A CENTRAL LIMIT THEOREM FOR CONVOLUTION EQUATIONS AND WEAKLY SELF-AVOIDING WALKS

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Abstract. The main result of this paper is a general central limit theorem for distributions defined by certain renewal type equations. We apply this to weakly self-avoiding random walks. We give good error estimates and Gaussian tail estimates which have not been obtained by other methods.

We use the 'lace expansion' and at the same time develop a new perspective on this method: We work with a fixed point argument directly in $\mathbb{Z}^d$ without using Laplace or Fourier transformation.

Contents

1. Introduction and Results

1.1. **Introduction.** The standard simple random walk on the hypercubic lattice $\mathbb{Z}^d$ is given by the uniform distribution on the set of nearest-neighbour paths starting in 0, and of length n. The law of the self-avoiding random walk is simply the uniform distribution on the set of walks having *no self-intersections*. An interpolation between the strictly self-avoiding walk and the standard random walk is the so-called weakly self-avoiding walk (also known as 'Domb-Joyce model'). Here, self-intersections are not completely forbidden, but penalized by a factor $1-\lambda$ for every self-intersection, where $\lambda \in (0,1)$ is a parameter. Not much is known rigorously for these self-avoiding random walks in dimensions $d = 2, 3$ and 4.

A quantity of basic interest is the so called connectivity $C_n(x)$, $x \in \mathbb{Z}^d$, $n \in \mathbb{N}$, which is obtained by summing the weights of all paths from 0 to x of length n: In the random walk case, the paths all get weight 1, in the strictly self-avoiding case, only the paths without self-intersections get weight 1, the others 0, and in the weakly self-avoiding case, the weight is given in terms of the number of self-intersections (and the parameter λ) as indicated above. After normalization, this defines the distribution of the end-point of the walk. The first results in the case $d \geq 5$ were obtained by Brydges and Spencer [2] in the middle of the eighties. They introduced a perturbative expansion technique, based on the so called lace expansion. With this, Brydges and Spencer proved a central limit theorem for the weakly self-avoiding walk in dimensions greater than four and for small parameter λ. The lace expansion

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is a kind of renewal equation for C_n in terms of the connectivities of the standard random walk of the following type:

$$C_n = C_{n-1} * 2d\, D + \sum_{m=1}^{n} \Pi_m * C_{n-m},\ n \geq 1. \tag{1.1}$$

Here D is the usual nearest-neighbour distribution $D(x) = 1/(2d)$ if $|x| = 1$, and 0 otherwise. The Π_m reflect the presence of the self-interaction of the paths. If they are not present, then C_n is of course just obtained by convoluting $2d\, D$. (1.1) can easily be derived by a kind of inclusion-exclusion. For the convenience of the reader, we give the derivation in Appendix B. One of the delicacies of the problem is that the Π_m are complicated expressions which cannot be written down explicitly, but the key point is that they can be estimated in terms of the C_k, $k \leq m$. Similar expansions have now found widespread applications in probability theory and mathematical physics, like in percolation theory [8], branched polymers and quite recently also for superprocess approximations of interactive particle systems.

The strictly self-avoiding walk was first treated by Slade [13] for large dimensions, where the large dimension serves in some sense as a small coupling parameter. Finally, in a tremendous effort, Hara and Slade [5, 7] were able to prove the diffusive behavior for $d \geq 5$ for the strictly self-avoiding case. However, their argument is still essentially perturbative and relies on computer-assisted estimates of a number of constants which have to be smaller than 1.

The earlier approaches to the lace expansion always depended on taking (complex) Laplace transforms in time $\sum_n z^n C_n$, and then inverting the transform. The latter step is notoriously difficult and leads to a number of problems. In particular, it seems hard to obtain good pointwise estimates for the connectivities in this way. In 1997, van der Hofstad, den Hollander and Slade [9] presented an inductive approach to the lace expansion avoiding Laplace inversion. In particular, these authors prove a local central limit theorem for the so-called 'elastic' self-avoiding walk, a model in which the penalty for self-intersections decreases (only polynomially) in time. Several further improvements have been obtained recently. In [10], van der Hofstad and Slade generalize and simplify the inductive approach. In a recent paper by Hara, van der Hofstad and Slade [6], a general method for deriving information on the generating two-point functions in $\mathbb{Z}^d$, i.e. expressions of the form $\sum_n z^n C_n(x)$ at the critical point z_c is developed.

In this paper we present a novel way to treat such problems. We first modify (1.1) slightly. To explain the main idea, one has to remark that the norming constants $c_n \overset{\text{def}}{=} \sum_x C_n(x)$ behave in first order exponentially in n : $\lim_{n\to\infty} (1/n) \log c_n$ exists. It turns out that, in leading order, the Π_n have the same (exponential) behavior as the C_n. It is therefore natural to define $B_n \overset{\text{def}}{=} \Pi_n / c_n$, and rewrite (1.1) as

$$C_n = C_{n-1} * 2d\, D + \sum_{m=1}^{n} c_m B_m * C_{n-m},\ n \geq 1. \tag{1.2}$$

The main result of our paper addresses the following problem: Given a sequence $(B_m)_{m\geq 1}$ of (possibly signed) distributions on $\mathbb{Z}^d$, we regard (1.2) as the defining equations for the sequence $(C_n)_{n\geq 0}$, of course with $C_0 \overset{\text{def}}{=} \delta_0$. We prove that if the sequence (B_m) has suitable smallness and decay properties, then C_n/c_n satisfies a central limit theorem. It is important that we require only polynomial decay properties of the (B_m). (In fact, one crucial difficulty for analyzing the (weakly) self-avoiding walks is that there is no mass gap between the C-sequence and the Π-sequence). Although, the main result is completely independent of the problem of self-avoiding walks, the version of the central limit theorem we prove is tailored much with this application in mind. We come as close as possible (we think) to a local central limit theorem, including good Gaussian tail estimates. The (weakly) self-avoiding walk can in fact not satisfy a local CLT in the strict sense.[1].

The main idea of our approach is to regard (1.2) as a fixed-point equation of an operator acting on sequences of distributions. There are of course many ways to do that, and the problem is to find the 'right' way. We prove a central limit theorem by the following reasoning: The operator we define has two crucial properties: First, it has contraction properties in suitable spaces, so that a fixed point exists by the Banach fixed point theorem. This fixed point has to satisfy (1.2), and as this has a unique solution (under appropriate conditions), we know that the fixed point of our operator is *the* solution. Secondly, we prove that the operator leaves the property of being 'asymptotically Gaussian' untouched: If a sequence is asymptotically Gaussian (in a sense which has to be quantified, and after normalization), then the same is true for the sequence transformed by the operator. If we start with a sequence being already asymptotically Gaussian (for instance just taking the discretized Gaussian distribution), then also the fixed point is asymptotically Gaussian, and therefore, the solution of (1.2) satisfies a central limit theorem (after normalization). This approach reveals in a transparent way what is behind the central limit theorem: The property of being asymptotically Gaussian is stable enough not to be destroyed by the presence of the B-sequence, provided this enjoys the appropriate properties.

As remarked above, the result we prove is tailored much to make the application to weakly self-avoiding walks very easy. The point is that the B's can be estimated in terms of the C's. These estimates are most naturally in x-space and not in Fourier-space. For this reason, we work all the time in x-space, except for deriving several crucial properties of convolutions of D. By these estimates, we prove by a simple 'circular' argument, that the 'true' B's coming from the weakly self-avoiding walk do in fact satisfy the conditions of our central limit theorem.

We believe that the method can be extended in various ways to treat many other problems as well, but here we restrict ourselves to the application to weakly self-avoiding walks. Even for this well studied problem, the results we obtain are sharper than those obtained by other methods.

1.2. **The Objects of the Weakly Self-Avoiding Walk.** We begin by introducing the connectivities C_n for the weakly self-avoiding walk. For $x \in \mathbb{Z}^d$ we set $C_0(x) \stackrel{\text{def}}{=} \delta_{0,x}$, and for $n \geq 1$ and $\beta \geq 0$, we define

$$C_n(x) \stackrel{\text{def}}{=} \sum_{\substack{\omega:0\rightsquigarrow x\\ |\omega|=n}} e^{-\beta\sum_{0\leq s<t\leq n} U_{st}(\omega)} = \sum_{\substack{\omega:0\rightsquigarrow x\\ |\omega|=n}} \prod_{0\leq s<t\leq n} (1-\lambda U_{st}(\omega)), \tag{1.3}$$

where the sum is over all n-step simple random walk paths ω from 0 to x, while

$$U_{st}(\omega) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } \omega(s)=\omega(t),\\ 0, & \text{if not,}\end{cases}$$

and

$$\lambda \stackrel{\text{def}}{=} 1-e^{-\beta}.$$

Here, loops of the walk are penalized with a factor $1-\lambda$. We have the fully self-avoiding walk for $\lambda=1$, while for $\lambda=0$ it is the simple random walk. We will usually suppress the λ-dependence in our notation.

[1]The walk starts at zero and will not forget this fact ever nor will the memory weaken. The resulting error term is of size $n^{-d/2}$, the same as the value of the approximating normal density at zero.

The lace expansion is a renewal type equation for the connectivities. It involves a function Π_m, defined in (B.5), which we will call 'lace function'. The expansion is the recursion formula stating that for all $x \in \mathbb{Z}^d$ we have

$$C_n(x) = 2d(D * C_{n-1})(x) + \sum_{m=2}^{n} \Pi_m * C_{n-m}(x), \tag{1.4}$$

where D denotes the law of one step of a simple random walk, that is,

$$D(y) \overset{\text{def}}{=} \begin{cases} 1/(2d), & \text{if } \|y\|_1 = 1, \\ 0, & \text{else.} \end{cases}$$

The star $*$ always refers to the discrete folding of two measures on $\mathbb{Z}^d$, that is $G * H(x) \overset{\text{def}}{=} \sum_{y \in \mathbb{Z}^d} G(y)H(x-y)$. Denote by c_n the total mass of C_n, that is,

$$c_n \overset{\text{def}}{=} \sum_{x \in \mathbb{Z}^d} C_n(x). \tag{1.5}$$

(We will always denote measures by capital letters and the corresponding total mass by the appropriate lower case letter.) Furthermore we will always write φ_η for the d-dimensional normal density with covariance matrix $\eta \cdot \mathrm{Id}_d$, that is,

$$\varphi_\eta(x) \overset{\text{def}}{=} (2\pi\eta)^{-d/2} \exp\left(-\tfrac{x^2}{2\eta}\right).$$

The general results proved in the sections 2 and 3 can be applied to the weakly self-avoiding walk (λ small enough) in dimensions above four. This is stated in the following theorem. The first part recovers the result of Brydges and Spencer [2]. The second part presents a pointwise comparison of the probabilities $C_n(x)/c_n$ with a normal density and gives strong error estimates.

Theorem 1.1. *Let the dimension d be greater or equal to five. Then there exists a $\lambda_0 > 0$ such that for all $0 \le \lambda \le \lambda_0$ and $n \in \mathbb{N}$,*

$$c_n = \alpha\mu^n\left(1 + O(n^{-1/2})\right). \tag{1.6}$$

For $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}$ such that $n - \|x\|_1$ is even,[2] *we have*

$$\left|\frac{C_n(x)}{c_n} - 2\varphi_{\delta n}(x)\right| \le K\left[n^{-1/2}\varphi_{n\nu}(x) + n^{-d/2}\sum_{j=1}^{n/2} j\varphi_{j\nu}(x)\right]. \tag{1.7}$$

The constants α, μ and δ are positive and depend on λ and d, whereas ν and K only depend on the dimension.

Note that for 'large' x the right-hand side of (1.7) is bounded by a multiple of $n^{-1/2}\varphi_{n\nu}(x)$ alone. For $|x| \le O(n^{1/2})$ the leading term in the error bound is the second part, which is of order $O(n^{-d/2})$ near zero. The constants α, μ and δ are identified in terms of c_n and π_m at the end of section 4.

[2] Because of the two-periodicity of nearest neighbour walks, $C_n(x)$ equals zero when $\|x\|_1$ and n don't have the same parity. The factor 2 for the normal density in (1.7) arises for the same reason – we have to double the density values to approximate a discrete probability measure on the two-periodic sublattices of $\mathbb{Z}^d$.

1.3. **Strategy and Motivation of the Approach.** We prove the theorem by splitting the problem into two parts: In the first step we show the existence of the connective constant μ and the diffusion constant δ and we determine their exact form. Given these parameters, we can write down the proper normal density to approximate the distributions themselves. So in the second step we only have to estimate the error of the approximation.

Several difficulties arise in this approach. The constants we want to determine in advance are given only implicitely as series in terms of c_n and π_n (the total mass constants of the 'lace functions' Π_n). In particular, we need the total mass values of the n-step weakly self-avoiding walk *for each natural* n. So at first it seems impossible to determine the constants without knowing quite a lot about the distributions themselves.

We deal with this difficulty by treating the whole sequence of mass constants as one object instead of studying the constants separately for each n. The idea of working in this kind of sequence spaces was taken from Grübel [4]. We can define an operator on some appropriate sequence space such that the sequence satisfying the recursion formula of the lace expansion is a fixed point of this operator. Once we have chosen space and operator properly, the Banach fixed point theorem yields the desired properties of the sequence.

The second step is organized as follows. To start we define a discrete random vector, whose variance is given by the diffusion constant from the first part. Then we consider the measures of the weakly self-avoiding walk as perturbations of the simple random walk whose single steps are given by the random vector defined above. We again use a fixed point argument to control the errors of the approximation, this time working in some sequence space of measures rather than of real numbers.

To give the fixed point arguments, the sequences we investigate shouldn't grow exponentially. So first we cancel the exponential growth of the mass constants for both the connectivities and the lace functions: For $m \geq 2$d we define the function B_m on $\mathbb{Z}^d$ by

$$B_m(x) \stackrel{\text{def}}{=} \frac{\Pi_m(x)}{\lambda c_m}. \tag{1.8}$$

Let b_m denote the total mass of B_m. Inserting (1.8) into (1.4), we obtain the following recursive identities (from now on suppressing x in the notation):

$$\begin{aligned} C_n =& 2dD * C_{n-1} + \lambda \sum_{m=2}^{n} c_m B_m * C_{n-m} \quad \text{and} \\ c_n =& 2dc_{n-1} + \lambda \sum_{m=2}^{n} c_m b_m c_{n-m}. \end{aligned}$$

Now suppose that C_n grows exponentially, that is, C_n equals $\mu^n A_n$ with some $\mu > 0$ such that $a_n = \sum_{x \in \mathbb{Z}^d} A_n(x)$ tends to some $\alpha \neq 0$ when n tends to infinity. This μ is called *connective constant.* The identities above lead to the following equations:

$$A_n = 2d\mu^{-1} D * A_{n-1} + \lambda \sum_{m=2}^{n} a_m B_m * A_{n-m}, \tag{1.9}$$

$$a_n = 2d\mu^{-1} a_{n-1} + \lambda \sum_{m=2}^{n} a_m b_m a_{n-m}. \tag{1.10}$$

If the sequence (a_n) is converging to a limit $\alpha > 0$ and if the b_n decay fast enough, we can let n tend to infinity in (1.10) to find

$$\alpha = 2d\mu^{-1}\alpha + \lambda \sum_{m=2}^{\infty} a_m b_m \alpha,$$

and therefore we have for the *connective constant* μ,

$$2d\mu^{-1} = 1 - \lambda \sum_{m=2}^{\infty} a_m b_m. \tag{1.11}$$

By substituting this into (1.10) we obtain

$$a_n = a_{n-1} - \lambda \left(\sum_{m=2}^{\infty} a_m b_m a_{n-1} - \sum_{m=2}^{n} a_m b_m a_{n-m} \right).$$

This derivation in mind, we will prove the existence and uniqueness of such a sequence by a Banach fixed point argument in section 2, under the hypothesis that the b_m decay fast enough.

Given the sequence (a_n) and specific pointwise estimates of $B_m(x)$, we will obtain pointwise approximations for $A_n(x)$ in dimension $d \geq 5$ in section 3. This central limit theorem with error estimates is our main result. The theorem is proven for sequences satisfying a slightly generalized version of (1.9).

In section 4 we prove the right behavior of the lace expansion terms, insuring that we can indeed apply the fixed point arguments to the weakly self-avoiding walk.

2. Determining the Mass Constants

2.1. Existence and Uniqueness.

Let $(b_m)_{m\in\mathbb{N}}$ be a realvalued sequence with

$$\beta \overset{\text{def}}{=} \sum_{m=1}^{\infty} m\,|b_m| < \infty.$$

In this section we will prove the following result:

Proposition 2.1. *There is a* $\lambda_0 = \lambda_0(\beta) > 0$ *such that for all* $\lambda \leq \lambda_0$ *there exists a unique sequence* $(a_n)_{n\in\mathbb{N}_0}$ *with*

$$a_0 = 1 \quad \text{and} \quad a_n = \big(1 - \lambda \sum_{m=1}^{\infty} a_m b_m\big) a_{n-1} + \lambda \sum_{m=1}^{n} a_m b_m a_{n-m}, \tag{2.1}$$

such that $\sum_{n=1}^{\infty} |a_n - a_{n-1}| < \infty$.

We will prove this proposition with a fixed point argument, but first we introduce some notation. Let $(l_\infty, \|.\|_\infty)$ be the Banach space of bounded real valued sequences $\boldsymbol{g} = (g_n)_{n\in\mathbb{N}_0}$ with the supremum norm. The difference operator $\Delta : \mathbb{R}^{\mathbb{N}_0} \to \mathbb{R}^{\mathbb{N}_0}$ is given by

$$(\Delta g)_0 \overset{\text{def}}{=} g_0 \qquad \text{and} \qquad (\Delta g)_n \overset{\text{def}}{=} g_n - g_{n-1} \qquad \text{for } n \in \mathbb{N}.$$

For $\boldsymbol{g} = (g_n)_{n\in\mathbb{N}_0}$ with $\sum_{n=0}^{\infty} |(\Delta g)_n| < \infty$ define the norm

$$\|\boldsymbol{g}\|_{\mathcal{D}} \overset{\text{def}}{=} \sum_{n=0}^{\infty} |(\Delta g)_n|.$$

Furthermore, define the operator $\sim$ on sequences by

$$\widetilde{g}_0 \overset{\text{def}}{=} g_0 \qquad \text{and}$$

$$\widetilde{g}_n \overset{\text{def}}{=} \widetilde{g}_{n-1} - \lambda \Big[\sum_{m=1}^{n} g_m b_m (g_{n-1} - g_{n-m}) + g_{n-1} \sum_{j=n+1}^{\infty} g_j b_j \Big]. \tag{2.2}$$

We will apply the Banach fixed point theorem to the operator $\sim$. In the three following lemmas we prove that the necessary conditions are fulfilled.

Lemma 2.2. *Let $\boldsymbol{g} \in l_\infty$ with $\|\boldsymbol{g}\|_{\mathcal{D}} < \infty$. Then we also have $\|\widetilde{\boldsymbol{g}}\|_{\mathcal{D}} < \infty$.*

Proof. Let $\|\boldsymbol{g}\|_{\mathcal{D}} < \infty$. We have to show that $\sum_{n=0}^{\infty}|(\Delta\widetilde{g})_n|$ is finite. First notice that

$$\|\boldsymbol{g}\|_\infty = \sup_{n\in\mathbb{N}_0} |g_n| = \sup_{n\in\mathbb{N}_0} \Big|\sum_{k=0}^{n}(\Delta g)_k\Big| \le \|g\|_{\mathcal{D}}. \tag{2.3}$$

From (2.2) we have

$$\begin{aligned}
\sum_{n=0}^{\infty}|(\Delta\widetilde{g})_n| &= |g_0| + \sum_{n=1}^{\infty}|\widetilde{g}_n - \widetilde{g}_{n-1}| \\
&= |g_0| + \lambda\sum_{n=1}^{\infty}\Big|\sum_{m=1}^{n} g_m b_m(\underbrace{g_{n-1}-g_{n-m}}_{=\sum_{l=1}^{m-1}(\Delta g)_{n-l}}) + \sum_{j=n+1}^{\infty} g_j b_j g_{n-1}\Big| \\
&\le |g_0| + \lambda\Big[\|\boldsymbol{g}\|_\infty \sum_{n=1}^{\infty}\sum_{l=1}^{n-1}|(\Delta g)_{n-l}| \underbrace{\sum_{m=l+1}^{n}|b_m|}_{\le\sum_{m=l+1}^{\infty}|b_m|} + \|\boldsymbol{g}\|_\infty^2 \underbrace{\sum_{n=1}^{\infty}\sum_{j=n+1}^{\infty}|b_j|}_{\le\sum_{j=2}^{\infty}j|b_j|\le\beta}\Big] \\
&\le |g_0| + \lambda\Big[\|\boldsymbol{g}\|_\infty \sum_{l=1}^{\infty}\sum_{m=l+1}^{\infty}|b_m|\sum_{n=1}^{\infty}|\Delta g_n| + \|\boldsymbol{g}\|_\infty^2\beta\Big] \\
&\le |g_0| + 2\lambda\beta\|\boldsymbol{g}\|_{\mathcal{D}}^2 < \infty, \qquad (2.4)
\end{aligned}$$

where we used (2.3) in the last line. ☐

Lemma 2.3. *Let $\mathcal{D}_L \overset{\text{def}}{=} \{g \in l_\infty : g_0 = 1 \text{ and } \|\boldsymbol{g}\|_{\mathcal{D}} \le L\}$, where L is a constant greater than or equal to $3/2$. Then for all $\lambda \le 1/(6\beta L)$ the operator $\sim$ is a contraction with respect to $\|.\|_{\mathcal{D}}$ on $\mathcal{D}_L$.*

Note that the value $3/2$ in the lemma is chosen to keep the constants simple. An analogous statement holds as long as L is bounded away from one.

Proof. We have to show:

(i) $\boldsymbol{g} \in \mathcal{D}_L \;\Rightarrow\; \widetilde{\boldsymbol{g}} \in \mathcal{D}_L$
(ii) There exists some $\kappa < 1$ such that$\|\widetilde{\boldsymbol{g}} - \widetilde{\boldsymbol{h}}\|_{\mathcal{D}} \le \kappa\|\boldsymbol{g} - \boldsymbol{h}\|_{\mathcal{D}}$ for all $\boldsymbol{g}, \boldsymbol{h} \in \mathcal{D}_L$.

To see (i), let $\boldsymbol{g} \in \mathcal{D}_L$ be given. We know $\widetilde{g}_0 = g_0 = 1$. According to (2.4) we have

$$\|\widetilde{\boldsymbol{g}}\|_{\mathcal{D}} \le 1 + 2\lambda\beta L^2 \le \frac{2}{3}L + \frac{1}{3}L,$$

whenever $\lambda \le \frac{1}{6\beta L}$.

To see (ii), take $\boldsymbol{g}, \boldsymbol{h} \in \mathcal{D}_L$. Since g_0 equals h_0, we have $\|\widetilde{\boldsymbol{g}} - \widetilde{\boldsymbol{h}}\|_{\mathcal{D}} = \sum_{n=1}^{\infty}|(\Delta(\widetilde{g} - \widetilde{h}))_n|$. We have (from (2.2)):

$$\begin{aligned}
\sum_{n=1}^{\infty}|(\Delta\widetilde{g})_n - (\Delta\widetilde{h})_n| = \lambda\sum_{n=1}^{\infty}\Big|&\sum_{m=1}^{n}(g_m - h_m)b_m(g_{n-1} - g_{n-m}) \\
&+ \sum_{m=1}^{n} h_m b_m\big[g_{n-1} - h_{n-1} - (g_{n-m} - h_{n-m})\big] \\
&+ \sum_{j=n+1}^{\infty} b_j\big[g_j(g_{n-1} - h_{n-1}) + (g_j - h_j)h_{n-1}\big]\Big|
\end{aligned}$$

We can estimate the absolute values of the three summands individually. The first one can be treated analogously to (2.4),

$$\lambda \sum_{n=1}^{\infty} \sum_{m=1}^{n} |b_m|\,|g_m - h_m|\,|g_{n-1} - g_{n-m}| \leq \lambda\beta \|\boldsymbol{g} - \boldsymbol{h}\|_{\infty} \|\boldsymbol{g}\|_{\mathcal{D}}.$$

Very similarly we obtain for the second one

$$\lambda \sum_{n=1}^{\infty} \sum_{m=1}^{n} |b_m|\,|h_m|\,|g_{n-1} - h_{n-1} - (g_{n-m} - h_{n-m})| \leq \lambda\beta \|\boldsymbol{h}\|_{\infty} \|\boldsymbol{g} - \boldsymbol{h}\|_{\mathcal{D}},$$

and for the third

$$\lambda \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} |b_j|\,|g_j(g_{n-1} - h_{n-1}) + (g_j - h_j)h_{n-1}| \leq \lambda\beta \|\boldsymbol{g} - \boldsymbol{h}\|_{\infty} (\|\boldsymbol{g}\|_{\infty} + \|\boldsymbol{h}\|_{\infty}).$$

Since both $\boldsymbol{g}$ and $\boldsymbol{h}$ are in $\mathcal{D}_L$ and $\lambda \leq \frac{1}{6\beta L}$, this yields

$$\|\widetilde{\boldsymbol{g}} - \widetilde{\boldsymbol{h}}\|_{\mathcal{D}} \leq 4\lambda\beta L \|\boldsymbol{g} - \boldsymbol{h}\|_{\mathcal{D}} \leq \frac{2}{3} \|\boldsymbol{g} - \boldsymbol{h}\|_{\mathcal{D}}. \qquad \square$$

Proof of Proposition 2.1. The elements of l_∞ with finite $\|.\|_{\mathcal{D}}$-norm form a Banach space with this norm, and $\mathcal{D}_L$ is a closed subset of this space.

Thus, using Lemma 2.2 and Lemma 2.3, the Banach fixed point theorem yields for small enough λ the existence and uniqueness of an element $\mathbf{a} \in \mathcal{D}_L$ with $\widetilde{\mathbf{a}} = \mathbf{a}$. Furthermore, the repeated iteration of $\sim$ with starting point $(1, 1, 1, \dots)$ converges to $\mathbf{a}$. As long as $L \geq 3/2$, the value of L has an influence only on the upper bounds for λ. This proves the proposition. $\square$

2.2. **Limit and Convergence Speed.** Now we investigate the limit and the convergence behavior of this 'fixed sequence' in a more particular setting. By choosing $L = 3/2$ we obtain for all $\lambda \leq 1/(9\beta)$ a sequence $\mathbf{a}$ with $a_0 = 1$, $\sum_{n=1}^{\infty} |(\Delta a)_n| \leq 1/2$ and for all $n \in \mathbb{N}$

$$a_n = u\mu^{-1} a_{n-1} + \lambda \sum_{m=1}^{n} a_m b_m a_{n-m},$$

where $u\mu^{-1} = 1 - \lambda \sum_{m=1}^{\infty} a_m b_m$ as in (1.11) for $u = 2d$. Since $\mathbf{a}$ is bounded and $\sum_{n=2}^{\infty} |b_n| < \infty$, $u\mu^{-1}$ is finite. Note also that for all $n \in \mathbb{N}_0$ we have

$$1/2 \leq a_n \leq 3/2.$$

We now want to investigate the limiting value $\alpha = \lim_{n\to\infty} a_n$. Since the difference sequence of $\mathbf{a}$ is absolutely summable, α exists, and we have

$$\alpha = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sum_{m=0}^{n} (\Delta a)_m.$$

Now consider for fixed $n \in \mathbb{N}$ (recall (2.1)):

$$\begin{aligned}
a_n &= 1 + \sum_{k=1}^{n} (\Delta a)_k \\
&= 1 - \lambda \sum_{k=1}^{n} \Big[\sum_{m=1}^{\infty} a_m b_m a_{k-1} - \sum_{m=1}^{k} a_m b_m a_{k-m} \Big] \\
&= 1 - \lambda \sum_{m=1}^{\infty} a_m b_m \sum_{k=1}^{n} a_{k-1} + \lambda \sum_{m=1}^{n} a_m b_m \sum_{k=m}^{n} a_{k-m} \\
&= 1 - \lambda \underbrace{\sum_{m=n+1}^{\infty} a_m b_m \sum_{k=1}^{n} a_{k-1}}_{=:F_1} - \lambda \sum_{m=1}^{n} a_m b_m \sum_{l=1}^{m-1} \underbrace{a_{n-l}}_{=\alpha-(\alpha-a_{n-l})} \\
&= 1 - \lambda F_1 - \lambda \sum_{m=1}^{n} a_m b_m (m-1)\alpha + \lambda \underbrace{\sum_{m=1}^{n} a_m b_m \sum_{l=1}^{m-1} \sum_{k=n-l+1}^{\infty} (\Delta a)_k}_{=:F_2}, \qquad (2.5)
\end{aligned}$$

where

$$|F_1| = \Big| \sum_{m=n+1}^{\infty} a_m b_m \sum_{k=1}^{n} a_{k-1} \Big| \le \sum_{m=n+1}^{\infty} L |b_m| n L \le L^2 \sum_{m=n+1}^{\infty} m |b_m| \xrightarrow{n\to\infty} 0$$

and

$$\begin{aligned}
|F_2| &= \Big| \sum_{m=1}^{n} a_m b_m \sum_{l=1}^{m-1} \sum_{k=n-l+1}^{\infty} (\Delta a)_k \Big| \le \sum_{l=1}^{n-1} \sum_{m=l+1}^{\infty} L |b_m| \sum_{k=n-l+1}^{\infty} |(\Delta a)_k| \\
&\le L \underbrace{\sum_{l=1}^{n/2} \sum_{m=l+1}^{\infty} |b_m|}_{\le \beta} \sum_{k=n/2}^{\infty} |(\Delta a)_k| + L \sum_{l=n/2}^{n-1} \sum_{m=l+1}^{\infty} |b_m| \underbrace{\sum_{k=1}^{\infty} |(\Delta a)_k|}_{\le \|\mathbf{a}\|_{\mathcal{D}}} \xrightarrow{n\to\infty} 0.
\end{aligned}$$

Letting n tend to infinity in (2.5), we obtain

$$\alpha = 1 - \lambda \sum_{m=1}^{\infty} (m-1) a_m b_m \alpha,$$

which yields

$$\alpha^{-1} = 1 + \lambda \sum_{m=1}^{\infty} (m-1)\, a_m b_m = u\mu^{-1} + \lambda \sum_{m=1}^{\infty} m\, a_m b_m. \qquad (2.6)$$

In case we know the rate of decay of the b_m, we can determine the speed of the convergence $a_n \longrightarrow \alpha$ more precisely. The following corollary states a result that we will need in the next section.

Corollary 2.4. *If there exist positive constants ε and β' such that*

$$|b_m| \le \beta' m^{-2-\varepsilon} \quad \textit{for all } m \in \mathbb{N},$$

then we get a decay of order $n^{-1-\varepsilon}$ for the difference sequence $\Delta\mathbf{a}$. More precisely we have

$$|(\Delta a)_n| \le \lambda \beta' K n^{-1-\varepsilon} \quad \textit{for all } n \in \mathbb{N},$$

where K is a positive constant not depending on λ or β'. In particular we have another constant K such that

$$|\alpha - a_n| \le \lambda \beta' K n^{-\varepsilon} \quad \textit{for all } n \in \mathbb{N}.$$

Proof. Using (2.5), both estimates can be easily seen by induction. □

3. Local Estimates in High Dimensions

We now turn to estimates not only for the normalization constants, but for the measures on $\mathbb{Z}^d$ themselves. In particular, we are interested in the following question: If we consider the measures as perturbations of the distribution of a sum of independent, identically distributed random vectors, then how big is the pointwise difference between the measure and the appropriate normal density? Having in mind the high-dimensional self-avoiding walk, we will not expect a proper local central limit theorem to hold: There will always be correction terms of order $n^{-d/2}$ near zero, since zero is the starting point of the walk. What we obtain is Gaussian decay for the perturbative errors on the whole $\mathbb{Z}^d$, improved by a factor of $n^{-1/2}$ for large x.

In this section, we will show the pointwise estimates in a more general context. Supposing some specific pointwise bounds for the distributions B_n, it is possible to show local estimates for the measures A_n in five or more dimensions. In the next section we will show that the lace functions in the weakly self-avoiding walk case have the necessary properties.

We begin by introducing some notations: Let the space $\mathcal{M}$ be defined as the set of the symmetric, rotationally invariant, signed real valued measures on $\mathbb{Z}^d$ with existing 'variance', that is,

$$\begin{aligned}\mathcal{M} := \{G : \mathbb{Z}^d \to \mathbb{R} \text{ such that } \sum_{x\in\mathbb{Z}^d} |G(x)| < \infty \text{ and } \sum_{x\in\mathbb{Z}^d} x^2|G(x)| < \infty;\\ G \text{ symmetric in each coordinate and rotationally invariant}\}.\end{aligned}$$

Here and hereafter, we denote by x^2 the square of the euclidean norm of $x \in \mathbb{R}^d$. For $G \in \mathcal{M}$ define

$$g \stackrel{\text{def}}{=} \sum_{x\in\mathbb{Z}^d} G(x) \qquad \text{and} \qquad \underline{g} \stackrel{\text{def}}{=} \sum_{x\in\mathbb{Z}^d} x^2\, G(x).$$

By $\mathcal{S}$ we denote the space of sequences with elements in $\mathcal{M}$, that is,

$$\mathcal{S} := \left\{ \boldsymbol{G} = (G_n)_{n\in\mathbb{N}_0} : G_n \in \mathcal{M} \ \text{ for all } n \in \mathbb{N}_0 \right\}.$$

¿From the first section recall the equation (1.9),

$$A_n = 2d\mu^{-1} D * A_{n-1} + \lambda \sum_{m=2}^{n} a_m B_m * A_{n-m}.$$

Here we will investigate a slightly different sequence (A_n) with

$$A_n = u\mu^{-1} S * A_{n-1} + \lambda \sum_{m=1}^{n} a_m B_m * A_{n-m}, \tag{3.1}$$

where $u > 0$ is a fixed constant ($u = 2d$ in (1.9)). We suppose $S \in \mathcal{M}$ is a non-degenerate[3] and aperiodic[4] probability measure of bounded range. More precisely, we have a constant $\ell' \geq 1$ such that $S(x) = 0$ for all x with $|x| > \ell'$. For simplicity we also assume that the 'variance' $\underline{s}$ is greater or equal to one. This condition is not necessary, but convenient, since many constants depend on the lower bound of the involved variance.

[3]that is, the probability $S(0) < 1$.

[4]Consider a probability measure S and the random walk defined as sum of independent steps, where each step is distributed according to S. Now take the greatest common divisor of all times n, at which the probability of staying at zero is not zero. This divisor is called *period* of the walk. If the period equals one, the random walk, and hence S itself, is called *aperiodic*.

Note that the distribution D from (1.9) is *not* aperiodic: For technical reasons we will give the proof for aperiodic measures only, and afterwards discuss the two-periodic case to which the self avoiding walk belongs.

For the whole section, the dimension d is greater or equal to five. K denotes a positive constant depending on d and ℓ' only. The value of K may change from line to line, whereas $\nu \geq 1/(2d)$ is an adjustable parameter and will be determined later.

Now let $\mathbf{B} \in \mathcal{S}$ be a sequence with $B_0 \equiv 0$ and with

$$|B_m(x)| \leq K m^{-d/2} \sum_{k=1}^{m/2} k^{1-d/2} \varphi_{k\nu}(x) \tag{3.2}$$

uniformly for all $m \geq 1$ and $x \in \mathbb{Z}^d$. We abbreviate the notation by writing

$$\psi_m(x) \stackrel{\text{def}}{=} m^{-d/2} \sum_{k=1}^{m/2} k^{1-d/2} \varphi_{k\nu}(x)$$

and define

$$\beta_\nu = \sup_{m,x} \frac{|B_m(x)|}{\psi_m(x)}.$$

We have (recall $d \geq 5$)

(3.3)

$$|b_m| \leq \sum_{x\in\mathbb{Z}^d} |B(x)| \leq \beta_\nu m^{-d/2} \sum_{k=1}^{m/2} k^{-3/2} \sum_{x\in\mathbb{Z}^d} \varphi_{k\nu}(x) \leq K\beta_\nu m^{-d/2} \quad \text{and}$$

$$|\underline{b}_m| \leq \sum_{x\in\mathbb{Z}^d} x^2 |B(x)| \leq \beta_\nu m^{-d/2} \sum_{k=1}^{m/2} k^{-3/2} \sum_{x\in\mathbb{Z}^d} x^2 \varphi_{k\nu}(x) \leq K\nu\beta_\nu m^{-(d-1)/2}.$$

Here we used that as long as $\eta \geq 1/2d$, we have

$$\sum_{x\in\mathbb{Z}^d} \varphi_\eta(x) \leq K \qquad \text{and} \qquad \sum_{x\in\mathbb{Z}^d} x^2 \varphi_\eta(x) \leq K\eta,$$

which is stated and proved in Lemma A.4 in the appendix. In particular, since $d \geq 5$, the condition on β from the last chapter is fulfilled, that is, $\beta = \sum_{m=1}^\infty m|b_m| \leq K\beta_\nu$. In fact, the b_n fulfill even the stronger condition from Corollary 2.4 for $\varepsilon = 1/2$ and $\beta' = \sup_m m^{d/2}|b_m|$.

For λ small enough, Proposition 2.1 and Corollary 2.4 now yield the existence of a unique sequence $(a_n)_{n\in\mathbb{N}_0}$ with $a_0 = 1$ and

$$a_n = u\mu^{-1} a_{n-1} + \lambda \sum_{m=1}^{n} a_m b_m a_{n-m},$$

where $u\mu^{-1} = 1 - \lambda \sum_{m=1}^\infty b_m a_m$ and $(\Delta a)_n = O(n^{-3/2})$. Using these a_n, we are now in the situation to define the sequence (A_n) of signed measures on $\mathbb{Z}^d$ properly by (see (3.1))

$$A_0 \stackrel{\text{def}}{=} \delta_0 \qquad \text{and} \qquad A_n \stackrel{\text{def}}{=} u\mu^{-1} S * A_{n-1} + \lambda \sum_{m=1}^{n} a_m B_m * A_{n-m}. \tag{3.4}$$

The reader might worry that there is a problem caused by the ambiguous use of a_n, which denotes the nth term of the given fixed point sequence on one hand and the total mass of A_n on the other hand. But by summing up (3.4) over $x \in \mathbb{Z}^d$, we see immediately that the normalization constant of A_n in fact *is* the given a_n.

We will use the following abbreviations:

$$\rho \stackrel{\text{def}}{=} \sum_{m=1}^{\infty} a_m\, b_m, \qquad \sigma \stackrel{\text{def}}{=} \sum_{m=1}^{\infty} (m-1) a_m\, b_m, \qquad \text{and} \qquad \tau \stackrel{\text{def}}{=} \sum_{m=1}^{\infty} a_m\, \underline{b}_m.$$

With these notations we have

$$u\mu^{-1} = 1 - \lambda\rho \qquad \text{and} \qquad \alpha^{-1} = 1 + \lambda\sigma,$$

where α is the limit of the sequence (a_n) (see (2.6)). In addition we know that $1/2 \le a_n \le 3/2$ for all $n \in \mathbb{N}_0$, and that $\alpha - a_n = O(n^{-1/2})$, which results from Corollary 2.4.

The key parameter of the following approximation is the constant

$$\delta \stackrel{\text{def}}{=} \frac{\underline{s}(1-\lambda\rho) + \lambda\tau}{d(1+\lambda\sigma)}, \tag{3.5}$$

which will turn out to be the right diffusion constant for the asymptotic probability law $A_n(x/\sqrt{n})/a_n$. We always assume λ to be small enough to ensure that

$$\underline{s}/2 \le d\delta \le 2\underline{s}. \tag{3.6}$$

Since $\underline{s} \ge 1$, we have in particular $\delta \ge 1/(2d)$. Now we can state the main result of this paper:

Theorem 3.1 (Local Estimates, aperiodic case)**.** *The sequence $(A_n)_{n\in\mathbb{N}_0}$ defined in* (3.4)*, has the following property: There exist $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ and for all $x \in \mathbb{Z}^d$:*

$$|A_n(x) - a_n \varphi_{n\delta}(x)| \le K\Big[n^{-1/2}\varphi_{n\nu}(x) + n^{-d/2}\sum_{j=1}^{n/2} j\, \varphi_{j\nu}(x)\Big], \tag{3.7}$$

where K and ν are positive constants depending on d and ℓ' only. (In particular they do not depend on the sequence (B_m) at all.)

We will prove this theorem by using the Banach fixed point theorem again. First we introduce the adequate Banach space. To keep the notation as simple as possible, we define χ_n by

$$\chi_n(x) = n^{-1/2}\varphi_{n\nu}(x) + n^{-d/2}\sum_{j=1}^{n/2} j\, \varphi_{j\nu}(x).$$

In particular, we suppress the ν-dependence of χ. For $\boldsymbol{G} \in \mathcal{S}$ we define the 'χ-weighted' norm

$$\|\boldsymbol{G}\|_{\mathcal{W}} := \sup_{x\in\mathbb{Z}^d} |G_0(x)| + \sup_{n\in\mathbb{N}, x\in\mathbb{Z}^d} \frac{|G_n(x)|}{\chi_n(x)},$$

whenever these suprema are finite. Finally we define the set

$$\mathcal{W} \stackrel{\text{def}}{=} \{\boldsymbol{G} \in \mathcal{S} : \|\boldsymbol{G}\|_{\mathcal{W}} < \infty\}.$$

Clearly $\mathcal{W}$ equipped with $\|.\|_{\mathcal{W}}$ is a Banach space. We now have to determine the contraction operator for the fixed point argument. This process will be more subtle than it was in section 2.

First we take an aperiodic probability measure E in $\mathcal{M}$ with covariance matrix $\delta \cdot \mathrm{Id}_d$. We can construct such a measure by taking the distribution S and shifting a small amount of the probability to vectors of length $\ell' + 1$ (if $d\delta \ge \underline{s}$) or to zero (if $d\delta < \underline{s}$). In this way, the range of E is bounded by $\ell \stackrel{\text{def}}{=} \ell' + 1$.

We use this E to define an appropriate contraction operator: For $\boldsymbol{G} \in \mathcal{S}$, the operator $\boldsymbol{G} \mapsto \widetilde{\boldsymbol{G}}$ is given by the following recursion:

$$\widetilde{G}_0 = G_0, \qquad \text{and for } n \geq 1$$

$$\widetilde{G}_n = E * \widetilde{G}_{n-1} + (S - E) * G_{n-1} - \lambda\big[\rho S * G_{n-1} - \sum_{m=1}^{n} a_m B_m * G_{n-m}\big]$$

for all sequences of signed measures $\boldsymbol{G} = (G_n)_{n\in\mathbb{N}_0}$ in $\mathcal{S}$. Clearly $\widetilde{\boldsymbol{G}}$ is an element of $\mathcal{S}$.

The sequence (A_n) that we have defined in (3.4), is obviously a fixed point of $\sim$. We will show that it is asymptotically close to the distribution of a sum of i.i.d. random vectors with law E. We will use the following lemmas:

Lemma 3.2. *Let $G_n := a_n E^{*n}$ and $\nu \geq 1/(2d)$ big enough. Then*

$$\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathcal{W}} \leq K(1 + \lambda\beta_\nu),$$

where the constant K depends on the dimension d and ℓ' only.

Lemma 3.3. *Let $\boldsymbol{G} \in \mathcal{W}$ with $G_0 = 0$. For λ small enough there exists a parameter $\kappa \in (0,1)$ such that*

$$\|\widetilde{\boldsymbol{G}}\|_{\mathcal{W}} \leq \kappa \|\boldsymbol{G}\|_{\mathcal{W}}.$$

The proofs of the two lemmas are very similar. Straightforward calculation allows us to rewrite $\widetilde{G}_n$ in two different ways:

(3.8)
$$\widetilde{G}_n = G_n - \sum_{l=1}^{n} E^{*n-l} * \big[G_l - (1 - \lambda\rho) S * G_{l-1} - \lambda \sum_{m=1}^{l} a_m B_m * G_{l-m}\big]$$

and

(3.9)
$$\widetilde{G}_n = E^{*n} * G_0 - \sum_{l=1}^{n} G_{n-l} * \big[E^{*l} - (1 - \lambda\rho) S * E^{*l-1} - \lambda \sum_{m=1}^{l} a_m B_m * E^{l-m}\big].$$

These expressions can be viewed as perturbations of the respective first term. The proofs of the lemmas now consist of error estimates for the sums appearing as perturbation terms.

Proof of Lemma 3.2. According to (3.8) it suffices to show that for all n

$$\sum_{l=1}^{n} \big|a_l E^{*n} - (1 - \lambda\rho) a_{l-1} S * E^{*n-1} - \lambda \sum_{m=1}^{l} a_m B_m a_{l-m} * E^{*n-m}\big| \leq (1 + \lambda\beta_\nu) K \chi_n. \tag{3.10}$$

The strategy of the proof is to approximate the discrete distributions E^{*n} by fitting normal densities and use their Taylor expansion to obtain the desired bounds. There are several error terms to control.

We use Lemma A.1 to obtain an approximation for E^{*n}. For $\nu' = \nu'(d, \ell)$ large enough, the lemma yields

$$\big|E^{*n}(x) - \big[1 + n^{-1} P_4(x/\sqrt{n})\big]\varphi_{n\delta}(x)\big| \leq K n^{-3/2} \varphi_{n\nu'}(x), \tag{3.11}$$

where P_4 is a polynomial of degree four. The coefficients of P_4 are rational functions of the moments of E up to order four. We now fix ν as the maximum of $\sqrt{2}\nu'$ and $6/d$. In particular, together with (3.6) this yields $\nu \geq 3\delta$, which we will use later for the error estimates.

To simplify the notation, we use the following abbreviations:

$$\breve{\varphi}_n(x) \stackrel{\text{def}}{=} \left[1 + n^{-1} P_4(x/\sqrt{n})\right] \varphi_{n\delta}(x) \quad \text{denotes the function above,}$$
$$Y_l \stackrel{\text{def}}{=} a_l E - (1-\lambda\rho) a_{l-1} S \qquad \text{and}$$
$$X_l \stackrel{\text{def}}{=} \Big| Y_l * \breve{\varphi}_{n-1} - \lambda \sum_{m=1}^{l\wedge n/2} a_m\, a_{l-m}\, B_m * \breve{\varphi}_{n-m} \Big|. \tag{3.12}$$

We split the left-hand side of (3.10) into several parts, which will be estimated separately:

$$\text{l.h.s. of (3.10)} \le \sum_{l=1}^{n} X_l \tag{3.13a}$$
$$+ \sum_{l=1}^{n} |Y_l| * |\breve{\varphi}_{n-1} - E^{*n-1}| \tag{3.13b}$$
$$+ \lambda \sum_{l=1}^{n} \sum_{m=1}^{l\wedge n/2} a_m\, a_{l-m}\, |B_m| * |\breve{\varphi}_{n-m} - E^{*n-m}| \tag{3.13c}$$
$$+ \lambda \sum_{l=n/2}^{n} \sum_{m=n/2}^{l} a_m\, a_{l-m}\, |B_m| * E^{*n-m}. \tag{3.13d}$$

Before we start to estimate the various sums, we want to state some facts that we will use extensively in this proof.

Sums of the form $\sum_{l=1}^{n} l^{\varepsilon}$ for some real ε are normally estimated by majorizing them with the appropriate integral $\int t^{\varepsilon} dt$. Double sums like $\sum_{l=1}^{n-1} l^{\varepsilon}(n-l)^{\varepsilon'}$ are bounded by splitting them in two parts $l \le n/2$ and $l \ge n/2$ and treating the halves separately.

Another often used inequality yields an upper bound on the discrete folding $\varphi_\eta * \varphi_\theta(x) \stackrel{\text{def}}{=} \sum_{y\in\mathbb{Z}^d} \varphi_\eta(y)\varphi_\theta(x-y)$ of two normal densities. We have

$$\varphi_\eta * \varphi_\theta(x) \le K \varphi_{\eta+\theta}(x)$$

uniformly for all $x \in \mathbb{Z}^d$ and $\eta, \theta \ge 1/(2d)$. This inequality is proven in Lemma A.5 in the appendix.

A last remark worth making is that for positive constants l, l' and m with $l \le l' \le m\, l$ we have

$$\varphi_{l\eta}(x) \le K(m) \varphi_{l'\eta}(x)$$

for all $x \in \mathbb{Z}^d$. This simple fact is obtained by bounding the first factor and the exponential term in $\varphi_{l\eta}(x)$ separately.

Now we come back to, or rather start with, (3.13). We go from bottom to top and start with bounding (3.13d).

A direct consequence of Lemma A.1 is that $E^{*n} \leq K\varphi_{n\nu'}$. So we obtain

$$
\begin{aligned}
(3.13\text{d}) &= \lambda \sum_{l=n/2}^{n} \sum_{m=n/2}^{l} a_m\, a_{l-m}\, |B_m| * E^{*n-m} \\
&\leq \lambda\beta_\nu K \sum_{l=n/2}^{n} \sum_{m=n/2}^{l} \underbrace{m^{-d/2}}_{\leq K\, n^{-d/2}} \sum_{k=1}^{m/2} k^{1-d/2} \underbrace{\varphi_{k\nu} * \varphi_{(n-m)\nu'}}_{\leq K\,\varphi_{(k+n-m)\nu}} \\
&\leq \lambda\beta_\nu K\, n^{-d/2} \sum_{m=n/2}^{n} (n-m) \sum_{k=1+n-m}^{n-m/2} (k-(n-m))^{1-d/2}\varphi_{k\nu} \\
&\leq \lambda\beta_\nu K\, n^{-d/2} \sum_{k=1}^{3n/4} \varphi_{k\nu} \sum_{m=n/2\vee n-k+1}^{n} (k-(n-m))^{1-d/2} \underbrace{(n-m)}_{\leq k} \\
&\leq \lambda\beta_\nu K\, \chi_n.
\end{aligned}
$$

In the last step we used the fact, that — in five and more dimensions — the sum $n^{-d/2}\sum_{k=1}^{n} k\varphi_{k\nu}$ is bounded above by χ_n, which can easily be seen by splitting the sum. We use (3.11) to see

$$
\begin{aligned}
(3.13\text{c}) &= \lambda \sum_{l=1}^{n} \sum_{m=1}^{l\wedge n/2} a_m\, a_{l-m}\, |B_m| * |\breve{\varphi}_{n-m} - E^{*n-m}| \\
&\leq \lambda K \sum_{l=1}^{n} \sum_{m=1}^{l\wedge n/2} (n-m)^{-3/2} |B_m| * \varphi_{(n-m)\nu'} \\
&\leq \lambda\beta_\nu K n^{-1/2} \sum_{m=1}^{n/2} m^{-d/2} \sum_{k=1}^{m/2} k^{1-d/2} \underbrace{\varphi_{k\nu} * \varphi_{(n-m)\nu'}}_{\leq K\,\varphi_{(n-(m-k))\nu}} \\
&\leq \lambda\beta_\nu K n^{-1/2}\varphi_{n\nu},
\end{aligned}
$$

where we used in the last line that $n/2 \leq n-(m-k) \leq n$ (and therefore $\varphi_{(n-(m-k))\nu} \leq K\varphi_{n\nu}$). Furthermore we have

$$
\begin{aligned}
(3.13\text{b}) &= \sum_{l=1}^{n} |Y_l| * |\breve{\varphi}_{n-1} - E^{*n-1}| \\
&\leq K(n-1)^{-3/2} \sum_{l=1}^{n} \underbrace{|Y_l| * \varphi_{(n-1)\nu'}}_{\leq K\,\varphi_{\sqrt{2}(n-1)\nu'}} \leq K\, n^{-1/2}\varphi_{n\nu},
\end{aligned}
$$

since $Y_l = a_l E - (1-\lambda\rho)a_{l-1} S$ is a signed measure of bounded steplength.

Now we come to (3.13a). Recall the definition of X_l in (3.12), that is

$$
X_l \stackrel{\text{def}}{=} \Big|Y_l * \breve{\varphi}_{n-1} - \lambda \sum_{m=1}^{l\wedge n/2} a_m\, a_{l-m}\, B_m * \breve{\varphi}_{n-m}\Big|.
$$

We analyze the terms in X_l separately, using Lemma A.2 and Lemma A.3. We write $P_2(z)$ for the polynomial $z^2/\delta - d$.

From (A.7) we obtain

$$
\begin{aligned}
Y_l * \breve{\varphi}_{n-1}(x) &= \big[a_l - (1-\lambda\rho)a_{l-1}\big] \cdot \breve{\varphi}_{n-1}(x) \\
&\quad + \Big[\frac{a_l d\delta - (1-\lambda\rho)a_{l-1}\underline{s}}{2(n-1)d\delta}\Big] \cdot P_2(x/\sqrt{n-1})\varphi_{(n-1)\delta}(x) \\
&\quad + R_4^{Y_l}(n-1,x) + (n-1)^{-1} R_2^{Y_l}(n-1,x;P_4).
\end{aligned}
\tag{3.14}
$$

Now we apply first (A.10) and then (A.7) to obtain [5]

$$
\begin{aligned}
B_m * \breve{\varphi}_{n-m}(x) &= B_m * \breve{\varphi}_{n-1}(x) - \frac{m-1}{2(n-1)}\big[B_m * P_2(./\sqrt{n-1})\varphi_{(n-1)\delta}\big](x) \\
&\quad + \big[B_m * S_2^{m-1}(n-1,.)\big](x) \\
&\quad + \big[B_m * S_1^{m-1}(n-1,.;(n-1)^{-1}P_4)\big](x) \\
&= b_m \cdot \breve{\varphi}_{n-1}(x) \\
&\quad + \Big[\frac{\underline{b}_m}{2(n-1)d\delta} - \frac{(m-1)b_m}{2(n-1)}\Big] \cdot P_2(x/\sqrt{n-1})\varphi_{(n-1)\delta}(x) \\
&\quad + R_4^{B_m}(n-1,x) + (n-1)^{-1} R_2^{B_m}(n-1,x;P_4) \\
&\quad - \frac{m-1}{2(n-1)} R_2^{B_m}(n-1,x;P_2) + \big[B_m * S_2^{m-1}(n-1,.)\big](x) \\
&\quad + \big[B_m * S_1^{m-1}(n-1,.;(n-1)^{-1}P_4)\big](x). \qquad (3.15)
\end{aligned}
$$

We insert (3.14) and (3.15) into (3.12) and obtain

$$
X_l(x) \le I_l \cdot \underbrace{\breve{\varphi}_{n-1}(x)}_{\le K\,\varphi_{2n\delta}(x)} + J_l \cdot \underbrace{|P_2(x/\sqrt{n-1})|\varphi_{(n-1)\delta}}_{\le K\,\varphi_{2n\delta}(x)} + R_l(x), \qquad (3.16)
$$

with

$$
I_l = \Big|a_l - (1-\lambda\rho)a_{l-1} - \lambda \sum_{m=1}^{l\wedge n/2} a_m b_m a_{l-m}\Big| \qquad \text{and}
$$

$$
J_l = \frac{1}{2(n-1)d\delta}\Big|a_l d\delta - (1-\lambda\rho)a_{l-1}\underline{s} - \lambda \sum_{m=1}^{l\wedge n/2} a_m a_{l-m}(\underline{b}_m - d\delta(m-1)b_m)\Big|.
$$

Using the recursion formula for a_l and the decay rate of $b_m = O(m^{-d/2})$ (see (3.3)), we obtain

$$
I_l \le \lambda \sum_{m=n/2}^{l} a_m |b_m| a_{l-m} \le \lambda\beta_\nu K n^{-3/2}. \qquad (3.17)
$$

Recall also (see Corollary 2.4) that $b_m = O(m^{-5/2})$ implies $(\Delta a)_n = O(n^{-3/2})$. More precisely, we have $|\alpha - a_n| \le \lambda\beta_\nu K n^{-1/2}$ and therefore

$$
\begin{aligned}
J_l &= \frac{1}{2(n-1)d\delta}\Big|a_l d\delta - (1-\lambda\rho)a_{l-1}\underline{s} - \lambda \sum_{m=1}^{l\wedge n/2} a_m a_{l-m}(\underline{b}_m - d\delta(m-1)b_m)\Big| \\
&\le \frac{\alpha}{2(n-1)d\delta}\Big|d\delta - (1-\lambda\rho)\underline{s} - \lambda \sum_{m=1}^{l\wedge n/2} a_m \underline{b}_m + d\delta\lambda \sum_{m=1}^{l\wedge n/2} (m-1)a_m b_m\Big| \\
&\quad + \lambda\beta_\nu K n^{-1} l^{-1/2} \\
&\le \frac{\alpha}{2(n-1)d\delta}\big|\underbrace{d\delta - (1-\lambda\rho)\underline{s} - \lambda\tau + d\delta\lambda\sigma}_{=0}\big| + \lambda\beta_\nu K n^{-1} l^{-1/2} \\
&\le \lambda\beta_\nu K n^{-1} l^{-1/2}, \qquad (3.18)
\end{aligned}
$$

[5] In the following formulas the dots denote the argument with respect to which we fold.

where we used (3.3) again and the fact that $\nu = \nu(d,\ell)$ has already been fixed and can thus be bounded by a constant K. The error term R_l in (3.16) is given by

$$R_l(x) = |R_4^{Y_l}(n-1,x)| + (n-1)^{-1}|R_2^{Y_l}(n-1,x;P_4)| \tag{3.19a}$$

$$+\lambda \sum_{m=1}^{l\wedge n/2} a_m a_{l-m} |R_4^{B_m}(n-1,x)| \tag{3.19b}$$

$$+\lambda (n-1)^{-1} \sum_{m=1}^{l\wedge n/2} a_m a_{l-m} |R_2^{B_m}(n-1,x;P_4)| \tag{3.19c}$$

$$+\lambda (2(n-1))^{-1} \sum_{m=1}^{l\wedge n/2} (m-1) a_m a_{l-m} |R_2^{B_m}(n-1,x;P_2)| \tag{3.19d}$$

$$+\lambda \sum_{m=1}^{l\wedge n/2} a_m a_{l-m} |S_2^{m-1}(n-1,.) * B_m(x)| \tag{3.19e}$$

$$+\lambda \sum_{m=1}^{l\wedge n/2} a_m a_{l-m} |S_1^{m-1}(n-1,.;(n-1)^{-1}P_4) * B_m(x)|. \tag{3.19f}$$

We use the local error estimates in lemmas A.2 and A.3 to estimate the various terms. We have

$$|R_4^{Y_l}(n-1,x)| \leq K n^{-2} \varphi_{2n\delta}(x) \qquad \text{by (A.8)},$$

since $|Y_l|$ has bounded steplength. Analogously we can show the same decay for the second term of (3.19a).

On the other hand, we bound (3.19b) using (A.9) to obtain

$$\begin{aligned}
&\lambda K \sum_{m=1}^{l\wedge n/2} \left|R_4^{B_m}(n-1,x)\right| \\
&\leq \lambda K \sum_{m=1}^{l\wedge n/2} n^{-2} \int_0^1 ds \sum_{z\in\mathbb{Z}^d} z^4 \, |B_m(z)| \, \varphi_{\sqrt{2}(n-1)\delta}(x-sz) \\
&\leq \lambda \beta_\nu K n^{-2} \sum_{m=1}^{l\wedge n/2} m^{-d/2} \sum_{k=1}^{m/2} k^{1-d/2} \\
&\qquad\qquad \times \int_0^1 ds \sum_{z\in\mathbb{Z}^d} \underbrace{z^4 \, \varphi_{k\nu}(z)}_{\leq K k^2 \varphi_{2k\nu}(z)} \quad \underbrace{\varphi_{\sqrt{2}(n-1)\delta}(x-sz)}_{\leq K s^{-d}\, \varphi_{\sqrt{2}(n-1)\delta/s^2}(x/s-z)} \\
&\leq \lambda \beta_\nu K n^{-2} \sum_{m=1}^{l\wedge n/2} m^{-d/2} \sum_{k=1}^{m/2} k^{3-d/2} \int_0^1 ds \, \underbrace{s^{-d} \varphi_{2k\nu+\sqrt{2}(n-1)\delta/s^2}(x/s)}_{\leq K \varphi_{2k\nu s^2+\sqrt{2}(n-1)\delta}(x)} \\
&\leq \lambda \beta_\nu K n^{-2} \varphi_{n\nu}(x) \underbrace{\sum_{m=1}^{l\wedge n/2} m^{-d/2} \sum_{k=1}^{m/2} k^{3-d/2}}_{\leq K \, \log(n)} \\
&\leq \lambda \beta_\nu K n^{-3/2} \varphi_{n\nu}(x),
\end{aligned}$$

where we used $\nu \geq 3\delta$ in the second to last step. Analogously we can show that (3.19c) and (3.19d) are bounded by $\lambda\beta_\nu K n^{-2}\varphi_{n\nu}(x)$ and $\lambda\beta_\nu K n^{-3/2}\varphi_{n\nu}(x)$, respectively. We bound (3.19e) with (A.11) by

$$\begin{aligned}
&\lambda\beta_\nu K \sum_{m=1}^{l\wedge n/2} m^{2-d/2} \underbrace{(n-m)^{-2}\big(\frac{n}{n-m}\big)^{d/2}}_{\leq K n^{-2}} \sum_{k=1}^{m/2} k^{1-d/2} \varphi_{\sqrt{2}(n-1)\delta} * \varphi_{k\nu}(x) \\
&\leq \lambda\beta_\nu K n^{-2} \sum_{m=1}^{l\wedge n/2} m^{2-d/2} \sum_{k=1}^{m/2} k^{1-d/2} \varphi_{k\nu+\sqrt{2}(n-1)\delta}(x) \\
&\leq \lambda\beta_\nu K n^{-2} \varphi_{n\nu}(x) \sum_{m=1}^{l\wedge n/2} m^{2-d/2} \underbrace{\sum_{k=1}^{m/2} k^{1-d/2}}_{\leq K} \\
&\leq \lambda\beta_\nu K n^{-2} l^{1/2} \varphi_{n\nu}(x) \leq \lambda\beta_\nu K n^{-3/2}\varphi_{n\nu}(x),
\end{aligned}$$

where we used $\nu \geq 3\delta$ to obtain the second to last line. For (3.19f) an even better bound without the $l^{1/2}$-term results analogously.

Combining the different estimates, we can bound R_l by

$$R_l \leq (1+\lambda\beta_\nu) K\, n^{-3/2} \varphi_{n\nu}.$$

Considering this together with (3.17) and (3.18), we can finally bound (3.13a):

$$\begin{aligned}
(3.13\text{a}) &= \sum_{l=1}^{n} X_l \\
&\leq \sum_{l=1}^{n} I_l\ \varphi_{2n\delta}(x) + J_l\ \varphi_{2n\delta}(x) +\ R_l(x) \\
&\leq (1+\lambda\beta_\nu) K\ \varphi_{n\nu}(x) \sum_{l=1}^{n} \Big[n^{-3/2} + n^{-1} l^{-1/2} \Big] \\
&\leq (1+\lambda\beta_\nu) K\, n^{-1/2} \varphi_{n\nu}(x).
\end{aligned}$$

This proves Lemma 3.2. ☐

Proof of Lemma 3.3. Let $\boldsymbol{G}$ with $G_0 = 0$ and $\|\boldsymbol{G}\|_w := \sup_{n,x} \chi_n(x)^{-1} |G_n(x)| < \infty$ be given.

We want to prove that for small enough λ we have $\|\widetilde{\boldsymbol{G}}\|_w \leq \kappa \|\boldsymbol{G}\|_w$ for some $\kappa < 1$. According to (3.9), it is sufficient to show that for all natural n

$$\sum_{l=1}^{n-1} \big|G_{n-l}\big| * \big|E^{*l} - (1-\lambda\rho) S * E^{*l-1} - \lambda \sum_{m=1}^{l} a_m\, B_m * E^{l-m}\big| \\ \leq \lambda(1+\beta_\nu) K \chi_n \|\boldsymbol{G}\|.$$

Once we know this, we can fix K and choose λ so small that $\lambda(1+\beta_\nu)K$ is strictly smaller than one. Since $\big|G_{n-l}\big| \leq \|\boldsymbol{G}\| \cdot \chi_{n-l}$, it suffices to show that

$$\sum_{l=1}^{n-1} \chi_{n-l} * \big| \big(E - (1-\lambda\rho)S\big) * E^{*l-1} - \lambda \sum_{m=1}^{l} a_m\, B_m * E^{*l-m} \big| \leq \lambda(1+\beta_\nu) K \chi_n. \tag{3.20}$$

The proof of this estimate is somewhat tedious, but similar to the preceding one. We take the same ν and we will split the sum in the very same way as before. Again we use some abbreviations to keep the notation as readable as possible:

$$\begin{aligned}\breve\varphi_n(x) &\overset{\text{def}}{=} \big[1+n^{-1}P_4(x/\sqrt{n})\big]\varphi_{n\delta}(x) \quad \text{as before,}\\ Y &\overset{\text{def}}{=} E-(1-\lambda\rho)S \qquad \text{and}\\ X'_l &\overset{\text{def}}{=} \Big|Y*\breve\varphi_{l-1}-\lambda\sum_{m=1}^{l/2} a_m\, B_m*\breve\varphi_{l-m}\Big|.\end{aligned}$$

The left hand side of (3.20) is split in the following parts, which will be estimated separately:

$$\text{l.h.s. of (3.20)} \le \sum_{l=1}^{n-1} \chi_{n-l}*X'_l \tag{3.21a}$$

$$+\sum_{l=1}^{n} \chi_{n-l}*|Y|*|\breve\varphi_{l-1}-E^{*l-1}| \tag{3.21b}$$

$$+\lambda\sum_{l=1}^{n-1} \chi_{n-l}*\sum_{m=1}^{l/2} a_m\,|B_m|*|\breve\varphi_{l-m}-E^{*l-m}| \tag{3.21c}$$

$$+\lambda\sum_{l=1}^{n-1} \chi_{n-l}*\sum_{m=l/2}^{l} a_m\,|B_m|*E^{*l-m}. \tag{3.21d}$$

We again start with the last term. First we split χ_{n-l} into $(n-l)^{-1/2}\varphi_{(n-l)\nu}$ and $(n-l)^{-d/2}\sum_{j=1}^{(n-l)/2} j\,\varphi_{j\nu}$. We treat the resulting parts of (3.21d) separately. For the first part this leads to

$$\begin{aligned}&\sum_{l=1}^{n-1}(n-l)^{-1/2}\varphi_{(n-l)\nu}*\sum_{m=l/2}^{l} a_m\,|B_m|*E^{*l-m}\\
&\le \beta_\nu K\sum_{l=1}^{n-1}(n-l)^{-1/2}\sum_{m=l/2}^{l} m^{-d/2}\sum_{k=1}^{m/2} k^{1-d/2}\underbrace{\varphi_{(n-l)\nu}*\varphi_{k\nu}*\varphi_{(l-m)\nu'}}_{\le K\varphi_{(n-m+k)\nu}}\\
&\le \beta_\nu K n^{-1/2}\varphi_{n\nu}\underbrace{\sum_{l=1}^{n/2}\sum_{m=l/2}^{l} m^{-d/2}}_{\le K}\underbrace{\sum_{k=1}^{m/2} k^{1-d/2}}_{\le K}\\
&\quad+\beta_\nu K n^{-d/2}\sum_{l=n/2}^{n-1}(n-l)^{-1/2}\sum_{m=l/2}^{l}\sum_{k=1}^{m/2} k^{1-d/2}\varphi_{(n-m+k)\nu}\\
&\le \beta_\nu K\Big[n^{-1/2}\varphi_{n\nu}+n^{-d/2}\sum_{m=n/4}^{n-1}\sum_{k=n-m+1}^{n-m/2}(k-(n-m))^{1-d/2}\varphi_{k\nu}\underbrace{\sum_{l=m}^{n-1}(n-l)^{-1/2}}_{\le n-m\,\le k}\Big]\\
&\le \beta_\nu K\Big[n^{-1/2}\varphi_{n\nu}+n^{-d/2}\sum_{k=1}^{7n/8} k\,\varphi_{k\nu}\underbrace{\sum_{m=n-k+1}^{n-1}(k-(n-m))^{1-d/2}}_{\le K}\Big]\\
&\le \beta_\nu K\chi_n.\end{aligned}$$

For the second part we split the sum and find

$$\begin{aligned}
&\sum_{l=1}^{n/2}(n-l)^{-d/2}\sum_{j=1}^{(n-l)/2} j\,\varphi_{j\nu} * \sum_{m=l/2}^{l} a_m\,|B_m| * E^{*l-m} \\
&\quad\le \beta_\nu K\, n^{-d/2}\sum_{l=1}^{n/2}\sum_{j=1}^{(n-l)/2} j \sum_{m=l/2}^{l} m^{-d/2}\sum_{k=1}^{m/2} k^{1-d/2}\varphi_{(j+k+l-m)\nu} \\
&\quad\le \beta_\nu K\, n^{-d/2}\sum_{m=1}^{n/2} m^{-d/2}\sum_{l=m}^{n/2\wedge 2m}\sum_{k=1}^{m/2} k^{1-d/2}\sum_{j=1+k}^{(n-l)/2+k} j\,\varphi_{(j+l-m)\nu} \\
&\quad\le \beta_\nu K\, n^{-d/2}\sum_{m=1}^{n/2} m^{-d/2}\sum_{l=m}^{n/2\wedge 2m}\sum_{j=1+l-m}^{(n+l-m)/2} j\,\varphi_{j\nu}\sum_{k=1}^{m/2} k^{1-d/2} \\
&\quad\le \beta_\nu K\, n^{-d/2}\sum_{j=1}^{3n/4} j\,\varphi_{j\nu}\sum_{m=1}^{n/2} m^{1-d/2} \\
&\quad\le \beta_\nu K \chi_n,
\end{aligned}$$

and finally

$$\begin{aligned}
&\sum_{l=n/2}^{n-1}(n-l)^{-d/2}\sum_{j=1}^{(n-l)/2} \underbrace{j}_{\le n-l}\,\varphi_{j\nu} * \sum_{m=l/2}^{l} a_m\,|B_m| * E^{*l-m} \\
&\quad\le \beta_\nu K\sum_{l=n/2}^{n-1}(n-l)^{1-d/2}\sum_{m=l/2}^{l} m^{-d/2}\sum_{k=1}^{m/2} k^{1-d/2}\sum_{j=1}^{(n-l)/2}\varphi_{(j+k+l-m)\nu} \\
&\quad\le \beta_\nu K\, n^{-d/2}\sum_{l=n/2}^{n-1}(n-l)^{1-d/2}\sum_{m=0}^{l/2}\sum_{k=1}^{(l-m)/2} k^{1-d/2}\sum_{j=1}^{(n-l)/2}\varphi_{(j+k+m)\nu} \\
&\quad\le \beta_\nu K\, n^{-d/2}\sum_{l=n/2}^{n-1}(n-l)^{1-d/2}\sum_{m=0}^{l/2}\sum_{j=1+m}^{(n+m)/2}\varphi_{j\nu} \\
&\quad\le \beta_\nu K\, n^{-d/2}\sum_{l=n/2}^{n-1}(n-l)^{1-d/2}\sum_{j=1}^{(2n+l)/4}\varphi_{j\nu}\sum_{m=0}^{j-1} \\
&\quad\le \beta_\nu K\, n^{-d/2}\sum_{j=1}^{3n/4} j\,\varphi_{j\nu} \\
&\quad\le \beta_\nu K \chi_n.
\end{aligned}$$

Thus (3.21d) $\le \lambda\beta_\nu K\chi_n$ is shown. In order to estimate (3.21c), we use (3.11) again, obtaining

$$\begin{aligned}
&\sum_{m=1}^{l/2} a_m\,|B_m| * |\breve{\varphi}_{l-m} - E^{*l-m}| \\
&\quad\le \beta_\nu K\sum_{m=1}^{l/2}(l-m)^{-3/2} m^{-d/2}\sum_{k=1}^{m/2} k^{1-d/2}\underbrace{\varphi_{k\nu} * \varphi_{(l-m)\nu'}}_{\le K\varphi_{l\nu}} \\
&\quad\le \beta_\nu K l^{-3/2}\varphi_{l\nu},
\end{aligned}$$

and therefore

$$
\begin{aligned}
(3.21\mathrm{c}) &= \lambda \sum_{l=1}^{n-1} \chi_{n-l} * \sum_{m=1}^{l/2} a_m \, |B_m| * |\breve{\varphi}_{l-m} - E^{*l-m}| \\
&\le \lambda\beta_\nu K \sum_{l=1}^{n-1} l^{-3/2} \, \chi_{n-l} * \varphi_{l\nu} \\
&\le \lambda\beta_\nu K \sum_{l=1}^{n-1} l^{-3/2} \Big[(n-l)^{-1/2}\varphi_{n\nu} + (n-l)^{-d/2} \sum_{j=1}^{(n-l)/2} j \, \varphi_{(j+l)\nu}\Big] \\
&\le \lambda\beta_\nu K n^{-1/2}\varphi_{n\nu} + \lambda\beta_\nu K n^{-d/2} \sum_{l=1}^{n/2} l^{-3/2} \sum_{j=1}^{(n-l)/2} j \, \varphi_{(j+l)\nu} \\
&\qquad\qquad + \lambda\beta_\nu K \sum_{l=n/2}^{n-1} l^{-3/2} \, (n-l)^{-d/2} \sum_{j=1}^{(n-l)/2} j \; \underbrace{\varphi_{(j+l)\nu}}_{\le K\varphi_{n\nu}} \\
&\le \lambda\beta_\nu K \chi_n. \qquad\qquad (3.22)
\end{aligned}
$$

Analogously, we have

$$
(3.21\mathrm{b}) = \sum_{l=1}^{n} \chi_{n-l} * |Y| * |\breve{\varphi}_{l-1} - E^{*l-1}| \le \lambda\beta_\nu K \chi_n.
$$

The remaining part, (3.21a), will be treated like (3.13a) in the proof of the last lemma, using Lemma A.2 and Lemma A.3. Again we write $P_2(z)$ for the polynomial $z^2/\delta - d$. From (A.7) we have

$$
\begin{aligned}
Y * \breve{\varphi}_{l-1}(x) &= \lambda\rho \cdot \breve{\varphi}_{l-1}(x) \\
&\quad + \Big[\frac{d\delta - (1-\lambda\rho)\underline{s}}{2(l-1)d\delta}\Big] \cdot P_2(x/\sqrt{l-1})\varphi_{(l-1)\delta}(x) \\
&\quad + R_4^Y(l-1,x) \; + (l-1)^{-1} R_2^Y(l-1,x;P_4).
\end{aligned}
$$

For the second sum, we use (3.15) after having replaced n by l everywhere. We obtain

$$
X_l'(x) \le I_l' \cdot \underbrace{\breve{\varphi}_{l-1}(x)}_{\le K\,\varphi_{2l\delta}(x)} + J_l' \cdot \underbrace{|P_2(x/\sqrt{l-1})|\varphi_{(l-1)\delta}}_{\le K\,\varphi_{2l\delta}(x)} + R_l'(x) \tag{3.23}
$$

with

$$
I_l' = \big|\lambda\rho - \lambda \sum_{m=1}^{l/2} a_m b_m\big| \le \lambda\beta_\nu K l^{-3/2} \qquad \text{and}
$$

$$
J_l' = \frac{1}{2(l-1)d\delta}\Big|d\delta + (1-\lambda\rho)\underline{s} - \lambda \sum_{m=1}^{l/2} a_m(\underline{b}_m - d\delta(m-1)b_m)\Big| \le \lambda\beta_\nu K l^{-3/2},
$$

which results from an argument very similar to that which led to (3.17) and (3.18). This time the error term R'_l in (3.23) is given by

$$R'_l(x) = |R_4^Y(l-1,x)| + (l-1)^{-1}|R_2^Y(l-1,x;P_4)| \tag{3.24a}$$

$$+\lambda \sum_{m=1}^{l/2} a_m |R_4^{B_m}(l-1,x)| \tag{3.24b}$$

$$+(l-1)^{-1}\lambda \sum_{m=1}^{l/2} a_m |R_2^{B_m}(l-1,x;P_4)| \tag{3.24c}$$

$$+(l-1)^{-1}\lambda \sum_{m=1}^{l/2} (m-1)a_m |R_2^{B_m}(l-1,x;P_2)| \tag{3.24d}$$

$$+\lambda \sum_{m=1}^{l/2} a_m |S_2^{m-1}(l-1,.) * B_m(x)| \tag{3.24e}$$

$$+\lambda \sum_{m=1}^{l/2} a_m a_{l-m} |S_1^{m-1}(l-1,.;(l-1)^{-1}P_4) * B_m(x)|. \tag{3.24f}$$

These terms are bounded in exactly the same way as the corresponding ones in formula (3.19). So we obtain

$$R'_l \leq \lambda(1+\beta_\nu)K\, l^{-3/2}\varphi_{l\nu}.$$

Combining these estimates we get

$$(3.21\text{a}) \leq \lambda(1+\beta_\nu)K \sum_{l=1}^{n-1} l^{-3/2}\, \chi_{n-l} * \varphi_{l\nu} \leq \lambda(1+\beta_\nu)K\chi_n$$

as in (3.22). □

Now we have the necessary tools to prove Theorem 3.1:

Proof of Theorem 3.1. First recall the space $\mathcal{W} = \{\boldsymbol{G} \in \mathcal{S} : \|\boldsymbol{G}\|_{\mathcal{W}} < \infty\}$ and write $\mathcal{W}_0$ for $\{\boldsymbol{G} \in \mathcal{W} : G_0 \equiv 0\}$. For the moment, we refer to the sequence $(a_n E^{*n})_{n\in\mathbb{N}_0}$ as $\mathbf{E}$. The following considerations always assume that λ is small enough. Lemma 3.2 yields $\big(\mathbf{E} - \widetilde{\mathbf{E}}\big) \in \mathcal{W}_0$. From Lemma 3.3 we know that $\sim$ is a contraction on $\mathcal{W}_0$. Since $\sim$ is linear, the Banach fixed point theorem now yields the existence of a unique fixed point in $\mathbf{E} + \mathcal{W}_0$. With other words, we have a unique sequence $\mathbf{A}$ of symmetric measures on $\mathbb{Z}^d$ with

(i) $A_0 = \delta_0$,
(ii) $\widetilde{A}_n(x) = A_n(x)$ for all $n \in \mathbb{N}$, $x \in \mathbb{Z}^d$,
(iii) $\|\mathbf{E} - \mathbf{A}\|_{\mathcal{W}} \leq K(1+\lambda\beta_\nu)$.

This sequence obviously *is* the sequence $(A_n)_{n\in\mathbb{N}_0}$ defined in (3.4). Since as a direct consequence of Lemma A.1 we have

$$|a_n E^{*n}(x) - a_n\varphi_{n\delta}(x)| \leq K n^{-1/2}\varphi_{n\nu}(x),$$

the estimate (3.7) now results by choosing $\lambda_0 = \lambda_0(d,\beta_\nu)$ small enough. □

To end this section, we briefly discuss the periodic case. Assume that S and the sequence $(B_m)_{m\in\mathbb{N}_0}$ are *two-periodic* (that is, $B_m(x) = 0$ whenever m and $\|x\|_1$ do not have the same parity). By (3.4), the periodicity transfers to the whole sequence (A_n). In order to use the same arguments as in the aperiodic case, we have to define a periodic probability measure E' so that we can approximate A_n by $(E')^{*n}$. If the diffusion constant δ (see (3.5)) is greater or equal to $1/d$, this can be done easily.

So let $S \in \mathcal{M}$ be a two-periodic distribution of bounded range less or equal ℓ'. Furthermore let $(B_m) \in \mathcal{S}$ be a two-periodic sequence which obeys (3.2) and has parameter δ (resulting from this sequence by (3.5)) greater or equal to $1/d$. Then we have the following theorem.

Theorem 3.4 (Local Estimates, two-periodic case)**.** *Under the above assumptions, the sequence (A_n) defined by (3.4), has the following property: There exist $\lambda_0 > 0$ small enough and $\nu > 0$ big enough such that for all $\lambda \in (0, \lambda_0)$*

$$|A_n(x) - 2a_n\varphi_{n\delta}(x)| \leq K\Big[n^{-1/2}\varphi_{n\nu}(x) + n^{-d/2}\sum_{j=1}^{n/2} j\,\varphi_{j\nu}(x)\Big],$$

where n is taken to have the same parity as $\|x\|_1$ and K is a positive constant depending only on d and ℓ'.

Proof. Replace the aperiodic E by the periodic E' everywhere. Instead of using (3.11), apply a periodic version of Lemma A.1, namely

$$\big|E'^{*n}(x) - 2\big[1 + n^{-1}P_4(x/\sqrt{n})\big]\varphi_{n\delta}(x)\big| \leq Kn^{-3/2}\varphi_{n\nu}(x),$$

whenever $\|x\|_1$ and n have the same parity. The rest of the proof for the aperiodic case carries over word by word (with the constants suitably adapted). $\square$

Note that if δ is smaller than $1/d$, we have a problem with our construction. A symmetric and rotationally invariant, twoperiodic probability measure on the lattice $\mathbb{Z}^d$ with variance smaller than $1/d$ simply does not exist. Possibly this case can be covered by choosing a more delicate contraction operator.

The weakly self-avoiding walk is spreading faster rather than slower compared with the simple random walk. Therefore the speed of its diffusion is greater or equal to the one of the simple random walk. This means $\delta \geq 1/d$. Instead of giving this heuristic argument one can calculate for small λ the leading term in δ (use (4.4) and (B.9) as well as (B.11)), which is larger than $1/d$.

Therefore it will be possible to apply Theorem 3.4 to the weakly self-avoiding walk as soon as we have shown that the lace functions have the desired decay property (3.2). This is the content of the next section.

4. Application to the Weakly Self-Avoiding Walk

We now come back to the specific context of the weakly self-avoiding walk, where the main objects of study are the two-point functions C_n with total mass c_n. Recall that they satisfy the lace expansion formula (1.4), that is

$$C_n = 2d(D * C_{n-1}) + \sum_{m=2}^{n} \Pi_m * C_{n-m}.$$

4.1. Decay Behavior.

We still have to show that the methods in the last chapters can in fact be applied to the weakly self-avoiding walk. This means that we have to show that $\Pi_m/(\lambda c_m)$ really has the behavior we assumed for B_m in section 3.

Lemma 4.1. *There are positive constants λ_0, ν and L, such that for all $\lambda \in (0, \lambda_0)$ we have*

$$\frac{|\Pi_m(x)|}{\lambda c_m} \leq Lm^{-d/2}\sum_{k=1}^{m/2} k^{1-d/2}\,\varphi_{k\nu}(x).$$

Proof. The argument uses induction on m. As in section 1, we define $B_m(x) \stackrel{\text{def}}{=} \Pi_m(x)/(\lambda c_m)$. We freely use the results of appendix B.

We have $\Pi_1(x) = 0$ and hence $B_1 \equiv 0$ too. Now consider Π_2: There is only one lace of length two, namely $\{02\}$. Therefore we have by (B.9) and (B.10)

$$\Pi_2(x) = -\lambda \sum_{\substack{\omega:0\rightsquigarrow x\\ |\omega|=2}} U_{02}(\omega) = -2d\lambda\delta_{0x}.$$

On the other hand, we calculate easily (recall (1.3) and (1.5))

$$c_2 = 2d(2d-\lambda).$$

So we have

$$B_2(x) = \frac{\Pi_2(x)}{\lambda 2d(2d-\lambda)} = -\frac{\delta_{0x}}{2d-\lambda}.$$

Since $\psi_2(0) = 2^{-d/2}\varphi_\nu(0) = (4\pi\nu)^{-d/2}$, we obtain

$$|B_2(x)| \le L\,\psi_2(x), \quad \text{whenever} \quad L \ge \frac{(4\pi\nu)^{d/2}}{2d-\lambda}.$$

Now we come to the induction step: Fix $m \ge 3$ and assume that $|B_k(x)| \le L\psi_k(x)$ for all $2 \le k < m$. Define the truncated sequence $(\bar{B}_n)_{n\ge 2}$ by

$$|\bar{B}_n(x)| \stackrel{\text{def}}{=} \begin{cases} B_n(x), & \text{if } |B_n(x)| \le L\psi_n(x), \\ L\psi_n(x), & \text{if } |B_n(x)| > L\psi_n(x). \end{cases}$$

This sequence satisfies the hypothesis of Theorem 3.4. Thus we obtain a sequence $(\bar{A}_n)_{n\in\mathbb{N}_0}$ of measures with

$$\left|\frac{\bar{A}_n(x)}{\bar{a}_n} - 2\varphi_{n\bar{\delta}}(x)\right| \le K\big[n^{-1/2}\varphi_{n\nu}(x) + n^{-d/2}\sum_{k=1}^{n/2} k\varphi_{k\nu}(x)\big], \tag{4.1}$$

whenever n has the same parity as $\|x\|_1$. As long as λ is small enough, the positive constants K and ν do not depend on L.

Defining $\bar{C}_n \stackrel{\text{def}}{=} \bar{\mu}^n\bar{A}_n$ and using (4.1) as well as the fact that $\bar{\delta} \le \nu$ and both are of comparable size, we have

$$\bar{C}_n(x) \le K\bar{c}_n\varphi_{n\nu}(x) \le K(\bar{\alpha} + Kn^{-1/2})\bar{\mu}^n\varphi_{n\nu}(x) \le L_1\bar{\mu}^n\varphi_{n\nu}(x),$$

where L_1 is a positive constant that we fix for the rest of the proof. Since $\bar{B}_n$ equals B_n for all $n < m$, we also have $\bar{C}_n = C_n$ for $n < m$. This can easily be seen by induction.

Now we consider Π_m. Recall from Lemma B.2 that

$$|\Pi_m(x)| \le \lambda K L_1\bar{\mu}^m m^{-d/2}\sum_{k=1}^{m/2} k^{1-d/2}\,\varphi_{k\nu}(x),$$

and therefore

$$|B_m(x)| \le KL_1\frac{\bar{\mu}^m}{c_m}m^{-d/2}\sum_{k=1}^{m/2} k^{1-d/2}\,\varphi_{k\nu}(x). \tag{4.2}$$

It remains to show that $\bar{\mu}^m/c_m$ is bounded. We know from the lace expansion formula that

$$\begin{aligned} c_m &= 2d\bar{c}_{m-1} + \sum_{k=2}^{m} \pi_k \bar{c}_{m-k} \\ &= 2d\bar{\mu}^{m-1}\bar{a}_{m-1} + \sum_{k=2}^{m} \pi_k \bar{\mu}^{m-k}\bar{a}_{m-k}, \end{aligned} \tag{4.3}$$

where π_k as usual denotes the total mass of Π_k. Now we apply Lemma B.2 to the π_k. We obtain $|\pi_k| \le \lambda K L_1 \bar{\mu}^k k^{-d/2}$ and insert this in equation (4.3). This leads to

$$\frac{c_m}{\bar{\mu}^m} \ge d\bar{\mu}^{-1} - \lambda K L_1 \ge K, \text{ if } \lambda = \lambda(d, L_1) \text{ small enough.}$$

Using this in equation (4.2) yields

$$|B_m(x)| \le K L_1 m^{-d/2} \sum_{k=1}^{m/2} k^{1-d/2} \varphi_{k\nu}(x).$$

By choosing L large enough we see that the lemma follows. □

Proof of Theorem 1.1. Using Lemma 4.1, the second part of the theorem follows directly from Theorem 3.4. In view of (3.3), the first part is a consequence of Corollary 2.4. □

4.2. **Identification of the Involved Parameters.** For the convenience of the reader we give a little survey of the parameters involved in Theorem 1.1. The following formulas were first derived by Brydges and Spencer [2] (see also [11] and [9]).

The connective constant μ satisfies the identity

$$2d\mu^{-1} = 1 - \sum_{m=2}^{\infty} \pi_m \, \mu^{-m}.$$

The limit α of the mass constants a_n is given by

$$\alpha^{-1} = 1 + \sum_{m=2}^{\infty} (m-1)\pi_m \, \mu^{-m},$$

and for the diffusion constant δ we have

$$\delta = \frac{1 - \sum_{m=2}^{\infty} (\pi_m - \underline{\pi}_m)\, \mu^{-m}}{d(1 + \sum_{m=2}^{\infty} (m-1)\pi_m \, \mu^{-m})}. \tag{4.4}$$

These formulas are obtained by substituting $b_m = \pi_m/(\lambda c_m)$ and $a_m = \mu^{-m} c_m$ into (1.11), (2.6) and (3.5), respectively.

Appendix A. A LCLT and Discretization Estimates

In order to make the fixed point argument in section 3, we need good local approximations of a general symmetric random walk and quite specific discretization estimates of d-dimensional normal densities. We state and prove these in the following lemmas. Lemma A.1 is a local central limit theorem. It controls the pointwise distance between a symmetric random walk and the appropriate density. This result is proven by standard large deviation techniques.

Lemma A.2 yields approximations of the discrete folding of a normal density with a symmetric signed measure on $\mathbb{Z}^d$. Analogously, Lemma A.3 approximates the density itself in the variance variable. Both results are easily calculated by using Taylor expansions.

In addition we give two estimates concerning the discretization of normal densities. The first one, stated in Lemma A.4, gives a simple bound for the total mass and the 'second moment' of a discretized normal density. The second one compares the discrete folding of two normal densities with their continuous folding. This is the content of Lemma A.5.

As in the whole paper, φ_η denotes the density of the centered normal distribution on $\mathbb{R}^d$ with covariance matrix $\eta \cdot \mathrm{Id}_d$, that is,

$$\varphi_\eta(x) = (2\pi\eta)^{-d/2} \exp\bigl(-\frac{x^2}{2\eta}\bigr).$$

¿

Lemma A.1. *Let G be the single step distribution of an aperiodic, nondegenerate, symmetric random walk on $\mathbb{Z}^d$ with bounded steplength, that is, $G(x) = 0$ for all $|x| > \ell$, where ℓ is fixed. Let $\mathcal{E} = \eta \cdot \mathrm{Id}_d$ denote the covariance matrix of G and assume $\eta \geq 1/(2d)$. Then there exist a polynomial P_4 of degree four and positive constants K and ν', such that for all x in $\mathbb{Z}^d$ and for all natural n,*

$$\bigl|G^{*n}(x) - \bigl[1 + n^{-1} P_4(x/\sqrt{n})\bigr]\varphi_{n\eta}(x)\bigr| \leq K n^{-3/2} \varphi_{n\nu'}(x). \tag{A.1}$$

The coefficients of the polynomial depend (rationally) on the moments of G up to order four, whereas K and ν' can be choosen independently of the specific law of G, depending only on d and ℓ.

Proof. The proof of the lemma combines standard large deviation properties with the approximation of $G^{*n}(x)$ obtained by tilting the measure.

Let $Z(t) \stackrel{\text{def}}{=} \sum_{x\in\mathbb{Z}^d} \exp(t\cdot x)G(x)$ and $I(\xi) \stackrel{\text{def}}{=} \sup_{t\in\mathbb{R}^d}\{t\cdot\xi - \log Z(t)\}$. Standard large deviation theory (see for example [3]) yields a large deviation principle with entropy function I for the laws of $G^{*n}(x/n)$. Let S_G denote the convex closure of the set of points with nonzero G measure. Then I is convex on $\mathbb{R}^d$ and even strictly convex on $\operatorname{int} S_G$, that is, the interior of S_G. Outside S_G, I equals $+\infty$.

The function $t \mapsto \nabla \log Z(t)$ is an analytic diffeomorphism from $\mathbb{R}^d$ onto $\operatorname{int} S_G$ (for a proof see [3], page 261). Therefore, for any $\xi \in \operatorname{int} S_G$, there exists a unique $t_\xi \in \mathbb{R}^d$ with $\nabla \log Z(t_\xi) = \xi$. Clearly $\nabla \log Z(0) = 0$ and $\nabla^2 \log Z(0) = \mathcal{E}$. For $\xi \in \operatorname{int} S_G$, we have $I(\xi) = t_\xi \cdot \xi - \log Z(t_\xi)$. Evidently, $I(0) = 0$. Because of symmetry, the odd partial derivatives of I vanish at zero. A simple computation yields $\nabla^2 I(0) = \mathcal{E}^{-1}$, and the fourth derivatives at zero depend only on the second and fourth moments of G.

Now denote by G_t for $t \in \mathbb{R}^d$ the tilted measure

$$G_t(x) \stackrel{\text{def}}{=} \frac{G(x)\exp(t\cdot x)}{Z(t)}.$$

Using this, we see that for $\xi \stackrel{\text{def}}{=} x/n \in \operatorname{int} S_G$, we can write

$$G^{*n}(x) = \exp(-nI(\xi)) \cdot G_{t_\xi}^{*n}(x). \tag{A.2}$$

Case $|\xi| \leq n^{-5/12}$:

Since G is symmetric and nondegenerate (remember $\eta \geq 1/(2d)$), the boundary of S_G is bounded away from zero. The covariance matrix $\mathcal{E}_\xi$ of G_{t_ξ} is depending analytically on ξ with $\mathcal{E}_0 = \mathcal{E} = \eta \cdot \mathrm{Id}_d$. It follows that the set R_G of all ξ such that

- $\xi \in \operatorname{int} S_G$ and
- the smallest eigenvalue of $\mathcal{E}_\xi$ is greater or equal $\eta/2$,

is a compact neighbourhood of zero. Thus for $|\xi| \le n^{-5/12}$, we have $\xi \in R_G$ for almost all n. It is sufficient to prove the estimate for these ξ, since we can cover the finite number of remaining cases by choosing K large enough.

So let $\xi = x/n \in R_G$ with $|\xi| \le n^{-5/12}$. To estimate the first factor in (A.2), we use Taylor expansion for I at zero. We obtain

$$\begin{aligned}\exp(-nI(\xi)) &= \exp(-\frac{x^2}{2n\eta}) \cdot \left[1 - nT^{(4)}(\xi) + nO(\xi^6)\right] \\ &= \exp(-\frac{x^2}{2n\eta}) \cdot \left[1 - n^{-1}T^{(4)}(x/\sqrt{n}) + O(n^{-3/2})\right],\end{aligned} \tag{A.3}$$

where $T^{(k)}$ denotes a polynomial containing kth order terms only. The coefficients of the polynomial are rational functions of the moments of G up to order four.

As a next step we will estimate the second factor in (A.2), $G_{t_\xi}^{*n}(x)$, using a local central limit theorem out of Bhattacharya/Rao [1]. Corollary 22.3 in [1] asserts

$$\Big|G_{t_\xi}^{*n}(x) - n^{-d/2}\sum_{r=0}^{3} n^{-r/2}Q_r((x-n\xi)/n)\Big)\Big| = o(n^{-(d+3)/2}), \tag{A.4}$$

where Q_r are the so called Edgeworth polynomials. They are formal polynomials, consisting of partial derivatives of the normal density in $\mathbb{R}^d$ with mean zero and covariance matrix $\mathcal{E}_\xi$ (to keep the notation simple we suppress the ξ-dependance of Q_r).

For our aims we need the constant implicit in the right hand side of (A.4) to be independent of ξ, which is a priori not guaranteed by [1]. Calculating the constants in the proof of [1], Corollary 22.3, however, shows that they can be choosen such that they only depend on the maximal steplength ℓ of G_{t_ξ} and on a lower bound for the smallest eigenvalue of $\mathcal{E}_\xi$ on the other. Therefore the error estimate in (A.4) is uniform on the compact set R_G.

Now we come back to the Edgeworth polynomials. In Q_r, only derivatives of order $r+2, r+4, \dots, 3r$ appear (see [1], Lemma 7.1). The coefficients of Q_r depend on the moments of G_{t_ξ} up to order $r+2$. Since $\xi = x/n$, there is only $Q_r(0)$ appearing in (A.4).

Q_1 and Q_3 vanish at zero, because the odd derivatives of centered normal densities do so. Q_0 is the centered normal density with covariance matrix $\mathcal{E}_\xi$ itself, so Taylor expansion yields $Q_0(0) = (2\pi\eta)^{-d/2} + T^{(2)}(\xi) + O(\xi^4)$. In the Taylor expansion of Q_2, the odd terms vanish likewise, and we obtain $Q_2(0) = K(2\pi\eta)^{-d/2} + O(\xi^2)$, where the constant K and the error term depend only on the moments of G up to order four. Therefore (A.4) simplifies to

$$\begin{aligned}G_{t_\xi}^{*n}(x) &= (2\pi n\eta)^{-d/2}[1 + T^{(2)}(\xi) + O(\xi^4) + n^{-1}T^{(0)}(\xi) + O(n^{-1}\xi^2) + o(n^{-3/2})] \\ &= (2\pi n\eta)^{-d/2}[1 + n^{-1}T^{(2)}(x/\sqrt{n}) + n^{-1}T^{(0)}(x/\sqrt{n}) + O(n^{-3/2})].\end{aligned} \tag{A.5}$$

Inserting (A.3) and (A.5) in (A.2) yields

$$G^{*n}(x) = \varphi_{n\eta}(x) \cdot \left[1 + n^{-1}P_4(x/\sqrt{n}) + O(n^{-3/2})\right],$$

where P_4 is a polynomial of degree four with even order terms only. This yields the desired estimate whenever $\nu' \ge \eta$.

Case $|\xi| \ge n^{-5/12}$*:*

For 'big' x we estimate $G^{*n}(x)$ and $[1 + n^{-1}P_4(x/\sqrt{n})]\varphi_{n\eta}(x)$ separately. Since for fixed natural k we have $|x|^k \exp(-x^2) \le K\exp(-x^2/\sqrt{2})$ for all $x \in \mathbb{Z}^d$, the latter is bounded by $K\varphi_{\sqrt{2}n\eta}(x)$. Using $n^{3/2} \le (x/\sqrt{n})^{18}$ in addition, this yields

$$|1 + n^{-1}P_4(x/\sqrt{n})|\varphi_{n\eta}(x) \le Kn^{-3/2}\varphi_{n\nu'}(x) \tag{A.6}$$

for $\nu' \geq 2\eta$ and for all n.

Now consider $G^{*n}(x)$. If $\xi \notin S_G$, $G^{*n}(x)$ equals zero. If $\xi \in S_G$, we bound $I(\xi)$ away from zero with $\theta\,\xi^2$, using Taylor expansion: We can do this in a neighborhood of zero with the constant $\theta = 1/(2\eta)$. Since S_G is compact (we even have $1/\sqrt{d} \leq |z| \leq \ell$ for all z in the boundary of S_G) and I convex, we have a nonzero minimum of I on ∂S_G and hence we find a constant $\theta > 0$ such that $I(z) \geq \theta z^2$ for all z. Since G is symmetric in each coordinate and rotationally invariant, θ can be choosen depending only on d and the range ℓ, but not on the specific law of G.

If $\xi \in \operatorname{int} S_G$, we use (A.2) to obtain $G^{*n}(x) \leq \exp(-\frac{x^2}{n\theta})$. The arguments leading to (A.6) yield the desired estimate for $\nu' \geq 2\theta$.

If ξ lies in the boundary ∂S_G, use the large deviation principle to obtain

$$G^{*n}(x) \leq G^{*n}(n \cdot \partial S_G) \quad \leq \exp(-n \inf_{z\in\partial S_G} I(z) + n\varepsilon) \leq \exp(-n\theta \inf_{z\in\partial S_G} z^2 + n\varepsilon)$$

for n large enough. Using $\inf_{z\in\partial S_G} z^2 \geq 1/d \geq \xi^2/(d\ell^2)$ and choosing ε small enough we obtain $G^{*n}(x) \leq \exp(-\frac{x^2}{2d\ell^2 n\theta})$. The rest of the argument proceeds as before.

Combining the different bounds and using $\eta \leq \ell^2/d$ we see that (A.1) holds whenever $\nu' \geq \max\{4d\ell^2\theta, 2\ell^2/d\}$. □

Lemma A.2. *Let $\eta > 1/(2d)$ and $G \in \mathcal{M}$ (recall the definition at the beginning of section 3). Then we have for all $x \in \mathbb{Z}^d$:*

$$\text{(A.7a)} \qquad G * \varphi_{n\eta}(x) = g\,\varphi_{n\eta}(x) + \frac{g}{2dn\eta}\Big[\frac{x^2}{n\eta} - d\Big]\varphi_{n\eta}(x) + R_4^G(n,x),$$

where $R_4^G(n,x) = \frac{1}{6}\int_0^1 ds\ (1-s)^3 \sum_{z\in\mathbb{Z}^d} G(z) D_x^4 \varphi_{n\eta}(x-sz)(z,z,z,z)$. If P_{2j} is a fixed polynomial of degree $2j$ for some $j \in \mathbb{N}_0$, then

$$\text{(A.7b)} \qquad G * [P_{2j}(x/\sqrt{n})\varphi_{n\eta}](x) = g\,[P_{2j}(x/\sqrt{n})\varphi_{n\eta}](x) + R_2^G(n,x;P_{2j}),$$

where $R_2^G(n,x;P_{2j}) = \int_0^1 ds\ (1-s)\sum_{z\in\mathbb{Z}^d} G(z) D_x^2[P_{2j}(./\sqrt{n})\varphi_{n\eta}](x-sz)(z,z)$. As local estimates for the error terms we get the following versions, depending on G: If there is a constant ℓ such that $G(x) = 0$ for all $|x| > \ell$, we can estimate the error terms by

$$\text{(A.8)} \qquad \begin{aligned} |R_4^G(n,x)| &\leq K(d,\eta,\ell)\, n^{-2} \sum_{z\in\mathbb{Z}^d} z^4\, |G(z)|\, \varphi_{2n\eta}(x) \quad \textit{and} \\ |R_2^G(n,x;P_{2j})| &\leq K(d,\eta,j,\ell)\, n^{-1} \sum_{z\in\mathbb{Z}^d} z^2\, |G(z)|\, \varphi_{2n\eta}(x). \end{aligned}$$

If no such ℓ exists, we still have the estimates

$$\text{(A.9)} \qquad \begin{aligned} |R_4^G(n,x)| &\leq K(d,\eta)\, n^{-2} \sum_{z\in\mathbb{Z}^d} z^4\, |G(z)| \int_0^1 ds\ \varphi_{\sqrt{2}n\eta}(x-sz) \quad \textit{and} \\ |R_2^G(n,x;P_{2j})| &\leq K(d,\eta,j)\, n^{-1} \sum_{z\in\mathbb{Z}^d} z^2\, |G(z)| \int_0^1 ds\ \varphi_{\sqrt{2}n\eta}(x-sz). \end{aligned}$$

Proof. Using Taylor expansion and the symmetry of G we obtain

$$\begin{aligned} G * \varphi_{n\eta}(x) &= \sum_{z\in\mathbb{Z}^d} G(z)\varphi_{n\eta}(x-z) \\ &= g\varphi_{n\eta}(x) + \frac{g}{2d}\Delta_x\varphi_{n\eta}(x) + R_4^G(n,x), \end{aligned}$$

which is (A.7a) after inserting $\Delta_x\varphi_{n\eta}(x) = \big[\frac{x^2}{n^2\eta^2} - \frac{d}{n\eta}\big]\varphi_{n\eta}(x)$. Analogously, first order Taylor approximation leads to equation (A.7b).

Now we come to the proof of the error estimates. We write $P_{(k)}$ to denote some polynomial of order k. The forth partial derivatives of $\varphi_{n\eta}$ are functions of the form $n^{-2}P_{(4)}(./\sqrt{n})\varphi_{n\eta}$ and therefore bounded by $K(d,\eta)n^{-2}\varphi_{\sqrt{2}n\eta}$. Similarly, the second partial derivatives of $P_{2j}(./\sqrt{n})\varphi_{n\eta}$ are of the form $n^{-1}P_{(2j+2)}(./\sqrt{n})\varphi_{n\eta}$ and bounded by $K(d,\eta,j)\, n^{-2}\varphi_{\sqrt{2}n\eta}$. This implies (A.9).

To prove (A.8), we use the fact that for z with $|z| \leq \ell$ we have $\varphi_{\sqrt{2}n\eta}(x-z) \leq K\varphi_{2n\eta}(x)$ with K depending on d, η and ℓ, but not on n. $\square$

Lemma A.3. *Let $\eta > 0$. Then for $x \in \mathbb{Z}^d$, $n \in \mathbb{N}$ and $k \in (0,n)$:*

$$\varphi_{(n-k)\eta}(x) = \varphi_{n\eta}(x) - \frac{k}{2n}\Big[\frac{x^2}{n\eta} - d\Big]\varphi_{n\eta}(x) + S_2^k(n,x), \tag{A.10a}$$

where $S_2^k(n,x) = \frac{1}{2}k^2\frac{\partial^2}{\partial\theta^2}\varphi_{\theta\eta}(x)$ for a $\theta \in (n-k,n)$. On the other hand, for a fixed polynomial P_{2j} of degree $2j$, $j \in \mathbb{N}_0$:

(A.10b)
$$[(n-k)^{-1}P_{2j}(x/\sqrt{n-k})\varphi_{(n-k)\eta}](x) = n^{-1}P_{2j}(x/\sqrt{n})\varphi_{n\eta}(x) + S_1^k(n,x;n^{-1}P_{2j}),$$

where $S_1^k(n,x;n^{-1}P_{2j}) = -k\frac{\partial}{\partial\theta}P_{2j}(x/\sqrt{\theta})\varphi_{\theta\eta}(x)$ for some $\theta \in (n-k,n)$. Estimates for the error terms are given by

$$\begin{aligned} |S_2^k(n,x)| &\leq K(d,\eta)\,\Big(\frac{k}{n-k}\Big)^2\Big(\frac{n}{n-k}\Big)^{d/2}\varphi_{\sqrt{2}n\eta}(x) \quad \textit{and} \\ |S_1^k(n,x;n^{-1}P_{2j})| &\leq K(d,\eta,j)\,\frac{k}{(n-k)^2}\Big(\frac{n}{n-k}\Big)^{d/2}\varphi_{\sqrt{2}n\eta}(x). \end{aligned} \tag{A.11}$$

Proof. Here we use one dimensional Taylor expansion for $\varphi_{n\eta}(x)$ as a function in n to write

$$\varphi_{(n-k)\eta}(x) = \varphi_{n\eta}(x) - k\frac{\partial}{\partial n}\varphi_{n\eta}(x) + S_2^k(n,x),$$

which implies (A.10a) by using $\frac{\partial}{\partial n}\varphi_{n\eta}(x) = \big[\frac{x^2}{2\eta n^2} - \frac{d}{2n}\big]\varphi_{n\eta}(x)$. Keeping only the constant term of the Taylor approximation leads to (A.10b).

To prove the error estimates, we first observe that the second derivative (with respect to θ) of $\varphi_{\theta\eta}(x)$ is of the form $\theta^{-2}P_{(4)}(x/\sqrt{\theta})\varphi_{\theta\eta}(x)$ and therefore bounded by $K(d,\eta)\theta^{-2}\varphi_{\sqrt{2}\theta\eta}(x)$, while the first derivative of $\theta^{-1}P_{2j}(x/\sqrt{\theta})\varphi_{\theta\eta}(x)$ has the form $\theta^{-2}P_{(2j+2)}(x/\sqrt{\theta})\varphi_{\theta\eta}(x)$ and bounded by $K(d,\eta,j)\theta^{-2}\varphi_{\sqrt{2}\theta\eta}$. Again, $P_{(k)}$ stands for a polynomial of order k.

Splitting the function φ and replacing θ separately by $n-k$ in the first factor and by n in the exponential term leads to (A.11). $\square$

Lemma A.4. *Assume $\eta \geq 1/(2d)$. Then there exists a constant K depending only on the dimension d, such that*

$$\sum_{x\in\mathbb{Z}^d}\varphi_\eta(x) \leq K \qquad \textit{and} \qquad \sum_{x\in\mathbb{Z}^d}x^2\varphi_\eta(x) \leq K\eta.$$

Proof. For $d=1$ we have

$$\sum_{x\in\mathbb{Z}}\varphi_\eta(x) = 2\sum_{n=1}^{\infty}\varphi_\eta(n) + \varphi_\eta(0) \leq 2\int_0^\infty \varphi_\eta(t)dt + \frac{1}{\sqrt{2\pi\eta}} \leq 1 + \frac{1}{\sqrt{2\pi\eta}} \leq K.$$

The first inequality for general dimension follows immediately from this estimate, because the sum over $\mathbb{Z}^d$ of the density values equals the dth power of the sum over $\mathbb{Z}$ of the values of the one dimensional normal density with variance η.

The second inequality follows from the first one by using

$$\sum_{x\in\mathbb{Z}^d} x^2\varphi_\eta(x) \le K\eta \sum_{x\in\mathbb{Z}^d}\varphi_\eta(x),$$

which comes from the fact that $(x^2/\eta)\exp(-x^2/(2\eta))$ can be bounded uniformly in $x\in\mathbb{Z}^d$ by $K\,\exp(-x^2/(4\eta))$. □

Lemma A.5. *Let $\eta,\theta \ge 1/(2d)$. Then for $x\in\mathbb{Z}^d$*

$$\sum_{y\in\mathbb{Z}^d}\varphi_\eta(y)\varphi_\theta(x-y) \le K\varphi_{\eta+\theta}(x).$$

Proof. Let $I^d \stackrel{\text{def}}{=} \left[-\frac{1}{2},\frac{1}{2}\right]^d$, and denote by $y+I^d$ the shifted cube. Then

$$\begin{aligned}
\varphi_{\eta+\theta}(x) &= \sum_{y\in\mathbb{Z}^d}\int_{y+I^d}\varphi_\eta(t)\varphi_\theta(x-t)\,dt\\
&= \frac{1}{(2\pi\eta)^{d/2}(2\pi\theta)^{d/2}}\sum_{y\in\mathbb{Z}^d}\int_{I^d}\exp\big[-\frac{1}{2\eta}(y+t)^2-\frac{1}{2\theta}(x-y-t)^2\big]\,dt\\
&\ge \frac{1}{\left(2\pi\eta\right)^{d/2}\left(2\pi\theta\right)^{d/2}}\sum_{y\in\mathbb{Z}^d}\exp\big[-\int_{I^d}\big(\frac{1}{2\eta}(y+t)^2+\frac{1}{2\theta}(x-y-t)^2\big)\,dt\big]
\end{aligned}$$

by Jensen's inequality. Note that for $t\in I^d$ we have $t^2\le d/4$, so that

$$\begin{aligned}
\int_{I^d}\Big(\frac{1}{2\eta}(y+t)^2+\frac{1}{2\theta}(x-y-t)^2\Big)\,dt &\le \frac{1}{2\eta}y^2+\frac{1}{2\theta}(x-y)^2+\frac{d}{8\eta}+\frac{d}{8\theta}\\
&\le \frac{1}{2\eta}y^2+\frac{1}{2\theta}(x-y)^2+K,
\end{aligned}$$

because we assumed that $\eta,\theta\ge 1/(2d)$. Therefore

$$\varphi_{\eta+\theta}(x)\ge \mathrm{e}^{-K}\sum_{y\in\mathbb{Z}^d}\varphi_\eta(y)\varphi_\theta(x-y).$$ □

Appendix B. The Lace Expansion

This section contains standard material on the lace expansion in the first subsection and bounds for the lace expansion terms in the second one. The lace expansion was introduced by Brydges and Spencer in [2] and discussed in detail by Madras and Slade in [11]. The following overview consists of the minimum necessary to make this thesis selfcontained. The first part is taken more or less literally from van der Hofstad, den Hollander and Slade [9].

B.1. Definition of the Lace Functions.

In this section we define the Lace Functions Π_m and prove the recursion formula (1.4), that is

$$C_n = 2d(D*C_{n-1})+\sum_{m=2}^{n}\Pi_m*C_{n-m}.$$

This requires the introduction of the following standard terminology. Given an interval $I=[a,b]\subset\mathbb{Z}$ of integers with $0\le a\le b$, we refer to a pair $\{s,t\}$ $(s<t)$ of elements of I as an *edge*. To abbreviate the notation, we write st for $\{s,t\}$. A set of edges is called a *graph*. A graph Γ on $[a,b]$ is said to be *connected* if both a and b are endpoints of edges in Γ and if, in addition, for any $c\in[a,b]$ there is an edge $st\in\Gamma$ such that $s<c<t$. The set of all graphs on $[a,b]$ is denoted $\mathcal{B}[a,b]$, and the subset consisting of all connected graphs is denoted $\mathcal{G}[a,b]$. A *lace* is a minimally connected graph, that is, a connected graph for which the removal of any edge would result

in a disconnected graph. The set of laces on $[a,b]$ is denoted $\mathcal{L}[a,b]$, and the set of laces on $[a,b]$ consisting of exactly N edges is denoted $\mathcal{L}^{(N)}[a,b]$.

Given a connected graph Γ, the following prescription associates to Γ an unique lace L_Γ: The lace consists of edges $s_1t_1, s_2t_2, \dots$, with t_1, s_1, t_2, s_2, ... determined (in that order) by

$$\begin{aligned} t_1 &= \max\{t : \, at \in \Gamma\}, & s_1 &= a, \\ t_{i+1} &= \max\{t : \, \exists s < t_i \text{ such that } st \in \Gamma\}, & s_{i+1} &= \min\{s : \, st_{i+1} \in \Gamma\}. \end{aligned}$$

Given a lace L, the set of all edges $st \notin L$ such that $L_{L\cup\{st\}} = L$ is denoted $\mathcal{C}(L)$. Edges in $\mathcal{C}(L)$ are said to be *compatible* with L.

Recall the definition of C_n,

$$C_n(x) = \sum_{\substack{\omega:0\rightsquigarrow x\\|\omega|=n}} \prod_{0\le s<t\le n} (1-\lambda U_{st}(\omega)),$$

where $U_{st}(\omega) = \delta_{\omega(s),\omega(t)}$. Now we define for integers $0 \le a < b$

$$K[a,b](\omega) \stackrel{\text{def}}{=} \prod_{a\le s<t\le b} (1-\lambda U_{st}(\omega)). \tag{B.1}$$

Then we can write

$$C_n(x) = \sum_{\substack{\omega:0\rightsquigarrow x\\|\omega|=n}} K[0,n](\omega), \tag{B.2}$$

where the sum is over all n step simple random walk paths from 0 to x. Expanding the product in the definition of $K[a,b](\omega)$, we get

$$K[a,b](\omega) = \sum_{\Gamma\in\mathcal{B}[a,b]} \prod_{st\in\Gamma} (-\lambda U_{st}(\omega)). \tag{B.3}$$

We also define an analogous quantity, in which the sum over graphs is restricted to connected graphs, namely,

$$J[a,b](\omega) \stackrel{\text{def}}{=} \sum_{\Gamma\in\mathcal{G}[a,b]} \prod_{st\in\Gamma} (-\lambda U_{st}(\omega)). \tag{B.4}$$

This allows us to define the 'lace functions', which are the key quantities in the lace expansion,

$$\Pi_m(x) = \sum_{\substack{\omega:0\rightsquigarrow x\\|\omega|=m}} J[0,m](\omega). \tag{B.5}$$

The identity (1.4) now follows from the following lemma.

Lemma B.1. *For* $n \ge 1$,

$$C_n(x) = \sum_{y:\|y\|_1=1} C_{n-1}(x-y) + \sum_{m=2}^{n} \sum_{z\in\mathbb{Z}^d} \Pi_m(z)\, C_{n-m}(x-z). \tag{B.6}$$

Proof. It suffices to show that for each path ω we have (suppressing ω in the formulas):

$$K[0,n] = K[1,n] + \sum_{m=2}^{n} J[0,m]\, K[m,n]. \tag{B.7}$$

Then (B.6) is obtained after insertion of (B.7) into (B.2) followed by factorization of the sum over ω. To prove (B.7), we note from (B.3) that the contribution to $K[0,n]$ from all graphs Γ for which 0 is not in an edge is exactly $K[1,n]$. To resum the contribution from the remaining graphs, we proceed as follows. When Γ does contain an edge ending at 0, we let $m[\Gamma]$ denote the largest value of m such that the set of edges in Γ with at least one end in the interval $[0,m]$ forms a connected graph on $[0,m]$. We lose nothing by taking $m \geq 2$, since $U_{a,a+1} = 0$ for all a. Then resummation over graphs on $[m,n]$ gives

$$K[0,n] = K[1,n] + \sum_{m=2}^{n} \sum_{\Gamma \in \mathcal{G}[0,m]} \prod_{st\in\Gamma} (-\lambda U_{st})\, K[m,n].$$

With (B.4) this proves (B.7). □

We next rewrite (B.5) in a form that can be used to obtain good bounds on $\Pi_m(x)$. For this, we begin by partially resumming the right-hand side of (B.4), to obtain

$$\begin{aligned} J[a,b] &= \sum_{L\in\mathcal{L}[a,b]} \sum_{\Gamma:\, L_\Gamma = L} \prod_{st\in L} (-\lambda U_{st}) \prod_{s't'\in\Gamma\setminus L} (-\lambda U_{s't'}) \\ &= \sum_{L\in\mathcal{L}[a,b]} \prod_{st\in L} (-\lambda U_{st}) \prod_{s't'\in\mathcal{C}(L)} (1-\lambda U_{s't'}). \end{aligned} \tag{B.8}$$

For $0 \leq a < b$, we define $J^{(N)}[a,b]$ to be, up to the factor $(-\lambda)^N$, the contribution to (B.8) coming from laces consisting of exactly N edges,

$$J^{(N)}[a,b] \stackrel{\text{def}}{=} \sum_{L\in\mathcal{L}^{(N)}[a,b]} \prod_{st\in L} U_{st} \prod_{s't'\in\mathcal{C}(L)} (1-\lambda U_{s't'}), \qquad N \geq 1.$$

Then

$$J[a,b] = \sum_{N=1}^{\infty} (-\lambda)^N J^{(N)}[a,b],$$

and by (B.5)

$$\Pi_m(x) = \sum_{N=1}^{\infty} (-\lambda)^N \Pi_m^{(N)}(x), \tag{B.9}$$

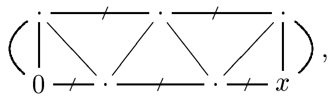
where we define

$$\begin{aligned} \Pi_m^{(N)}(x) &\stackrel{\text{def}}{=} \sum_{\substack{\omega:0\rightsquigarrow x\\ |\omega|=m}} J^{(N)}[0,m](\omega) \\ &= \sum_{\substack{\omega:0\rightsquigarrow x\\ |\omega|=m}} \sum_{L\in\mathcal{L}^{(N)}[0,m]} \prod_{st\in L} U_{st} \prod_{s't'\in\mathcal{C}(L)} (1-\lambda U_{s't'}). \end{aligned} \tag{B.10}$$

B.2. **Bounds on the Lace Functions.** In this section, we obtain bounds on $\Pi_m(x)$. A first part will provide the standard bounds in terms of $C_n(x)$, the two-point functions of the appropriate weakly self avoiding walk after n steps. In the second part we will obtain specific bounds by assuming a Gaussian decay of the $C_n(x)$. In this case good bounds result easily from the Cauchy-Schwarz inequality. There is only one lace on $[0,m]$ consisting of exactly one edge, namely $\{0m\}$. Therefore we have for $N=1$:

$$\begin{aligned} \Pi_m^{(1)}(x) &= \delta_{0x} \sum_{\substack{\omega:0\rightsquigarrow 0\\ |\omega|=m}} \prod_{\substack{0\leq s'<t'\leq m\\ s't'\neq 0m}} (1-\lambda U_{s't'}(\omega)) \\ &\leq \delta_{0x} \sum_{y:\, \|y\|_1=1} \sum_{\substack{\omega:y\rightsquigarrow 0\\ |\omega|=m-1}} K[1,m](\omega) \qquad \text{using } 1-U_{0t'} \leq 1 \quad \forall\, t' \text{ and (B.1)} \\ &= \delta_{0x}\, 2dD * C_{m-1}(0) \qquad \text{by (B.2).} \end{aligned} \tag{B.11}$$

Now for $N \geq 2$: A walk giving a nonzero contribution to (B.10) must intersect itself N times, to ensure that $U_{st} \neq 0$ for each $st \in L$. For example, when $N = 7$, the walk must undergo a trajectory of the form



where the slashed lines denote subwalks that may have length zero, whereas non-slashed lines denote subwalks containing at least one step.

Using $1 - \lambda U_{s't'} \leq 1$ in (B.10) whenever s' and t' belong to different subwalks, we get an upper bound in which distinct subwalks no longer interact. However, each subwalk remains weakly self-avoiding. We thus have

$$\text{(B.12)}\quad \Pi_m^{(N)}(x) \leq \sum_{y_i, m_i} C_{m_1}(y_1) C_{m_2}(y_1) C_{m_3}(y_2) C_{m_4}(y_1 - y_2) C_{m_5}(y_3 - y_1) \cdots$$
$$\cdots C_{m_{2N-3}}(x - y_4) C_{m_{2N-2}}(y_5 - x) C_{m_{2N-1}}(x - y_5),$$

where the sum is over $y_1, \ldots, y_{N-2} \in \mathbb{Z}^d$ and over $m_1, \ldots, m_{2N-1}$ with $\sum m_i = m$. The m_i are nonnegative integers, and only $m_3, m_5, \ldots, m_{2N-3}$ can equal zero.

Lemma B.2. *Fix $m \geq 2$ and $d \geq 5$. Assume that for all $x \in \mathbb{Z}^d$ and natural $n < m$, we have*

$$\text{(B.13)}\qquad |C_n(x)| \leq L_1 \mu^n \varphi_{n\nu}(x)$$

with constants $\mu > 0$, $L_1 \geq 1$ and $\nu \geq 1/(2d)$. Then, for $\lambda = \lambda(d, L_1)$ small enough, we have

$$\text{(B.14)}\qquad |\Pi_m(x)| \leq \lambda K L_1 \mu^m m^{-d/2} \sum_{k=1}^{m/2} k^{1-d/2}\, \varphi_{k\nu}(x).$$

Proof. Again we abbreviate the notation by writing

$$\psi_m(x) \stackrel{\text{def}}{=} m^{-d/2} \sum_{k=1}^{m/2} k^{1-d/2} \varphi_{k\nu}(x).$$

We consider the terms $\Pi_m^{(N)}$ separately. For $N = 1$, equation (B.11) yields

$$\text{(B.15)}\qquad \Pi_m^{(1)}(x) \leq \delta_{0x}\, 2dL_1\mu^m\, D * \varphi_{(m-1)\nu}(0) \leq KL_1\mu^m\, \psi_m(x),$$

since $D * \varphi_{(m-1)\nu}(0) \leq K\varphi_{2(m-1)\nu}(0) \leq Km^{-d/2}\varphi_\nu(0)$.

To keep the notation simple, we set $\varphi_0(x) \stackrel{\text{def}}{=} \delta_{0,x}$. For $N \geq 2$ we define

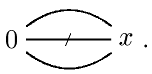
$$P_m^{(N)}(x) \stackrel{\text{def}}{=} \sum_{y_i, m_i} \varphi_{m_1\nu}(y_1) \varphi_{m_2\nu}(y_1) \varphi_{m_3\nu}(y_2) \varphi_{m_4\nu}(y_1 - y_2) \varphi_{m_5\nu}(y_3 - y_1) \cdots$$
$$\cdots \varphi_{m_{2N-3}\nu}(x - y_4) \varphi_{m_{2N-2}\nu}(y_5 - x) \varphi_{m_{2N-1}\nu}(x - y_5),$$

with the same summation as in (B.12) up to the fact that we allow an additional term for $m_2 = 0$ (this corresponds to slashing the line from zero vertically up and is necessary to give the induction below). By (B.13), we have for all n:

$$\text{(B.16)}\qquad \Pi_m^{(N)} \leq L_1^{2N-1}\, \mu^m\, P_m^{(N)}.$$

Now we show by induction that there is a constant L_2 depending only on the dimension d such that

$$\text{(B.17)}\qquad |P_m^{(N)}| \leq L_2^N \psi_m.$$

For $N = 2$ the lace diagram is three-legged:  . We have

$$P_m^{(2)}(x) = \sum_{\substack{k+l+j=m \\ l,j\geq 1}} \varphi_{k\nu}(x)\varphi_{l\nu}(x)\varphi_{j\nu}(x) = \delta_{0,x} I + J,$$

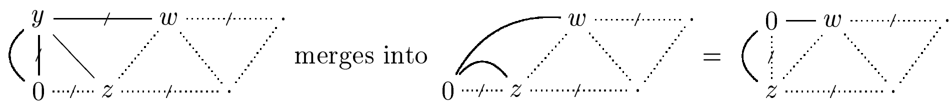
where

$$\begin{aligned} I &= \sum_{l=1}^{m-1} \varphi_{l\nu}(0)\varphi_{(m-l)\nu}(0) \leq K(2\pi\nu)^{-d} \sum_{l=1}^{m-1} l^{-d/2}(m-l)^{-d/2} \\ &\leq K \underbrace{(2\pi\nu)^{-d/2}}_{=\varphi_\nu(0)} m^{-d/2} \leq K\psi_m(0) \\ \text{and} \quad J &\leq K \sum_{\substack{k+l+j=m \\ 1\leq k\leq l\leq j}} \varphi_{k\nu}(x)\, l^{-d/2}\, m^{-d/2} \\ &\leq K m^{-d/2} \sum_{k=1}^{m/3} \varphi_{k\nu}(x) \sum_{l=k}^{m} l^{-d/2} \leq K\psi_m(x). \end{aligned}$$

So together we obtain

$$P_m^{(2)}(x) \leq K\psi_m(x). \tag{B.18}$$

Now we come to the induction step. For $N \geq 3$ we will reduce $P_m^{(N)}$ to $P_m^{(N-1)}$ by merging four subwalks in the lace into two. The following figure illustrates this process:



We use Cauchy-Schwarz to obtain

$$\begin{aligned} &\sum_{\substack{u_1+u_2=u \\ u_1\geq 1}} \sum_{\substack{t_1+t_2=t \\ t_2\geq 1}} \sum_{y\in\mathbb{Z}^d} \varphi_{u_1\nu}(y)\varphi_{u_2\nu}(w-y)\varphi_{t_1\nu}(y)\varphi_{t_2\nu}(y-z) \\ &\leq \sum_{\substack{u_1+u_2=u \\ t_1+t_2=t \\ u_1,t_2\geq 1}} \Big[\sum_{y\in\mathbb{Z}^d} \varphi^2_{u_1\nu}(y)\varphi^2_{u_2\nu}(w-y)\Big]^{1/2} \Big[\sum_{y\in\mathbb{Z}^d} \varphi^2_{t_1\nu}(y)\varphi^2_{t_2\nu}(y-z)\Big]^{1/2}. \end{aligned} \tag{B.19}$$

Note that for $u_2 \geq 1$ we have

$$\big[\varphi^2_{u_1\nu} * \varphi^2_{u_2\nu}\big]^{1/2} \leq K\nu^{-d/2}(u_1u_2)^{-d/4}[\varphi_{u\nu/2}]^{1/2} \leq K(u_1u_2)^{-d/4}u^{d/4}\varphi_{u\nu},$$

and therefore (recall $d \geq 5$)

$$\sum_{\substack{u_1+u_2=u \\ u_1\geq 1}} \big[\varphi^2_{u_1\nu} * \varphi^2_{u_2\nu}\big]^{1/2} \leq \big(1 + u^{d/4} \sum_{u_1=1}^{u-1} u_1^{-d/4}(u-u_1)^{-d/4}\big)\,\varphi_{u\nu} \leq K\varphi_{u\nu}.$$

Inserting this into (B.19) yields

$$\sum \varphi_{u_1\nu}(y)\varphi_{u_2\nu}(w-y)\varphi_{t_1\nu}(y)\varphi_{t_2\nu}(y-z) \leq K\varphi_{u\nu}(w)\,\varphi_{t\nu}(z). \tag{B.20}$$

Using (B.20) with y_1, y_2, y_3, m_1, m_5, m_2 and m_4 in place of y, z, w, u_1, u_2, t_1 and t_2, respectively, gives

$$\begin{aligned} P_m^{(N)}(x) &= \sum_{y_i, m_i} \varphi_{m_2\nu}(y_1)\varphi_{m_4\nu}(y_1 - y_2)\varphi_{m_1\nu}(y_1)\varphi_{m_5\nu}(y_3 - y_1)\varphi_{m_3\nu}(y_2)\cdots \\ &\leq K \sum \varphi_{(m_2+m_4)\nu}(y_2)\varphi_{m_3\nu}(y_2)\varphi_{(m_1+m_5)\nu}(y_3)\cdots \\ &\leq K P_m^{(N-1)}(x). \qquad \text{(B.21)} \end{aligned}$$

Now we choose L_2 to be the maximum of the constants appearing in (B.18) and (B.21). We obtain (B.17). Now (B.15) and (B.16) together imply

$$\Pi_m^{(N)} \leq K L_2^N \, L_1^{2N-1} \, \mu^m \, \psi_m. \qquad \text{(B.22)}$$

The lemma now follows by inserting (B.22) into (B.9) and choosing λ small enough. □

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