \in \mathcal{C}_+$, i.e., if $\langle \widehat{v}_1, v_2\rangle > 0$ then

$$0 < \langle Bv_2, C\widehat{v}_1 + Dv_2\rangle = -\langle \widehat{v}_1, v_2\rangle + \langle v_2, B^*Dv_2\rangle.$$

Choosing $v_2 = \epsilon\widehat{v}_1, \epsilon > 0$ we obtain

$$\langle \widehat{v}_1, \widehat{v}_1\rangle < \epsilon\langle \widehat{v}_1, B^*D\widehat{v}_1\rangle,$$

which leads to $\widehat{v}_1 = 0$. $\square$

The symplectic operators of the form

$$\begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}$$

are also $\mathcal{Q}$–isometries.

By Lemma 2.2 given a $\mathcal{Q}$–separated symplectic S we can factor out the following $\mathcal{Q}$-isometry on the right

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & K \\ H & \cdot \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}$$

Symplecticity of S forces H and K to be symmetric and allows the further unique factorization

$$S = \begin{pmatrix} I & 0 \\ H & I \end{pmatrix} \begin{pmatrix} I & K \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}. \tag{2.1}$$

Moreover H and K must be positive semidefinite ($H \geq 0$, $K \geq 0$). It follows from the following

Proof of Theorem 2.1. Using the factorization (2.1) we get for $v = (v_1, v_2)$

$$\mathcal{Q}(Sv) = \langle v_1, v_2\rangle + \langle Kv_2, v_2\rangle + \langle H(v_1 + Kv_2), v_1 + Kv_2\rangle.$$

Taking $v_2 = 0$ we obtain $\mathcal{Q}(v) = 0$ and hence $\mathcal{Q}(Sv) = \langle Hv_1, v_1\rangle \geq 0$.

Taking $v_1 = -Kv_2$ we obtain $\mathcal{Q}(Sv) = 0$ which implies that $\mathcal{Q}(v) = -\langle Kv_2, v_2\rangle \leq 0$.

□

As a byproduct of the proof we get the following useful observation

Proposition 2.3. *A $\mathcal{Q}$–monotone symplectic operator S maps vectors from L_1 (L_2) either into $\mathcal{C}_+$ or into L_1 (L_2). Moreover if S maps L_1 and L_2 into $\mathcal{C}_+$ then it is strictly $\mathcal{Q}$–monotone.*

□

It turns out that the modulus of a symplectic $\mathcal{Q}$-monotone operator is also symplectic and hence the positive and negative singular values are inverses of each other. Indeed, from (2.1) we obtain the following factorization of the $\mathcal{Q}$–monotone symplectic operator $S\begin{pmatrix} A^{-1} & 0 \\ 0 & A^* \end{pmatrix}$.

$$\begin{pmatrix} I & K \\ H & I+HK \end{pmatrix} = \begin{pmatrix} (I+KH)^{\frac{1}{2}} & K(I+HK)^{-\frac{1}{2}} \\ H(I+KH)^{\frac{1}{2}} & (I+HK)^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} (I+KH)^{-\frac{1}{2}} & 0 \\ 0 & (I+HK)^{\frac{1}{2}} \end{pmatrix}.$$

Since the last matrix is a $\mathcal{Q}$–isometry and the next to last matrix is $\mathcal{Q}$–symmetric we obtain that the modulus of S is equal to the next to last matrix. We proceed to calculate its eigenvalues.

Proposition 2.4. *The positive singular values of a $\mathcal{Q}$–monotone operator $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ are equal to $\sqrt{1+t_i} + \sqrt{t_i} = \exp(\sinh^{-1}\sqrt{t_i})$, where $t_1, \dots, t_p$ are the eigenvalues of CB^*, which are real and nonnegative.*

Proof. To calculate the pseudo Euclidean singular values of S we will use the fact that they are not changed if we multiply S on left and on the right by $\mathcal{Q}$–isometries. In the factorization (2.1) $CA^{-1} = H$ and $BA^* = K = AB^*$ so that HK has the same eigenvalues as CB^*, in particular they must be all real nonnegative (since $H \geq 0, K \geq 0$). Let us assume that $K > 0$, the general case will follow by continuity. We get

$$\begin{pmatrix} K^{-\frac{1}{2}} & 0 \\ 0 & K^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} I & K \\ H & I+HK \end{pmatrix} \begin{pmatrix} K^{\frac{1}{2}} & 0 \\ 0 & K^{-\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} I & I \\ F & I+F \end{pmatrix}$$

where $F = K^{\frac{1}{2}}HK^{\frac{1}{2}}$ has the same eigenvalues as HK. Finally if Z is the orthogonal matrix which diagonalizes F, i.e., $Z^{-1}FZ = T$ is diagonal, then

$$\begin{pmatrix} Z^{-1} & 0 \\ 0 & Z^{-1} \end{pmatrix} \begin{pmatrix} I & I \\ F & I+F \end{pmatrix} \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix} = \begin{pmatrix} I & I \\ T & I+T \end{pmatrix} =$$
$$\begin{pmatrix} (I+T)^{-\frac{1}{2}} & (I+T)^{-\frac{1}{2}} \\ T(I+T)^{\frac{1}{2}} & (I+T)^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} (I+T)^{\frac{1}{2}} & I \\ T & (I+T)^{\frac{1}{2}} \end{pmatrix}.$$

It shows that the singular values of S are equal to the eigenvalues of next to last matrix and they are easy to calculate. $\square$

It follows from the proof of Proposition 2.4 that

Proposition 2.5. *If a $\mathcal{Q}$–monotone symplectic operator S maps all vectors in L_1 and L_2 into $\mathcal{C}_+$ except for a subspace of L_1 of codimension k then at least k largest singular values of S are strictly larger than one.*

§3. The Lyapunov exponents.

We are going to translate the properties of $\mathcal{J}$-separated operators into the properties of Lyapunov exponents of matrix cocycles with values in the semigroup of $\mathcal{J}$-separated matrices.

Let X be a measurable space with a probabilistic measure ν and let $\Phi : X \to X$ be an ergodic map preserving the measure ν.

Let further $S : X \to Gl(V)$ be a measurable map such that $\log \|S(x)\|$ and $\log \|S(x)^{-1}\|$ are ν–integrable functions. We will refer to this requirement as *integrability condition.* We define the matrix valued cocycle

$$
\begin{aligned}
S^m(x) &= S(\Phi^{m-1}x)\dots S(\Phi x)S(x), \quad \text{for} \quad m \geq 0,\\
S^m(x) &= S(\Phi^m x)^{-1}\dots S(\Phi^{-1}x)^{-1}, \quad \text{for} \quad m \leq -1.
\end{aligned}
$$

By the Oseledets Multiplicative Ergodic Theorem, [Ra],[Ru], there are numbers $\lambda_1 < \cdots < \lambda_N$, called the Lyapunov exponents of the measurable cocycle $S(x),\ x \in X$, and for almost all x there is an invariant splitting

$$V = V_1(x) \oplus \cdots \oplus V_N(x),$$

such that for all vectors $0 \neq v \in V_k(x)$

$$\lim_{m\to\pm\infty} \frac{1}{|m|} \log \|S^m(x)v\| = \pm\lambda_k.$$

The invariance of the splitting means that $S(x)V_k(x) = V_k(\Phi x)$.

To simplify the notation we will omit the dependence of the subspaces on $x \in X$,

The dimensions d_k of the subspaces V_k are called the multiplicities of the Lyapunov exponents.

Moreover, if we denote by $W_k = \bigoplus_{i=1}^k V_i$, we have that for any linear subspace $L \subset W_k$ having the trivial intersection with W_{k-t-1} and such that $W_{k-t} + L = W_k$ the exponential rate of growth of volume on S is equal to the sum of appropriate Lyapunov exponents taken with multiplicities,

$$
\begin{aligned}
&\lim_{m\to+\infty} \frac{1}{m} \log |det\,(S^m(x)|_L)\,| = \widetilde{d}\lambda_{k-t} + d_{k-t+1}\lambda_{k-t+1} + \cdots + \cdots + d_k\lambda_k,\\
&\widetilde{d} = \dim W_{k-t} \cap L = \dim L - (d_{k-t+1} + \cdots + d_k).
\end{aligned}
\tag{3.1}
$$

We will be assuming that the matrices $S(x), x \in X$, are all $\mathcal{J}$–separated. Let us denote by $r_p^+(x) \geq \cdots \geq r_1^+(x) \geq r_1^-(x) \geq \cdots \geq r_q^-(x) > 0$ the pseudo Euclidean singular values of $S(x)$.

Lemma 3.1. *All the functions* $\log r_k^{\pm}(x)$ *are* ν*-integrable.*

Proof. It is enough to check integrability for $\log r_p^+(x)$ and $\log r_q^-(x)$. We have $(r_p^+)^2 = \|R^2\| = \|SJS^*J\| \le \|S\|^2$, $(r_q^-)^{-2} = \|R^{-2}\| = \|J(S^*)^{-1}JS^{-1}\| \le \|S^{-1}\|^2$, which proves the claim due to the assumed integrability of $\log \|S(x)\|$ and $\log \|S(x)^{-1}\|$. □

The Lyapunov exponents $\lambda^{\pm}(v)$ (forward and backward in time) are defined for almost every initial point $x \in X$ and for all nonzero vectors $v \in V$. Let us denote by

$l_{\pm} = \int \log r_1^{\pm}(x) d\nu(x)$.

Proposition 3.2. *For any vector* $v \in \mathcal{C}_{\pm}, v \neq 0$ *we have* $\lambda^{\pm}(v) \geq \pm l_{\pm}$.

Proof. Since $\frac{1}{r_1^+(x)}S$ is $\mathcal{J}$-monotone we have $\mathcal{J}(S(x)v) \geq \left(r_1^+(x)\right)^2 \mathcal{J}(v)$ for all vectors v. In the positive cone the square of norm is not smaller than $\mathcal{J}$. Hence the exponential growth of $\mathcal{J}$ implies the exponential growth of the norm. More precisely we have for $v \in \mathcal{C}_+$,

$$\|S^m(x)v\|^2 \geq \mathcal{J}(S^m(x)v) \geq \prod_{i=0}^{m-1} \left(r_1^+(\Phi^i x)\right)^2 \mathcal{J}(v).$$

It follows from Lemma 3.1 and the Birkhoff Ergodic Theorem that

$$\lambda^+(v) \geq \int \log r_1^+(x) d\nu(x) = l_+.$$

To obtain the inequality $\lambda^-(v) \geq -l_-$. we need to use the fact that the operators $S(x)^{-1}$ are $(-\mathcal{J})$–separated and apply the argument above to $S(x)^{-1}$. □

Proposition 3.3. *If* $l_+ > l_-$ *then for almost every* $x \in X$ *there is* $m \geq 1$ *such that* $S^m(x)$ *is strictly* $\mathcal{J}$*-separated.*

A matrix cocycle with the last property will be called *eventually strictly* $\mathcal{J}$*-separated.*

Proof. It follows immediately from Corollary 1.4 since a $\mathcal{J}$-separated operator S is strictly $\mathcal{J}$-separated if and only if $r_1^+(S) > r_1^-(S)$. □

We introduce another labeling of the Lyapunov exponents and the subspaces of the splitting.

If $d_1 > q$ then we define $s_- = 0$. Similarly if $d_N > p$ we define $s_+ = 0$. Otherwise we let

$$s_- = \max\{k | \sum_{i=1}^{k} d_i \leq q\} \quad \text{and} \quad s_+ = \max\{k | \sum_{i=N-k+1}^{N} d_i \leq p\}.$$

When $s_- + s_+ = N$ we relabel the subspaces as

$$V_{-s_-} \oplus \cdots \oplus V_{-1} \oplus V_1 \oplus \cdots \oplus V_{s_+},$$

and when $s_- + s_+ = N - 1$ we use also the zero label

$$V_{-s_-} \oplus \cdots \oplus V_{-1} \oplus V_0 \oplus V_1 \oplus \cdots \oplus V_{s_+}.$$

Hence the subspace V_0 may be present or absent.

The interval of separation $[r_1^-(x), r_1^+(x)]$ varies with the operator $S(x)$. We get a gap in the Lyapunov spectrum if $l_+ = \int \log r_1^+ d\nu > \int \log r_1^- d\nu = l_-$ and in such a case V_0 is absent.

Proposition 3.4. *If V_0 is present in the new labeling, then $l_- = l_+ = \lambda_0$. If $l_+ > l_-$, then V_0 is absent in the new labeling, $\lambda_1 \geq l_+ > l_- \geq \lambda_{-1}$ and*

$$V_{-s_-} \oplus \cdots \oplus V_{-1} \subset \mathcal{C}_-, \quad V_1 \oplus \cdots \oplus V_{s_+} \subset \mathcal{C}_+. \tag{3.2}$$

Proof. If V_0 is present then the dimensions of $V_{-s_-} \oplus \cdots \oplus V_{-1} \oplus V_0$ and $V_0 \oplus V_1 \oplus \cdots \oplus V_{s_+}$ exceed q and p respectively. It follows that the former subspace must contain positive vectors and the latter negative vectors. Using Proposition 3.2 we conclude that $\lambda_0 \geq l_+$ and $-\lambda_0 \geq -l_-$.

In the same way we obtain that if V_0 is absent then $\lambda_1 \geq l_+$ and $l_- \geq \lambda_{-1}$.

It remains to show the inclusions (3.2). First of all we have the weaker inclusions

$$V_{-s_-} \oplus \cdots \oplus V_{-1} \subset \mathcal{C}_- \cup \mathcal{C}_0, \quad V_1 \oplus \cdots \oplus V_{s_+} \subset \mathcal{C}_+ \cup \mathcal{C}_0. \tag{3.3}$$

Indeed, if for example $V_{-s_-} \oplus \cdots \oplus V_{-1}$ contains any positive vectors then by Proposition 3.2 $\lambda_{-1} \geq l_+$, which violates the assumption that $l_+ > l_-$.

Further by Proposition 3.3 the assumption $l_+ > l_-$ implies that the matrix cocycle is eventually strictly $\mathcal{J}$-separated. By the invariance of the splitting we get (3.2) almost everywhere. □

It seems plausible that the weaker inclusions (3.3) must hold without any further assumptions, but we were unable to prove it.

Proposition 3.5. *If the matrix cocyle is eventually strictly $\mathcal{J}$-separated then V_0 is absent, $\lambda_1 > \lambda_{-1}$ and the inclusions* (3.2) *hold.*

Proof. By Proposition 3.4 the claims must hold for strictly $\mathcal{J}$-separated cocycles. The general case of the eventually strictly $\mathcal{J}$-separated cocycles reduces to the special case by considering the subset $X_m = \{x \in X | S^m(x) \text{ is strictly } \mathcal{J}\text{-separated}\}$. For large $m \geq 1$ the set X_m must have measure close to 1(the full measure). We further consider the return map to such a set X_m and the induced matrix cocycle which is automatically strictly $\mathcal{J}$-separated. But the splitting for the induced cocycle coincides with the original splitting restricted to the subset X_m and the Lyapunov exponents are changed by the factor equal to the average return time (cf. [K-H], Lemma S.2.8, page 665). □

Let us introduce $d_0^- = q - d_{-1} - \cdots - d_{s_-}$ and $d_0^+ = p - d_1 - \cdots - d_{s_+}$. Clearly $d_0^- + d_0^+ = \dim V_0$.

Theorem 3.6. *If a measurable cocycle $S(x)$, $x \in X$, satisfies the integrability condition and it has values in the semigroup of $\mathcal{J}$–separated matrices then for $0 \leq k \leq s_+$*

$$d_0^+ \lambda_0 + d_1 \lambda_1 + \cdots + d_k \lambda_k \geq \sum_{i=1}^{D} \int \log r_i^+ d\nu,$$

where $D = d_0^+ + d_1 + \cdots + d_k$ and for $0 \leq k \leq s_-$

$$d_0^- \lambda_0 + d_{-1} \lambda_{-1} + \cdots + d_{-k} \lambda_{-k} \leq \sum_{i=1}^{D} \int \log r_i^- d\nu.$$

where $D = d_0^- + d_{-1} + \cdots + d_{-k}$.

Proof of Theorem 3.6. We will use the formula (3.1) for the rate of volume growth on subspaces. We proceed as in the proof of Proposition 3.2. Let $L \subset W_k$ be a subspace transversal to W_{-1} and such that $W_0 + L = W_k$. An important point is that we can choose such a subspace in the positive cone $\mathcal{C}_+$, if its dimension D is equal to

$D = \dim W_k - q = d_0^+ + d_1 + \cdots + d_k$. Indeed, the family of subspaces of W_k of dimension D (or equivalently of codimension q) which are transversal to W_{-1}, is open and dense in the respective Grassmanian. At the same time the intersection of W_k with any maximal positive subspace (of dimension p) must be of dimension at least D. Hence there are positive subspaces in W_k of dimension D. Moreover the family of such subspaces is open in the respective Grassmanian. It follows that there must be positive subspaces of W_k of dimension D transversal to W_{-1} (actually there must be an open family of such subspaces in the respective Grassmanian).

If the subspace L is in the positive cone then the Euclidean volume on it is not smaller than the $\mathcal{J}$–volume. Using Proposition 1.3 we get

$$\begin{aligned}\log |det\,(S^m(x)|_L)\,| &\geq \log \alpha_D(L, S^m(x)) + c = \sum_{i=0}^{m-1} \log \alpha_D(L, S(\Phi^i x)) + c \\ &\geq \sum_{i=0}^{m-1} \log \sigma_D(S(\Phi^i x)) + c = \sum_{i=0}^{m-1} (\log r_1^+(\Phi^i x) + \cdots + \log r_D^+(\Phi^i x)) + c,\end{aligned}$$

where the constant c is the logarithm of the ratio of the $\mathcal{J}$-volume and the Euclidean volume on the subspace L. Hence in view of (3.1) we obtain the first inequality. The other inequality is obtained by observing that $S(x)^{-1}$ are $(-\mathcal{J})$–separated and applying the same argument. $\square$

If we introduce the n Lyapunov exponents repeated according to the multiplicities and labeled as $\mu_{-q} \leq \mu_{-q+1} \leq \cdots \leq \mu_{-1} \leq \mu_1 \leq \cdots \leq \mu_p$ we can formulate the following

Corollary 3.7. *For* $1 \leq k_1 \leq p$ *and* $1 \leq k_2 \leq q$

$$\mu_1 + \cdots + \mu_{k_1} \geq \sum_{i=1}^{k_1} \int \log r_i^+ d\nu \quad \textit{and} \quad \sum_{i=1}^{k_2} \int \log r_i^- d\nu \geq \mu_{-1} + \cdots + \mu_{-k_2}.$$

Proof. It follows formally from all the inequalities of Theorem 3.7. One can also repeat the proof of this theorem starting from a full noninvariant flag of subspaces as introduced in the proof of Theorem 3.9 below. $\square$

Let us now formulate the symplectic version of the estimates. Hence we assume that the matrix cocycle $S(x), x \in X$, has values in the symplectic group. In such a case the Lyapunov exponents come in pairs of opposite numbers. The following proof of this fact is taken from [W-L].

Lemma 3.8. *For an integrable cocycle $S(x)$, $x \in X$, with values in the symplectic group $Sp(\mathbb{R}^{2p})$, for any pair of non skew-orthogonal vectors $u, v \in \mathbb{R}^{2p}$, i.e., $\omega(u,v) \neq 0$, we have*

$$\lambda(u) + \lambda(v) \geq 0.$$

Proof. For the standard Euclidean norm $\|\cdot\|$ we have $|\omega(u,v)| \leq \|u\|\|v\|$. By symplecticity

$$\log|\omega(S^m(x)u, S^m(x)v)| = \log|\omega(u,v)|,$$

and

$$\frac{1}{m}\log|\omega(S^m(x)u, S^m(x)v)| \leq \frac{1}{m}\log\|S^m(x)u\| + \frac{1}{m}\log\|S^m(x)v\|.$$

□

Theorem 3.9. *If an integrable cocycle $S(x)$, $x \in X$, has values in the symplectic group $Sp(\mathbb{R}^{2p})$ then we have the following symmetry of the Lyapunov spectrum*

$$\lambda_0 = 0, \quad \textit{if } V_0 \textit{ is present}, \quad s_+ = s_- = s, \ \lambda_{-k} + \lambda_k = 0,$$

and the multiplicities of λ_{-k} and λ_k are equal, for $k = 1, 2, \ldots, s$. Moreover the subspace $W_k = \oplus_{i=-s}^{k} V_i$ is the skew-orthogonal complement of $W_{-(k+1)} = \oplus_{i=-s}^{-(k+1)} V_i$.

Proof. Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{2p}$ be the Lyapunov exponents listed with repetitions according to their multiplicities. We have $\mu_1 + \mu_2 + \cdots + \mu_{2p} = 0$.

We can choose a flag of subspaces

$$\{0\} = Z_0 \subset Z_1(x) \subset \cdots \subset Z_{2p-1}(x) \subset Z_{2p} = \mathbb{R}^{2p},$$

such that $\dim Z_l = l$ and for all vectors $v \in Z_l(x) \setminus Z_{l-1}(x)$ the Lyapunov exponent $\lambda(v) = \mu_l$, for $l = 1, 2, \ldots, 2p$. (Note that except in the case of all multiplicities equal to 1 there is a continuum of such flags.)

Since for any $l \leq p$, $\dim Z_l + \dim Z_{2p-l+1} = 2p + 1$, there are vectors $u \in Z_l$ and $v \in Z_{2p-l+1}$ such that $\omega(u,v) \neq 0$. By continuity there must be also vectors $\tilde{u} \in Z_l \setminus Z_{l-1}$ and $\tilde{v} \in Z_{2p-l+1} \setminus Z_{2p-l}$ such that $\omega(\tilde{u}, \tilde{v}) \neq 0$. It follows from Lemma 3.8 that

$$\mu_l + \mu_{2p-l+1} \geq 0,$$

for $l = 1, 2, \ldots, p$. Adding these inequalities together, we get

$$0 = \sum_{l=1}^{p} (\mu_l + \mu_{2p-l+1}) \geq 0,$$

which shows that all the inequalities must be actually equalities. It follows immediately that for any $k = 1, \ldots, s$, the multiplicities of λ_k and λ_{-k} are equal and $\lambda_k + \lambda_{-k} = 0$.

To show that the subspace W_k is the skew-orthogonal complement of the subspace $W_{-(k+1)}$ (denoted by $W_{-(k+1)}^{\angle}$) we observe that $\omega(u, v) = 0$ for any $u \in W_k$

and $v \in W_{-(k+1)}$. Indeed, if this is not the case we could use Lemma 3.8 to claim that $\lambda_k + \lambda_{-(k+1)} \geq 0$, which leads to the contradiction

$$0 = \lambda_k + \lambda_{-k} > \lambda_k + \lambda_{-(k+1)} \geq 0.$$

We obtain $W_k \subset W^{\angle}_{-(k+1)}$. Since the dimensions of these subspaces are equal they must coincide. $\square$

If all the symplectic matrices $S(x) = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}, x \in X$, are also $\mathcal{Q}$–monotone (with the standard form $\mathcal{Q}$) then using Proposition 2.4 we can reformulate Theorem 3.6 as

Theorem 3.10. *For any* $1 \leq k \leq p$

$$\mu_1 + \cdots + \mu_k \geq \sum_{i=1}^{k} \int \sinh^{-1} \sqrt{t_i} d\nu,$$

where $0 \leq t_1(x) \leq \cdots \leq t_p(x)$ *are the eigenvalues of* CB^*.

§4. The $\mathcal{J}$–separated diffeomorphisms and flows.

Let us move the discussion to a manifold of dimension n and a diffeomorhism $\Phi : M^n \to M^n$. We assume that the manifold is equipped with a continuous pseudo Riemannian metric of type $(p, q), p + q = n$, which we will denote again by $\mathcal{J}$.

We obtain the fields of positive and negative cones $\mathcal{C}_{\pm}(x), x \in M$. A diffeomorphism Φ is called $\mathcal{J}$–separated if $D\Phi\mathcal{C}_+(x) \subset \mathcal{C}_+(\Phi x)$. Similarly we define diffeomorphisms which are strictly $\mathcal{J}$–separated, $\mathcal{J}$–monotone and strictly $\mathcal{J}$–monotone.

If a diffeomorphism is $\mathcal{J}$–separated then the pseudo Riemanian singular values are well defined for the derivative $D\Phi$ at any point $x \in M$. We will denote them by

$0 < r_q^-(x) \leq \cdots \leq r_1^-(x) \leq r_1^+(x) \leq \cdots \leq r_p^+(x)$.

Proposition 4.1. *If a diffeomorphism Φ is strictly $\mathcal{J}$–separated then it has a dominated splitting, i.e,, it has a continuous invariant splitting of the tangent bundle*

$$T_xM = E^-(x) \oplus E^+(x),$$

such that for some $0 < \lambda < 1, c > 0$, *and all* $m \geq 1$

$$\|D\Phi^m{}_{|E^-(x)}\| \|D\Phi^{-m}{}_{|E^+(\Phi^m x)}\| \leq c\lambda^m. \tag{4.1}$$

Proof. At every point $x \in M$ we have the manifold of positive subspaces $\mathcal{L}_+(x)$. We will use the contraction property of the action of $D\Phi$ on $\mathcal{L}_+(x)$. The diameter of $D\Phi\mathcal{L}_+(x)$ is finite because $D\Phi$ is strictly $\mathcal{J}$–separated, and hence it is uniformly bounded by a constant c_0. It follows from Theorem 1.5 that the diameter of $D\Phi^m\mathcal{L}_+(D\Phi^{-m}x)$ does not exceed $c_0 \prod_{i=1}^{m-1} \frac{r_1^-(\Phi^{-i}x)}{r_1^+(\Phi^{-i}x)}$. Since the subsets

$$D\Phi^{m+1}\mathcal{L}_+(D\Phi^{-m-1}x) \subset D\Phi^m\mathcal{L}_+(D\Phi^{-m}x)$$

are nested and their diameter decays exponentially we get a unique point of intersection

$$E^+(x) = \bigcap_{m=1}^{\infty} D\Phi^m \mathcal{L}_+(D\Phi^{-m}x).$$

It is clear that the subspace depends continuously on x (by the standard argument it is actually Hölder continuous). To obtain the other subspace of the splitting we need to observe that Φ^{-1} is strictly $(-\mathcal{J})$–separated and repeat the above argument.

Finally we get (4.1) with $\lambda = \sup_x \frac{r_1^-(x)}{r_1^+(x)} < 1$. Indeed by the definition of the pseudo Euclidean singular values we have for every vector $v \in T_{\Phi x}M$ such that $D\Phi^{-1}v$ is positive

$$\frac{\|v\|}{\|D\Phi^{-1}v\|} \geq r_1^+(x).$$

Since the subspaces $E^+(x)$ are by construction positive and invariant we conclude that

$$\|D\Phi^{-1}{}_{|E^+(\Phi x)}\| \leq \frac{1}{r_1^+(x)}.$$

Similarly we can estimate $\|D\Phi_{|E^-(x)}\|$. $\square$

As a corollary we obtain the theorem of Lewowicz, [L],

Theorem 4.2. *If a diffeomorphism is strictly $\mathcal{J}$–monotone then it is Anosov.*

Proof. By Proposition 4.1 we obtain the continuous invariant splitting. On the unstable subspace the $\mathcal{J}$–form increases exponentially and it can be compared with any norm.

Establishing that a dynamical system is $\mathcal{J}$–separated is an important step, but this property alone allows varied and complicated behavior different from an Anosov system. For example the DA attractors can be made $\mathcal{J}$–separated, [Ro]. Also the examples studied by Hu and Young, [H-Y], are $\mathcal{J}$–separated without being Anosov.

In the piecewise differentiable setup with a chosen invariant measure ν we get for a strictly $\mathcal{J}$–separated mapping Φ a measurable invariant splitting $T_xM = E^-(x) \oplus E^+(x)$ defined ν almost everywhere and $E^+(x) \subset \mathcal{C}_+,\ E^-(x) \subset \mathcal{C}_-$. Now the contraction in the manifold of positive subspaces is not uniform but by the Birkhoff Ergodic Theorem it is pointwise exponential almost everywhere.

For an ergodic invariant measure ν we can obtain the estimates of the Lyapunov exponents in terms of the pseudo-Riemannian singular values of the $\mathcal{J}$–separated $D\Phi$. We need to assume only that $D\Phi$ is $\mathcal{J}$–separated almost everywhere with respect to the measure.

Theorem 4.3.

For a piecewise differentiable $\mathcal{J}$–separated Φ and an an ergodic invariant measure ν, if $\log \|D\Phi^{\pm 1}\|$ are integrable functions then the Lyapunov exponents

$$\mu_{-q} \leq \mu_{-q-1} \leq \cdots \leq \mu_{-1} \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_p,$$

satisfy

$$\mu_1 + \cdots + \mu_{k_1} \geq \sum_{i=1}^{k_1} \int \log r_i^+ d\nu \quad and \quad \sum_{i=1}^{k_2} \int \log r_i^- d\nu \geq \mu_{-1} + \cdots + \mu_{-k_2},$$

for any $k_1 \leq p, k_2 \leq q$*, where the coefficients* $0 < r_q^-(x) \leq \cdots \leq r_1^-(x) \leq r_1^+(x) \leq \cdots \leq r_p^+(x)$ *are the singular values of the* $\mathcal{J}$*–separated operators* $D_x\Phi$*.*

Proof. We have an auxiliary Riemannian metric on M which defines the norm $\|\cdot\|$ in the integrability condition (in $\log\|D\Phi^{\pm 1}\|$).

We introduce a measurable Riemannian metric associated to the pseudo Riemannian metric $\mathcal{J}$ by choosing a measurable field of positive subspaces $P(x), x \in M$. and considering their $\mathcal{J}$–orthogonal complements. which we denote by $Q(x), x \in M$,. The Riemannian metric is equal to $\mathcal{J}$ on P and $-\mathcal{J}$ on Q, and the two subspaces are orthogonal. Let $\|\cdot\|_J$ denote the respective norm. If we had the integrability of $\log\|D\Phi^{\pm 1}\|_J$, we could apply Theorem 3.6 directly. It turns out that we do not need to impose any further integrability conditions. Indeed, there is a measurable function $c(x) > 0$ such that

$$\frac{1}{c(x)}\|\cdot\| \leq \|\cdot\|_J \leq c(x)\|\cdot\|.$$

For every $\epsilon > 0$ consider the subset $M_\epsilon = \{x \in M \mid c(x) \geq \epsilon\}$ and the return map $\Phi_\epsilon : M_\epsilon \to M_\epsilon$. We clearly have the integrability of $\log\|D\Phi_\epsilon^{\pm 1}\|$ which leads to the integrability of $\log\|D\Phi_\epsilon^{\pm 1}\|_J$. Hence we can apply to $D\Phi_\epsilon$ the estimates in Theorem 3.6. The Lyapunov exponents for $D\Phi_\epsilon$ and $D\Phi$ differ by the average return time to M_ϵ which is close to 1 for small ϵ. The final observation is that the average of the coefficient $\sigma_d(D\Phi_\epsilon)$ is not smaller than the average of the coefficient $\sigma_d(D\Phi)$. It follows from the supermultiplicativity (1.1) of σ_d. Hence in the limit of $\epsilon \to 0$ we obtain the desired estimate. □

To develop the theory of $\mathcal{J}$–separated flows $\Phi^t : M^n \to M^n, t \in \mathbb{R}$, we need the velocity vector field $F(x)$ of the flow, $F(x) = \frac{d}{dt}\Phi^t(x)_{|t=0}$, to be nonzero everywhere and the form $\mathcal{J}$ to be degenerate. More precisely the positive index of inertia of $\mathcal{J}$ is equal to $p \geq 1$, the negative index of inertia is equal to $q \geq 1$ and $p + q = n - 1$, and for the bilinear form $\mathcal{J}$ the linear form $\mathcal{J}(F, \cdot)$ vanishes identically.

Equivalently we consider the quotient tangent spaces $\widetilde{T}_x M^n = T_x M^n / span\{F(x)\}$ and the nondegenerate forms $\mathcal{J}$ on them. The derivative $D\Phi^t$ projects naturally to the quotient spaces and we can apply the theory developed previously to the derivative cocycles on the quotient spaces.

In particular if for a smooth flow the derivative cocycle is strictly $\mathcal{J}$–separated then by Proposition 4.1 we get a splitting of the quotient space, but in general it does not lift to the splitting of the tangent space. We can lift the splitting in the case of strict $\mathcal{J}$–monotonicity by the classical proof of Anosov [A] (see also [W8]) and we obtain the counterpart of the Lewowicz's Theorem 4.2 for flows.

Theorem 4.4. *If the derivative cocycle of a smooth flow is strictly* $\mathcal{J}$*–monotone then the flow is Anosov.*

§Appendix. Boltzmann – Sinai Gas of Hard Spheres.

In our discussion we will follow the approach developed in [W2] and [W5]. We consider the system of N spherical particles in a d-dimensional vessel, either a d-dimensional torus or a box. The particles have arbitrary masses: $m_1, m_2, \ldots, m_N$, and they collide elastically with each other (and the walls of the box). The Hamiltonian of the system is

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i}$$

where $p_i \in \mathbb{R}^d$ is the momentum of i-th particle, $i = 1, \ldots, N, d \geq 2$.

The Hamiltonian differential equations are linear

$$\dot{q_i} = \frac{p_i}{m_i}, \quad \dot{p_i} = 0, \quad i = 1, ..., N.$$

Our phase space is a linear symplectic space. All its tangent spaces are naturally identified with it but for the sake of clarity we will distinguish between a point in the phase space and a tangent vector by placing δ in front of q and p. We choose the constant Lagrangian subspaces in the $(\delta q, \delta p)$-space $\mathbb{R}^{dN} \times \mathbb{R}^{dN}$

$$L_1 \equiv \mathbb{R}^{dN} \times \{0\} = \{dp_1 = \cdots = dp_N = 0\}$$

and

$$L_2 \equiv \{0\} \times \mathbb{R}^{dN} = \{dq_1 = \cdots = dq_N = 0\}.$$

This is the choice which makes the integrable system of noninteracting particles $\mathcal{Q}$–monotone.

We see that the velocity vector field is in L_1. The quadratic form $\mathcal{Q}$ is equal to

$$\mathcal{Q} = \sum_{k=1}^{N} \langle dq_k, dp_k \rangle,$$

where $\langle dq_k, dp_k \rangle = \sum_{i=1}^{d} dq_{k,i} dp_{k,i}$.

The linearized equations are

$$\frac{d}{dt}\delta q_i = \frac{\delta p_i}{m_i}, \qquad \frac{d}{dt}\delta p_i = 0.$$

We obtain immediately that between collisions

$$\frac{d}{dt}\mathcal{Q} = \sum_{i=1}^{N} \frac{(dp_i)^2}{m_i} \geq 0,$$

which means that the system is $\mathcal{Q}$–monotone. Moreover it follows that L_2 enters $\mathcal{C}_+$ immediately and completely.

We proceed to investigate collisions of two particles in dimension $d \geq 2$. Mathematically speaking the collisions are described by a symplectomorphism defined on the boundary of the phase space. More precisely not all positions of the balls are allowed since they cannot overlap. The configurations with at least two particles touching each other form the boundary of the phase space. The intersection of this boundary with a given total energy level, say $\{H = \frac{1}{2}\}$, has a canonical symplectic structure (its tangent can be identified with the factor space by the line spanned by the velocity vector). Collisions are then described by a symplectomorphism of this symplectic manifold (a gross simplification – this boundary has many discontinuities and so we only get piecewise differentiability, but it is still possible to talk about Lyapunov exponents). So we have here a combination of a Hamiltonian flow and a symplectomorphism. Such systems were discussed abstractly in [W3] and they are called there flows with collisions, the boundary being called the collision manifold.

Let the colliding particles have positions q_1, q_2, momenta p_1, p_2, masses m_1, m_2 and radii r_1, r_2. The collision manifold is given locally by $\|q_2 - q_1\| = r = r_1 + r_2$ and its tangent space by

$$\langle e, dq_2 - dq_1\rangle = 0 \quad \text{where} \quad e = \frac{1}{r}(q_2 - q_1), \|e\| = 1.$$

To facilitate the calculations we introduce locally new symplectic coordinates (x, y), the center of mass, relative position and the respective momenta

$$\begin{aligned} x_1 &= \frac{m_1}{m_1+m_2}q_1 + \frac{m_2}{m_1+m_2}q_2, & p_1 &= \frac{m_1}{m_1+m_2}y_1 - y_2 \\ x_2 &= q_2 - q_1, & p_2 &= \frac{m_2}{m_1+m_2}y_1 + y_2. \end{aligned}$$

In these coordinates the collision manifold is $\|x_2\| = r$, its tangent space $\langle x_2, dx_2\rangle = 0$ and the collision map is given by the formulas

$$y_1^+ = y_1^-, \quad y_2^+ = y_2^- - \frac{2}{r^2}\langle y_2^-, x_2\rangle x_2.$$

Differentiating we obtain

$$dy_2^+ = dy_2^- - \frac{2}{r^2}\langle dy_2^-, x_2\rangle x_2 - \frac{2}{r^2}\langle y_2^-, dx_2\rangle x_2 - \frac{2}{r^2}\langle y_2^-, x_2\rangle dx_2 \tag{A.1}$$

Since we have that

$$Q = \langle dq_1, dp_1\rangle + \langle dq_2, dp_2\rangle + \dots = \langle dx_1, dy_1\rangle + \langle dx_2, dy_2\rangle + \dots,$$

we obtain on the collision manifold

$$\begin{aligned} Q^+ - Q^- &= \langle dx_2, dy_2^+\rangle - \langle dx_2, dy_2^-\rangle = -\frac{2}{r^2}\langle y_2^-, x_2\rangle\langle dx_2, dx_2\rangle \\ &= -\frac{2m_1m_2}{(m_1+m_2)r^2}\langle \frac{p_2^-}{m_2} - \frac{p_1^-}{m_1}, q_2 - q_1\rangle\langle dq_2 - dq_1, dq_2 - dq_1\rangle \geq 0. \end{aligned} \tag{A.2}$$

Thus monotonicity is verified also for the collisions. (The inequality $\langle \frac{p_2^-}{m_2} - \frac{p_1^-}{m_1}, q_2 - q_1\rangle > 1$ holds automatically for all nondegenerate collisions. It simply means that the particles are indeed colliding and not flying away from each other.)

Let us note that when approaching collision before we can apply (A.1) and (A.2) to a vector $(\delta q, \delta p)$ we have to project it onto the tangent space of the collision manifold, $\langle e, dq_2 - dq_1\rangle = 0$, along the velocity vector of the flow.

$$(\delta q_i, \delta p_i) \mapsto \left(\delta q_i - \lambda \frac{p_i}{m_i}, \delta p_i\right), \quad \lambda = \frac{\langle \delta q_2 - \delta q_1, e\rangle}{\langle \frac{p_2}{m_2} - \frac{p_1}{m_1}, e\rangle}.$$

This projection reflects the fact that nearby trajectories do not arrive at the collision at the same time. It does not change the value of the form $\mathcal{Q}$ but it does change the tangent vector.

The formula for the increase of the form $\mathcal{Q}$ shows that if a vector in L_1 does not enter $\mathcal{C}_+$ as a result of a nondegenerate collision of the first two particles then

$$\delta q_2^- - \delta q_1^- = 0.$$

If we take into account the necessary projection preceding the application of the derivative of the collision map it means that $\delta q_2 - \delta q_1$ is parallel to $\frac{p_2}{m_2} - \frac{p_1}{m_1}$ (then the projection makes the two components equal). Thus in every nondegenerate collision of two particles all vectors from L_1 enter $\mathcal{C}_+$ except for the subspace of codimension $d-1$ which stays in L_1.

It is clear that there are many special orbits and vectors in L_1 which do not enter $\mathcal{C}$ for a long time (ever).

Establishing strict monotonicity for almost all orbits in the system is difficult. This result was obtained by Simányi and Szász, [S-Sz]. One obstacle to strict monotonicity is the possibility that the particles are split into two families with all the collisions occuring inside one of the two families. Let us denote the union of orbits on which no such splitting is possible by Z. Simányi and Szász, [S-Sz], Theorem 5.1, discovered a short and transparent proof of the fact that for the system on the torus the set Z has full measure.

Using Proposition 2.5 and Theorem 3.10 we can easily prove the following

Proposition A.1. *There are at least $(N-1)(d-1)$ positive Lyapunov exponents almost everywhere on the set Z.*

Proof. We have established that each nondegenerate collision pushes vectors from L_1 into $\mathcal{C}_+$ except for a codimension $d-1$ subspace which stays in L_1. This exceptional subspace evolves then inside L_1 until the next collision. At that time some vectors from this evolved exceptional subspace are pushed into $\mathcal{C}_+$ unless the new exceptional subspace coincides with it. In general we will obtain a new exceptional subspace of higher codimension. The question is what is the guaranteed codimension of this exceptional subspace after many collisions.

Each collision contributes $d-1$ linear conditions (equations) only in variables corresponding to the particles involved in the collision. During the evolution between collisions these equations change but no new variables may be introduced into them.

On orbits from Z after some time all the particles are connected by a chain of collisions. Let us record the equations determining the evolved exceptional subspaces generated by a minimal chain of $N-1$ collisions connecting all the particles.

We obtain $(N-1)(d-1)$ equations. We claim that they are linearly independent. Indeed, suppose k particles have collided generating $(k-1)(d-1)$ equations. These equations contain only the variables associated with the k particles. Note that in the forward evolution these equations are changed but no new variables may be added into them until a new particle collides with one of these k particles. A collision with a new particle generates $d-1$ equations linearly independent in the d variables associated with the new particle. Hence these new $d-1$ equations are linearly independent of the previous equations (because they do not depend on these new variables). We conclude that after the $N-1$ collisions in the minimal chain of collisions we will have $(N-1)(d-1)$ linearly independent equations.

It follows from the above discussion that on orbits from Z after sufficiently long time all the vectors from L_1 are pushed into $\mathcal{C}_+$ except possibly for a subspace of codimension at least $(N-1)(d-1)$. Our Propositions follows now from Proposition 2.5 and Theorem 3.10. $\square$

Note that in the above proof no distinction is made between the cases of the torus and the cube as vessels. In the case of the torus the total momentum is the first integral of the system which leads to $2d$ zero Lyapunov exponents. Complete hyperbolicity has not yet been established for the case of particles in a box, although there is no doubt that it does hold. The special case of two particles was studied by Simányi, [S2].

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