VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS, AND ASYMPTOTICS FOR HAMILTONIAN SYSTEMS

DIOGO AGUIAR GOMES

U.C. Berkeley - CA, US and I.S.T. - Lisbon, Portugal
email:dgomes@math.ist.utl.pt

Abstract. In this paper we apply the theory of viscosity solutions of Hamilton-Jacobi equations to understand the structure of certain Hamiltonian flows. In particular, we describe the asymptotic behavior of minimizing orbits of Hamiltonian flows by proving a weak KAM theorem which holds under very general conditions. Then, using Mather measures, we prove results on the uniform continuity, difference quotients and non-uniqueness of solutions of time-independent Hamilton-Jacobi equations.

1. Introduction

Consider the Hamiltonian differential equations

$$\dot{x} = D_p H(p, x) \qquad \dot{p} = -D_x H(p, x), \tag{1}$$

where $H(p, x) : \mathbb{R}^{2n} \to \mathbb{R}$ is smooth function. The first objective of this paper is to understand the asymptotic behavior of certain trajectories of (1). The second is to make clear the connections between Mather measures [Mat91] (invariant measures under the flow of (1) with certain minimizing properties) and viscosity solutions of Hamilton-Jacobi equations

$$H(P + D_x u, x) = \overline{H}(P). \tag{2}$$

The third and last objective is to study the regularity properties of viscosity solutions of the previous equation.

We make the assumption that H is strictly convex in p, and $\mathbb{Z}^n$ periodic in x, i.e.

$$H(p, x + k) = H(p, x)$$

for all $(p, x) \in \mathbb{R}^{2n}$ and $k \in \mathbb{Z}^n$. This hypothesis is satisfied in many important applications, for instance, the motion of particles in a lattice potential or perturbations of Hamiltonian systems in action-angle coordinates.

It is well known [Arn99] that by solving a Hamilton-Jacobi PDE (2) it is possible to construct a change of variables that simplifies the dynamics of (1). Suppose for each $P \in \mathbb{R}^n$ there are smooth functions $\overline{H}(P)$ and $u(x, P)$ solving the partial differential equation (2). Furthermore assume that the equations

$$p = P + D_x u(x, P) \qquad X = x + D_P u(x, P) \tag{3}$$

define a smooth change of coordinates $(x, p) \to (X, P)$. Then in the new coordinates (X, P) the ODE (1) is

$$\dot{X} = D_P \overline{H}(P) \qquad \dot{P} = 0. \tag{4}$$

Therefore, since (4) is trivial to solve, we would have solved (1), up to changes of coordinates.

Unfortunately there are several points where this method can fail. Firstly (2) may not have any classical solution. Secondly, for fixed P there is not uniqueness and therefore u may not be differentiable in P. Finally, in the very special situation where u is smooth both in P and in x, (4) may not be solvable or may not define a gobal smooth change of coordinates.

Ignoring the previous remarks, we point out the following facts:

- Since $\dot{P} = 0$, for each P there exists an invariant set, the graph $p = P + D_x u$.
- In this set the trajectories are straight lines (up to a change of coordinates), because $\dot{X} = D_P \overline{H}(P)$.
- Since $D_P u$ is bounded, solutions with initial conditions on the invariant set have the asymptotic property
$$\lim_{t \to \infty} \frac{x(t)}{t} = D_P \overline{H}.$$

It turns out that these statements (or analogs of them) are still true as long as classical solutions are replaced by viscosity solutions.

Recall that u is a viscosity solution of (2), provided that whenever ϕ is a smooth function such that $u - \phi$ has a local maximum at a point x_0 (resp. minimum) then $H(P + D_x \phi(x_0), x_0) \le \overline{H}(P)$ (resp. $\ge$). A classical-yet-unpublished theorem from Lions, Papanicolaou, and Varadhan [LPV88] (see also [Con95], [Con97], [Con96], or [BD98]) guarantees the existence of a viscosity solution of (2). More precisely, for each P there exists a unique $\overline{H}(P)$ and a function $u(x,P)$ (possibly not unique), solving (2) in the viscosity sense. Furthermore $\overline{H}$ is convex in P and $u(x,P)$ Lipschitz in x.

If u is a viscosity solution of (2) then it satisfies the following equation [FS93]

$$u(x,P) = \inf_{x(0)=x} \int_0^t \left[L(x,\dot{x}) + P\dot{x} + \overline{H}(P)\right] ds + u(x(t),P), \tag{5}$$

where the infimum is taken over all Lipschitz trajectories $x(\cdot)$ with initial condition x, and $L(x,v) = \sup_p \left[-v\cdot p - H(p,x)\right]$ is the Legendre transform of H. Furthermore there exists an optimal trajectory $x^*(s)$, $0 \leq s \leq t$. Let $p^* = P + D_v L(x^*, \dot{x}^*)$. Then (x^*, p^*) is a solution of the backwards Hamilton equations

$$\dot{x}^* = -D_p H(x^*, p^*) \qquad \dot{p}^* = D_x H(x^*, p^*). \tag{6}$$

For $0 < s < t$, $p^*(s) = P + D_x u(x^*(s), P)$, in particular u is differentiable along the optimal trajectory and if u is differentiable at x, $p^*(0) = P + D_x u(x,P)$.

The results by A. Fathi [Fat97a], [Fat97b], [Fat98a], [Fat98b], and W. E [E99] make clear the connection between viscosity solutions and Hamiltonian dynamics. The main idea is that if $u(x,P)$ is a viscosity solution of (2) then there exists an invariant set $\mathcal{I}$ contained on the graph

$$\{(x, P + D_x u(x,P))\}.$$

Furthermore, $\mathcal{I}$ is a subset of a Lipschitz graph, i.e. $D_x u(x,P)$ is a Lipschitz function on $\pi(\mathcal{I})$, where $\pi(x,p) = x$. If $\overline{H}$ is differentiable at P, then any solution $(x(t), p(t))$ of (1) with initial conditions on $\mathcal{I}$ satisfies

$$\lim_{t\to\infty} \frac{|x(t) - D_P\overline{H}t|}{t} = 0. \tag{7}$$

We improve the asymptotic estimate (7) using viscosity solutions methods (theorem 1). Then we prove that there is a one-to-one correspondence between Mather measures and viscosity solutions (see also [EG99]). This, for instance, explains why viscosity solutions of (2) may not be unique. Then we give conditions under which the solution

$u(x,P)$ is uniformly continuous in P. The proof relies in understanding how viscosity solutions encode information about Mather measures. Finally, we prove that change of coordinates (3) satisfies a weak nondegeneracy condition if $\overline{H}$ is uniformly convex.

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2. Improved Asymptotics

Suppose that u is a periodic viscosity solution of (2). Define

$$\mathcal{G} = \{(x, P + D_x u) : u \, \text{is differentiable at} \, x\}.$$

Let Ξ_t be the flow corresponding to (6). Observe that, for all $t > 0$, $\Xi_t(\mathcal{G}) \subset \mathcal{G}$. Define $\mathcal{G}_t = \Xi_t(\overline{\mathcal{G}})$. Let

$$\mathcal{I} = \cap_{t>0} \mathcal{G}_t.$$

Then [E99] $\mathcal{I}$ is a nonempty closed invariant set for the Hamiltonian flow. Furthermore, if $\overline{H}(P)$ is differentiable at P, the trajectories $(x(t), p(t))$ of the (forward) Hamiltonian flow with initial conditions on the invariant set $\mathcal{I}(P)$ satisfy

$$\lim_{t\to\infty} \frac{x(t)}{t} = D_P \overline{H}(P).$$

The main result in this section improves the previous asymptotic estimate.

Theorem 1. *Suppose $(x(t), p(t))$ is a solution of (1) with initial conditions on the invariant set $\mathcal{I}$. Furthermore assume $\overline{H}$ is twice differentiable at P. Let*

$$\|x - y\| \equiv \min_{k\in\mathbb{Z}^n} |x - y + k|,$$

i.e., the "periodic distance" between x and y. Then there exists a constat C such that

$$|x(t) - x(0) - D_P \overline{H} t| \leq C\sqrt{\|x(t) - x(0)\| t}.$$

Suppose there exists a continuous function ω, with $\omega(0) = 0$, such that

$$|u(x,P) - u(x,P')| \leq \omega(|P - P'|).$$

Then

$$|x(t)-x(0)-D_P\overline{H}t| \leq \min_{\delta} \frac{\|x(t)-x(0)\| \wedge \omega(\delta)}{\delta} + Ct\delta. \tag{8}$$

Finally, if u is uniformly differentiable in P in $\mathcal{I}$,

$$x(t)+D_Pu(x(t),P)-x(0)-D_Pu(x(0),P)-D_P\overline{H}t=0.$$

Proof. Let $u(x,P)$ be a viscosity solution of (2). Then, for some C^1 function $x^*(s)$ with $x^*(0)=x$

$$u(x,P)=u(y,P)+\int_0^t \left[L(x^*,\dot{x}^*)+P\cdot\dot{x}^*+\overline{H}(P)\right]ds,$$

where $y=x^*(t)$. Then, for any other P'

$$u(x,P')\leq u(y,P')+\int_0^t \left[L(x^*,\dot{x}^*)+P'\cdot\dot{x}^*+\overline{H}(P')\right]ds.$$

Thus

$$\begin{aligned} u(x,P)-u(x,P') \geq & u(y,P)-u(y,P')+ \\ & +\int_0^t \left[(P-P')\cdot\dot{x}^*+\overline{H}(P)-\overline{H}(P')\right]ds. \end{aligned}$$

If $\overline{H}$ is twice differentiable (or at least $C^{1,1}$) at P we have

$$\overline{H}(P')\leq\overline{H}(P)-\omega\cdot(P'-P)+C|P'-P|^2,$$

where $\omega=-D_P\overline{H}(P)$. Thus

$$\begin{gathered} u(x,P)-u(x,P')+u(y,P')-u(y,P)\geq \\ (P-P')\cdot\int_0^t[\dot{x}^*-\omega]-Ct|P'-P|^2. \end{gathered}$$

The left hand side can be estimated by

$$u(x,P)-u(x,P')+u(y,P')-u(y,P)\leq\|x-y\|.$$

Choose $|P-P'|=\sqrt{\frac{\|x-y\|}{t}}$ then

$$|\int_0^t[\dot{x}^*-\omega]|\leq C\sqrt{\|x-y\|t}.$$

Now assume that there exists a continuous function ω, with $\omega(0)=0$, such that

$$|u(x,P)-u(x,P')|\leq\omega(|P-P'|)$$

then we have

$$u(x,P)-u(x,P')+u(y,P')-u(y,P)\leq \|x-y\|\wedge\omega(|P-P'|).$$

Finally if u is uniformly differentiable in P we get

$$x(t)+D_Pu(x(t),P)-x(0)-D_Pu(x(0),P)-D_P\overline{H}t=0.$$

This last equality shows that whenever the expression

$$X=x+D_Pu$$

is well defined in the invariant set, we have $\dot{X}=D_P\overline{H}$. ∎

In section 4 we investigate sufficient conditions for the existence of a modulus of continuity $\omega(\delta)$ for u. Such conditions in conjunction with estimate (8) yield a sharper asymptotic estimate.

3. Mather measures

Let $(x(t),p(t))$ be a trajectory with initial conditions on the invariant set $\mathcal{I}$. For $E\subset T^n\times\mathbb{R}^n$ define the measure

$$\mu_T(E)=\frac{1}{T}\int_0^T 1_E(x(t),p(t)).$$

μ_T is a probability measure and since $p(t)$ is bounded we can extract a weakly converging subsequence to some measure μ. Since $\mathcal{I}$ is closed this measure will be supported on $\mathcal{I}$. Such measures are called a Mather measures. The two theorems in this section prove the equivalence of our definition of Mather measure and the usual one [Mat91], i.e., that μ minimizes

$$\int L+P\cdot v d\eta,$$

over all probability measures η that are invariant under the flow Ξ_t.

Theorem 2. *Suppose μ a Mather measure, associated with a periodic viscosity solution of*

$$H(P+D_xu,x)=\overline{H}(P).$$

Then μ minimizes

$$\int L+P\cdot v d\eta,$$

over all probability measures η that are invariant under the flow Ξ_t.

Proof. If the claim were false, there would be an invariant probability measure ν such that

$$-\overline{H} = \int L + P \cdot v d\mu > \int L + P \cdot v d\nu = -\lambda.$$

We may assume that ν is ergodic, otherwise we could choose an ergodic component of ν for which the previous inequality holds. Take a generic point of (x, v) in the support of ν and consider its orbit $x(s)$. Then

$$u(x(0), P) - \overline{H}(P)t \leq \int_0^t L(x(s), \dot{x}(s)) + P \cdot \dot{x}(s) ds + u(x(t), P).$$

As $t \to \infty$

$$\frac{1}{t} \int_0^t L(x(s), \dot{x}(s)) + P\dot{x}(s) ds \to -\lambda,$$

by the ergodic theorem. Hence

$$-\overline{H} \leq -\lambda,$$

which is a contradiction. ■

Next we prove that any of Mather's measures is "embedded" in a viscosity solution of a Hamilton-Jacobi equation. To do so recall the result from [Mn96]: suppose μ is a ergodic minimizing measure. Then there exists a Lipschitz function $W : \operatorname{supp}(\mu) \to \mathbb{R}$ and a constant $\overline{H}(P) > 0$ such that

$$-L - P \cdot v = \overline{H}(P) + D_x W v + D_p W D_x H.$$

By taking Was initial condition (interpreting W as a function of x alone instead of (x, p) - which is possible because $\operatorname{supp} \mu$ is a Lipschitz graph) we can embed this minimizing measure in one of our measures ν_t. More precisely we have:

Theorem 3. *Suppose μ is a ergodic Mather measure. Then there exists a viscosity solution u of (2) such that $u = W$ on $\operatorname{supp}(\mu)$. Furthermore, for almost every $x \in \operatorname{supp}(\mu)$ the measures ν_t obtained by taking minimizing trajectories that pass trough x coincides with μ.*

Proof. Consider the terminal value problem $V(x, 0) = W(x)$ if $x \in \operatorname{supp}(\mu)$ and $V(x, 0) = +\infty$ elsewhere, with

$$-D_t V + H(P + D_x V, x) = \overline{H}(P).$$

Then, for $x \in \operatorname{supp}(\mu)$

$$V(x, -t) = W(x).$$

Also if $x \notin \operatorname{supp}(\mu)$ then

$$V(x,-t) \leq V(x,-s),$$

if $s < t$. Hence, as $t \to \infty$ the function $V(x,-t)$ decreases pointwise. Since V is bounded and uniformly Lipschitz in x it must converge uniformly (because V is periodic) to some function u. Then u will be a viscosity solution of

$$H(P + D_x u, x) = \overline{H}(P).$$

Since $u = W$ on the support of μ, the second part of the theorem is a consequence of the ergodic theorem. ■

Finally, to complete the picture, recall a theorem from [EG99] that states that any Mather measure is supported on the graph $p = P + D_x u$, for any u viscosity solution of (2). This theorem shows that any viscosity solution of (2) encodes all the information about all possible Mather measures.

In case in which, for the same P, there are distinct Mather measures $\nu_1 \dots \nu_k$ with disjoint supports, we could use the functions $W_1 \dots W_k$ as initial condition (with $+\infty$ outside the union of $\operatorname{supp} \nu_i$) to construct a viscosity solution of (2). By adding arbitrary constants to W_i we can change the solution, therefore proving non-uniqueness.

4. Uniform Continuity of Viscosity Solutions

This section adresses the question whether viscosity solutions of (2) are continuous or not in P. Obviously, adding an arbitrary function of P to u produces another viscosity solution. We could think that by defining a new family of solutions $v = u + f(P)$, with an appropriate choice for f (for instance such that $v(0, P) = 0$) we would get a continuous family of solutions v. However, the non-uniqueness observation from the previous section implies that such results are not to be expected, in general. However, as we prove bellow, when there is a unique Mather measure ν (unique ergodicity) then, u is uniformly continuous in P on the support of ν.

Proposition 1. *Suppose ν is a Mather measure as in the previous section. Let $P_n \to P$. Then there exists a point x in the support of ν such that for any T*

$$\sup_{0 \leq t \leq T} |u(x^*(t), P) - u(x^*(t), P_n)| \to 0,$$

as $n \to \infty$, provided $u(x, P_n) = u(x, P)$.

Proof. We start by proving an auxiliary lemma

Lemma 1. *There exist a point (x, p) in the support of ν, and sequences $x_n, \tilde{x}_n \to x$, $p_n, \tilde{p}_n \to p$, with $(x_n, p_n) \in \operatorname{supp} \nu$ optimal pair for P, and $(\tilde{x}_n, \tilde{p}_n)$ optimal pairs for P_n.*

Proof. Take a generic point (x_0, p_0) in the support of ν. Let $x^*(t)$ be the optimal trajectory for P with initial condition (x_0, p_0). Then for all $t > 0$

$$H(P + D_x u(x^*(t), P), x^*(t)) = \overline{H}(P).$$

Also, for almost every y, we have

$$H(P + D_x u(x^*(t) + y, P_n), x^*(t)) = \overline{H}(P_n) + O(|y|),$$

for almost every t. Choose y_n with $|y_n| \leq |P - P_n|$ such that the previous identity holds. By strict convexity of H in p, we get

$$\dot{x}^*(t)\xi + \theta\xi^2 \leq C|P_n - P|,$$

where

$$\xi = \left[P - P_n + D_x u(x^*(t), P) - D_x u(x^*(t) + y_n, P_n)\right],$$

and

$$\dot{x}^*(t) = -D_p H(P + D_x u(x^*(t), P), x^*(t)).$$

Note that

$$\left|\frac{1}{T}\int_0^T \dot{x}^*(t)\xi\right| | \leq |P - P_n| + \frac{|u(x^*(0), P) - u(x^*(T), P)|}{T} + \\ + \frac{|u(x^*(0) + y_n, P_n) - u(x^*(T) + y_n, P_n)|}{T}.$$

Therefore we may choose T (depending on n) such that

$$\left|\frac{1}{T}\int_0^T \dot{x}^*(t)\xi\right| \leq 2|P - P_n|.$$

Thus

$$\frac{1}{T}\int_0^T |P + D_x u(x^*(t), P) - P_n - D_x u(x^*(t) + y_n, P_n)|^2 \leq \\ \leq C|P - P_n|.$$

Choose t_n for which

$$|P + D_x u(x^*(t_n), P) - P_n - D_x u(x^*(t_n) + y_n, P_n)|^2 \leq C|P - P_n|.$$

Let $x_n = x^*(t_n)$, $\tilde{x}_n = x^*(t_n) + y_n$, $p_n = P + D_x u(x^*(t_n), P)$, and $\tilde{p}_n = P_n + D_x u(x^*(t_n) + y_n, P)$. By extracting a subsequence, if necessary, we may assume $x_n \to x$, $\tilde{x}_n \to x$, etc. ▪

To see that the lemma proves the proposition, let $x_n^*(t)$ be the optimal trajectory for P with initial conditions (x_n, p_n). Similarly let $\tilde{x}_n^*(t)$ be the optimal trajectory for P_n with initial conditions $(\tilde{x}_n, \tilde{p}_n)$. Then

$$u(x_n, P) = \int_0^t L(x_n^*, \dot{x}_n^*) + P \cdot \dot{x}_n^* + \overline{H}(P) ds + u(x_n^*(t), P),$$

and

$$u(\tilde{x}_n, P_n) = \int_0^t L(\tilde{x}_n^*, \dot{\tilde{x}}_n^*) + P_n \cdot \dot{\tilde{x}}_n^* + \overline{H}(P_n) ds + u(\tilde{x}_n^*(t), P_n).$$

On $0 \leq t \leq T$ both x_n^* and $\tilde{x}_n^*$ converge uniformly to x^*. and, since by hypothesis $u(x_n, P), u(\tilde{x}_n, P_n) \to u(x, P)$, we conclude that $u(\tilde{x}_n^*(t), P_n) - u(x_n^*(t), P) \to 0$ uniformly on $0 \leq t \leq T$. Therefore $u(x^*(t), P_n) - u(x^*(t), P) \to 0$ uniformly on $[0, T]$. ■

Theorem 4. *Suppose ν is an ergodic Mather measure with $\nu|_{\mathrm{supp}(\nu)}$ uniquely ergodic with respect to the restricted flow. Assume $P_n \to P$. Then*

$$u(x, P_n) \to u(x, P),$$

uniformly on the support of ν, provided that an appropriate constant $C(P_n)$ is added to $u(x, P_n)$.

PROOF. Fix $\epsilon > 0$. We need to show that if n is sufficiently large then

$$\sup_{x \in \mathrm{supp}(\nu)} |u(x, P_n) - u(x, P)| < \epsilon.$$

Choose M such that $\|D_x u(x, P)\|, \|D_x u(x, P_n)\| \leq M$. Let $\delta = \frac{\epsilon}{8M}$. Cover $\mathrm{supp}\,\nu$ with finitely many balls B_i with radius $\leq \delta$. Choose (x, p) as in the previous proposition. Let $(x^*(t), p^*(t))$ be the optimal trajectory for P with initial condition (x, p). Then there exists T_δ and $0 \leq t_i \leq T_\delta$ such that $x_i = x^*(t_i) \in B_i$. Choose n sufficiently small such that

$$\sup_{0 \leq t \leq T_\delta} |u(x^*(t), P) - u(x^*(t), P_n)| \leq \frac{\epsilon}{2}.$$

Then, on each y in B_i

$$|u(y, P) - u(y, P_n)| \leq |u(y, P) - u(y_i, P)| + |u(y_i, P) - u(y_i, P_n)| + \\ + |u(y_i, P_n) - u(y, P_n)| \leq 4M\delta + \frac{\epsilon}{2} \leq \epsilon.$$

■

Actually, the hypothesis that ν is uniquely ergodic is not too restrictive since by Mane's result [Mn96] "most" Mather measures are uniquely ergodic (in the sense that after applying small generic perturbations to the Lagrangian there is a uniquely ergodic Mather measure).

5. Non-Degeneracy

In this section we will show that under the hypothesis that $\overline{H}$ is strictly convex the change of coordinates (3) satisfies a non-degeneracy condition.

As motivation for our computations consider the following proposition

Proposition 2. *Suppose both $\overline{H}(P)$ and $u(x,P)$ are smooth functions and $\overline{H}(P)$ is strictly convex at P. Then for any vector ξ*

$$c|\xi|^2 \leq \lim_{T\to\infty} \frac{1}{T}\int_0^T \left|\left[I + D^2_{xP}u(x(t),P)\right]\xi\right|^2 \leq C|\xi|^2, \tag{9}$$

with $0 < c \leq C$, and $(x(t), p(t))$ a solution of (1) with initial conditions on the invariant set $\mathcal{I}$. In particular $0 < |\det\left[I + D^2_{xP}u(x(t),P)\right]| < \infty$ a.e..

Proof. Let $D_\xi u = D_P u\xi$. Applying $D^2_{\xi\xi}$ to equation (2) we get

$$c\left|I + D^2_{x\xi}u\right|^2 + D_pHD^3_{x\xi\xi}u = D^2_{\xi\xi}\overline{H},$$

since by uniform convexity $D^2_pH > c$. Integrating, we conclude

$$\int_0^T D_pHD^3_{x\xi\xi}u = O(1),$$

uniformly in T. Thus

$$c|\xi|^2 \leq \lim_{T\to\infty} \frac{1}{T}\int_0^T \left|\left[I + D^2_{xP}u(x(t),P)\right]\xi\right|^2.$$

The proof of the other inequality is similar, using the second derivative bound $D^2_{pp}H \leq C$. ■

With the help of difference quotients we can make the previous proposition precise in the case where u is a viscosity solution. An analog of the inequality

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left|\left[I + D^2_{xP}u(x(t),P)\right]\xi\right|^2 dt \leq C|\xi|^2$$

was proved in [EG99]; the next theorem is a slightly different version such estimate.

Theorem 5. *Suppose $\overline{H}(P)$ is twice differentiable at P. Then for almost every y sufficently small (for instance, we may take $|y| \leq |P - P'|^2$)*

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T |P' + D_x u(x(t)+y, P') - P - D_x u(x(t), P)|^2 dt \leq$$
$$\leq C|P' - P|^2,$$

where $(x(t), p(t))$ is a solution of (1) with initial conditions on $\mathcal{I}$.

Remark. The idea of considering difference quotients in P with "slightly" shifted functions in x has to do with the fact that $u(x(t), P')$ may not be differentiable along $x(t)$. However for almost every $y \in \mathbb{R}^n$ $u(x(t)+y, P')$ will be differentiable for almost every t.

Proof. Note that

$$H(P + D_x u(x(t), P), x(t)) = \overline{H}(P) \tag{10}$$

and for almost every y sufficiently small,

$$H(P' + D_x u(x(t)+y, P'), x(t)) = \overline{H}(P') + O(|P - P'|^2). \tag{11}$$

Subtracting (11) from (10) we obtain the inequality

$$D_p H(P + D_x u(x(t), P), P)\zeta + \theta|\zeta|^2 \leq D_P\overline{H}(P' - P) + C|P - P'|^2,$$

where

$$\zeta = P' + D_x u(x(t)+y, P') - P + D_x u(x(t), P), \tag{12}$$

using the strict convexity of H and twice differentiablity of $\overline{H}$. Observe that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T D_p H(P + D_x u(x(t), P), P)\zeta = D_P\overline{H}(P' - P),$$

since $\dot{x} = D_p H$. Thus

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T |\zeta|^2 \leq C|P - P'|^2,$$

as we claim. ■

The converse result is

Theorem 6. *Suppose $\overline{H}(P)$ is strictly convex at P. Then for almost every y sufficently small (for instance $|y| \leq |P - P'|^2$ will do)*

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T |P' + D_x u(x(t) + y, P') - P - D_x u(x(t), P)|^2 dt \geq$$
$$\geq c|P' - P|^2.$$

Proof. Using the notation from the previous theorem and the hypothesis that $\overline{H}$ is strictly convex at P we obtain the inequality

$$D_p H(P + D_x u(x(t), P), P)\zeta + \theta|\zeta|^2 \geq D_P \overline{H}(P' - P) + c|P - P'|^2.$$

Thus, by integration, we conclude

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T |\zeta|^2 \geq c|P - P'|^2,$$

as we claim. ■

This last theorem shows that strict convexity of $\overline{H}$ implies that at least in a measure-theoretic sense, the graphs $(x, P + D_x u(x, P))$ and $(x, P' + D_x u(x, P'))$ are distinct. Therefore the change of coordinates (3) is in a weak sense non-degenerate.

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