THE PROPAGATOR OF THE RADIAL DIRAC EQUATION

TAKASHI ICHINOSE AND BRIAN JEFFERIES

ABSTRACT. The propagator for the radial Dirac equation is explicitly constructed. It turns out to be a distribution of order zero, but it is shown that there exists no path-space measure associated with this equation.

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1. Introduction

It is always of interest to know whether or not there is a path-space measure associated with an initial-value problem for an evolution equation in mathematical physics, so that we may represent its solution by a path integral, that is, an integral with respect to a σ-additive set function defined on some σ-algebra of subsets of a suitable function space of 'paths'. Then perturbations to the original equation have solutions represented by a path integral too and we can derive information about the solution by means of this representation.

A broad class of initial-value problems can be treated by semigroup theory, but the precise relationship of the idea of a path integral to these problems is somewhat ill-defined. Unfortunately, no such path-space measure exists as a mathematically rigorous measure associated with the Schrödinger equation, although it was with this equation in mind that Feynman conceived the very concept of *path integration* [Feynman (1948)].

However, as is well-known, for the imaginary-time Schrödinger equation, that is, the heat equation or more general evolution equations appearing in a Markov process in probability theory, the solution semigroup $S(t)$, $t \geq 0$, is a positivity preserving operator on some space $L^\infty(\Sigma)$ of bounded measurable functions on a certain set Σ, and there exists an associated path integral or path-space measure.

It is, roughly speaking, because there exists an integral kernel or transition function $p_t(x, dy)$ such that $S(t)$ is given by

$$(S(t)f)(x) = \int_\Sigma p_t(x, dy) f(y), \quad x \in \Sigma,\ t \geq 0,$$

for $f \in L^\infty(\Sigma)$ and that the bound $\|S(t)f\|_\infty \leq \|f\|_\infty$ holds for all $f \in L^\infty(\Sigma)$.

The aim of this paper is to discuss this issue for the Dirac equation, in particular, the radial Dirac equation. Before going to the situation in three space dimensions, we first consider the Dirac operator D_1 with mass $m > 0$ in one space dimension, defined in the space $L^2(\mathbb{R}, \mathbb{C}^2)$ of $\mathbb{C}^2$-valued square-integrable functions in $\mathbb{R}$ by

$$D_1 = \alpha \frac{1}{i} \frac{\partial}{\partial x} + m\beta, \tag{1}$$

for two (2×2) hermitian matrices α and β satisfying $\alpha^2 = \beta^2 = 1$ and $\alpha\beta + \beta\alpha = 0$. Here and in the following, we are using a coordinate system where the speed of light c and Planck's constant $\hbar$ are equal to one. In [Ichinose (1982), Ichinose (1984), Ichinose and Tamura (1984), Ichinose and Tamura (1987) and Ichinose and Tamura (1988)] T. Ichinose and H. Tamura showed that the group of operators $e^{-iD_1 t}$, $t \in \mathbb{R}$, acting on $L^2(\mathbb{R}, \mathbb{C}^2)$, is similar to a strongly continuous group $S(t)$ of operators for which the bound

$$\|S(t)f\|_\infty \leq e^{m|t|} \|f\|_\infty \tag{2}$$

holds for all $t \in \mathbb{R}$ and all bounded continuous functions $f = {}^t(f_1, f_2)$ in $L^2(\mathbb{R}, \mathbb{C}^2)$, where $\|f\|_\infty = \max\{\|f_1\|_\infty, \|f_2\|_\infty\}$ stands for the norm of f as an element of the space $L^\infty(\mathbb{R}, \mathbb{C}^2)$ of $\mathbb{C}^2$-valued bounded functions in $\mathbb{R}$, endowed with ℓ^∞-norm on $\mathbb{C}^2$. As in the case of a Markov process above, the bound (2) is sufficient to ensure that there exist matrix-valued path-space measures $\nu_{x,t}$ such that

$$\int_\Omega \nu_{x,t}(d\omega) f(\omega(0)) = \left(e^{-itD_1} f\right)(x), \quad \text{almost all } x \in \mathbb{R},\ t \geq 0, \tag{3}$$

for every bounded function f in $L^2(\mathbb{R}, \mathbb{C}^2)$. It should be mentioned here that the integral kernel of e^{-itD_1} is a matrix-valued distribution of order zero or a matrix-valued *measure* in $\mathbb{R}\times\mathbb{R}$. For each $x \in \mathbb{R}$ and $t \geq 0$, the measure $\nu_{x,t}$, defined on the cylindrical σ-algebra of the space $\Omega_{x,t}$ of all paths $\omega : [0,t] \to \mathbb{R}$ with $\omega(t) = x$, takes values in the space $M_2(\mathbb{C})$ of (2×2) complex matrices. Further analysis in [Ichinose and Tamura (1988) or Ichinose and Tamura (1987)] shows that $\nu_{x,t}$ is a σ-additive Borel measure concentrated on the space $\Omega'_{x,t}$ of all Lipschitz-continuous paths $\omega : [0,t] \to \mathbb{R}$ with $\omega(t) = x$ and with velocity ± 1 and finitely many changes of direction in each finite time interval.

Initial-value problems for which the associated evolution $S(t)$, $t \geq 0$, satisfies a bound of the form (2) are comparatively rare. Together with the wave equation in one space dimension, the examples mentioned above are the most studied in the literature.

This paper is concerned with $\mathbb{C}^2$-valued solutions $u(r,t)$ of the initial-value problem for the *radial Dirac equation:*

$$\frac{\partial u}{\partial t}(r,t) = -i(\tau_k u)(r,t), \quad u(r,0) = u_0(r), \tag{4}$$

for all $r \in \mathbb{R}_+ := (0,\infty)$ and $t \in \mathbb{R}$. The differential operator τ_k is defined for $k = \pm 1, \pm 2, \ldots$ by

$$\tau_k : u \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_1'(r) \\ u_2'(r) \end{pmatrix} + \begin{pmatrix} m & -k/r \\ -k/r & -m \end{pmatrix}\begin{pmatrix} u_1(r) \\ u_2(r) \end{pmatrix}, \tag{5}$$

for suitably smooth $u \in L^2(\mathbb{R}_+, \mathbb{C}^2)$. The nonzero integer k represents an eigenvalue of the "spin-orbit operator", see [Thaller (1992) pp. 125–130]. The radial Dirac operator (5) arises from the spin-angular momentum decomposition of the free Dirac operator in three space dimensions, defined as follows.

The *Dirac operator* D_3 in three space dimensions is defined in the space $L^2(\mathbb{R}^3, \mathbb{C}^4)$ of $\mathbb{C}^4$-valued square-integrable functions in $\mathbb{R}^3$ by means of the differential expression

$$D_3 = \sum_{j=1}^{3} \alpha_j \mathbf{p}_j + \alpha_4 m,$$

where $m > 0$ is the mass of the particle as in (1),

$$\mathbf{p}_j = \frac{1}{i}\frac{\partial}{\partial x_j}, \qquad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \qquad \text{for } j = 1,2,3,$$

with the *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\alpha_4 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \qquad \text{with } \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Here we have taken the standard representation of the Dirac operator [Thaller (1992), Eq. (1.213) p.36]. Then via the spin-angular momentum decomposition, the operator D_3 is equal to the direct sum of operators unitarily equivalent to the operators (5) with $k = \pm 1, \pm 2, \ldots$.

The fact that τ_k defines a selfadjoint operator acting in $L^2(\mathbb{R}_+, \mathbb{C}^2)$ follows from the spin-angular momentum decomposition of D_3 ([Thaller (1992), Theorem 4.14]), hence, $e^{-it\tau_k}$, $t \in \mathbb{R}$, is a strongly continuous unitary group of operators acting on $L^2(\mathbb{R}_+, \mathbb{C}^2)$.

The initial-value problem (4) is a worthy object of study, because on adding a Coulomb potential $\sigma_0 e/r$, $e \in \mathbb{R}$, to τ_1, we get an essentially selfadjoint operator if and only if $|e| \leq \sqrt{3}/2$ ([Weidmann (1987), Theorem 6.9]. For $\sqrt{3}/2 < |e| < 1$, there is a distinguished selfadjoint extension. In this respect, the operator $\tau_1 + \sigma_0 e/r$ has properties similar to the

Laplacian plus an attractive potential proportional to $1/r^2$ treated by a Feynman path integral in [Nelson, (1964)]. For larger values of $|e|$ close to one, the boundary conditions near $r = 0$ need to take into account the effects of the nuclear dimensions and particle pair-production in quantum field theory [Popov (1971)].

The initial-value problem (4) has a superficial appeal from the point of view of analysis by path integrals because of the following line of reasoning. The operator

$$i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial x}$$

is unitarily similar to the generator

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial}{\partial x} \tag{6}$$

of the direct sum of groups of translations acting on $L^\infty(\mathbb{R}, \mathbb{C}^2)$. This is obviously a group of isometries of $L^\infty(\mathbb{R}, \mathbb{C}^2)$.

For each $t \in \mathbb{R}$ and $r > 0$, the matrix

$$\exp\left[-it\begin{pmatrix} m & -k/r \\ -k/r & -m \end{pmatrix}\right] \tag{7}$$

is unitary. The operator of multiplication by a function taking its values in the unitary (2×2) matrices is obviously an isometry of $L^\infty(\mathbb{R}_+, \mathbb{C}^2)$.

Thus, one might hope to apply the Trotter product formula

$$e^{-it\tau_k} = \lim_{N\to\infty}\left[\exp\left(\frac{-it}{N}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial r}\right)\exp\left(\frac{-it}{N}\begin{pmatrix} m & -k/r \\ -k/r & -m \end{pmatrix}\right)\right]^N$$

to show that $e^{-it\tau_k}$ is $L^\infty(\mathbb{R}_+, \mathbb{C}^2)$-bounded and the operator norm is one. Standard arguments would then enable one to construct a σ-additive path-space measure associated with $e^{-it\tau_k}$, $t \in \mathbb{R}$. As seen in [Ichinose and Tamura (1984)], such a scalar measure representing $(e^{-it\tau_k}f, g)$ actually exists for states f and g with supports sufficiently separated from zero. The evidence seems to suggest that the initial-value problem (4) possesses a path-space measure.

However, the argument above is *false*. The mistake is as follows. In order that multiplication by the matrix-valued function (7) be unitary, we need to give $\mathbb{C}^2$ the usual Euclidean norm, that is, the ℓ^2-norm. But equipped with this norm, the operator

$$\exp\left[-it\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial r}\right]$$

is *not* a contraction on $L^\infty(\mathbb{R}_+, \mathbb{C}^2)$ (to make sense of this exponential, a boundary condition at $r = 0$ must be added). The group generated by (6) is a continuous group of isometries on $L^\infty(\mathbb{R}, \mathbb{C}^2)$ only if we endow $\mathbb{C}^2$ with the ℓ^∞-*norm*.

The conflict between the two norms for $\mathbb{C}^2$ means that the Trotter product formula cannot be used in $L^\infty(\mathbb{R}_+, \mathbb{C}^2)$. Intuitively, we would *not* expect a path-space measure for (4) to exist, otherwise path-integrals would select a distinguished selfadjoint extension for *every* perturbation by a Coulomb potential $\sigma_0 e/r$ by virtue of a type of Feynman-Kac formula, whereas we know that an essentially selfadjoint perturbed operator exists only for $|e| \le \sqrt{3}/2$. The main purpose of this work is to clarify the situation: we show in Section 4, Corollary 1, that the operator $e^{-it\tau_1}$ is not actually bounded on $L^\infty(\mathbb{R}_+, \mathbb{C}^2)$ for any $t \neq 0$.

It is not hard to see that $e^{-it\tau_1}$ *is* bounded on $L^p(\mathbb{R}_+, \mathbb{C}^2)$ for all $1 < p < \infty$ and $t \in \mathbb{R}$. By contrast, for the Laplacian Δ in $\mathbb{R}^n$ and the Dirac operator D_3 in $\mathbb{R}^3$, for each $t \neq 0$, the operator $e^{it\Delta}$ is bounded on $L^p(\mathbb{R}^n)$ and the operator e^{-itD_3} is bounded on $L^p(\mathbb{R}^3, \mathbb{C}^4)$, only for $p = 2$ [Hörmander (1960), Brenner (1966)].

In order to obtain information about the group $e^{-it\tau_1}$, $t \in \mathbb{R}$, of operators, we need an explicit formula for the propagator, that is, the integral kernel $\mathcal{K}_t$ of the operator $e^{-it\tau_1}$ for each $t \in \mathbb{R}$. Although $e^{-it\tau_1}$ is a unitary operator acting on $L^2(\mathbb{R}_+, \mathbb{C}^2)$ for each $t \in \mathbb{R}$, its kernel $\mathcal{K}_t$ may well be a matrix-valued distribution defined on $\mathbb{R}_+ \times \mathbb{R}_+$. Explicit formulae are known for the propagator of the group e^{-itD_3}, $t \in \mathbb{R}$ [Prosser (1963), Vladimirov (1971), Rosen (1983), Brzeźniak (1990)]. For each $t \neq 0$, this is a matrix-valued distribution of order one, from which it follows that there is no path-space measure associated with the Dirac equation in four space-time dimensions.

The main result of this paper, given in Theorem 1 of Section 4, is an explicit formula for the propagator $\mathcal{K}_t$ associated with the radial Dirac equation. From this we deduce that there is no path-space measure associated with the radial Dirac equation like the one that exists for the Dirac equation in two space-time dimensions. More formally, we show that $e^{-it\tau_1}$ does not map $L^2(\mathbb{R}_+, \mathbb{C}^2) \cap L^\infty(\mathbb{R}_+, \mathbb{C}^2)$ into $L^\infty(\mathbb{R}_+, \mathbb{C}^2)$ for any $t \neq 0$. Nevertheless, a path integral approach to this equation is given in [Jefferies (1994), (1996a)] along the mathematical lines developed in [Jefferies (1996b)] for integration with respect to unbounded set functions.

Although the propagator associated with D_3 is a matrix-valued distribution of order one on $\mathbb{R}^3 \times \mathbb{R}^3$, it is somewhat surprising to learn that $\mathcal{K}_t$ is a matrix-valued distribution of order *zero* on $\mathbb{R}_+ \times \mathbb{R}_+$. In the course of the derivation of the formula for $\mathcal{K}_t$, it is apparent that averaging over the unit sphere smooths out the kernel.

It can be verified directly that $\partial/\partial t\, \mathcal{K}_t(r, \rho) = -i\tau_1 \mathcal{K}_t(r, \rho)$, for all $r > 0$, $\rho > 0$, with $\mathcal{K}_0(r, \rho) = \delta(r - \rho)\sigma_0$ from the formula (19) for the distribution $\mathcal{K}_t$ we obtain in Section 4, but the proof of Theorem 1 may be applied to obtain the kernel of $e^{-it\tau_k}$ for integer values of k other than one.

A key part of our derivation is the representation (14) of the kernel of e^{-itD_3} obtained in Section 3. The representation first appears in [Rosen (1983)] but we give a formal proof in Proposition 2. Using this representation, we obtain a regularisation of e^{-itD_3} which preserves the reducing subspaces of D_3 with respect to the "spin-orbit" operator [Thaller (1992), Eq. (4.105), p. 125]. The regularisation simplifies our computations because we can deal with functions rather than distributions.

2. Preliminaries

We first briefly fix some notation and terminology concerning function spaces and distributions.

2.1. **Function Notation.** Let n be a positive integer. The sesquilinear inner product of $\mathbb{C}^n$ is denoted by $(\,\cdot\,,\,\cdot\,)$, which is linear in the first component and anti-linear in the second. The corresponding inner product of two $\mathbb{C}^n$-valued functions $f(x) = {}^t(f_1(x), \ldots, f_n(x))$ and $g(x) = {}^t(g_1(x), \ldots, g_n(x))$ in $\mathbb{R}^d$ is denoted by

$$(f, g) = \int_{\mathbb{R}^d} (f(x), g(x))\, dx = \int_{\mathbb{R}^d} \sum_{j=1}^n f_j(x)\overline{g_j(x)}\, dx.$$

We shall also use the bilinear inner product $\langle \cdot, \cdot \rangle$.

For an open subset U of $\mathbb{R}^d$, $C_c^\infty(U, \mathbb{C}^n)$ denotes the space of $\mathbb{C}^n$-valued C^∞-functions with compact support in U. $\mathcal{S}(\mathbb{R}^d, \mathbb{C}^n)$ is the space of $\mathbb{C}^n$-valued rapidly decreasing C^∞ functions on $\mathbb{R}^d$. For $n = 1$, they are the usual function spaces $C_c^\infty(U)$ and $\mathcal{S}(\mathbb{R}^d)$. The notation $C_0(\mathbf{R}^d, \mathbf{C}^n)$ is used for the space of $\mathbf{C}^n$-valued continuous functions vanishing at infinity on $\mathbb{R}^d$. It has the supremum norm.

For two locally convex topological linear spaces X and Y, $\mathcal{L}(X, Y)$ denotes the topological linear space of all continuous linear mappings of X into Y.

Since $\mathbb{C}^{n^2}$ is identified with the space $M_n(\mathbb{C})$ of $(n \times n)$ matrices, we denote the space of all $M_n(\mathbb{C})$-valued distributions with compact support in U by $\mathcal{L}(C_c^\infty(U), M_n(\mathbb{C}))$, which the dual space of the space $C_c^\infty(U, M_n(\mathbb{C}))$. We denote also the space of $M_n(\mathbb{C})$-valued tempered distributions on $\mathbb{R}^d$ by $\mathcal{L}(\mathcal{S}(\mathbb{R}^d), M_n(\mathbb{C}))$, which is the dual space of the space $\mathcal{S}(\mathbb{R}^d, M_n(\mathbb{C}))$. For $n = 1$, they are nothing but the usual spaces $C_c^\infty(U)'$ of Schwartz distributions in U, and $\mathcal{S}(\mathbb{R}^d)'$ of tempered distributions on $\mathbb{R}^d$.

2.2. **Kernels of Bounded Linear Operators.** Let U be an open subset of $\mathbb{R}^d$. An $M_n(\mathbb{C})$-valued distribution K defined in $U \times U$, i.e. an element in $\mathcal{L}(C_c^\infty(U \times U), M_n(\mathbb{C}))$ is called the *integral kernel* or just the *kernel*, of a bounded linear operator T of $L^2(U, \mathbb{C}^n)$ into itself, if for every $f, g \in C_c^\infty(U)$ and $j, l = 1, \ldots, n$, the equality

$$(T(fe_j), \overline{g}e_l) = (K(g \otimes f)e_j, e_l)$$

holds for the standard basis vectors $e_1, \ldots, e_n$ of $\mathbb{C}^n$. Here $g \otimes f$ is the function $(x, y) \mapsto g(x)f(y)$, $x, y \in U$. Thus, if $T : L^2(U, \mathbb{C}^n) \to L^2(U, \mathbb{C}^n)$ is a bounded linear operator and the $M_n(\mathbb{C})$-valued function $(x, y) \mapsto k(x, y)$, $x, y \in U$, is measurable and locally integrable in $U \times U$, then k is the kernel of T if, for every $f \in C_c^\infty(U)$ and $j = 1, \ldots, n$, we have

$$T(fe_j)(x) = \int_U [k(x, y)e_j]f(y)\, dy$$

for almost all $x \in U$.

It follows that if k is a matrix-valued distribution defined in $\mathbb{R}$, i.e. an element of the space $\mathcal{L}(C_c^\infty(\mathbb{R}), M_n(\mathbb{C}))$, then the operator of convolution with respect to k has the distribution $k(x-y)$ defined in $\mathbb{R}^2$ as its kernel. The identity operator acting on $L^2(U, \mathbb{C}^n)$ has the distribution $\delta(x - y)I$ as its kernel. Here I is the $(n \times n)$ identity matrix and $\delta : f \mapsto f(0)$, $f \in C_c^\infty(\mathbb{R})$, is the Dirac distribution at zero.

3. Representation of the propagator of the Dirac equation

The kernel of e^{-itD_3} is known from [Prosser (1963), Vladimirov (1971), Rosen (1983), Brzeźniak (1990)]. It is simply obtained by applying the Dirac operator to the known propagator for the Klein-Gordon equation in four space-time dimensions [for the Klein–Gordon propagator, see for example, De Jager (1963)].

The derivation of the kernel of $e^{-it\tau_1}$ is achieved by computing the restriction of the unitary operator e^{-itD_3} to the eigenspace corresponding to the eigenvalue $k = 1$ of the "spin-orbit" operator [Thaller (1992), Eq. (4.105), p. 125]. As we see below, the calculation is not entirely straightforward and the formula for the kernel is not simple.

We first obtain a representation of the kernel of e^{-itD_3} that simplifies the task of calculating its restriction to the eigenspace of the "spin-orbit" operator by looking at the Dirac equation in 2 space-time dimensions. The idea originates in [Rosen (1983)] where a certain one-dimensional character of the motion of a Dirac particle is emphasised.

3.1. **The 1-D Dirac Equation.** The first statement is a simple consequence of perturbation theory for linear operators. It is the basis of the proof that there is a path-space measure associated with the Dirac equation in two space-time dimensions [Ichinose and Tamura (1984)]. Although we have formula (9) below for the kernel of e^{-itD_1}, it is not obvious that it gives the operator bounds set out in Proposition 1 below.

Proposition 1. *Let $\partial/\partial\xi$ be the operator acting in $L^2(\mathbb{R},\mathbb{C}^2)$, with domain $\mathcal{D}(\partial/\partial\xi) = H^1(\mathbb{R},\mathbb{C}^2)$ consisting of all functions $f = {}^t(f_1, f_2) \in L^2(\mathbb{R},\mathbb{C}^2)$, for which f_1 and f_2 are absolutely continuous and f' is square-integrable.*

Then the operator $-[\sigma_3\partial/\partial\xi + im\sigma_1] : \mathcal{D}(\partial/\partial\xi) \to L^2(\mathbb{R},\mathbb{C}^2)$ is the generator of a unitary group $e^{-t(\sigma_3\partial/\partial\xi+im\sigma_1)}$, $t \in \mathbb{R}$, such that for every $1 \le p \le \infty$,

$$\left\|e^{-t(\sigma_3\partial/\partial\xi+im\sigma_1)}f\right\|_p \le e^{m|t|}\|f\|_p\,, \quad t\in\mathbb{R}, \tag{8}$$

for all $f \in L^2(\mathbb{R},\mathbb{C}^2) \cap L^p(\mathbb{R},\mathbb{C}^2)$.

Furthermore, the operator $e^{-t(\sigma_3\partial/\partial\xi+im\sigma_1)}$ leaves (the subspace) $C_0(\mathbb{R},\mathbb{C}^2)$ in $L^\infty(\mathbb{R},\mathbb{C}^2)$ invariant for each $t \in \mathbb{R}$.

Here note that whether a bound like (8) holds for every $1 \le p \le \infty$ amounts essentially to the same thing as whether it holds for $p = 1$ or $p = \infty$. Because if it holds for $p = 1$ or $p = \infty$, by duality it also holds for the other p, and then for every $1 \le p \le \infty$ by interpolation.

Proof of Proposition 1. The operator $\partial/\partial\xi$ is the infinitesimal generator of the group of translations on $L^2(\mathbb{R})$, so $\sigma_3\partial/\partial\xi$ acting in $L^2(\mathbb{R},\mathbb{C}^2)$ is the generator of that continuous group $T(t)$ of isometries on $L^2(\mathbb{R},\mathbb{C}^2)$ which maps the function $\xi \mapsto {}^t(f_1(\xi), f_2(\xi))$, $\xi \in \mathbb{R}$, to the function $\xi \mapsto {}^t(f_1(\xi + t), f_2(\xi - t))$, $\xi \in \mathbb{R}$.

Then $T(t)$ also defines a group of isometries $T_p(t)$ of $L^p(\mathbb{R},\mathbb{C}^2)$, for $1 \le p \le \infty$. Clearly $T_p(t)$ is continuous for $1 \le p < \infty$. The group of isometries $T_\infty(t)$ leaves the space $C_0(\mathbb{R},\mathbb{C}^2)$ invariant, and defines a continuous group of isometries there.

Because $-i\,m\,\sigma_1$ is a bounded perturbation of the operator $-\sigma_3\partial/\partial\xi$ acting in $L^p(\mathbb{R},\mathbb{C}^2)$ (or $C_0(\mathbb{R},\mathbb{C}^2)$), it follows from [Reed and Simon (1975), Section X.2] that $-\sigma_3\partial/\partial\xi - i\,m\,\sigma_1$ is the generator of a group $S(t) = e^{-t(\sigma_3\partial/\partial\xi+im\sigma_1)}$, $t \in \mathbb{R}$, of operators satisfying $\|S(t)\| \le e^{m|t|}$, for all $t \in \mathbb{R}$. Here the same symbol $S(t)$ is used for the group of operators so defined on each space $L^p(\mathbb{R},\mathbb{C}^2)$, $1 \le p < \infty$ and $C_0(\mathbb{R},\mathbb{C}^2)$, and $\|\cdot\|$ denotes the operator norm of $\mathcal{L}(L^p(\mathbb{R},\mathbb{C}^2))$, for $1 \le p < \infty$, or of $\mathcal{L}(C_0(\mathbb{R},\mathbb{C}^2))$. □

In the following, we consider the case for $t > 0$. Similar considerations apply to the case $t < 0$. The integral kernel of the operator $e^{-t(\sigma_3\partial/\partial\xi+im\sigma_1)}$ is given [Ichinose and Tamura (1984), (3.2)] by convolution with respect to the distribution

$$H(\xi,t) = \frac{1}{2}\left(\sigma_0\frac{\partial}{\partial t} - \sigma_3\frac{\partial}{\partial\xi} - im\sigma_1\right)J_0\big(m(t^2-\xi^2)^{1/2}\big)\theta(t-|\xi|).$$

Here $J_0(t)$ is the Bessel function of order zero and θ is the Heaviside function — the characteristic function of the interval $[0,\infty)$. The distributional notation means that if $\phi : \mathbb{R} \to \mathbb{C}$ is a rapidly decreasing function, then $H(\,\cdot\,,t)(\phi) \in M_2(\mathbb{C})$ is the matrix

$$\begin{aligned} H(\,\cdot\,,t)(\phi) \;=\; & \frac{1}{2}\int_{-t}^{t}\left[\left(\sigma_0\frac{\partial}{\partial t} - \sigma_3\frac{\partial}{\partial\xi} - im\sigma_1\right)J_0\big(m(t^2-\xi^2)^{1/2}\big)\right]\phi(\xi)\,d\xi \\ & +\frac{1}{2}(\phi(t)+\phi(-t))\sigma_0 + \frac{1}{2}(\phi(t)-\phi(-t))\sigma_3. \end{aligned} \tag{9}$$

Now the derivatives after the integral sign are interpreted in the pointwise sense. Then for each $t > 0$, the convolution $H(\,\cdot\,,t) * \psi$ belongs to $C_0(\mathbb{R},\mathbb{C}^2)$ for every ψ belonging to the space $\mathcal{S}(\mathbb{R},\mathbb{C}^2)$ of rapidly decreasing $\mathbb{C}^2$-valued functions, and

$$[H(\,\cdot\,,t)*\psi](\xi) = \big[e^{-t(\sigma_3\partial/\partial\xi+im\sigma_1)}\psi\big](\xi), \qquad \xi\in\mathbb{R}. \tag{10}$$

Since $dJ_0(t)/dt = -J_1(t)$ is bounded, it is clear from the representation (9) that $H(\,\cdot\,,t)$ is a matrix-valued distribution of order zero, that is, a matrix-valued measure, so we shall sometimes write $\int_{\mathbb{R}} H(d\xi,t)\phi(\xi)$ for (9) and $\int_{\mathbb{R}} H(d\eta,t)\psi(\xi-\eta)$ for (10). It follows from the representation (9) that for each $\xi \in \mathbb{R}$, the matrix-valued measure $H(d\xi,t)$ is supported by the interval $[-t,t]$.

3.2. **The Kernel of e^{-itD_3}.** A bound like (8) with $p=\infty$ in Proposition 1 does not hold for the 3-D Dirac operator D_3, as noted in [Ichinose (1984)], because the initial-value problem for the hyperbolic system **(with non-commuting matrix coefficients)** of the first order in $d+1$ space-time dimensions is L^∞ well-posed if and only if $d=1$ (cf. [Brenner (1966), Zastawniak (1989), Brzeźniak (1990)]). It is simply because the kernel of e^{-itD_3} is a distribution of order *one*, as is easily seen from the explicit representation obtained in Proposition 2 below.

It is convenient to take an equivalent representation of the Dirac operator D_3, so that

$$\alpha'_j = \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}, \qquad \text{for } j=1,2,3, \qquad \alpha'_4 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \tag{11}$$

obtained from the Dirac matrices α_j by the similarity transformation $\alpha'_j = Q\alpha_j Q$, $j = 1,2,3,4$, with

$$Q = 2^{-1/2}\begin{pmatrix} \sigma_0 & \sigma_0 \\ \sigma_0 & -\sigma_0 \end{pmatrix}. \tag{12}$$

Let $\{e_1,e_2\}$ be the standard basis of $\mathbb{C}^2$. The tensor product $A\otimes B$ of two 2×2 matrices A and B is the 4×4 matrix of the associated linear transformation with respect to the basis $\mathcal{E} = \{e_1\otimes e_1, e_2\otimes e_1, e_1\otimes e_2, e_2\otimes e_2\}$ of $\mathbb{C}^4$. Linear transformations are identified with matrices with respect to these fixed bases. Then $\alpha'_j = \sigma_j\otimes\sigma_3$ for $j=1,2,3$ and $\alpha'_4 = \sigma_0\otimes\sigma_1$ and in this representation, the Dirac operator becomes

$$D_3 = \sum_{j=1}^{3}(\sigma_j\otimes\sigma_3)\mathbf{p}_j + m\sigma_0\otimes\sigma_1. \tag{13}$$

The following result appears in [Rosen (1983)] in a somewhat compressed form. In the remainder of this section, we translate into the present notation and give a proof of this statement. We first give some additional notation.

Let μ_r be the uniform surface probability on the sphere S_r of radius $r>0$ in $\mathbb{R}^3$. For $r>0$, the μ_r-integral of a function f over S_r is denoted by $\mu_r(f)$. For a smooth function ϕ on $\mathbb{R}^3$, we have $\partial/\partial r\phi(x) = |x|^{-1}\langle x,\nabla\rangle\phi(x)$, $r=|x|\neq 0$. For $r=0$, we understand $\mu_0=\delta$, the delta measure at $x=0$. For any $M_2(\mathbb{C})$-valued Borel measure λ on $\mathbb{R}$ and $M\in M_2(\mathbb{C})$, the set function $M\otimes\lambda : A\mapsto M\otimes[\lambda(A)]$, with A a Borel set in $\mathbb{R}$, is an $M_4(\mathbb{C})$-valued measure.

Proposition 2. *Let D_3 be Dirac operator in the representation (13). Then for each $t\in\mathbb{R}$, the operator e^{-itD_3} is convolution with respect to the matrix-valued distribution $K_t\in\mathcal{L}(\mathcal{S}(\mathbb{R}^3), M_4(\mathbb{C}))$ of order one, given for each test function $\phi\in\mathcal{S}(\mathbb{R}^3)$, by*

$$\begin{aligned} K_t(\phi) = &\int_{-t}^{t}\left[\mu_{|\xi|}(\phi) + |\xi|\mu_{|\xi|}(\partial/\partial r\phi)\right][\sigma_0\otimes H(d\xi,t)] \\ &+ \int_{-t}^{t}\xi\,\mu_{|\xi|}(\langle\sigma,\nabla\phi\rangle)\otimes H(d\xi,t), \end{aligned} \tag{14}$$

or, as an $M_4(\mathbb{C})$-valued distribution in $\mathbb{R}^3$, by

$$
\begin{aligned}
K_t(x) = &-\frac{1}{4\pi r}\sigma_0 \otimes \left(\frac{\partial H(r,t)}{\partial r} + \frac{\partial H(-r,t)}{\partial r}\right) \\
&- \frac{1}{4\pi}\langle \sigma, \nabla\rangle \otimes \left(\frac{H(r,t)}{r} - \frac{H(-r,t)}{r}\right), \quad \textit{for } r = |x|.
\end{aligned} \tag{15}
$$

The representation (14) is proved by applying the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$, defined for every $f \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ by

$$
\mathcal{F}f(p) = \int_{\mathbb{R}^3} e^{-i\langle x,p\rangle} f(x)\, dx, \quad \text{for all } p \in \mathbb{R}^3.
$$

Then $\mathcal{F}^{-1}f(x) = (2\pi)^{-3}\int_{\mathbb{R}^3} e^{i\langle x,p\rangle} f(p)\, dp$ defines the inverse operator. The same symbols are used for the operators defined on $L^2(\mathbb{R}^3, \mathbb{C}^n)$ by taking tensor products with the identity map on $\mathbb{C}^n$. Sometimes $\mathcal{F}f$ is written as $\hat{f}$ and $\mathcal{F}^{-1}f$ as $\check{f}$.

If $\Phi : \mathbb{R}^3 \to M_n(\mathbb{C})$ is a matrix-valued Borel measurable function, then the operator of *multiplication* by Φ sends the element $f \in L^2(\mathbb{R}^3, \mathbb{C}^n)$ to $x \mapsto \Phi(x)f(x)$, $x \in \mathbb{R}^3$. This is a bounded linear operator on $L^2(\mathbb{R}^3, \mathbb{C}^n)$ if and only if Φ is essentially bounded.

The integral $\int_{\mathbb{R}^3} \Phi(x) \otimes \mu(dx) \in M_4(\mathbb{C})$ of a matrix-valued function $\Phi : \mathbb{R}^3 \to M_2(\mathbb{C})$ with respect to an $M_2(\mathbb{C})$-valued Borel measure μ on $\mathbb{R}^3$ is taken componentwise with respect to the basis $\mathcal{E}$ of $\mathbb{C}^4$.

Lemma 1. *The operator $\mathcal{F}e^{-itD_3}\mathcal{F}^{-1}$ acting on $L^2(\mathbb{R}^3, \mathbb{C}^4)$ is the operator of multiplication by the matrix-valued function $p \mapsto \int_{\mathbb{R}} e^{-i\langle\sigma,p\rangle\xi} \otimes H(d\xi, t)$, $p \in \mathbb{R}^3$, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $\langle\sigma, p\rangle = \sum_{j=1}^3 \sigma_j p_j$.*

Proof. The integral converges because $H(d\xi, t)$ is a matrix-valued Borel measure on $\mathbb{R}$ and in the matrix norm $\|e^{-i\langle\sigma,p\rangle\xi}\| = 1$ for all $\xi \in \mathbb{R}$ and $p \in \mathbb{R}^3$. By the functional calculus for the selfadjoint operator D_3, the operator $\mathcal{F}e^{-itD_3}\mathcal{F}^{-1}$ acting on $L^2(\mathbb{R}^3, \mathbb{C}^4)$ is the operator of multiplication by the matrix-valued function $p \mapsto e^{-it(\langle\sigma,p\rangle\otimes\sigma_3 + m\sigma_0\otimes\sigma_1)}$, so it suffices to prove that

$$
e^{-it(\langle\sigma,p\rangle\otimes\sigma_3 + m\sigma_0\otimes\sigma_1)} = \int_{\mathbb{R}} e^{-i\langle\sigma,p\rangle\xi} \otimes H(d\xi, t). \tag{16}
$$

To do so, rewrite the $M_4(\mathbb{C})$-valued function on the right-hand side by a formal integral with respect to $d\xi$:

$$
w(p,t) := \int_{\mathbb{R}} e^{-i\langle\sigma,p\rangle\xi} \otimes H(\xi, t) d\xi,
$$

each of whose (j, k) entries with j, $k = 1, 2$, makes sense as the bilinear inner product between a C^∞ function and a distribution with compact support in $\xi \in \mathbb{R}$. Then we have by (10)

$$
\begin{aligned}
\frac{\partial w(p,t)}{\partial t} &= -\int_{\mathbb{R}} e^{-i\langle\sigma,p\rangle\xi} \otimes \left(\sigma_3\frac{\partial}{\partial\xi} + im\sigma_1\right) H(\xi, t)d\xi \\
&= -i\int_{\mathbb{R}} [\langle\sigma, p\rangle \otimes \sigma_3 + m\sigma_0 \otimes \sigma_1](e^{-i\langle\sigma,p\rangle\xi} \otimes H(\xi, t))d\xi \\
&= -i[\langle\sigma, p\rangle \otimes \sigma_3 + m\sigma_0 \otimes \sigma_1]w(p, t).
\end{aligned}
$$

It follows that $w(p, t)$ is nothing but the left-hand side of equation (16). $\square$

Lemma 2. *The inverse Fourier transform of $p \mapsto e^{-i\langle\sigma,p\rangle\xi}$ in the sense of distributions is $\mu_{|\xi|} - \langle\nabla, x\rangle\mu_{|\xi|} - \xi\langle\sigma, \nabla\rangle\mu_{|\xi|}$ for all $\xi \in \mathbb{R}$.*

Proof. The Fourier transform $p \mapsto \hat{\mu}_r(p)$, $p \in \mathbb{R}^3$, of μ_r is the function $p \mapsto \frac{\sin(r|p|)}{r|p|}$ [Vladimirov (1971), p.135 (38)] as may be seen by calculating $\hat{\mu}_r(p) = \int_{\mathbb{R}^3} e^{-i\langle x,p\rangle}\mu_r(dx)$ directly.

Suppose that $\xi \neq 0$. It follows from the commutation relations of the Pauli matrices $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ that $e^{-i\langle\sigma,p\rangle\xi} = \cos(|\xi||p|) - i\langle\sigma, p\rangle\xi\frac{\sin(|\xi||p|)}{|\xi||p|}$. The operation of multiplication of the Fourier transform $\hat{T}(p)$ of a distribution T by ip corresponds to taking the gradient ∇T of T, so $\left(i\langle\sigma, p\rangle\xi\frac{\sin(|\xi|\,|p|)}{|\xi|\,|p|}\right)\check{} = \xi\langle\sigma,\nabla\rangle\mu_{|\xi|}$.

The term $\cos(|\xi|\,|p|)$ can be expressed as

$$\cos(|\xi|\,|p|) = \left[\frac{\partial}{\partial r} r\frac{\sin(|\xi|\,r)}{|\xi|\,r}\right]_{r=|p|} = \frac{\sin(|\xi|\,|p|)}{|\xi|\,|p|} + \langle p,\nabla\rangle\frac{\sin(|\xi|\,|p|)}{|\xi|\,|p|}.$$

The operation of forming the distribution $\frac{1}{i}\nabla\hat{T}$ from the Fourier transform $\hat{T}$ of a distribution T corresponds to multiplying the distribution $T(x)$ by $-x$, so taking the inverse Fourier transform gives $\cos(|\xi|\,|p|)\check{} = \mu_{|\xi|} - \langle\nabla, x\rangle\mu_{|\xi|}$ and we obtain the stated formula. In the case that $\xi = 0$, we have $-[\langle\nabla, x\rangle\mu_0](\phi) = \delta(r\partial/\partial r\phi) = 0$ for all $\phi \in \mathcal{S}(\mathbb{R}^3)$ and the formula still holds. □

Proof of Proposition 2. Let $t \in \mathbb{R}$. According to Lemma 1 the operator e^{-itD_3} corresponds to convolution with respect to the inverse Fourier transform $K_t \in \mathcal{L}(\mathcal{S}(\mathbb{R}^3), M_4(\mathbb{C}))$ of $p \mapsto \int_{\mathbb{R}} e^{-i\langle\sigma,p\rangle\xi} \otimes H(d\xi, t)$, $p \in \mathbb{R}^3$. The distribution K_t is evaluated as follows.

The map $\Phi : \xi \mapsto e^{-i\langle\sigma,\cdot\rangle\xi}$, $\xi \in \mathbb{R}$, may be viewed as a function taking its values in the space $\mathcal{L}(\mathcal{S}(\mathbb{R}^3), M_2(\mathbb{C}))$ of matrix-valued tempered distributions. An application of (the vector-valued) Fubini's theorem shows that for each $\phi \in \mathcal{S}(\mathbb{R}^3)$

$$K_t(\phi) = \int_{\mathbb{R}^3} \check{\phi}(p)\left[\int_{\mathbb{R}} e^{-i\langle\sigma,p\rangle\xi} \otimes H(d\xi, t)\right] dp = \int_{\mathbb{R}} [\Phi(\xi)\check{}(\phi)] \otimes H(d\xi, t).$$

By Lemma 2, $\Phi(\xi)\check{} = \mu_{|\xi|} - \langle\nabla, x\rangle\mu_{|\xi|} - \xi\langle\sigma,\nabla\rangle\mu_{|\xi|}$ for every $\xi \in \mathbb{R}$.

If $\phi \in \mathcal{S}(\mathbb{R}^3)$ is a test function, then

$$K_t(\phi) = \int_{\mathbb{R}} [\mu_{|\xi|}(\phi) + |\xi|\mu_{|\xi|}(\partial/\partial r\phi)]\sigma_0 \otimes H(d\xi, t) + \int_{\mathbb{R}} \xi\mu_{|\xi|}(\langle\sigma,\nabla\rangle\phi) \otimes H(d\xi, t),$$

so that equation (14) holds.

We now verify the representation (15) of the distribution K_t. Let $\phi \in \mathcal{S}(\mathbb{R}^3)$. In terms of the uniform probability measure μ_r on the sphere S_r with radius $r > 0$, we have

$$\begin{aligned}
K_t(\phi) &= \int_{\mathbb{R}}\int_{S_{|\xi|}} \sigma_0 \otimes H(\xi, t)\phi(x)\, d\mu_{|\xi|}(x)d\xi \\
&\qquad + \int_{\mathbb{R}}\int_{S_{|\xi|}} |\xi|\sigma_0 \otimes H(\xi, t)\frac{\partial\phi}{\partial r}(x)\, d\mu_{|\xi|}(x)d\xi \\
&\qquad\qquad + \int_{\mathbb{R}} \xi \int_{S_{|\xi|}} [(\langle\sigma,\nabla\rangle\phi)(x) \otimes H(\xi, t)]\, d\mu_{|\xi|}(x)d\xi \\
&= \int_0^\infty \int_{S_1} \sigma_0 \otimes [H(r, t) + H(-r, t)]\left[\frac{\partial}{\partial r} r\phi(r\omega)\right] d\mu_1(\omega)dr \\
&\qquad + \int_0^\infty r\int_{S_1} [\langle\sigma,\nabla\rangle\phi](r\omega) \otimes [H(r, t) - H(-r, t)]\, d\mu_1(\omega)dr \\
&= -\int_{S_1}\int_0^\infty \sigma_0 \otimes \left[\frac{\partial H(r, t)}{\partial r} + \frac{\partial H(-r, t)}{\partial r}\right]\phi(r\omega)r\, dr d\mu_1(\omega)
\end{aligned}$$

$$-\int_0^\infty \int_{S_1} \phi(r\omega)\left(\langle\sigma,\nabla\rangle \otimes [r^{-1}H(r,t) - r^{-1}H(-r,t)]\right) r^2 d\mu_1(\omega) dr,$$

using the change of variables $r = |x|$, $\omega = x/r$ for $x \in \mathbb{R}^3$ and Fubini's theorem. Here note that $d\mu_1(\omega) = \frac{1}{4\pi} d\omega$ with $d\omega$ the surface measure on S_1. □

4. The propagator of the radial Dirac operator

In this section, we present the calculation of the kernel of the group generated by (the closure of) the operator $i\tau_1$ defined by (5). The characteristic function of the interval $[0,\infty)$ is denoted by θ. The open interval $(0,\infty)$ is written as $\mathbb{R}_+$.

The function $D(\xi,t) = \frac{1}{2}J_0(m\sqrt{t^2-\xi^2})\theta(t^2-\xi^2)$, $\xi \in \mathbb{R}$, $t \in \mathbb{R}$, is the solution, in the sense of distributions, of the initial-value problem

$$\frac{\partial^2 g}{\partial t^2}(\xi,t) - \frac{\partial^2 g}{\partial \xi^2}(\xi,t) + m^2 g(\xi,t) = 0, \quad g(\xi,0) = 0, \quad \frac{\partial g}{\partial t}(\xi,0) = \delta(\xi), \tag{17}$$

for the 1-dimensional Klein-Gordon equation [Vladimirov (1971)].

For each $t \in \mathbb{R}$, let

$$\begin{aligned} E(\xi,t) &= \left(\frac{\partial}{\partial t} + im\right) D(\xi,t), \\ \overline{E}(\xi,t) &= \left(\frac{\partial}{\partial t} - im\right) D(\xi,t), \\ F(\xi,t) &= \frac{\partial D}{\partial \xi}(\xi,t) \end{aligned} \tag{18}$$

in the sense of distributions belonging to $C_c^\infty(\mathbb{R})'$. These distributions are of order zero and involve unit point masses at $\pm t$. Note that as $D(\cdot,t)$ is an even function, so are $E(\cdot,t)$ and $\overline{E}(\cdot,t)$, while $F(\cdot,t)$ is an odd function.

The following representation for the integral kernel $\mathcal{K}_t$ of $e^{-it\tau_1}$ holds.

Theorem 1. *For each $t \in \mathbb{R}\backslash\{0\}$, $\mathcal{K}_t$ is an $M_2(\mathbb{C})$-valued distribution defined in $\mathbb{R}_+\times\mathbb{R}_+$, i.e. an element in $\mathcal{L}(C_c^\infty(\mathbb{R}_+\times\mathbb{R}_+), M_2(\mathbb{C}))$ of order zero such that for all $f, g \in C_c^\infty(\mathbb{R}_+)$, the equality $\langle e^{-it\tau_1}(fe_j), \overline{g}e_k\rangle = \langle \mathcal{K}_t(g\otimes f)e_j, e_k\rangle$ holds for $j,k = 1,2$ with respect to the standard basis vectors e_1, e_2 of $\mathbb{C}^2$.*

The distribution $e^{-it\tau_1}(r,\rho) := \mathcal{K}_t(r,\rho)$ is given by the matrix

$$\begin{pmatrix} \overline{E}(r-\rho,t) - \overline{E}(r+\rho,t) & iW_{12}(r,\rho;t) \\ -iW_{21}(r,\rho;t) & E(r-\rho,t) + E(r+\rho,t) - W_{22}(r,\rho;t) \end{pmatrix} \tag{19}$$

of distributions of order zero belonging to $C_c^\infty(\mathbb{R}_+\times\mathbb{R}_+)'$, where

$$\begin{aligned} W_{12}(r,\rho;t) &= \left[F(r-\rho,t) + F(r+\rho,t) + \frac{1}{\rho}(D(r-\rho,t) - D(r+\rho,t))\right], \\ W_{21}(r,\rho;t) &= \left[F(r-\rho,t) - F(r+\rho,t) - \frac{1}{r}(D(r-\rho,t) - D(r+\rho,t))\right], \\ W_{22}(r,\rho;t) &= (r\rho)^{-1}\left(\frac{\partial}{\partial t} + im\right)\int_{|r-\rho|}^{r+\rho} D(\xi,t)\xi\, d\xi. \end{aligned}$$

The differentiation in the t-variable is interpreted in the sense of distributions.

4.1. **Regularisation.** Let $\xi \mapsto v(\xi)$ be a nonnegative smooth even function in $\mathbb{R}$, with support in the interval $[-1,1]$ and with the property that $\int_{\mathbb{R}} v(\xi)\,d\xi = 1$. Set $v(\xi) = \varepsilon^{-1}v(\xi/\varepsilon)$ for $\varepsilon > 0$. Then the function $D_\varepsilon(\,\cdot\,,t) = v_\varepsilon * D(\,\cdot\,,t)$ is the solution of the Klein-Gordon equation (17) but with the initial conditions $g(\xi,0) = 0$ and $\partial g/\partial t(\xi,0) = v_\varepsilon(\xi)$, $\xi \in \mathbb{R}$. The function $(\xi,t) \mapsto D_\varepsilon(\xi,t)$, $\xi \in \mathbb{R}$, $t > 0$, is smooth in ξ and t and for each $t > 0$, the function $D_\varepsilon(\,\cdot\,,t)$ has support in the interval $[-(t+\varepsilon), t+\varepsilon]$.

Then, as $\varepsilon \to 0+$, the smooth matrix-valued function H_ε defined by

$$H_\varepsilon(\xi,t) = \left(\sigma_0\frac{\partial}{\partial t} - \sigma_3\frac{\partial}{\partial \xi} - im\sigma_1\right)D_\varepsilon(\xi,t), \quad \xi \in \mathbb{R},\ t \geq 0 \tag{20}$$

converges in the space $\mathcal{L}(\mathcal{S}(\mathbb{R}), M_2(\mathbb{C}))$ of matrix-valued distributions to the distribution $H(\xi,t)$ defined by formula (9). Furthermore, because $H(\xi,t)$ is actually a distribution of order zero with compact support, as $\varepsilon \to 0+$, the matrix-valued measure $H_\varepsilon(\xi,t)\,d\xi$ converges to $H(d\xi,t)$ in the topology of the Banach space $\mathcal{L}(C_b(\mathbb{R}), M_2(\mathbb{C}))$ of all continuous linear mappings of $C_b(\mathbb{R})$ into $M_2(\mathbb{C})$. Here $C_b(\mathbb{R})$ is the Banach space of all bounded continuous functions on $\mathbb{R}$.

Let $K_t^{(\varepsilon)} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^3), M_4(\mathbb{C}))$ be defined by

$$\begin{aligned} K_t^{(\varepsilon)}(\phi) = &\int_{\mathbb{R}} [\mu_{|\xi|}(\phi) + |\xi|\mu_{|\xi|}(\partial/\partial r\phi)]\sigma_0 \otimes H_\varepsilon(\xi,t)\,d\xi \\ &+ \int_{\mathbb{R}} \xi\mu_{|\xi|}(\langle\sigma,\nabla\rangle\phi) \otimes H_\varepsilon(\xi,t)\,d\xi, \end{aligned} \tag{21}$$

for every test function $\phi \in \mathcal{S}(\mathbb{R}^3)$. As a consequence of the smoothing operation of convolution with respect to v_ε, we can use integration by parts in the integral above. In order to derive the representation (19) of the kernel of $e^{-it\tau_1}$, we calculate the kernel of $K_t^{(\varepsilon)}$ and take the limit as $\varepsilon \to 0+$.

Lemma 3. *As $\varepsilon \to 0+$,*

i) *the distribution $K_t^{(\varepsilon)}$ converges to K_t in $\mathcal{L}(\mathcal{S}(\mathbb{R}^3), M_4(\mathbb{C}))$,*
ii) *for any function $\psi \in \mathcal{S}(\mathbb{R}^3,\mathbb{C}^4)$, the convolution $K_t^{(\varepsilon)} * \psi$ belongs to $\mathcal{S}(\mathbb{R}^3,\mathbb{C}^4)$ and converges to $e^{-itD_3}\psi = K_t * \psi$ in the topology of $\mathcal{S}(\mathbb{R}^3,\mathbb{C}^4)$, and*
iii) *for any function $\psi \in L^2(\mathbb{R}^3,\mathbb{C}^4)$, the convolution $K_t^{(\varepsilon)} * \psi$ belongs to $L^2(\mathbb{R}^3,\mathbb{C}^4)$ and converges to $e^{-itD_3}\psi = K_t * \psi$ in the topology of $L^2(\mathbb{R}^3,\mathbb{C}^4)$. The operator norm of $\psi \mapsto K_t^{(\varepsilon)} * \psi$, $\psi \in L^2(\mathbb{R}^3,\mathbb{C}^4)$, is bounded by e^{mt}.*

Proof. Given $\phi \in \mathcal{S}(\mathbb{R}^3)$, the function $\xi \mapsto [\mu_{|\xi|}(\phi) + |\xi|\mu_{|\xi|}(\partial/\partial r\phi)]$, $\xi \in \mathbb{R}$, is a bounded, continuous scalar-valued function and $\xi \mapsto \mu_{|\xi|}(\langle\sigma,\nabla\rangle\phi)$, $\xi \in \mathbb{R}$, is a bounded, continuous $M_2(\mathbb{C})$-valued function. The first assertion (i) now follows from the observation that $H_\varepsilon(\xi,t)d\xi$ converges to $H(d\xi,t)$ in $\mathcal{L}(C_b(\mathbb{R}), M_2(\mathbb{C}))$.

Observe that the distribution $K_t^{(\varepsilon)}$ is obtained from K_t by replacing H by H_ε. Let $T_\varepsilon(t)$ denote the operator of convolution with respect to $K_t^{(\varepsilon)}$ acting on $\mathcal{S}(\mathbb{R}^3,\mathbb{C}^4)$. The proof of Lemma 1 shows that $\mathcal{F}T_\varepsilon(t)\mathcal{F}^{-1}$ is the operator of multiplication by the matrix-valued function $p \mapsto \int_{\mathbb{R}} e^{-i\langle\sigma,p\rangle\xi} \otimes H_\varepsilon(\xi,t)\,d\xi$.

Then for every $\psi \in \mathcal{S}(\mathbb{R}^3,\mathbb{C}^4)$, the function $p \mapsto \left[\int_{\mathbb{R}} e^{-i\langle\sigma,p\rangle\xi} \otimes H_\varepsilon(\xi,t)\,d\xi\right]\psi(p)$, $p \in \mathbb{R}^3$, also belongs to $\mathcal{S}(\mathbb{R}^3,\mathbb{C}^4)$ and $\mathcal{F}T_\varepsilon(t)\mathcal{F}^{-1}\psi \to \mathcal{F}e^{-itD_3}\mathcal{F}^{-1}\psi$ in $\mathcal{S}(\mathbb{R}^3,\mathbb{C}^4)$ as $\varepsilon \to 0+$ because $H_\varepsilon(\xi,t)d\xi$ converges to $H(d\xi,t)$ in $\mathcal{L}(C_b(\mathbb{R}), M_2(\mathbb{C}))$. Statement (ii) now follows from the observation that the Fourier transform leaves the space $\mathcal{S}(\mathbb{R}^3,\mathbb{C}^4)$ invariant. A similar argument proves (iii). The bound

$$\left\| \int_{\mathbb{R}} e^{-i\langle\sigma,p\rangle\xi} \otimes H_\varepsilon(\xi,t)\,d\xi \right\|_{\mathcal{L}(\mathbb{C}^4)} \le \|v_\varepsilon * H(\,\cdot\,,t)\|_{L^1(\mathbb{R},M_2(\mathbb{C}))} \le e^{mt}$$

for each $p \in \mathbb{R}^3$ follows from (20) and Proposition 1, so $\|T_\varepsilon(t)\| \le e^{mt}$. $\square$

Let $\phi \in \mathcal{S}(\mathbb{R}^3)$. In terms of the uniform probability measure μ_r on the sphere S_r with radius $r > 0$, we have

$$\begin{aligned} K_t^{(\varepsilon)}(\phi) \;=\; & -\int_0^\infty r \int_{S_1} \sigma_0 \otimes \left[\left.\frac{\partial H_\varepsilon(\xi,t)}{\partial \xi}\right|_{\xi=r} - \left.\frac{\partial H_\varepsilon(\xi,t)}{\partial \xi}\right|_{\xi=-r} \right] \phi(r\omega)\,d\mu_1(\omega)dr \\ & + \int_0^\infty r \int_{S_1} [\langle\sigma,\nabla\rangle\phi](r\omega) \otimes [H_\varepsilon(r,t) - H_\varepsilon(-r,t)]\,d\mu_1(\omega)dr, \end{aligned}$$

by an appeal to Fubini's theorem and integrating by parts as in the proof of Proposition 2. On converting to cartesian coordinates, the convolution $K_t^{(\varepsilon)} * \psi(x)$ of the distribution $K_t^{(\varepsilon)}$ with $\psi \in \mathcal{S}(\mathbb{R}^3,\mathbb{C}^4)$ is equal to

$$K_t^{(\varepsilon)} * \psi(x)$$

$$= -1/4\pi \int_{\mathbb{R}^3} |x-y|^{-1}\sigma_0 \otimes \left[\left.\frac{\partial H_\varepsilon(\xi,t)}{\partial \xi}\right|_{\xi=|x-y|} - \left.\frac{\partial H_\varepsilon(\xi,t)}{\partial \xi}\right|_{\xi=-|x-y|} \right] \psi(y)\,dy \tag{22}$$

$$-1/4\pi \int_{\mathbb{R}^3} |x-y|^{-1}\sigma_0 \otimes [H_\varepsilon(|x-y|,t) - H_\varepsilon(-|x-y|,t)][\langle\sigma,\nabla\rangle \otimes \sigma_0]\psi(y)\,dy, \tag{23}$$

for all $x \in \mathbb{R}^3$.

4.2. **Reducing Subspaces.** According to [Thaller (1992), Section 4.6.4], [see also Bjorken and Drell (1964), p.52ff], for every $m_k = -|k|, \dots, |k|-1$ and $k = \pm1, \pm2, \dots$, the operators e^{-itD_3}, $t \in \mathbb{R}$, leave the subspace $\mathcal{H}(k,m_k)$ of $L^2(\mathbb{R}^3,\mathbb{C}^4)$ invariant. The subspace $\mathcal{H}(k,m_k)$ equals the orthogonal sum $\mathcal{H}(k,m_k)_+ \oplus \mathcal{H}(k,m_k)_-$ with

$$\begin{aligned} \mathcal{H}(k,m_k)_+ &= \left\{ \Phi^{(+)}(k,m_k) \otimes \frac{u(r)}{r} : u \in L^2(\mathbb{R}_+) \right\}, \\ \mathcal{H}(k,m_k)_- &= \left\{ \Phi^{(-)}(k,m_k) \otimes \frac{u(r)}{r} : u \in L^2(\mathbb{R}_+) \right\}, \end{aligned}$$

for certain $\mathbb{C}^4$-valued eigenfunctions $\Phi^{(\pm)}(k,m_k)$ on S_1. If $u = {}^t(u_1,u_2) \in L^2(\mathbb{R}_+,\mathbb{C}^2)$ are functions and c_+, c_- are complex numbers such that the element

$$\psi(x) = c_+\Phi^{(+)}(k,m_k) \otimes \frac{u_1(r)}{r} + c_-\Phi^{(-)}(k,m_k) \otimes \frac{u_2(r)}{r}, \qquad r = |x|,$$

of $L^2(\mathbb{R}^3,\mathbb{C}^4)$ belongs to the domain of D_3, then

$$(D_3\psi)(x) = c_+\Phi^{(+)}(k,m_k) \otimes \frac{(\tau_k u)_1(r)}{r} + c_-\Phi^{(-)}(k,m_k) \otimes \frac{(\tau_k u)_2(r)}{r}, \qquad r = |x|,$$

for the differential operator τ_k given by formula (5). Moreover,

$$\left(e^{-itD_3}\psi\right)(x) = c_+\Phi^{(+)}(k,m_k) \otimes \frac{(e^{-it\tau_k}u)_1(r)}{r} + c_-\Phi^{(-)}(k,m_k) \otimes \frac{(e^{-it\tau_k}u)_2(r)}{r}, \qquad r = |x|,$$

for all $\psi \in \mathcal{H}(k,m_k)$. The function $e^{-itD_3}\psi$ is also expressed as a convolution $K_t * \psi$.

It turns out that convolution with respect to the regularised kernel $K_t^{(\varepsilon)}$ also leaves the reducing subspaces $\mathcal{H}(k,m_k)$ invariant — the first integral (22) leaves the subspaces $\mathcal{H}(k,m_k)_+$ and $\mathcal{H}(k,m_k)_-$ invariant, and the second integral (23) maps $\mathcal{H}(k,m_k)_+$ to

$\mathcal{H}(k,m_k)_-$ and $\mathcal{H}(k,m_k)_-$ to $\mathcal{H}(k,m_k)_+$. We shall only prove this for the case $k=1$ and $m_k=0$. A similar argument works for $k=-1$ with only changes in signs. We compute the distributional kernel of $e^{-it\tau_1}$ by taking the limit as $\varepsilon\to 0+$ of $K_t^{(\varepsilon)}*\psi$ for $\psi\in\mathcal{H}(1,0)$.

In the usual representation of the Dirac matrices,

$$\Phi^{(+)}(k,m_k)=\begin{pmatrix} i\psi(k,m_k)\\ 0\\ 0\end{pmatrix},\qquad \Phi^{(-)}(k,m_k)=\begin{pmatrix}0\\ 0\\ \psi(-k,m_k)\end{pmatrix}$$

$$\begin{aligned}\psi(k,m_k) &= \tfrac{1}{\sqrt{2k-1}}\begin{pmatrix}\sqrt{k+m_k}\,Y(k-1,m_k)\\ \sqrt{k-m_k-1}\,Y(k-1,m_k+1)\end{pmatrix}, &&\text{for } k>0,\\ \psi(k,m_k) &= \tfrac{1}{\sqrt{2|k|+1}}\begin{pmatrix}\sqrt{|k|-m_k}\,Y(|k|,m_k)\\ -\sqrt{|k|+m_k+1}\,Y(|k|,m_k+1)\end{pmatrix}, &&\text{for } k<0,\end{aligned}$$

for the spherical harmonic functions $Y(l,j)$.

For $k=1$ and $m_1=-1,0$, we have

$$\Phi^{(+)}(1,0)=\begin{pmatrix} iY(0,0)\\ 0\\ 0\\ 0\end{pmatrix},\quad \Phi^{(-)}(1,0)=\frac{1}{\sqrt{3}}\begin{pmatrix}0\\ 0\\ Y(1,0)\\ -\sqrt{2}\,Y(1,1)\end{pmatrix}$$

$$\Phi^{(+)}(1,-1)=\begin{pmatrix} 0\\ iY(0,0)\\ 0\\ 0\end{pmatrix},\quad \Phi^{(-)}(1,-1)=\frac{1}{\sqrt{3}}\begin{pmatrix}0\\ 0\\ \sqrt{2}\,Y(1,-1)\\ -Y(1,0)\end{pmatrix}$$

The spherical harmonic functions in this formula are given, with $0\le\theta\le\pi$ and $0\le\varphi\le 2\pi$, by

$$Y(0,0)=\frac{1}{\sqrt{4\pi}},\quad Y(1,0)=\frac{\sqrt{\frac{3}{\pi}}\cos\theta}{2}$$

$$Y(1,1)=-\frac{e^{i\varphi}\sqrt{\frac{3}{2\pi}}\sin\theta}{2},\quad Y(1,-1)=\frac{e^{-i\varphi}\sqrt{\frac{3}{2\pi}}\sin\theta}{2}$$

so that

$$\Phi^{(+)}(1,0)=\frac{i}{2\sqrt{\pi}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \Phi^{(-)}(1,0)=\frac{1}{2\sqrt{\pi}}\begin{pmatrix}0\\ 0\\ \cos\theta\\ e^{i\varphi}\sin\theta\end{pmatrix}$$

With respect to the block-diagonal matrix representation (11) of the Dirac matrices, the new eigenvectors are $Q\Phi^{\pm}(1,0)$, $Q\Phi^{\pm}(1,-1)$, with Q in equation (12) used to define the α_j' in Section 3.2. For the purposes of the following computation, it is simpler to represent the tensor products $\sigma_j\otimes\sigma_3$, $j=1,2,3$ and $\sigma_0\otimes\sigma_1$ with respect to the permuted basis $\mathcal{E}'=\{e_1\otimes e_1, e_1\otimes e_2, e_2\otimes e_1, e_2\otimes e_2\}$ of $\mathbb{C}^4$ so that $\sigma_0\otimes\sigma_j=\begin{pmatrix}\sigma_j & 0\\ 0 & \sigma_j\end{pmatrix}$ for $j=1,2,3$. With respect to this new basis, the eigenvectors $Q\Phi^{\pm}(1,0)$, $Q\Phi^{\pm}(1,-1)$ are represented by the column vectors

$$\Psi^{(+)}(1,0) = \frac{i}{2\sqrt{2\pi}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \Psi^{(-)}(1,0) = \frac{1}{2\sqrt{2\pi}} \begin{pmatrix} \cos\theta \\ -\cos\theta \\ e^{i\varphi}\sin\theta \\ -e^{i\varphi}\sin\theta \end{pmatrix}$$

$$\Psi^{(+)}(1,-1) = \frac{i}{2\sqrt{2\pi}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \Psi^{(-)}(1,-1) = \frac{1}{2\sqrt{2\pi}} \begin{pmatrix} e^{-i\varphi}\sin\theta \\ -e^{-i\varphi}\sin\theta \\ -\cos\theta \\ \cos\theta \end{pmatrix}$$

Proof of Theorem 1. According to [Thaller (1992), Theorem 4.1.4], the matrix elements of (19) we are trying to calculate are the limiting distributions as $\varepsilon \to 0+$ corresponding to the action of the convolutions (22) and (23) on the subspaces $\mathcal{H}(1,0)_+$ and $\mathcal{H}(1,0)_-$ of $L^2(\mathbb{R}^3, \mathbb{C}^4)$. We look at each subspace separately.

4.2.1. *The subspace $\mathcal{H}(1,0)_+$.* On setting $\psi(y) = \Psi^{(+)}(1,0) \otimes \rho^{-1}u(\rho)$ in polar cordinates (ρ,θ,φ) in the integral (22), we have

$$\begin{aligned}
-1/4\pi \int_0^\infty \int_0^\pi \int_0^{2\pi} & |r^2+\rho^2-2r\rho\cos\theta|^{-1/2}\sigma_0 \otimes \Bigg[\frac{\partial H_\varepsilon(\xi,t)}{\partial\xi}\bigg|_{\xi=|r^2+\rho^2-2r\rho\cos\theta|^{1/2}} \\
&- \frac{\partial H_\varepsilon(\xi,t)}{\partial\xi}\bigg|_{\xi=-|r^2+\rho^2-2r\rho\cos\theta|^{1/2}} \Bigg] \Psi^{(+)}(1,0)u(\rho)\,\rho\sin\theta d\varphi d\theta d\rho \\
= -1/2 \int_0^\infty \int_0^\pi & |r^2+\rho^2-2r\rho\cos\theta|^{-1/2}\sigma_0 \otimes \Bigg[\frac{\partial H_\varepsilon(\xi,t)}{\partial\xi}\bigg|_{\xi=|r^2+\rho^2-2r\rho\cos\theta|^{1/2}} \\
&- \frac{\partial H_\varepsilon(\xi,t)}{\partial\xi}\bigg|_{\xi=-|r^2+\rho^2-2r\rho\cos\theta|^{1/2}} \Bigg] \Psi^{(+)}(1,0)u(\rho)\,\rho\sin\theta d\theta d\rho
\end{aligned}$$

Let $\xi = (r^2+\rho^2-2r\rho\cos\theta)^{1/2}$. Then $\xi = r+\rho$ when $\theta=\pi$ and $\xi = |r-\rho|$ when $\theta = 0$. The integral (22) becomes

$$\begin{aligned}
-\frac{1}{2r}\int_0^\infty \Bigg[\int_{|r-\rho|}^{r+\rho} \sigma_0 \otimes \bigg(\left(\frac{\partial H_\varepsilon}{\partial\xi}\right)(\xi,t) - \left(\frac{\partial H_\varepsilon}{\partial\xi}\right)(-\xi,t) \bigg) d\xi \Bigg] & \Psi^{(+)}(1,0)u(\rho)\,d\rho \\
= -\frac{1}{2r}\int_0^\infty \sigma_0 \otimes \big[H_\varepsilon(r+\rho,t) - H_\varepsilon(|r-\rho|,t) & \\
+ H_\varepsilon(-(r+\rho),t) - H_\varepsilon(-|r-\rho|,t)\big] & \Psi^{(+)}(1,0)u(\rho)\,d\rho.
\end{aligned}$$

Now $D(\,\cdot\,,t)$ is an even function and we chose v_ε to be even, so $D_\varepsilon(\,\cdot\,,t)$ is also an even function. In the expression (20) for H_ε, the components of $\big(\sigma_0\partial/\partial t - im\sigma_1\big)D_\varepsilon(\xi,t)$ are even functions of ξ and the components of $\sigma_3\partial D_\varepsilon(\xi,t)/\partial\xi$ are odd functions of ξ. Furthermore, in the chosen basis, the application of either of the matrices $\sigma_0\otimes\sigma_0$ and $\sigma_0\otimes\sigma_1$ does not change the vector $\Psi^{(+)}(1,0)$.

On setting $\overline{E}_\varepsilon(\xi,t) = \big(\partial/\partial t - im\big)D_\varepsilon(\xi,t)$ for $\xi\in\mathbb{R}$ and $t>0$, the integral (22) is therefore equal to

$$-\frac{\Psi^{(+)}(1,0)}{r}\int_0^\infty \big[\overline{E}_\varepsilon(r+\rho,t) - \overline{E}_\varepsilon(r-\rho,t)\big]u(\rho)\,d\rho.$$

The matrix element corresponding to (22) is therefore $\overline{E}_\varepsilon(r-\rho,t) - \overline{E}_\varepsilon(r+\rho,t)$.

The integrand of (23) consists of the odd components of $H_\varepsilon(\,\cdot\,,t)$. Set $F_\varepsilon(\xi,t) = \partial D_\varepsilon(\xi,t)/\partial\xi$, for all $\xi\in\mathbb{R}$ and $t\geq 0$. Then note that

$$H_\varepsilon(\xi,t)-H_\varepsilon(-\xi,t) = -\sigma_3(F_\varepsilon(\xi,t)-F_\varepsilon(-\xi,t)) = -2\sigma_3F_\varepsilon(\xi,t).$$

Then in polar coordinates, the integral (23) becomes

(24)
$$2\int_0^\infty\left[\int_{S_1}\langle\sigma,\omega\rangle\otimes\sigma_3\Psi^{(+)}(1,0)\frac{F_\varepsilon([r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2},t)}{[r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2}}\,\mu_1(d\omega)\right]\left[\frac{d}{d\rho}\frac{u(\rho)}{\rho}\right]\rho^2\,d\rho$$

for $r=|x|$ and $\hat r = x/r$. Here we have used the fact that $\Psi^{(+)}(1,0)$ is a *constant* vector.

Suppose that $\{\hat e_1,\hat e_2,\hat r\}$ is an orthonormal basis of $\mathbb{R}^3$ and $e_j = a_{j1}\hat e_1 + a_{j2}\hat e_2 + a_{j3}\hat r$ for the standard basis vectors e_j, $j=1,2,3$, of $\mathbb{R}^3$. Then

$$\langle\sigma,\omega\rangle = \sum_{j=1}^3\sigma_j[a_{j1}\langle\omega,\hat e_1\rangle + a_{j2}\langle\omega,\hat e_2\rangle + a_{j3}\langle\omega,\hat r\rangle].$$

Because $\int_{S_1}f(\langle\omega,\hat r\rangle)\langle\omega,\hat e_j\rangle d\mu(\omega)=0$, for $j=1,2$ and any bounded measurable function f, only the integral over the term $\sum_{j=1}^3\sigma_j a_{j3}\langle\omega,\hat r\rangle$ of $\langle\sigma,\omega\rangle$ in (24) remains. But $\hat r = \sum_{j=1}^3 e_j a_{j3}$ because $[a_{j\kappa}]_{j,\kappa=1}^3$ is an orthogonal matrix, that is, $a_{j3}=\hat r_j$ for $j=1,2,3$, so the integral (23) is equal to

$$2\int_0^\infty\left[\int_{S_1}\langle\omega,\hat r\rangle\frac{F_\varepsilon([r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2},t)}{[r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2}}\,\mu_1(d\omega)\right]\left[\frac{d}{d\rho}\frac{u(\rho)}{\rho}\right]\rho^2\,d\rho$$

times the vector $[\langle\sigma,\hat r\rangle\otimes\sigma_3]\Psi^{(+)}(1,0)$.

With the substitution $\xi = [r^2+\rho^2-2r\rho\cos\theta]^{1/2}$, $\cos\theta = \langle\omega,\hat r\rangle$ and integration by parts, we have

$$\begin{aligned}
-2\int_0^\infty d\rho\frac{u(\rho)}{\rho}\frac{\partial}{\partial\rho}\rho^2\int_{S_1}\langle\omega,\hat r\rangle\frac{F_\varepsilon([r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2},t)}{[r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2}}\,\mu_1(d\omega)&\\
= -\frac{2}{4\pi}\int_0^\infty d\rho\frac{u(\rho)}{\rho}\frac{\partial}{\partial\rho}\rho^2\int_0^\pi\int_0^{2\pi}\cos\theta\frac{F_\varepsilon([r^2+\rho^2-2r\rho\cos\theta]^{1/2},t)}{[r^2+\rho^2-2r\rho\cos\theta]^{1/2}}\sin\theta d\varphi d\theta&\\
= -\int_0^\infty d\rho\frac{u(\rho)}{\rho}\frac{\partial}{\partial\rho}\rho^2\int_{|r-\rho|}^{r+\rho}\frac{r^2+\rho^2-\xi^2}{2r^2\rho^2}F_\varepsilon(\xi,t)\,d\xi&\\
= -\frac{1}{2r^2}\int_0^\infty d\rho\frac{u(\rho)}{\rho}\frac{\partial}{\partial\rho}\int_{|r-\rho|}^{r+\rho}(r^2+\rho^2-\xi^2)F_\varepsilon(\xi,t)\,d\xi.&
\end{aligned}$$

In the region $\rho>r$, we have

$$\begin{aligned}
-\frac{1}{2r^2}\int_0^\infty d\rho\frac{u(\rho)}{\rho}\frac{\partial}{\partial\rho}\int_{\rho-r}^{r+\rho}(r^2+\rho^2-\xi^2)F_\varepsilon(\xi,t)\,d\xi&\\
= -\frac{1}{2r^2}\int_0^\infty d\rho\frac{u(\rho)}{\rho}\left(\left[(r^2+\rho^2-\xi^2)F_\varepsilon(\xi,t)\right]_{\rho-r}^{r+\rho}+2\rho\int_{\rho-r}^{r+\rho}F_\varepsilon(\xi,t)\,d\xi\right)&\\
= -\frac{1}{r}\int_0^\infty d\rho u(\rho)\big[-(F_\varepsilon(r+\rho,t)+F_\varepsilon(\rho-r,t))+\frac{1}{r}(D_\varepsilon(r+\rho,t)-D_\varepsilon(\rho-r,t))\big]&\\
= -\frac{1}{r}\int_0^\infty d\rho u(\rho)\big[F_\varepsilon(r-\rho,t)-F_\varepsilon(r+\rho,t)+\frac{1}{r}(D_\varepsilon(r+\rho,t)-D_\varepsilon(r-\rho,t))\big],&
\end{aligned}$$

because F_ε is an odd function. The same expression is obtained for $\rho<r$.

A calculation [cf. Thaller (1992), Eq. (4.122) p. 127] shows that

$$[\langle\sigma,\hat r\rangle\otimes\sigma_3]\Psi^{(+)}(1,0)=i\Psi^{(-)}(1,0)(\hat r). \tag{25}$$

Indeed,

$$\langle\sigma,\hat r\rangle\otimes\sigma_3=\begin{pmatrix}\hat r_3 & \hat r_1-i\hat r_2\\ \hat r_1+i\hat r_2 & -\hat r_3\end{pmatrix}\otimes\sigma_3=\begin{pmatrix}\hat r_3\sigma_3 & (\hat r_1-i\hat r_2)\sigma_3\\ (\hat r_1+i\hat r_2)\sigma_3 & -\hat r_3\sigma_3\end{pmatrix}.$$

The sum of the first two columns of $\langle\sigma,\hat r\rangle\otimes\sigma_3$ is

$$\begin{pmatrix}\hat r_3\\ -\hat r_3\\ \hat r_1+i\hat r_2\\ -(\hat r_1+i\hat r_2)\end{pmatrix}=\begin{pmatrix}\cos\theta\\ -\cos\theta\\ e^{i\varphi}\sin\theta\\ -e^{i\varphi}\sin\theta\end{pmatrix}$$

which is a constant multiple of $\Psi^{(-)}(1,0)(\hat r)$. The integral (23) is equal to

$$-i\frac{\Psi^{(-)}(1,0)(\hat r)}{r}\int_0^\infty d\rho u(\rho)\big[F_\varepsilon(r-\rho,t)-F_\varepsilon(r+\rho,t)-\frac{1}{r}(D_\varepsilon(r-\rho,t)-D_\varepsilon(r+\rho,t))\big].$$

so that the corresponding matrix element is

$$-iW_{21}^{(\varepsilon)}(r,\rho;t)=-i\big[F_\varepsilon(r-\rho,t)-F_\varepsilon(r+\rho,t)-\frac{1}{r}(D_\varepsilon(r-\rho,t)-D_\varepsilon(r+\rho,t))\big].$$

4.2.2. *The subspace $\mathcal H(1,0)_-$.* On the other hand, for $\psi(y)=\Psi^{(-)}(1,0)\otimes\rho^{-1}u(\rho)$, (22) is

$$-2\int_0^\infty\left[\int_{S_1}\sigma_0\otimes\frac{\frac{\partial G_\varepsilon}{\partial\xi}([r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2},t)}{[r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2}}\Psi^{(-)}(1,0)(\omega)\,d\mu_1(\omega)\right]u(\rho)\rho\,d\rho$$

for $G_\varepsilon(\xi,t)=\big(\sigma_0\partial/\partial t-im\sigma_1\big)D_\varepsilon(\xi,t)$, for all $\xi\in\mathbb R$ and $t>0$. Note here that

$$\frac{\partial}{\partial\xi}H_\varepsilon(\xi,t)-\frac{\partial}{\partial\xi}H_\varepsilon(-\xi,t)=2(\sigma_0\frac{\partial}{\partial t}-im\sigma_1)\frac{\partial}{\partial\xi}D_\varepsilon(\xi,t)=2\frac{\partial}{\partial\xi}G_\varepsilon(\xi,t).$$

Now

$$\begin{aligned}
&\int_{S_1}\sigma_0\otimes\frac{\frac{\partial G_\varepsilon}{\partial\xi}([r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2},t)}{[r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2}}\Psi^{(-)}(1,0)(\omega)\,d\mu_1(\omega)\\
=\;&-i\int_{S_1}\sigma_0\otimes\frac{\frac{\partial G_\varepsilon}{\partial\xi}([r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2},t)}{[r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2}}[\langle\sigma,\omega\rangle\otimes\sigma_3]\Psi^{(+)}(1,0)\,d\mu_1(\omega)\\
=\;&-i\int_{S_1}\sigma_0\otimes\frac{\frac{\partial G_\varepsilon}{\partial\xi}([r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2},t)}{[r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2}}[\langle\sigma,\hat r\rangle\otimes\sigma_3]\Psi^{(+)}(1,0)\langle\omega,\hat r\rangle\,d\mu_1(\omega)\\
=\;&\int_{S_1}\sigma_0\otimes\frac{\frac{\partial G_\varepsilon}{\partial\xi}([r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2},t)}{[r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2}}\Psi^{(-)}(1,0)(\hat r)\langle\omega,\hat r\rangle\,d\mu_1(\omega)\\
=\;&\Psi^{(-)}(1,0)(\hat r)\int_{S_1}\frac{\frac{\partial E_\varepsilon}{\partial\xi}([r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2},t)}{[r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2}}\langle\omega,\hat r\rangle\,d\mu_1(\omega)\\
=\;&4^{-1}\Psi^{(-)}(1,0)(\hat r)(r\rho)^{-2}\int_{|r-\rho|}^{r+\rho}\frac{\partial E_\varepsilon(\xi,t)}{\partial\xi}(r^2+\rho^2-\xi^2)\,d\xi,
\end{aligned}$$

where we have used the identity (25) and the fact that $(\sigma_0\otimes\sigma_0)\Psi^{(-)}(1,0)=\Psi^{(-)}(1,0)$ and $(\sigma_0\otimes\sigma_1)\Psi^{(-)}(1,0)=-\Psi^{(-)}(1,0)$. Integrating by parts, we have

$$-2^{-1}(r\rho)^{-1}\int_{|r-\rho|}^{r+\rho}\frac{\partial E_\varepsilon(\xi,t)}{\partial\xi}(r^2+\rho^2-\xi^2)\,d\xi$$

$$= E_\varepsilon(r+\rho) + E_\varepsilon(r-\rho) - (r\rho)^{-1}\int_{|r-\rho|}^{r+\rho} E_\varepsilon(\xi)\xi\,d\xi,$$

which is the matrix element corresponding to the convolution (22) acting on $\mathcal{H}(1,0)_-$.

For the other integral (23), appealing to the identity (25), we have

$$\begin{aligned}
[\langle\sigma,\nabla\rangle]\otimes\sigma_3]\psi(y) &= [\langle\sigma,\nabla\rangle\otimes\sigma_3][\Psi^{(-)}(1,0)(\omega)\rho^{-1}u(\rho)] \\
&= \frac{u(\rho)}{\rho}[\langle\sigma,\nabla\rangle\otimes\sigma_3]\Psi^{(-)}(1,0)(\omega) + \langle\sigma,\omega\rangle\otimes\sigma_3\Psi^{(-)}(1,0)(\omega)\frac{d}{d\rho}\frac{u(\rho)}{\rho} \\
&= \frac{1}{i}\left[\frac{u(\rho)}{\rho}[\langle\sigma,\nabla\rangle\otimes\sigma_3][\langle\sigma,\omega\rangle\otimes\sigma_3]\Psi^{(+)}(1,0) + \Psi^{(+)}(1,0)\frac{d}{d\rho}\frac{u(\rho)}{\rho}\right],
\end{aligned}$$

in coordinates $\omega = y/|y|$ and $\rho = |y|$.

The function $\Psi^{(+)}(1,0)$ is a constant vector and a calculation shows that

$$[\langle\sigma,\nabla\rangle\otimes\sigma_3]\langle\sigma,\omega\rangle\otimes\sigma_3 = \frac{2}{\rho}\sigma_0\otimes\sigma_0.$$

One integral in (23) is therefore

$$\begin{aligned}
&-2i\Psi^{(+)}(1,0)\int_0^\infty\left[\int_{S_1}\frac{F_\varepsilon([r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2},t)}{[r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2}}\,\mu_1(d\omega)\right][\frac{d}{d\rho}\frac{u(\rho)}{\rho}]\rho^2\,d\rho \\
&= 2i\Psi^{(+)}(1,0)\int_0^\infty\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho^2\left[\int_{S_1}\frac{F_\varepsilon([r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2},t)}{[r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2}}\,\mu_1(d\omega)\right]u(\rho)\,d\rho \\
&= \frac{i\Psi^{(+)}(1,0)}{r}\int_0^\infty\frac{u(\rho)}{\rho}\frac{\partial}{\partial\rho}\rho\left[\int_{|r-\rho|}^{r+\rho}F_\varepsilon(\xi,t)\,d\xi\right]d\rho \\
&= \frac{i\Psi^{(+)}(1,0)}{r}\int_0^\infty\frac{u(\rho)}{\rho}\frac{\partial}{\partial\rho}\rho[D_\varepsilon(r+\rho,t)-D_\varepsilon(r-\rho,t)]\,d\rho \\
&= \frac{i\Psi^{(+)}(1,0)}{r}\int_0^\infty[F_\varepsilon(r-\rho,t)+F_\varepsilon(r+\rho,t) \\
&\qquad -\frac{1}{\rho}[D_\varepsilon(r-\rho,t)-D_\varepsilon(r+\rho,t)]u(\rho)\,d\rho
\end{aligned}$$

and the other is

$$\begin{aligned}
&-4i\Psi^{(+)}(1,0)\int_0^\infty\left[\int_{S_1}\frac{F_\varepsilon([r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2},t)}{[r^2+\rho^2-2r\rho\langle\omega,\hat r\rangle]^{1/2}}\,\mu_1(d\omega)\right]u(\rho)\,d\rho \\
&= -\frac{2i\Psi^{(+)}(1,0)}{r}\int_0^\infty\frac{u(\rho)}{\rho}\int_{|r-\rho|}^{r+\rho}F_\varepsilon(\xi,t)\,d\xi\,d\rho \\
&= \frac{2i\Psi^{(+)}(1,0)}{r}\int_0^\infty\frac{u(\rho)}{\rho}[D_\varepsilon(r-\rho,t)-D_\varepsilon(r+\rho,t)]\,d\rho
\end{aligned}$$

Adding the two terms, the integral (23) becomes

$$\frac{i\Psi^{(+)}(1,0)}{r}\int_0^\infty[F_\varepsilon(r-\rho,t)+F_\varepsilon(r+\rho,t)+\frac{1}{\rho}[D_\varepsilon(r-\rho,t)-D_\varepsilon(r+\rho,t)]u(\rho)\,d\rho,$$

so that the corresponding matrix element is

$$iW_{12}^{(\varepsilon)}(r,\rho;t) = i[F_\varepsilon(r-\rho,t)+F_\varepsilon(r+\rho,t)+\frac{1}{\rho}[D_\varepsilon(r-\rho,t)-D_\varepsilon(r+\rho,t)].$$

The operator of convolution with respect to $K_t^{(\varepsilon)}$ therefore leaves the subspace $\mathcal{H}(1,0)$ invariant and its action on the radial components is the operator $e^{-it\tau_1}$ whose kernel is given by

$$\begin{pmatrix} \overline{E_\varepsilon}(r-\rho,t) - \overline{E_\varepsilon}(r+\rho,t) & iW_{12}^{(\varepsilon)}(r,\rho;t) \\ -iW_{21}^{(\varepsilon)}(r,\rho;t) & E_\varepsilon(r-\rho,t) + E_\varepsilon(r+\rho,t) - W_{22}^{(\varepsilon)}(r,\rho;t) \end{pmatrix} \tag{26}$$

where

$$\begin{aligned} W_{12}^{(\varepsilon)}(r,\rho;t) &= \left[F_\varepsilon(r-\rho,t) + F_\varepsilon(r+\rho,t) + \frac{1}{\rho}(D_\varepsilon(r-\rho,t) - D_\varepsilon(r+\rho,t))\right] \\ W_{21}^{(\varepsilon)}(r,\rho;t) &= \left[F_\varepsilon(r-\rho,t) - F_\varepsilon(r+\rho,t) - \frac{1}{r}(D_\varepsilon(r-\rho,t) - D_\varepsilon(r+\rho,t))\right] \\ W_{22}^{(\varepsilon)}(r,\rho;t) &= (r\rho)^{-1}\int_{|r-\rho|}^{r+\rho} E_\varepsilon(\xi,t)\xi\, d\xi. \end{aligned}$$

By virtue of Lemma 3 and the observation that the operator of convolution with respect to $K_t^{(\varepsilon)}$ leaves the subspace $\mathcal{H}(1,0)$ invariant, the operator $e^{-it\tau_1}$ is the limit in the strong operator topology on $L^2(\mathbb{R}_+,\mathbb{C}^2)$, as $\varepsilon \to 0+$, of bounded linear operators with kernels given by (26).

It is readily verified that as $\varepsilon \to 0+$, the components of (26) converge in the space $C_c^\infty(\mathbb{R}_+\times\mathbb{R}_+)'$ of distributions to the components of (19). The proof of Theorem 1 is complete. ☐

The explicit form of the kernel $e^{-it\tau_1}(r,\rho)$ given in Theorem 1 enables us to show that $e^{-it\tau_1}$ is not a bounded operator on $L^\infty(\mathbb{R}_+,\mathbb{C}^2)$ for $t \neq 0$.

Proposition 3. *For each $\delta > 0$ and $t \neq 0$, $e^{-it\tau_1}(\chi_{(0,\delta)}.e_2) \notin L^\infty(\mathbb{R}_+,\mathbb{C}^2)$.*

Proof. Let $t > 0$. A similar argument holds for $t < 0$. It is enough to show that the function

$$r \mapsto \int_0^\delta W_{12}(r,\rho;t)\, d\rho, \quad 0 < r < t, \tag{27}$$

is not bounded near $r = t$. The distribution of order zero with the kernel $F(r-\rho,t) + F(r+\rho,t)$ is associated with the sum of convolutions with respect to measures, giving a uniformly bounded function of r, so we examine the integral

$$\int_0^\delta \frac{D(r-\rho,t) - D(r+\rho,t)}{\rho}\, d\rho.$$

In the region $\{(r,\rho) \in (\mathbb{R}_+)^2 : |r+\rho| < t\}$, there exists a constant $C > 0$ such that

$$\begin{aligned} |D(r-\rho,t) - D(r+\rho,t)| &\le C\left|\sqrt{t^2-(r-\rho)^2} - \sqrt{t^2-(r+\rho)^2}\right| \\ &= \frac{4Cr\rho}{\sqrt{t^2-(r-\rho)^2} + \sqrt{t^2-(r+\rho)^2}} \\ &\le \frac{4Cr\rho}{\sqrt{t^2-(r-\rho)^2}}\,. \end{aligned}$$

It follows that $r \mapsto \int_0^{\delta\wedge(t-r)} \rho^{-1}[D(r-\rho,t) - D(r+\rho,t)]\, d\rho$ is a uniformly bounded function for all $0 < r < t$.

In the region $\rho > t-r$, the term $\rho^{-1}D(r+\rho,t)$ in the kernel $W_{12}(r,\rho;t)$ is zero. Suppose that $t-\delta < r < t$. Then

$$\int_{t-r}^\delta \frac{D(r-\rho,t)}{\rho}\, d\rho = \frac{1}{2}\int_{t-r}^{\delta\wedge(t+r)} \frac{J_0\left(m\sqrt{t^2-(\rho-r)^2}\right)}{\rho}\, d\rho. \tag{28}$$

Because $J_0(0) = 1$ and J_0 is continuous, it follows that the function (28) and hence (27) has a logarithmic divergence at $r = t$. □

Corollary 1. *The operator $e^{-it\tau_1}$ is not bounded from $L^\infty(\mathbb{R}_+, \mathbb{C}^2)$ to $L^\infty(\mathbb{R}_+, \mathbb{C}^2)$ for any $t \neq 0$.*

According to the following corollary, the group $e^{-it\tau_1}$, $t > 0$, does not have path-space measures, somewhat in the same sense that the heat equation $e^{tD\Delta}$ with non-real diffusion constant D as well as the Schrödinger group $e^{it\Delta/2}$ do not possess path-space measures [cf. Cameron (1960/61)].

The path-space $\Omega_{r,t}$ consists of all functions $\omega : [0, t] \to \mathbb{R}_+$ with $\omega(t) = r$ and we are looking for a family of $M_2(\mathbb{C})$-valued measures $\nu_{r,t}$ defined on the σ-algebra generated by cylinder sets in $\Omega_{r,t}$, such that the equations (29) below hold.

Corollary 2. *There does not exist a family $\{\nu_{r,t}\}$ of matrix-valued path-space measures on the path-space $\Omega_{r,t}$ such that*

$$\int_{\Omega_{r,t}} \nu_{r,t}(d\omega) f(\omega(0)) = \left(e^{-it\tau_1} f\right)(r), \quad \text{for all } r > 0,\ t > 0, \tag{29}$$

for every continuous function $f : [0, \infty) \to \mathbb{C}^2$ with compact support.

Proof. By smoothing out the function $\chi_{(0,\delta)}.e_2$ used in Proposition 3, we can find a continuous function $f : [0, \infty) \to \mathbb{C}^2$ with compact support such that $(e^{-it\tau_1} f)(r)$ diverges as $r \to t$. Hence, a formula like (29) cannot hold for *all* $r > 0$. □

By duality, $e^{-it\tau_1}$ cannot be bounded from $L^1(\mathbb{R}_+, \mathbb{C}^2)$ to $L^1(\mathbb{R}_+, \mathbb{C}^2)$ either. Moreover, a construction similar to that above shows one can find a function f with an integrable singularity at $\rho = t$ such that the function

$$r \mapsto \int_{t-\delta}^{t} W_{21}(r, \rho; t) f(\rho)\, d\rho, \quad 0 < r < t,$$

is not integrable at $r = 0$.

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Department of Mathematics, Kanazawa University, Kanazawa, 920-1192, JAPAN
E-mail address: ichinose@kappa.s.kanazawa-u.ac.jp

School of Mathematics, The University of New South Wales, NSW 2052 AUSTRALIA
E-mail address: b.jefferies@unsw.edu.au