A remark on L^p-boundedness of wave operators for two dimensional Schrödinger operators

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Abstract

Let $H = -\Delta + V$ be a two dimensional Schrödinger operator with a real potential $V(x)$ satisfying the decay condition $|V(x)| \leq C\langle x\rangle^{-\delta}$, $\delta > 6$. Let $H_0 = -\Delta$. We show that the wave operators $s\text{-}\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}$ are bounded in $L^p(\mathbf{R}^2)$ under the condition that H has no zero resonances or bound states. In this paper thecondition $\int_{\mathbf{R}^2} V(x)dx \neq 0$, imposed in a previous paper (K. Yajima, Commun. Math. Phys. **208** (1999), 125–152), is removed.

1 Introduction

Let $H = -\Delta + V$ and $H_0 = -\Delta$ be Schrödinger operators in $L^2(\mathbf{R}^2)$. We assume that V is multiplication by a function $V(x)$, which satisfies the following condition:

Assumption 1.1. $V(x)$ is real-valued and $|V(x)| \leq C\langle x\rangle^{-\delta}$, $x \in \mathbf{R}^2$, for some $\delta > 6$.

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It is well-known that under this assumption the wave operators $W_\pm$ defined by the limits

$$W_\pm u = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}u, \quad u \in L^2(\mathbf{R}^2),$$

exist and are complete, i.e. $\operatorname{Ran} W_\pm = L^2_{\mathrm{ac}}(H)$, the absolutely continuous subspace of $L^2(\mathbf{R}^2)$ for H, and the singular continuous spectrum of H is absent.

In this note we prove the following theorem:

Theorem 1.2. *Let Assumption* 1.1 *be satisfied. Suppose that* 0 *is neither an eigenvalue nor a resonance of* H*, viz. there are no solutions* $u \in H^2_{\mathrm{loc}}(\mathbf{R}^2)$ *of* $-\Delta u + Vu = 0$*, which satisfy for* $|\alpha| \le 1$

$$\partial_x^\alpha \left(u - a - \frac{b_1x_1 + b_2x_2}{|x|^2} \right) = O(|x|^{-1-\varepsilon-|\alpha|}), \quad |x| \to \infty. \tag{1.1}$$

Then the wave operators $W_\pm$ *are bounded in* $L^p(\mathbf{R}^2)$ *for all* p*,* $1 < p < \infty$*.*

In [2], one of the authors has shown Theorem 1.2 under the additional assumption that $\int_{\mathbf{R}^2} V(x)dx \neq 0$. This additional assumption was made to simplify the asymptotic analysis as $\lambda \to 0$ of the boundary values $R^\pm(\lambda) = \lim_{\varepsilon\downarrow 0} R(\lambda \pm i\varepsilon)$ on the reals of the resolvent $R(z) = (H-z)^{-1}$ of H. By applying the recent results [1] of the other author with G. Nenciu on precisely this asymptotic problem, we show that this additional assumption is unnecessary.

2 Proof of the Theorem

We choose $c > 0$ sufficiently small and let $\chi(t) \in C_0^\infty([0,\infty))$ be a cut-off function such that $\chi(t) = 1$ for $t \le c/2$ and $\chi(t) = 0$ for $t \ge c$. We set $\tilde\chi(t) = 1 - \chi(t)$. The argument in Sections 2 and 3 of [2] does not use the assumption $\int_{\mathbf{R}^2} V(x)dx \neq 0$, and it implies that the high energy part of the wave operators $W_\pm\tilde\chi(H_0)$ are bounded in $L^p(\mathbf{R}^2)$ for $1 < p < \infty$. Thus we have only to prove that the low energy part $W_\pm\chi(H_0)$ are bounded in $L^p(\mathbf{R}^2)$ for $1 < p < \infty$.

2.1 Preliminaries

It suffices to consider W_+. We record some results from [1] and [2] which we need in what follows.

The following three results are Proposition 2.1, Lemma 4.4 and Lemma 4.1 of [2], respectively. We define the operator $W^{(1)}(V)$ depending on a function V by

$$W^{(1)}(V)u = -\frac{1}{2\pi i}\int_0^\infty R_0^-(\lambda)V\{R_0^+(\lambda) - R_0^-(\lambda)\}u\,d\lambda \tag{2.1}$$

for $u \in \mathcal{S}(\mathbf{R}^2)$. Here $R_0^\pm(\lambda) = \lim_{\varepsilon\downarrow 0} R_0(\lambda \pm i\varepsilon)$ denote the boundary values of the free resolvent. As is well known, these boundary values exist for $\lambda > 0$ in $\mathcal{B}(L^{2,s}(\mathbf{R}^2), L^{2,-s}(\mathbf{R}^2))$ for $s > 1/2$.

Lemma 2.1. *If $V \in L^{2,s}(\mathbf{R}^2)$ for some $s > 1$, then $W^{(1)}(V)$ extends to a bounded operator in $L^p(\mathbf{R}^2)$ for any p, $1 < p < \infty$, and*

$$\|W^{(1)}(V)\|_{\mathcal{B}(L^p)} \le C_{sp}\|\langle x\rangle^s V\|_2. \tag{2.2}$$

Corollary 2.2. *Suppose that K is an integral operator with the integral kernel $K(x,y)$ and that K satisfies*

$$\int_{\mathbf{R}^2}\left(\int_{\mathbf{R}^2}\langle x\rangle^{2s}|K(x,x-y)|^2dx\right)^{1/2}dy \equiv \|K\|_s < \infty \tag{2.3}$$

for some $s > 1$. Then the operator Z, defined by

$$Zu = -\frac{1}{2\pi i}\int_0^\infty R_0^-(\lambda)K\{R_0^+(\lambda) - R_0^-(\lambda)\}u\,d\lambda \tag{2.4}$$

for $u \in \mathcal{S}(\mathbf{R}^2)$, can be extended to a bounded operator in $L^p(\mathbf{R}^2)$ for any p, $1 < p < \infty$, and furthermore $\|Zu\|_p \le C_{sp}\|K\|_s\|u\|_p$.

Lemma 2.3. *Suppose that $N(k)$ satisfies for some $s > 3$*

$$\|(d/dk)^j N(k)\|_{\mathcal{B}(L^{2,-s},L^{2,s})} \le C_j k^{2-j}\langle \log k\rangle \tag{2.5}$$

for $j = 0,1,2$ and for $0 < k < c$. Then the operator A, defined by

$$Au = -\frac{1}{\pi i}\int_0^\infty R_0^-(k^2)N(k)\{R_0^+(k^2) - R_0^-(k^2)\}\chi(k^2)u\,k\,dk \tag{2.6}$$

for $u \in \mathcal{S}(\mathbf{R}^2)$, can be extended to a bounded operator in $L^p(\mathbf{R}^2)$ for any p, $1 \le p \le \infty$.

For studying the low energy behavior of $R^\pm(k^2)$ we define, following [1],

$$U(x) = \begin{cases} 1 & \text{if } V(x) \ge 0, \\ -1 & \text{if } V(x) < 0, \end{cases}$$

and

$$v(x) = |V(x)|^{1/2}, \quad w(x) = U(x)v(x).$$

We also need

$$M^{\pm}(k) = U + vR_0^{\pm}(k^2)v, \quad k > 0.$$

Define the orthogonal projections in $L^2(\mathbf{R}^2)$ by

$$P = \|V\|_1^{-1} v \otimes v, \quad Q = 1 - P.$$

It follows from the results in [1] and Assumption 1.1 that

$$M^{\pm}(k) = U + c^{\pm}(k)P + vG_0v + O(k^2 \log k) \tag{2.7}$$

in the operator norm of $\mathcal{B}(L^2)$, where $c^{\pm}(k) = a^{\pm} + b^{\pm} \log k$, and G_0 is the integral operator with the integral kernel

$$G_0(x, y) = -\frac{1}{2\pi} \log |x - y|.$$

The term $O(k^2 \log k)$ stands for a $\mathcal{B}(L^2)$-valued C^2 function $\tilde{N}(k)$, which satisfies

$$\|d^j/dk^j \tilde{N}(k)\|_{\mathcal{B}(L^2)} \leq Ck^{2-j}\langle \log k \rangle, \quad 0 < k < c, \tag{2.8}$$

for $j = 0, 1, 2$. The differentiability of the expansion (2.7) is easily verified using the results in [1]. Note that the decay rate $V(x) = O(\langle x \rangle^{-\delta})$, $\delta > 6$, suffices in order to differentiate twice. The error term is handled using an appropriate version of the remainder in Taylor's formula and the results in [1]. Hereafter we denote operators which satisfy (2.8) indiscriminately by $O(k^2 \log k)$.

Let $M_0 = U + vG_0v$. It is known (cf. [1, Theorem 6.2]) that

$$QM_0Q \text{ is invertibel in } QL^2(\mathbf{R}^2),$$

if and only if 0 is neither an eigenvalue nor a resonance of H and, in that case,

$$\begin{aligned} M^{\pm}(k)^{-1} = g^{\pm}(k)^{-1}\{&P - PM_0QD_0Q - QD_0QM_0P \\ &+ QD_0QM_0PM_0QD_0Q\} \\ &+ QD_0Q + O(k^2 \log k), \end{aligned} \tag{2.9}$$

where $g^{\pm}(k) = c^{\pm} \log k + d^{\pm}$ with non-vanishing constant $c^{\pm}$, and where we introduced the notation $D_0 = (QM_0Q)^{-1}$, see formula (6.27) of [1]. Notice that each of the operators in the braces is a rank one operator. With $\alpha = \|V\|_1$, and $v_1 = QD_0QM_0v$ we have

$$P = \alpha^{-1} v \otimes v, \qquad PM_0QD_0Q = \alpha v \otimes v_1, \tag{2.10}$$

$$QD_0QM_0P = \alpha v_1 \otimes v, \qquad QD_0QM_0PM_0QD_0Q = \alpha v_1 \otimes v_1. \tag{2.11}$$

Lemma 2.4. *The operator $QD_0Q - QUQ$ is an operator of Hilbert-Schmidt type.*

Proof. Since QM_0Q is invertible in $QL^2(\mathbf{R}^2)$, the operator $T = P + QM_0Q$ is invertible in $L^2(\mathbf{R}^2)$ and $D_0 = QT^{-1}Q$. Clearly

$$T = U + \{vG_0v + P + PM_0P - PM_0Q - QM_0P\} \equiv U(1 + S).$$

Here P, PM_0P, PM_0Q, and QM_0P are rank one operators, and vG_0v is of Hilbert-Schmidt type, since $v(x) = O(\langle x \rangle^{-\delta/2})$, $\delta/2 > 3$. Thus S is a Hilbert-Schmidt operator. Since U is invertible, we have that $1 + S$ is also invertible. Using

$$(1 + S)^{-1} = 1 - S(1 + S)^{-1},$$

it follows that $T^{-1} - U$ is a Hilbert-Schmidt operator, which implies the result in the lemma. □

2.2 The Proof

By the stationary representation formula for the wave operators we have

$$W_+\chi(H_0)u = \chi(H_0)u - \frac{1}{2\pi i} \int_0^\infty R^-(\lambda)V\{R_0^+(\lambda) - R_0^-(\lambda)\}\chi(\lambda)u \, d\lambda. \tag{2.12}$$

The operator $\chi(H_0)$ has a smooth and rapidly decreasing integral kernel, so it is bounded in $L^p(\mathbf{R}^2)$ for any $1 \leq p \leq \infty$. Hence, we need to study the operator W_1 defined by the integral on the right of (2.12). Change to the variable k determined by $\lambda = k^2$, and use the formula

$$R^{\pm}(k^2)V = R_0^{\pm}(k^2)vM^{\pm}(k)v, \tag{2.13}$$

cf. Section 4 in [1]. Then

$$W_1u = -\frac{1}{\pi i} \int_0^\infty R_0^-(k^2)vM^-(k)^{-1}v\{R_0^+(k^2) - R_0^-(k^2)\}\chi(k^2)u \, k \, dk. \tag{2.14}$$

By virtue of (2.9), (2.10), (2.11), and Lemma 2.4, we have

$$M^-(k)^{-1} = d(k)F + L + U + O(k^2 \log k), \quad d(k) = g^-(k)^{-1}, \tag{2.15}$$

where F is of rank 3, and L is of Hilbert-Schmidt type. It follows that the integral kernels $K_1(x, y)$ and $K_2(x, y)$ of vFv and $v(L + U)v$ satisfy the condition (2.3) of Corollary 2.2. Thus,

$$W_{11}u = -\frac{1}{\pi i} \int_0^\infty R_0^-(k^2)vFv\{R_0^+(k^2) - R_0^-(k^2)\}\chi(k^2)u\, k\, dk, \tag{2.16}$$

$$W_{12}u = -\frac{1}{\pi i} \int_0^\infty R^-(k^2)v(L + U)v\{R_0^+(k^2) - R_0^-(k^2)\}\chi(k^2)u\, k\, dk, \tag{2.17}$$

are bounded in $L^p(\mathbf{R}^2)$ for $1 < p < \infty$. On the other hand $vO(k^2 \log k)v$ satisfies the condition (2.5) of Lemma 2.3, since the error term in (2.15) is found using the Neumann series, cf. [1], and since the error term in (2.7) satisfies (2.8). Therefore we can apply Lemma 2.3 to conclude that

$$W_{13}u = -\frac{1}{\pi i} \int_0^\infty R_0^-(k^2)vO(k^2 \log k)v\{R_0^+(k^2) - R_0^-(k^2)\}\chi(k^2)u\, k\, dk \tag{2.18}$$

is bounded in $L^p(\mathbf{R}^2)$ for $1 \le p \le \infty$. Thus,

$$W_1 = W_{11}d(|D|) + W_{12} + W_{13}$$

is bounded in $L^p(\mathbf{R}^2)$ for $1 < p < \infty$, since $d(|D|)$ is bounded in $L^p(\mathbf{R}^2)$ for $1 < p < \infty$ by the standard Fourier multiplier theorem.

References

[1] A. Jensen and G. Nenciu, *A unified approach to resolvent expansion at thresholds*. Rev. Math. Phys. **13** (2001), 717–754.

[2] K. Yajima, *L^p-boundedness of wave operators for two dimensional Schrödinger Operators*, Commun. Math. Phys. **208** (1999), 125–152.