Aharonov–Bohm Effect in Scattering by a Chain of Point–like Magnetic Fields

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Abstract According to the Aharonov–Bohm effect, magnetic potentials have a direct significance to the motion of particles in quantum mechanics. We study this quantum effect through the scattering by several point–like magnetic fields at large separation in two dimensions. We derive the asymptotic formula for scattering amplitudes as the distances between centers of fields go to infinity. The result heavily depends on the location of centers. A special emphasis is placed on the case of scattering by fields with centers on an even line. The obtained formula depends on fluxes of fields and on ratios of distances between adjacent centers. We figure the approximate values of differential cross sections to see how the pattern of interferences changes with flux parameters.

1. Introduction

We study the magnetic scattering by several point–like fields at large separation in two dimensions. The aim is to derive the asymptotic formula for scattering amplitudes as the distances between centers of fields go to infinity. Even if a magnetic field is compactly supported, the corresponding vector potential does not necessarily fall off rapidly, and in general, it has the long–range property at infinity. We discuss from a mathematical point of view how the scattering by separate fields interacts with one another through long-range magnetic potentials. A special emphasis is placed on the case of scattering by fields with centers on an even line. The obtained result depends on fluxes of fields and on ratios of distances between adjacent centers. It is known as the Aharonov–Bohm effect ([3]) that magnetic potential has a direct significance to the motion of quantum particles. An extensive list of physical literatures on the Aharonov–Bohm scattering can be found in the book [2]. We refer to the recent article [9] for the Aharonov–Bohm effect in many point–like fields, where the term *magnetic vortex* is used for point–like magnetic field. We also note that a similar problem has been studied by [8] in the case of scattering by short–range scalar potentials. We are going to make a further comment on the result obtained there at the end of the section.

Throughout the whole exposition, we work in the two dimensional space $\boldsymbol{R}^2$ with generic point $x = (x_1, x_2)$. We write

$$H(A) = (-i\nabla - A)^2 = \sum_{j=1}^{2}(-i\partial_j - a_j)^2, \qquad \partial_j = \partial/\partial x_j,$$

for the magnetic Schrödinger operator with vector potential $A(x) = (a_1(x), a_2(x)) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$. The magnetic field $b(x)$ is defined as

$$b = \nabla \times A = \partial_1 a_2 - \partial_2 a_1$$

and the quantity $\alpha = (2\pi)^{-1} \int b(x)\, dx$ is called the total flux of field b, where the integration with no domain attached is taken over the whole space. We often use this abbreviation.

We first consider the case of a single point–like field. The Hamiltonian with such a field is regarded as one of solvable models and the explicit representation for scattering amplitude has been already obtained by [2, 3, 11]. We are going to discuss the subject in some detail in section 2. Let $2\pi\alpha\delta(x)$ be the magnetic field with flux α and center at the origin. Then the magnetic potential $A_\alpha(x)$ associated with the field is given by

$$A_\alpha(x) = \alpha\,(-\partial_2 \log|x|, \partial_1 \log|x|) = \alpha\left(-x_2/|x|^2, x_1/|x|^2\right). \tag{1.1}$$

In fact, we can easily see that

$$\nabla \times A_\alpha = \alpha\,\Delta \log|x| = 2\pi\alpha\delta(x).$$

We should note that $A_\alpha(x)$ does not fall off rapidly at infinity and it has the long–range property. We write $H_\alpha = H(A_\alpha)$. The potential $A_\alpha(x)$ has a strong singularity at the origin, so that H_α is not necessarily essentially self–adjoint in $C_0^\infty(\boldsymbol{R}^2 \setminus \{0\})$ ([1, 4]). We have to impose some boundary conditions at the origin to define H_α as a self–adjoint extension in $L^2 = L^2(\boldsymbol{R}^2)$. We denote by the same notation H_α the operator with domain

$$\mathcal{D}(H_\alpha) = \{u \in L^2 : H(A_\alpha)u \in L^2,\ \lim_{|x|\to 0} |u(x)| < \infty\},$$

where $H(A_\alpha)u$ is understood in $\mathcal{D}'$ (distribution sense). Then H_α is known to be self–adjoint in L^2 and this operator is called the Aharonov–Bohm Hamiltonian. If, in particular, $\alpha \notin \boldsymbol{Z}$ is not an integer, $u \in \mathcal{D}(H_\alpha)$ is convergent to zero as $|x| \to 0$. As stated above, the amplitude $f_\alpha(\omega \to \tilde{\omega}; E)$ for the scattering from initial direction $\omega \in S^1$ to final one $\tilde{\omega}$ at energy $E > 0$ has been already calculated. If we identify the coordinates over the unit circle S^1 with the azimuth angles from the positive x_1 axis, then $f_\alpha(\omega \to \tilde{\omega}; E)$ is explicitly represented as

$$f_\alpha = c(E)\Big((\cos\alpha\pi - 1)\delta(\tilde{\omega} - \omega) - (i/\pi)\sin\alpha\pi\, e^{i[\alpha](\tilde{\omega}-\omega)} F_0(\tilde{\omega} - \omega)\Big) \tag{1.2}$$

with $c(E) = (2\pi/i\sqrt{E})^{1/2}$, where the Gauss notation $[\alpha]$ denotes the maximal integer not exceeding α, and $F_0(\theta)$ is defined by $F_0 = \text{v.p.}\, e^{i\theta}/(e^{i\theta} - 1)$.

We move to the scattering by point–like field supported on N points $d_j \in \boldsymbol{R}^2$, $1 \le j \le N$. We make large the distance $|d_{jk}| = |d_k - d_j|$ between centers d_j and d_k under the assumption that

$$\text{the direction } \widehat{d}_{jk} = d_{jk}/|d_{jk}| \text{ remains fixed} \tag{1.3}$$

for all pairs (j,k) with $j \ne k$, $1 \le j, k \le N$. We further assume that

$$\max |d_{jk}| \le c \min |d_{jk}| \tag{1.4}$$

for some $c > 1$. These two assumptions on the location of centers are always assumed to be fulfilled. By translation, we may assume d_1 to remain fixed, so that all the centers are in a disk $\{|x| < c\,d\}$ with another $c > 1$, where $d = \min |d_{jk}|$. We write

$$H_d = H(A_d), \qquad A_d(x) = \sum_{j=1}^{N} A_j(x) = \sum_{j=1}^{N} A_{\alpha_j}(x - d_j), \tag{1.5}$$

for the Schrödinger operator with field $\sum_{j=1}^{N} 2\pi\alpha_j \delta(x - d_j)$, where $A_\alpha(x)$ is defined by (1.1). According to the results in [7], H_d becomes a self–adjoint operator with domain

$$\mathcal{D}(H_d) = \{u \in L^2 : H(A_d)u \in L^2, \ \lim_{|x-d_j| \to 0} |u(x)| < \infty, \ 1 \le j \le N\}.$$

We also know from [7] that the wave operators

$$W_\pm(H_d, H_0) = s - \lim_{t \to \pm\infty} \exp(itH_d)\exp(-itH_0)$$

exist and are asymptotically complete, where $H_0 = -\Delta$ is the free Hamiltonian. We denote by $f_d(\omega \to \tilde{\omega}; E)$ the scattering amplitude of pair (H_d, H_0). The aim is to analyze the asymptotic behavior as $d \to \infty$ of $f_d(\omega \to \tilde{\omega}; E)$.

We fix the notation to state the obtained results. We denote by $\gamma(\hat{x}; \omega)$, $\hat{x} = x/|x|$, the azimuth angle from direction $\omega \in S^1$. Let $A_j(x)$, $1 \le j \le N$, be as in (1.5) and set

$$H_j = H(A_j), \qquad 1 \le j \le N. \tag{1.6}$$

The operator H_j admits a self–adjoint realization under the boundary condition $\lim_{|x-d_j| \to 0} |u(x)| < \infty$ at center $x = d_j$ and the scattering amplitude of pair (H_j, H_0) is given by

$$f_j(\omega \to \tilde{\omega}; E) = \exp(-i\sqrt{E} d_j \cdot (\tilde{\omega} - \omega)) f_{\alpha_j}(\omega \to \tilde{\omega}; E),$$

where f_α is defined by (1.2). The first main theorem is formulated as follows.

Theorem 1.1 *Let the notation be as above. Assume* (1.3) *and* (1.4). *If* $\omega \neq \hat{d}_{jk}$ *and* $\tilde{\omega} \neq \hat{d}_{jk}$ *for all pairs* (j,k) *with* $j \neq k,\ 1 \leq j,\ k \leq N$, *and* $\omega \neq \tilde{\omega}$, *then* $f_d(\omega \to \tilde{\omega}; E)$ *obeys*

$$f_d(\omega \to \tilde{\omega}; E) = \sum_{j=1}^{N} \exp\left(i(\tau_j - \tilde{\tau}_j)\right) f_j(\omega \to \tilde{\omega}; E) + o(1), \quad d \to \infty,$$

where $\tau_j = \sum_{k=1,\ k\neq j}^{N} \alpha_k \gamma(\widehat{d}_{kj}; \omega)$ *and* $\tilde{\tau}_j = \sum_{k=1,\ k\neq j}^{N} \alpha_k \gamma(\widehat{d}_{kj}; -\tilde{\omega})$.

We can find the Aharonov–Bohm effect in the theorem above. As is seen from the asymptotic formula, the scattering by field $2\pi\alpha_j\delta(x - d_j)$ is influenced by other fields through the coefficient $\exp(i\tau_j)$, although the centers of fields are far away from one another. This means that vector potentials have a direct significance to quantum particles moving in magnetic fields. The magnetic effect is more strongly reflected in the case when $\omega = \hat{d}_{jk}$ or $\tilde{\omega} = \hat{d}_{jk}$. We add the new notation. We interpret $\exp(i\alpha\gamma(\omega;\omega))$ as

$$\exp(i\alpha\gamma(\omega;\omega)) := (1 + \exp(i2\alpha\pi))/2 = \cos\alpha\pi \exp(i\alpha\pi).$$

Then the same asymptotic formula as in Theorem 1.1 can be shown to remain true even for $\omega = \hat{d}_{kj}$ or $\tilde{\omega} = \hat{d}_{kj}$ under the assumption that

$$\text{there is no other center on } l_{jk} \text{ for all pairs } (j,k), \tag{1.7}$$

where l_{jk} is the line joining the two centers d_j and d_k. We do not intend to prove this result here. We have studied the case $N = 2$ in [7]. If $N = 2$, (1.7) is automatically satisfied. For example, we have obtained that the backward scattering amplitude obey

$$f_d(\omega \to -\omega; E) = f_1(\omega \to -\omega; E) + (\cos\alpha_1\pi)^2 f_2(\omega \to -\omega; E) + o(1) \tag{1.8}$$

for $\omega = \widehat{d}_{12}$.

Our emphasis is placed on the case without (1.7). As a typical case, we study the scattering by point–like fields with centers on an even line. For brevity, we confine ourselves to the simple case $N = 3$. The argument extends to the general case $N \geq 4$. What is interesting is that the asymptotic formula depends not only on fluxes of fields but also on ratios of distances between adjacent centers. We assume that three centers are along the direction $\omega_1 = (1,0)$ in the order of $d_1,\ d_2$ and d_3. We further assume that

$$\text{the ratio } |d_{23}|/|d_{12}| = \delta_0 \text{ remains fixed} \tag{1.9}$$

for some $\delta_0 > 0$. This assumption can be weakened as $\lim_{d\to\infty} |d_{23}|/|d_{12}| = \delta_0$. We define $\theta_\pm\pi$ as

$$\theta_\pm\pi = \text{ angle between two vectors } (0, \pm 1) \text{ and } (1, -\delta_0^{1/2}). \tag{1.10}$$

It is obvious that $\theta_+ + \theta_- = 1$ and $0 < \theta_- < \theta_+ < 1$. If, for example, three centers are at even intervals, then $\delta_0 = 1$, so that $\theta_\pm$ are determined as $\theta_+ = 3/4$ and $\theta_- = 1/4$. We are now in a position to state the second main theorem.

Theorem 1.2 *Let the notation be as above. Assume that* (1.9) *is satisfied. If $\tilde{\omega} \neq \pm\omega_1$ for the incident direction $\omega_1 = (1,0)$, then $f_d = f_d(\omega_1 \to \tilde{\omega}; E)$ behaves like*

$$\begin{aligned} f_d &= e^{i\alpha_2(\pi-\gamma(-\omega_1;-\tilde{\omega}))} e^{i\alpha_3(\pi-\gamma(-\omega_1;-\tilde{\omega}))} f_1 \\ &+ (\cos\alpha_1\pi)\, e^{i\alpha_1(\pi-\gamma(\omega_1;-\tilde{\omega}))} e^{i\alpha_3(\pi-\gamma(-\omega_1;-\tilde{\omega}))} f_2 \\ &+ (\theta_+ \cos(\alpha_1+\alpha_2)\pi + \theta_- \cos(\alpha_1-\alpha_2)\pi)\, e^{i\alpha_1(\pi-\gamma(\omega_1;-\tilde{\omega}))} e^{i\alpha_2(\pi-\gamma(\omega_1;-\tilde{\omega}))} f_3 + o(1) \end{aligned}$$

as $d \to \infty$, where $f_j = f_j(\omega_1 \to \tilde{\omega}; E)$ for $1 \le j \le 3$. Moreover the backward scattering amplitude $f_d(\omega_1 \to -\omega_1; E)$ obeys

$$f_d = f_1 + (\cos\alpha_1\pi)^2 f_2 + (\theta_+ \cos(\alpha_1+\alpha_2)\pi + \theta_- \cos(\alpha_1-\alpha_2)\pi)^2 f_3 + o(1)$$

with $f_j = f_j(\omega_1 \to -\omega_1; E)$.

We make several comments on the two theorems above.

Remark 1.1 The quantity $|f_d(\omega_1 \to \tilde{\omega}; E)|^2$ is called the differential cross section. We figure the approximate values of cross sections obtained from the asymptotic formula on the right side in the appendix at the end of the paper and we see how the pattern of interferences changes with three flux parameters α_1, α_2 and α_3.

Remark 1.2 The idea in the proof of Theorem 1.2, in principle, enables us to prove Theorem 1.1 without assuming that $\omega \neq \hat{d}_{kj}$ and $\tilde{\omega} \neq \hat{d}_{kj}$. For example, it is possible to extend Theorem 1.2 to the case of scattering by several chains of point–like fields. However the formula takes a rather complicated form and we do not have yet obtained a unified form of representation.

Remark 1.3 If we make a change of variables $x \to dy$, then Theorems 1.1 and 1.2 can be easily seen to yield the asymptotic behavior at high energy of scattering amplitudes when the distances between centers of fields remain fixed.

We prove Theorem 1.1 in section 4 and Theorem 1.2 in sections 6 and 7. We write $R(z;H) = (H-z)^{-1} : L^2 \to L^2$, $\operatorname{Im} z \neq 0$, for the resolvent of self–adjoint operator H. We know ([7, Propositions 7.2 and 7.3]) that H_d has no bound states and that the boundary values to the positive axis

$$R(E \pm i0; H_d) = \lim_{\varepsilon \downarrow 0} R(E \pm i\varepsilon; H_d) : L^2_s(\boldsymbol{R}^2) \to L^2_{-s}(\boldsymbol{R}^2) \tag{1.11}$$

exist as a bounded operator from the weighted L^2 space $L^2_s(\boldsymbol{R}^2) = L^2(\boldsymbol{R}^2; \langle x\rangle^{2s}\, dx)$ into $L^2_{-s}(\boldsymbol{R}^2)$ for $s > 1/2$, where $\langle x\rangle = (1+|x|^2)^{1/2}$. We take $0 < \sigma \ll 1$ small enough and denote by $s_j(x)$ the characteristic function of set

$$S_j = \{x \in \boldsymbol{R}^2 : |x - d_j| < Cd^\sigma\}, \quad 1 \le j \le N, \tag{1.12}$$

with $C > 1$. The proof of the main theorems is based on the resolvent estimate

$$\|s_j R(E \pm i0; H_d) s_k\| = O(d^{-1/2+c\sigma}) \tag{1.13}$$

for $j \neq k$, where $\| \quad \|$ denotes the norm of bounded operators acting on L^2. This implies

$$\|s_j \left(R(E \pm i0; H_d) - R(E \pm i0; H_j)\right) s_j\| = O(d^{-1/2+c\sigma})$$

and enables us to split the amplitude $f_d(\omega \to \tilde{\omega}; E)$ into the sum of amplitudes $f_j(\omega \to \tilde{\omega}; E)$ of $H_j = H(A_j)$ as in the theorems.

As stated at the beginning, the work [8] has studied the same problem in the case of potential scattering for the operator $-\Delta + \sum_{j=1}^{N} V_j(x - d_j)$ with potentials falling off rapidly at infinity. The obtained result is that the scattering amplitude

$$f_d(\omega \to \tilde{\omega}; E) = \sum_{j=1}^{N} f_j(\omega \to \tilde{\omega}; E) + o(1)$$

is completely split into the sum of amplitudes $f_j(\omega \to \tilde{\omega}; E)$ corresponding to potentials $V_j(\cdot - d_j)$, and we do not have to modify the phase factors. A new difficulty arises in the case of magnetic scattering. Roughly speaking, this is due to the long–range property of magnetic potentials and several new devices are required to overcome such a difficulty. We work in the phase space and the microlocal analysis plays an important role in proving the theorems. We conclude the section by stating that in the scattering by point–like magnetic fields, the fields interact with one another through long–range magnetic potentials by the Aharonov–Bohm effect, although the trapping effect between fields is weak, as is seen from resolvent estimate (1.13).

2. Representation for scattering amplitude

In this section we study the scattering theory for the magnetic Schrödinger operator with point–like field supported on a single point. The aim here is to represent the scattering amplitude in terms of resolvent.

We begin by making a brief review on the results from [3, 7, 11]. Let A_α be defined by (1.1). This has the δ–like field $2\pi\alpha\delta(x)$ at the origin. We consider the Hamiltonian $H_\alpha = H(A_\alpha)$, which is self–adjoint with domain

$$\mathcal{D}(H_\alpha) = \{u \in L^2 : H_\alpha u \in L^2, \ \lim_{|x| \to 0} |u(x)| < \infty\}.$$

We know that the wave operators

$$W_\pm(H_\alpha, H_0) = s - \lim_{t \to \pm\infty} \exp(itH_\alpha)\exp(-itH_0) : L^2 \to L^2$$

exist and are asymptotically complete : $\operatorname{Ran} W_\pm(H_\alpha, H_0) = L^2$. Hence the scattering operator

$$S(H_\alpha, H_0) = W_+^*(H_\alpha, H_0) W_-(H_\alpha, H_0) : L^2 \to L^2$$

can be defined as a unitary operator. Let

$$\varphi_0(x;\lambda,\omega)=\exp(i\sqrt{\lambda}\,x\cdot\omega),\quad \lambda>0,\ \omega\in S^1,$$

be the generalized eigenfunction of the free Hamiltonian $H_0=-\Delta$, where the notation $\cdot$ denotes the scalar product in $\boldsymbol{R}^2$. The unitary mapping

$$(Fu)(\lambda,\omega)=2^{-1/2}(2\pi)^{-1}\int\bar{\varphi}_0(x;\lambda,\omega)u(x)\,dx : L^2\to L^2((0,\infty)\times S^1) \tag{2.1}$$

decomposes $S(H_\alpha,H_0)$ into the direct integral

$$S(H_\alpha,H_0)\simeq FS(H_\alpha,H_0)F^*=\int_0^\infty \oplus S(\lambda;H_\alpha,H_0)\,d\lambda,$$

where the fibre $S(\lambda;H_\alpha,H_0) : L^2(S^1)\to L^2(S^1)$ is called the scattering matrix at energy λ, and it acts as

$$\left(S(\lambda;H_\alpha,H_0)(Fu)(\lambda,\,\cdot\,)\right)(\omega)=\left(FS(H_\alpha,H_0)u\right)(\lambda,\omega)$$

for $u\in L^2$. If we denote by $S_\alpha(\tilde{\omega},\omega;\lambda)$ the integral kernel of $S(\lambda;H_\alpha,H_0)$, then the scattering amplitude $f_\alpha(\omega\to\tilde{\omega};E)$ for the scattering from initial direction ω into final one $\tilde{\omega}$ at energy $E>0$ is defined by

$$f_\alpha(\omega\to\tilde{\omega};E)=c(E)\left(S_\alpha(\tilde{\omega},\omega;E)-\delta(\tilde{\omega}-\omega)\right), \tag{2.2}$$

where $c(E)=(2\pi/i\sqrt{E})^{1/2}$. The amplitude $f_\alpha(\omega\to\tilde{\omega};E)$ can be explicitly calculated and it is represented as (1.2) by identifying the coordinates over S^1 with the azimuth angles from the positive x_1 axis. By (1.11), we know that the boundary values $R(E\pm i0;H_\alpha)$ to the positive real axis exist as an operator from the weighted L^2 space $L^2_s(\boldsymbol{R}^2)$ into $L^2_{-s}(\boldsymbol{R}^2)$ for $s>1/2$. We represent $f_\alpha(\omega\to\tilde{\omega};E)$ in terms of resolvent $R(E+i0;H_\alpha)$.

Let $\chi\in C_0^\infty[0,\infty)$ be a smooth cut–off function such that $\chi(s)\ge 0$ and

$$\chi(s)=1\quad\text{for}\ 0\le s\le 1,\qquad \chi(s)=0\quad\text{for}\ s>2. \tag{2.3}$$

We now fix energy $E>0$ and initial direction $\omega\in S^1$. We define

$$\beta(\xi)=\chi(|\xi-\sqrt{E}\omega|/\delta^2)$$

for δ, $0<\delta\ll 1$, small enough. We further take a nonnegative function $e_0\in C^\infty(\boldsymbol{R}^2)$ such that

$$\operatorname{supp} e_0\subset\Sigma(R,\omega,\delta),\qquad e_0=1\ \text{ on }\ \Sigma(2R,\omega,2\delta), \tag{2.4}$$

and $\partial_x^\beta e_0(x)=O(|x|^{-|\beta|})$ at infinity, where

$$\Sigma(R,\omega,\delta)=\{x:|x|>R,\ |\hat{x}-\omega|>\delta\},\quad R>0. \tag{2.5}$$

The azimuth angle $\gamma(x;\omega)$ from $\omega\in S^1$ satisfies $\nabla\gamma=(-x_2/|x|^2,x_1/|x|^2)$, so that we have

$$\exp(-i\alpha\gamma(x;\omega))H_\alpha\exp(i\alpha\gamma(x;\omega))=H(A_\alpha-\alpha\nabla\gamma)=H_0 \tag{2.6}$$

on $\Sigma(R,\omega,\delta)$. The next lemma is well known ([10, Theorem IX.31]).

Lemma 2.1 *Let $f \in L^2$. Then the free solution $\exp(-itH_0)f$ behaves like*

$$(\exp(-itH_0)f)(x) = (2it)^{-1}\exp(i|x|^2/4t)\widehat{f}(x/2t) + o(1), \quad |t| \to \infty,$$

in L^2, where $\widehat{f}(\xi) = (2\pi)^{-1}\int e^{-ix\cdot\xi} f(x)\,dx$ is the Fourier transform.

We introduce the new notation. We write $W_\pm(K_2, K_1; J)$ for the limit

$$W_\pm(K_2, K_1; J) = s - \lim_{t\to\pm\infty} \exp(itK_2)J\exp(-itK_1),$$

where K_j, $j = 1, 2$, are self–adjoint in L^2 and $J : L^2 \to L^2$ is bounded. Let $\beta(\xi)$ and $e_0(x)$ be as above, and set $J = e_0^2\beta^2$ with $\beta = \beta(D_x)$. Then

$$W_-(H_\alpha, H_0)\beta^2 = W_-(H_\alpha, H_0; J)$$

by Lemma 2.1, and hence we have the decomposition

$$W_-(H_\alpha, H_0)\beta^2 = W_-(H_\alpha, H_0; J_+)W_-(H_0, H_0; J_-), \tag{2.7}$$

where $J_\pm = e_0 \exp(\pm i\alpha\gamma(x;\omega))\beta$. The existence of $W_-(H_0, H_0; J_-)$ follows from Lemma 2.1, while the existence of $W_-(H_\alpha, H_0; J_+)$ is verified by (2.6). The same argument applies to final direction $\tilde{\omega}$. We define

$$\tilde{\beta}(\xi) = \chi(|\xi - \sqrt{E}\tilde{\omega}|/\delta^2)$$

and we take a function $\tilde{e}_0 \in C^\infty(\boldsymbol{R}^2)$ such that

$$\operatorname{supp}\tilde{e}_0 \subset \Sigma(R, -\tilde{\omega}, \delta), \qquad \tilde{e}_0 = 1 \text{ on } \Sigma(2R, -\tilde{\omega}, 2\delta). \tag{2.8}$$

If we set $\tilde{J}_\pm = \tilde{e}_0 \exp(\pm i\alpha\gamma(x; -\tilde{\omega}))\tilde{\beta}$, then we obtain

$$W_+(H_\alpha, H_0)\tilde{\beta}^2 = W_+(H_\alpha, H_0; \tilde{J}_+)W_+(H_0, H_0; \tilde{J}_-). \tag{2.9}$$

We combine (2.7) and (2.9) to obtain that

$$\tilde{\beta}^2 S(H_\alpha, H_0)\beta^2 = W_+^*(H_0, H_0; \tilde{J}_-)S_0(H_\alpha, H_0)W_-(H_0, H_0; J_-), \tag{2.10}$$

where $S_0(H_\alpha, H_0) = W_+^*(H_\alpha, H_0; \tilde{J}_+)W_-(H_\alpha, H_0; J_+)$. The operator $S_0(H_\alpha, H_0)$ also has the direct integral decomposition, because it commutes with H_0. We denote by $S_0(\theta', \theta; \lambda)$ the kernel of fibre $S_0(\lambda; H_\alpha, H_0) : L^2(S^1) \to L^2(S^1)$. By Lemma 2.1, $W_-(H_0, H_0; J_-)$ acts as the multiplication

$$FW_-(H_0, H_0; J_-)F^* = \exp(-i\alpha\gamma(-\theta;\omega))\beta(\sqrt{\lambda}\theta)\times$$

on $L^2((0,\infty) \times S^1)$, where $F : L^2 \to L^2((0,\infty) \times S^1)$ is the unitary mapping defined by (2.1). Similarly

$$FW_+(H_0, H_0; \tilde{J}_-)F^* = \exp(-i\alpha\gamma(\theta; -\tilde{\omega}))\tilde{\beta}(\sqrt{\lambda}\theta) \times .$$

Since $\beta(\sqrt{E}\omega) = \tilde{\beta}(\sqrt{E}\tilde{\omega}) = 1$ and $\gamma(-\omega;\omega) = \gamma(\tilde{\omega};-\tilde{\omega}) = \pi$, we have

$$S(\tilde{\omega},\omega;E) = S_0(\tilde{\omega},\omega;E) \tag{2.11}$$

by (2.10).

We shall derive the representation for the kernel $S_0(\theta',\theta;E)$ on the right side of (2.11). The derivation is based on the idea due to [6, 12]. Set $T = H_\alpha J_+ - J_+ H_0$. Then T is calculated as

$$T = \exp(i\alpha\gamma(x;\omega))\,(H_0 e_0 - e_0 H_0)\,\beta = \exp(i\alpha\gamma(x;\omega))[H_0, e_0]\beta$$

by (2.6). Similarly

$$\tilde{T} = H_\alpha \tilde{J}_+ - \tilde{J}_+ H_0 = \exp(i\alpha\gamma(x;-\tilde{\omega}))[H_0,\tilde{e}_0]\tilde{\beta}.$$

Since $e_0(x)$ vanishes for $|x| \gg 1$ with $|\hat{x}-\omega| < \delta$ and since $\beta(\xi)$ has support in $\{|\xi - \sqrt{E}\omega| < 2\delta^2\}$ only, it follows from Lemma 2.1 that $W_+(H_\alpha, H_0; J_+) = 0$ and hence

$$W_-(H_\alpha, H_0; J_+) = -i\int \exp(itH_\alpha)T\exp(-itH_0)\,dt.$$

If we make use of this relation, then we obtain

$$S_0(\lambda; H_\alpha, H_0) = 2\pi i F(\lambda)\left(-\tilde{J}_+^* T + \tilde{T}^* R(\lambda + i0; H_\alpha)T\right)F^*(\lambda) \tag{2.12}$$

in exactly the same way as [6, Theorem 3.3] (see also [12, Section 7]), where $F(\lambda) : L^2_s(\boldsymbol{R}^2) \to L^2(S^1),\ s > 1/2$, is the trace operator defined by

$$(F(\lambda)u)(\theta) = (Fu)(\mu,\theta)|_{\mu=\lambda}.$$

Thus it follows from (2.11) and (2.12) that

$$\begin{aligned} f_\alpha(\omega \to \tilde{\omega}; E) &= -(ic(E)/4\pi)(T\varphi_0(\omega,E), \tilde{J}_+\varphi_0(\tilde{\omega},E)) \\ &+ (ic(E)/4\pi)(R(E+i0;H_\alpha)T\varphi_0(\omega,E), \tilde{T}\varphi_0(\tilde{\omega},E)) \end{aligned} \tag{2.13}$$

for $\omega \ne \tilde{\omega}$, where we denote by $(\ ,\ )$ the L^2 scalar product and write $\varphi_0(\omega,E)$ for $\varphi_0(x;\omega,E) = \exp(i\sqrt{E}x\cdot\omega)$.

We now fix σ, $0 < \sigma \ll 1$, small enough and take $R = d^\sigma$, $d \gg 1$, in (2.4) and (2.8). We may assume that e_0 obeys $\partial_x^\beta e_0(x) = O(|x|^{-|\beta|})$ uniformly in d and similarly for $\tilde{e}_0$. The operators $\tilde{J}_+$, T and $\tilde{T}$ are all pseudodifferential operators. If $\omega \ne \tilde{\omega}$, then we can choose δ so small that the support of symbols $T(x,\xi)$ and $\tilde{J}_+(x,\xi)$ does not intersect with each other. Hence the first term on the right side of (2.13) obeys

$$(T\varphi_0(\omega,E), \tilde{J}_+\varphi_0(\tilde{\omega},E)) = O(d^{-L}), \quad d \to \infty,$$

for any $L \gg 1$, and we have

$$f_\alpha(\omega \to \tilde{\omega}; E) = (ic(E)/4\pi)(R(E+i0; H_\alpha)T\varphi_0(\omega, E), \tilde{T}\varphi_0(\tilde{\omega}, E)) + o(1). \quad (2.14)$$

We continue to analyze the term on the right side. We decompose $T = T(x, D_x)$ into

$$T = \chi_0 T + (1-\chi_0)T = T_0 + T_\infty,$$

where $\chi_0(x) = \chi(|x|/2d^\sigma)$ for basic cut–off function $\chi \in C_0^\infty[0,\infty)$ with property (2.3). By (2.4), ∇e_0 and Δe_0 vanish on $\Sigma(2d^\sigma, \omega, 2\delta)$. Hence the symbol $T_\infty(x,\xi)$ has the support in the outgoing region

$$\operatorname{supp} T_\infty \subset \{(x,\xi) : |x| > 2d^\sigma,\ \ |\xi - \sqrt{E}\omega| < 2\delta^2,\ \ \hat{x}\cdot\hat{\xi} > 1 - 3\delta\}.$$

The classical particle starting from $(x,\xi) \in \operatorname{supp} T_\infty$ at $t = 0$ moves like the free particle and it does not pass around the origin for $t > 0$. In fact, we have

$$|x + 2t\xi|^2 \ge |x|^2 - 4t(1-\delta/3)|x|\,|\xi| + 4t^2|\xi|^2 \ge c\,(|x| + t|\xi|)^2\,, \quad c > 0.$$

Hence the particle does not take momentum around $\sqrt{E}\tilde{\omega}$. This yields

$$(R(E+i0; H_\alpha)T_\infty\varphi_0(\omega, E), \tilde{T}\varphi_0(\tilde{\omega}, E)) = O(d^{-L}).$$

If we decompose $\tilde{T}$ into $\tilde{T} = \tilde{T}_0 + \tilde{T}_\infty$ in a similar way, then we have

$$(R(E+i0; H_\alpha)T_0\varphi_0(\omega, E), \tilde{T}_\infty\varphi_0(\tilde{\omega}, E)) = O(d^{-L}).$$

Thus we have obtained the following lemma.

Lemma 2.2 *Let e_0 and $\tilde{e}_0$ be as in (2.4) and (2.8) respectively and let $\chi_0(x)$ be defined by $\chi_0 = \chi(|x|/2d^\sigma)$. If $\omega \ne \tilde{\omega}$, then*

$$f_\alpha(\omega \to \tilde{\omega}; E) = (ic(E)/4\pi)(R(E+i0; H_\alpha)T_0\varphi_0(\omega, E), \tilde{T}_0\varphi_0(\tilde{\omega}, E)) + o(1)$$

as $d \to \infty$, where T_0 acts as

$$T_0\varphi_0(\omega, E) = \exp(i\alpha\gamma(x;\omega))\chi_0[H_0, e_0]\varphi_0(\omega, E)$$

on $\varphi_0(\omega, E) = \exp(i\sqrt{E}x\cdot\omega)$, and $\tilde{T}_0$ acts as

$$\tilde{T}_0\varphi_0(\tilde{\omega}, E) = \exp(i\alpha\gamma(x;-\tilde{\omega}))\chi_0[H_0, \tilde{e}_0]\varphi_0(\tilde{\omega}, E).$$

We skip the proof of the lemma (see Lemma 2.3 of [7]). The proof is done by constructing approximations to $R(E+i0; H_\alpha)T_\infty$ and $R(E-i0; H_\alpha)\tilde{T}_\infty$. The approximations take the forms (3.19) and (3.20) in the next section. We repeat a similar argument in the proof of Theorem 1.1.

3. Auxiliary operators and approximation to resolvents

The present section is devoted to a preliminary step toward the proof of resolvent estimate (1.13). We set

$$H_a = H(A_a), \qquad A_a(x) = \sum_{j \in a} A_j(x), \tag{3.1}$$

for subset $a \subset \{1, 2, \dots, N\}$, which is self–adjoint with domain

$$\mathcal{D}(H_a) = \{u \in L^2 : H(A_a)u \in L^2, \ \lim_{|x-d_j| \to 0} |u(x)| < \infty, \ j \in a\}.$$

We study how to construct an approximation to $R(E \pm i0; H_a)$ microlocally.

Let δ, $0 < \delta \ll 1$, be as in (2.4) and (2.8), and let $\zeta_j \in C^\infty(\boldsymbol{R})$, $1 \le j \le N$, be a real periodic function with period 2π such that $\zeta_j(s) = \alpha_j s$ on the interval $(\delta, 2\pi - \delta)$. We fix $\theta \in S^1$ and define $q_j \in C^\infty(\boldsymbol{R}^2)$ by

$$q_j(x) = q_j(x; \theta) = \zeta_j(\gamma(x - d_j; \theta)) \tag{3.2}$$

for $|x - d_j| > d^\sigma$ and by $q_j(x) = 0$ for $|x - d_j| < d^\sigma/2$. We may assume that

$$|\partial_x^\beta q_j(x)| \le C_\beta \left(|x - d_j| + d^\sigma\right)^{-|\beta|} \tag{3.3}$$

By definition, it follows that

$$\nabla q_j(x) = A_j(x) \tag{3.4}$$

on the complement $\Pi_j(\theta)^{\mathrm{c}}$ of

$$\Pi_j(\theta) = \{|x - d_j| < d^\sigma\} \cup \{|x - d_j| \ge d^\sigma, \ |(\widehat{x - d_j}) - \theta| < \delta\} \tag{3.5}$$

with $(\widehat{x - d_j}) = (x - d_j)/|x - d_j|$. Assume that $m \in a$. Then we introduce the auxiliary operator

$$K_{am}(\theta) = \exp(iq_m) H(A_a - A_m) \exp(-iq_m) = H(A_a - A_m + \nabla q_m). \tag{3.6}$$

This is self–adjoint with domain

$$\mathcal{D}(K_{am}(\theta)) = \{u \in L^2 : K_{am}(\theta)u \in L^2, \ \lim_{|x-d_j| \to 0} |u(x)| < \infty, \ j \in a \setminus \{m\}\}$$

and it follows from (3.4) that

$$K_{am}(\theta) = H_a \tag{3.7}$$

on $\Pi_m(\theta)^{\mathrm{c}}$. The domain $\mathcal{D}(K_{am}(\theta))$ is different from that of H_a, so that the ordinary resolvent identity is not expected to hold true. Nevertheless we can show the following relation between resolvents of H_a and $K_{am}(\theta)$.

Lemma 3.1 *Let $\psi_j \in C^\infty(\boldsymbol{R}^2)$ be a function vanishing around $x = d_j$. Assume that ψ_j has bounded derivatives. If $\varepsilon > 0$, then*

$$R(E \pm i\varepsilon; H_a)\psi_m = \psi_m R(E \pm i\varepsilon; K_{am}) + R(E \pm i\varepsilon; H_a) W_{am} R(E \pm i\varepsilon; K_{am})$$

on L^2, where $K_{am} = K_{am}(\theta)$ and

$$W_{am} = W_{am}(\theta) = [\psi_m, K_{am}] - (H_a - K_{am})\psi_m.$$

Proof. We write ψ and K_a for ψ_m and K_{am}. Then

$$R(E + i\varepsilon; H_a)\psi - \psi R(E + i\varepsilon; K_a) = R(E + i\varepsilon; H_a) W_{am} R(E + i\varepsilon; K_a),$$

where $W_{am} = \psi(K_a - E) - (H_a - E)\psi$. The desired relation follows after a simple calculation. □

The differential operator W_{am} takes the form $W_{am} = w_1 \cdot \nabla + w_0$. We mostly use the lemma with

$$\psi_j = 1 - \chi(2|x - d_j|/d^\sigma). \tag{3.8}$$

Then we denote W_{am} by the distinct notation

$$V_{am}(\theta) = [\psi_m, K_{am}(\theta)] - (H_a - K_{am}(\theta))\psi_m. \tag{3.9}$$

By (3.7), the coefficients of V_{am} have support in $D_m = \Pi_m(\theta) \cap \{|x - d_m| > d^\sigma/2\}$. If there are no centers d_j, $j \in a$, in D_m, then it follows from (1.3) and (1.4) that $|x - d_m| \le c\,|x - d_j|$ over D_m, and hence the coefficients satisfy

$$|\partial_x^\beta w_1(x)| + |\partial_x^\beta w_0(x)| \le C_\beta \left(|x - d_m| + |d|^\sigma\right)^{-1-|\beta|} \tag{3.10}$$

uniformly in $d \gg 1$. The later discussion uses the relation obtained by taking the limit $\varepsilon \to 0$. According to this relation, $R(E \pm i0; H_a)$ is approximated by $R(E \pm i0; K_{am}(\theta))$. We analyze the second operator on the right side. To do this, we work in the phase space $\boldsymbol{R}_x^2 \times \boldsymbol{R}_\xi^2$.

Let $\{\beta_\pm, \beta_\infty\}$ be a smooth nonnegative partition of unity normalized by

$$\beta_+(\xi) + \beta_-(\xi) + \beta_\infty(\xi) = 1 \tag{3.11}$$

such that $\operatorname{supp}\beta_\infty \subset \{|\xi|^2 < E/2 \ \text{or} \ |\xi|^2 > 2E\}$ and

$$\operatorname{supp}\beta_\pm \subset \{E/3 < |\xi|^2 < 3E, \ \pm\hat{\xi} \cdot \theta > -\delta\}.$$

We now assume that $m \in a$ and $\theta \in S^1$ satisfy

$$d_j \notin \Gamma_m(d^\sigma, \theta, c_0), \quad j \in a, \tag{3.12}$$

for some $c_0,\ 0 < c_0 < 1$, where

$$\Gamma_j(R,\theta,c) = \{x \in \boldsymbol{R}^2 : |x-d_j| > R,\ \widehat{(x-d_j)} \cdot \theta > -c\} \tag{3.13}$$

for $R \gg 1$ and $0 < c < 1$. Let $V_{am}(\theta)$ be defined by (3.9). According to (3.11), it admits the decomposition

$$V_{am}(\theta) = X_{am}(\theta) + X^+_{am}(\theta) + X^-_{am}(\theta) + X^\infty_{am}(\theta), \tag{3.14}$$

where $X_{am} = g_m V_{am}$ with $g_m = \chi(|x-d_m|/Md^\sigma),\ M \gg 1$, and

$$X^\pm_{am} = (1-g_m)V_{am}\beta_\pm, \qquad X^\infty_{am} = (1-g_m)V_{am}\beta_\infty.$$

The symbol $X^+_{am}(x,\xi)$ of pseudodifferential operator X^+_{am} has support in

$$\{|x-d_m| > Md^\sigma,\ |\widehat{(x-d_m)} - \theta| < \delta,\ E/3 < |\xi|^2 < 3E,\ \hat{\xi}\cdot\theta > -\delta\} \tag{3.15}$$

and $(x,\xi) \in \operatorname{supp} X^+_{am}$ has the outgoing property $\widehat{(x-d_m)} \cdot \hat{\xi} > -2\delta$. On the other hand, $X^-_{am}(x,\xi)$ has support in the incoming region

$$\{|x-d_m| > Md^\sigma,\ |\widehat{(x-d_m)} - \theta| < \delta,\ E/3 < |\xi|^2 < 3E,\ \hat{\xi}\cdot\theta < \delta\} \tag{3.16}$$

and $\widehat{(x-d_m)} \cdot \hat{\xi} < 2\delta$ for $(x,\xi) \in \operatorname{supp} X^-_{am}$.

We construct an approximation for $R(E \pm i0; H_a)X^\pm_{am}$. A similar argument applies to $R(E \pm i0; K_{am}(\theta))$. Let $p_m(x)$ be a smooth function such that

$$\operatorname{supp} p_m \subset \Gamma_m(2d^\sigma, \theta, c_0/2), \qquad p_m = 1 \ \text{ on } \ \Gamma_m(3d^\sigma, \theta, c_0/3)$$

for c_0 in (3.12) and that $|\partial_x^\beta p_m(x)| \le C_\beta \left(|x-d_m| + d^\sigma\right)^{-|\beta|}$. We take $\tilde{\beta}_\pm \in C_0^\infty(\boldsymbol{R}^2)$ in such a way that $\operatorname{supp} \tilde{\beta}_\pm$ is slightly wider than $\operatorname{supp} \beta_\pm$ and that $\tilde{\beta}_\pm = 1$ on $\operatorname{supp} \beta_\pm$. Then $\left(1 - \tilde{\beta}_\pm(\xi)\right) X^\pm_{am}(x,\xi) = 0$ as a symbol, and hence $\left(1-\tilde{\beta}_\pm\right) X^\pm_{am} = \tilde{r}_L$ as an operator for any $L \gg 1$, where $\tilde{r}_L$ denotes an operator

$$\left(|x|+d\right)^L \tilde{r}_L, \quad \tilde{r}_L \left(|x|+d\right)^L : L^2 \to L^2 \tag{3.17}$$

bounded uniformly in $d \gg 1$. We set $\theta_a(x) = \sum_{j\in a} \alpha_j \gamma(x-d_j; -\theta)$. Since θ satisfies (3.12), θ_a is smooth on $\operatorname{supp} p_m$ and $\nabla\theta_a = A_a$ there. Hence it follows from (2.6) that

$$(H_a - E)e^{i\theta_a} p_m = e^{i\theta_a}(H_0 - E)p_m = e^{i\theta_a}\left(p_m(H_0-E) + [H_0, p_m]\right).$$

Note that $p_m X^+_{am} = X^+_{am}$. Thus we obtain the relation

$$\begin{aligned} &(H_a - E)e^{i\theta_a} p_m R(E+i0; H_0)\tilde{\beta}_+ e^{-i\theta_a} X^+_{am} \\ &\quad = X^+_{am} + \tilde{r}_L + e^{i\theta_a}[H_0, p_m] R(E+i0; H_0)\tilde{\beta}_+ e^{-i\theta_a} X^+_{am}. \end{aligned} \tag{3.18}$$

Since $X^+_{am}(x,\xi)$ has support in (3.15) and since ∇p_m vanishes on $\Gamma_m(3d^\sigma,\theta,c_0/3)$, we can take $0<\delta\ll 1$ so small and $M\gg 1$ so large that the free particle starting from $(y,\xi)\in \operatorname{supp} X^+_{am}$ at time $t=0$ never passes over $\operatorname{supp}\nabla p_m$ for $t>0$. In fact, we can show that

$$|x-y-2t\xi|\ge c\,(t+|x-d_m|+|y-d_m|+d^\sigma)$$

when $x\in\operatorname{supp}\nabla p_m$ and $(y,\xi)\in\operatorname{supp}X^+_{am}$. This enables us to put

$$[H_0,p_m]R(E+i0;H_0)\tilde\beta_+e^{-i\theta_a}X^+_{am}=\tilde r_L$$

for the remainder term on the right side of (3.18). Thus $R(E+i0;H_a)X^+_{am}$ is approximated as

$$R(E+i0;H_a)X^+_{am}\sim e^{i\theta_a}p_mR(E+i0;H_0)\tilde\beta_+e^{-i\theta_a}X^+_{am} \tag{3.19}$$

with error term $R(E+i0;H_a)\tilde r_L$ for any $L\gg 1$. Similarly we have

$$R(E-i0;H_a)X^-_{am}\sim e^{i\theta_a}p_mR(E-i0;H_0)\tilde\beta_-e^{-i\theta_a}X^-_{am}. \tag{3.20}$$

We require a slight modification to construct an approximation for

$$X^-_{am}R(E+i0;H_a)=\left(R(E-i0;H_a)\left(X^-_{am}\right)^*\right)^*.$$

We expand $(X^-_{am})^*$ as $(X^-_{am})^*=X^-_{am,L}+\tilde r_L$ by use of calculus of pseudodifferential operators, where $X^-_{am,L}(x,\xi)$ has support in (3.16) and satisfies

$$|\partial_\xi^\alpha\partial_x^\beta X^-_{am,L}(x,\xi)|\le C_{\alpha\beta L}\,(|x-d_m|+d^\sigma)^{-|\beta|}\,\langle\xi\rangle^{-|\alpha|} \tag{3.21}$$

uniformly in $d\gg 1$. We apply the same argument as above to $R(E-i0;H_a)X^-_{am,L}$ and we obtain

$$X^-_{am}R(E+i0;H_a)\sim\left(e^{i\theta_a}p_mR(E-i0;H_0)\tilde\beta_-e^{-i\theta_a}X^-_{am,L}\right)^* \tag{3.22}$$

with error term $\tilde r_LR(E+i0;H_a)$. Similarly we have

$$X^+_{am}R(E-i0;H_a)\sim\left(e^{i\theta_a}p_mR(E+i0;H_0)\tilde\beta_+e^{-i\theta_a}X^+_{am,L}\right)^*,$$

where $X^+_{am,L}(x,\xi)$ has support in (3.15) and obeys (3.21).

An approximation for $R(E\pm i0;H_a)X^\infty_{am}$ is constructed in the same way as parametrices for elliptic operators. The symbol $X^\infty_{am}(x,\xi)$ has support in

$$\{|x-d_m|>Md^\sigma,\ |\widehat{(x-d_m)}-\theta|<\delta,\ |\xi|^2<E/2\ \text{ or }\ |\xi|^2>2E\}. \tag{3.23}$$

If θ satisfies (3.12), then the symbol $H_a(x,\xi)$ of H_a behaves like $H_a(x,\xi)\sim|\xi|^2$ when $|\widehat{(x-d_m)}-\theta|\ll 1$ and $|x-d_m|\gg 1$. Hence $(H_a(x,\xi)-E)^{-1}$ is uniformly bounded on $\operatorname{supp}X^\infty_{am}$. Thus the approximation for $R(E\pm i0;H_a)X^\infty_{am}$ takes the form

$$R(E+i0;H_a)X^\infty_{am}=X^\infty_{am,L}+R(E+i0;H_a)\tilde r_L \tag{3.24}$$

for some pseudodifferential operator $X^{\infty}_{am,L}(x,D_x)$, where $X^{\infty}_{am,L}(x,\xi)$ has support in (3.23) and obeys (3.21). The approximation for $X^{\infty}_{am}R(E\pm i0;H_a)$ can be constructed in a similar way.

We use another decomposition

$$V_{am}(\theta)=Y_{am}(\theta)+Y^{+}_{am}(\theta)+Y^{-}_{am}(\theta)+Y^{\infty}_{am}(\theta) \tag{3.25}$$

to prove (1.13), where $Y_{am}=h_mV_{am}$ with $h_m=\chi(|x-d_m|/Md^{1/2+\sigma}),\ M\gg 1$, and

$$Y^{\pm}_{am}=(1-h_m)V_{am}\beta_{\pm},\qquad Y^{\infty}_{am}=(1-h_m)V_{am}\beta_{\infty}.$$

We can construct approximations for $Y^{\pm}_{am}$ and Y^{∞}_{am} in exactly the same way as above. For example, we have

$$R(E+i0;H_a)Y^{+}_{am}\sim e^{i\theta_a}p_mR(E+i0;H_0)\tilde{\beta}_+e^{-i\theta_a}Y^{+}_{am}. \tag{3.26}$$

We consider the case when (3.12) is not fulfilled. In later discussion, this exceptional case occurs when all the centers d_j are on the even segment

$$d_j\in\{\widehat{(x-d_m)}\cdot\theta=0\},\quad j\in a, \tag{3.27}$$

with d_m as an interior point. Let $\{\eta_{\pm},\beta_{\infty}\}$ be a smooth partition normalized by $\eta_++\eta_-+\beta_\infty=1$, where $\eta_\pm$ has support in

$$\operatorname{supp}\eta_{\pm}\subset\{E/3<|\xi|^2<3E,\ \pm\hat{\xi}\cdot\theta>-d^{-1/2}\}$$

and satisfies $|\partial^{\alpha}_{\xi}\eta_{\pm}|\le C_\alpha d^{|\alpha|/2}$ uniformly in d. We decompose $V_{am}(\theta)$ into

$$V_{am}(\theta)=Z_{am}(\theta)+Z^{+}_{am}(\theta)+Z^{-}_{am}(\theta)+Z^{\infty}_{am}(\theta), \tag{3.28}$$

where $Z_{am}=h_mV_{am}$ and

$$Z^{\pm}_{am}=(1-h_m)V_{am}\eta_{\pm},\qquad Z^{\infty}_{am}=(1-h_m)V_{am}\beta_{\infty}$$

for h_m as above. We construct an approximation for $R(E+i0;H_a)Z^{+}_{am}$. The symbol $Z^{+}_{am}(x,\xi)$ has support in

$$\{|x-d_m|>Md^{1/2+\sigma},\ |\widehat{(x-d_m)}-\theta|<\delta,\ E/3<|\xi|^2<3E,\ \hat{\xi}\cdot\theta>-d^{-1/2}\}$$

and it satisfies

$$|\partial^{\alpha}_{\xi}\partial^{\beta}_{x}Z^{+}_{am}(x,\xi)|\le C_{\alpha\beta}\left(|x-d_m|+d^{1/2+\sigma}\right)^{-|\beta|}d^{|\alpha|/2}.$$

Let $\tilde{p}_j(x)\in C^{\infty}(\boldsymbol{R}^2)$ be a function such that

$$\operatorname{supp}\tilde{p}_j\subset\Gamma_j(2d^{1/2+\sigma},\theta,1/2),\qquad \tilde{p}_j=1\ \text{ on }\ \Gamma_j(3d^{1/2+\sigma},\theta,1/3)$$

and that $|\partial_x^\beta \tilde{p}_j(x)| \le C_\beta \left(|x-d_j| + d^{1/2+\sigma}\right)^{-|\beta|}$. We set $p_a(x) = \prod_{j\in a} \tilde{p}_j(x)$. Then $\nabla\theta_a = A_a$ on $\operatorname{supp} p_a$ and we have the relation

$$(H_a - E)e^{i\theta_a} p_a = e^{i\theta_a} \left(p_a(H_0 - E) + [H_0, p_a]\right),$$

where θ_a is still defined as above. We can take $M \gg 1$ so large that $|x - y - 2t\xi| \ge c\, d^{1/2+\sigma}$ for $t > 0$, where $(y, \xi) \in \operatorname{supp} Z_{am}^+$ and $x \in \operatorname{supp} \nabla p_a$. This implies that the free particle starting from $\operatorname{supp} Z_{am}^+$ at time $t = 0$ does not pass over $\operatorname{supp} \nabla p_a$ for $t > 0$. In fact, it follows that

$$|x - y - 2t\xi| \ge c\,(t + |x| + |y| + d)^{1/2+\sigma}$$

with another $c > 0$ for $y,\ \xi$ and x as above, and hence we have

$$[H_0, p_a] R(E + i0; H_0)\tilde{\eta}_+ e^{-i\theta_a} Z_{am}^+ = \tilde{r}_L$$

for any $L \gg 1$, where $\tilde{\eta}_\pm \in C_0^\infty(\boldsymbol{R}^2)$ satisfies $\tilde{\eta}_\pm = 1$ on $\operatorname{supp} \eta_\pm$ and obeys the same estimate as $\eta_\pm$. Since $p_a Z_{am}^+ = Z_{am}^+$, we see that $R(E + i0; H_a)Z_{am}^+$ is approximated as

$$R(E + i0; H_a)Z_{am}^+ \sim e^{i\theta_a} p_a R(E + i0; H_0)\tilde{\eta}_+ e^{-i\theta_a} Z_{am}^+ \tag{3.29}$$

with error term $R(E + i0; H_a)\tilde{r}_L$. A similar argument applies to $R(E - i0; H_a)Z_{am}^-$ and $Z_{am}^\mp R(E \pm i0; H_a)$.

We end the section by noting that the approximation for

$$R(E \pm i0; K_{am}(\theta)) = e^{iq_m} R(E \pm i0; H_b)e^{-iq_m} \tag{3.30}$$

with $b = a \setminus \{m\}$ can be constructed by replacing the phase function θ_a by

$$\theta_{am}(x) = \sum_{j\in b} \alpha_j \gamma(x - d_j; -\theta) + q_m(x; \theta).$$

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Recall that $s_j(x)$ denotes the characteristic function of set S_j defined by (1.12), while $b_j(x)$ denotes the characteristic function of set

$$B_j = \{x \in \boldsymbol{R}^2 : |x - d_j| < Cd^{1/2+\sigma}\}, \quad 1 \le j \le N, \tag{4.1}$$

with $C > 1$. We write $r_{\pm\nu}$ for the multiplication operator

$$r_{\pm\nu} = (|x|^2 + d^2)^{\mp\nu}\times \tag{4.2}$$

with $\nu \gg 1$. The proof is based on the following theorem on the resolvent estimate for $H_a = H(A_a)$ defined by (3.1) for subset $a \subset \{1, 2, \ldots, N\}$.

Theorem 4.1 *Assume* (1.3) *that* (1.4)*. Set*

$$s_{j1}(x) = s_j(x)\langle x - d_j\rangle^{-1/2}, \qquad b_{j1}(x) = b_j(x)\langle x - d_j\rangle^{-1/2}.$$

Then one has the following statements :

(1) *If* $\nu \gg 1$ *is large enough, then*

$$\begin{aligned} \|r_\nu R(E \pm i0; H_a) r_\nu\| &= O(d^{-\nu}), && (4.3)\\ \|r_\nu R(E \pm i0; H_a) b_{j1}\| &= O(d^{-\nu/2}). && (4.4) \end{aligned}$$

(2) *If* $0 < \sigma \ll 1$ *is small enough, then there exists* $c > 0$ *independent of* σ *such that*

$$\|b_{j1} R(E \pm i0; H_a) b_{k1}\| = O(d^{c\sigma}) \tag{4.5}$$

and that

$$\begin{aligned} \|b_{j1} R(E \pm i0; H_a) s_{k1}\| &= O(d^{-1/4+c\sigma}), && (4.6)\\ \|s_{j1} R(E \pm i0; H_a) s_{k1}\| &= O(d^{-1/2+c\sigma}) && (4.7) \end{aligned}$$

for $j \neq k$*.*

The resolvent estimate (1.13) is obtained as an immediate consequence of (4.7) in the above theorem. We proceed to the proof of Theorem 1.1, accepting Theorem 4.1 as proved. The proof is done in the next section.

Proof of Theorem 1.1. The scattering amplitude $f_d = f_d(\omega \to \tilde{\omega}; E)$ is defined by (2.2) through the scattering kernel. We follow the argument in section 2 to represent the amplitude in terms of resolvent $R(E + i0; H_d)$. Recall that $\beta(\xi)$ and $\tilde{\beta}(\xi)$ are defined by

$$\beta(\xi) = \chi(|\xi - \sqrt{E}\omega|/\delta^2), \qquad \tilde{\beta}(\xi) = \chi(|\xi - \sqrt{E}\tilde{\omega}|/\delta^2)$$

and that $e_0(x)$ and $\tilde{e}_0(x)$ have the properties

$$\begin{aligned} \operatorname{supp} e_0 \subset \Sigma(d^\sigma, \omega, \delta), &\qquad e_0 = 1 \ \text{ on } \ \Sigma(2d^\sigma, \omega, 2\delta),\\ \operatorname{supp} \tilde{e}_0 \subset \Sigma(d^\sigma, -\tilde{\omega}, \delta), &\qquad \tilde{e}_0 = 1 \ \text{ on } \ \Sigma(2d^\sigma, -\tilde{\omega}, 2\delta) \end{aligned}$$

with $d = \min |d_{jk}| \gg 1$, where the notation $\Sigma(R, \omega, \delta)$ is defined by (2.5). We set $e_j(x) = e_0(x - d_j)$ and $e(x) = \prod_{j=1}^N e_j(x)$, and similarly for $\tilde{e}_j(x)$ and $\tilde{e}(x)$. We define $J_+ = e \exp(i\theta_d)\beta$ and $\tilde{J}_+ = \tilde{e} \exp(i\tilde{\theta}_d)\tilde{\beta}$, where $\beta = \beta(D_x)$, $\tilde{\beta} = \tilde{\beta}(D_x)$ and

$$\theta_d(x) = \sum_{j=1}^N \alpha_j \gamma(x - d_j; \omega), \qquad \tilde{\theta}_d = \sum_{j=1}^N \alpha_j \gamma(x - d_j; -\tilde{\omega}).$$

We can calculate

$$T = H_d J_+ - J_+ H_0 = \exp(i\theta_d)[H_0, e]\beta, \qquad \tilde{T} = \exp(i\tilde{\theta}_d)[H_0, \tilde{e}]\tilde{\beta}$$

by making use of relation (2.6), and we obtain

$$f_d = (ic(E)/4\pi)(R(E+i0;H_d)T\varphi_0(\omega,E),\tilde{T}\varphi_0(\tilde{\omega},E)) + o(1) \tag{4.8}$$

by repeating the same argument as used to derive (2.14). We set

$$\chi_j(x) = \chi(|x-d_j|/2d^\sigma), \qquad \chi_\infty(x) = 1 - \sum_{j=1}^{N} \chi_j(x).$$

If $k \ne j$, then $e_k = 1$ on $\operatorname{supp}\chi_j$, and we can decompose T into the sum

$$T = \sum_{j=1}^{N} (T_j + T_{j\infty}),$$

where $T_j = \chi_j \exp(i\theta_d)[H_0, e_j]\beta$ and

$$T_{j\infty} = \chi_\infty \exp(i\theta_d) e_1 \dots e_{j-1}[H_0, e_j] e_{j+1} \dots e_N \beta.$$

Similarly we decompose $\tilde{T}$ into $\tilde{T} = \sum_{j=1}^{N} \left(\tilde{T}_j + \tilde{T}_{j\infty}\right)$. The symbol $T_{j\infty}(x,\xi)$ has support in the outgoing region

$$\operatorname{supp} T_{j\infty} \subset \{(x,\xi) : |x-d_j| > 2d^\sigma,\ |\xi - \sqrt{E}\omega| < 2\delta^2,\ \widehat{(x-d_j)}\cdot\hat{\xi} > 1-3\delta\},$$

while $\tilde{T}_{j\infty}(x,\xi)$ has support in the incoming region

$$\operatorname{supp} \tilde{T}_{j\infty} \subset \{(x,\xi) : |x-d_j| > 2d^\sigma,\ |\xi - \sqrt{E}\tilde{\omega}| < 2\delta^2,\ \widehat{(x-d_j)}\cdot\hat{\xi} < -1+3\delta\}.$$

We now use the assumption that $\omega \ne \hat{d}_{jk}$ and $\tilde{\omega} \ne \hat{d}_{jk}$. This assumption implies that the outgoing classical particle starting from $\operatorname{supp} T_{j\infty}$ at $t=0$ passes far away from all the centers d_k for $t>0$ and moves like the free particle. This enables us to construct an approximation to $R(E+i0;H_d)T_{j\infty}$ in a form similar to (3.19). Similarly the incoming particle starting from $\operatorname{supp} \tilde{T}_{j\infty}$ does not pass around the centers for $t<0$, and $R(E-i0;H_d)\tilde{T}_{j\infty}$ can be approximated as in (3.20). Since the outgoing and incoming particles do not take momentum around $\sqrt{E}\tilde{\omega}$ and $\sqrt{E}\omega$ for $t>0$ and $t<0$ respectively, we obtain by Theorem 4.1 that

$$\tilde{T}_k^* R(E+i0;H_d)T_{j\infty} = \tilde{r}_L, \qquad T_k^* R(E-i0;H_d)\tilde{T}_{j\infty} = \tilde{r}_L \tag{4.9}$$

for any $L \gg 1$, where $\tilde{r}_L : L^2 \to L^2$ has the property (3.17). We can also show that

$$\tilde{T}_{k\infty}^* R(E+i0;H_d)T_{j\infty} = \tilde{r}_L \tag{4.10}$$

and hence we have

$$(R(E+i0;H_d)T_{j\infty}\varphi_0(\omega,E), \tilde{T}_{k\infty}\varphi_0(\tilde{\omega},E)) = o(1). \tag{4.11}$$

The rigorous proof of (4.9) and (4.10) requires the lemma below (Lemma 4.1). We prove them after completing the proof of the theorem. If we use (4.7) in Theorem 4.1, then

$$(R(E+i0;H_d)T_j\varphi_0(\omega,E),\tilde{T}_k\varphi_0(\tilde{\omega},E))=o(1)$$

for $j\neq k$, and hence it follows from (4.8) that

$$f_d=(ic(E)/4\pi)\sum_{j=1}^{N}(R(E+i0;H_d)T_j\varphi_0(\omega,E),\tilde{T}_j\varphi_0(\tilde{\omega},E))+o(1).$$

Let $H_j=H(A_j)$ be as in (1.6). We claim that

$$\|s_j\,(R(E\pm i0;H_d)-R(E\pm i0;H_j))\,s_j\|=O(d^{-1+c\sigma}) \tag{4.12}$$

for some $c>0$. This is also verified after the completion of the proof. The bound yields

$$f_d=(ic(E)/4\pi)\sum_{j=1}^{N}(R(E+i0;H_j)T_j\varphi_0(\omega,E),\tilde{T}_j\varphi_0(\tilde{\omega},E))+o(1).$$

If $k\neq j$, then

$$\gamma(x-d_k;\omega)=\gamma(\hat{d}_{kj};\omega)+O(d^{-1+\sigma}),\quad d_{kj}=d_j-d_k,$$

on $\operatorname{supp}\chi_j$, so that $T_j\varphi_0(\omega,E)$ behaves like

$$T_j\varphi_0(\omega,E)=\exp(i\tau_j)\exp(i\alpha_j\gamma(x-d_j;\omega))\chi_j[H_0,e_j]\varphi_0(\omega,E)+O(d^{-1+\sigma})$$

with τ_j as in the theorem. Similarly

$$\tilde{T}_j\varphi_0(\tilde{\omega},E)=\exp(i\tilde{\tau}_j)\exp(i\alpha_j\gamma(x-d_j;-\tilde{\omega}))\chi_j[H_0,e_j]\varphi_0(\tilde{\omega},E)+O(d^{-1+\sigma}).$$

Thus Lemma 2.2 completes the proof. □

We proceed to the proof of (4.9), (4.10) and (4.12) which remain unproved. The proof of (4.9) and (4.10) uses the following lemma. The lemma is a special case of [5, Theorem 1].

Lemma 4.1 *Let $g_\pm(x)$ and $f_\pm(\xi)$ be smooth bounded functions such that*

$$\operatorname{supp} g_\pm\subset\{|x|>c\},\qquad \operatorname{supp} f_\pm\subset\{1/c<|\xi|<c\}$$

for some $c>1$. Assume that there exists μ, $0<\mu<1$, such that

$$\pm\hat{x}\cdot\hat{\xi}>-\mu,\quad (x,\xi)\in\operatorname{supp} g_\pm\times\operatorname{supp} f_\pm.$$

If $\nu\gg 1$, then $R(E\pm i0;H_0)$ satisfies

$$\|r_{\nu/2}R(E+i0;H_0)f_+g_+r_{-\nu/3}\|+\|r_{\nu/2}R(E-i0;H_0)f_-g_-r_{-\nu/3}\|=O(1)$$

as $d\to\infty$, where $f_\pm=f_\pm(D_x)$.

Proof. We prove the lemma for $R(E+i0;H_0)$ only. Set

$$G(t) = r_{\nu/2}\exp(-itH_0)f_+g_+r_{-\nu/3}$$

for $t>0$. Then

$$r_{\nu/2}R(E+i0;H_0)f_+g_+r_{-\nu/3} = i\int_0^\infty e^{itE}G(t)\,dt.$$

The operator $G(t)$ has the kernel

$$G(t,x,y) = (2\pi)^{-2}r_{\nu/2}(x)\left(\int e^{ix\cdot\xi}e^{-i\psi(t,\xi,y)}f_+(\xi)\,d\xi\right)g_+(y)r_{-\nu/3}(y),$$

where $\psi(t,\xi,y) = t|\xi|^2 + y\cdot\xi$. By assumption, $|\nabla_\xi\psi(t,\xi,y)| \ge c\,(t+|y|)$ when $(y,\xi)\in \operatorname{supp} g_+ \times \operatorname{supp} f_+$. If $\nu \gg 1$, then partial integration yields

$$|G(t,x,y)| \le c\,(t+1)^{-2}\,(|x|+1)^{-2}\,(|y|+1)^{-2}$$

uniformly in $d\gg 1$. This proves the lemma. $\square$

We shall prove (4.9). As stated above, the outgoing classical particle starting from $\operatorname{supp} T_{j\infty}$ at $t=0$ moves like the free particle and do not take momentum around $\sqrt{E}\tilde{\omega}$ for $t>0$. Let $\beta_1(\xi)$ be a smooth symbol such that it has support around $\sqrt{E}\omega$ and $\beta_1(\xi)\beta(\xi)=\beta(\xi)$. We write β_1 for $\beta_1(D_x)$. Then we have

$$\tilde{T}_k^* R(E+i0;H_d)T_{j\infty} = \tilde{r}_L R(E+i0;H_0)\beta_1 e^{-i\theta_d}T_{j\infty} + \tilde{T}_k^* R(E+i0;H_d)\tilde{r}_L$$

by (3.19). We use Lemma 4.1 for the first operator on the right side and (4.4) in Theorem 4.1 for the second one to obtain that $\tilde{T}_k^* R(E+i0;H_d)T_{j\infty} = \tilde{r}_L$. A similar argument applies to $T_k^* R(E-i0;H_d)\tilde{T}_{j\infty}$, and (4.9) is verified. The proof of (4.10) is done in almost the same way as (4.9). We construct approximations for $R(E+i0;H_d)T_{j\infty}$ in the form (3.19) and for $R(E-i0;H_d)\tilde{T}_{k\infty}$ in the form (3.20). Then we have

$$\tilde{T}_{k\infty}^* R(E+i0;H_d)T_{j\infty} = \tilde{r}_L + \left(R(E-i0;H_d)\tilde{T}_{k\infty}\right)^*\tilde{r}_L = \tilde{r}_L$$

by making use of Lemma 4.1 and of (4.3) in Theorem 4.1.

We move to the proof of (4.12). This is obtained as an immediate consequence of the fact that

$$\|s_j\,(R(E\pm i0;H_a) - R(E\pm i0;H_j))\,s_j\| = O(d^{-1+c\sigma}), \quad j\in a, \tag{4.13}$$

for some $c>0$ independent of σ, and the proof of this fact is based on the following lemma.

Lemma 4.2 *Assume that $m \in a$ and $\theta \in S^1$ fulfill* (3.12). *Let $X^{\pm}_{am} = X^{\pm}_{am}(\theta)$ be defined by* (3.14) *and let $b_j(x)$ be the characteristic function of B_j defined by* (4.1).

(1) *Assume that H_a satisfy* (4.4). *If*

$$|x - y - 2t\xi| \ge c\, d^{\sigma}, \quad (y, \xi) \in \operatorname{supp} X^{+}_{am}, \ x \in B_l,$$

for $t > 0$, then $b_l R(E + i0; H_a) X^{+}_{am} = \tilde{r}_L$ for any $L \gg 1$, and if

$$|x - y - 2t\xi| \ge c\, d^{\sigma}, \quad (y, \xi) \in \operatorname{supp} X^{-}_{am}, \ x \in B_l,$$

for $t < 0$, then $X^{-}_{am} R(E + i0; H_a) b_l = \tilde{r}_L$.

(2) *Let $b = a \setminus \{m\}$. Assume that H_b satisfy* (4.4). *Then* (1) *holds true for the auxiliary operator $K_{am}(\theta)$ defined by* (3.6).

(3) *The same statements as* (1) *and* (2) *remain true for $Y^{\pm}_{am} = Y^{\pm}_{am}(\theta)$ defined by* (3.25).

Proof. We show (1) only. The same argument applies to the other statements. In particular, (2) follows from relation (3.30). The operator $R(E + i0; H_a) X^{+}_{am}$ is approximated by (3.19). By assumption, the free particle starting from $\operatorname{supp} X^{+}_{am}$ does not pass over B_l for $t > 0$. This, together with (4.4), yields the desired result. $\square$

We shall prove (4.13). We take $m \in a, \ m \ne j$, so as to satisfy

$$|d_{jm}| = \max \{ |d_{jk}| : k \in a \}$$

for $j \in a$ fixed, although m is not necessarily determined uniquely. We define $K_{am} = K_{am}(\theta)$ by (3.6) with $\theta = \hat{d}_{jm}$. Let ψ_m be defined by (3.8). Then $s_j \psi_m = s_j$ and we have

$$s_j \left(R(E + i0; H_a) - R(E + i0; K_{am}) \right) s_j = s_j R(E + i0; H_a) V_{am} R(E + i0; K_{am}) s_j$$

by Lemma 3.1. It follows from (1.3) and (1.4) that $m \in a$ and $\theta = \hat{d}_{jm}$ satisfy (3.12). We decompose $V_{am} = V_{am}(\theta)$ as in (3.14) and apply Lemma 4.2 to two operators $s_j R(E + i0; H_a) X^{+}_{am}$ and $X^{-}_{am} R(E + i0; K_{am}) s_j$. It can be easily seen that the free particles starting from $\operatorname{supp} X^{\pm}_{am}$ fulfill the assumptions in Lemma 4.2. Hence we have by (4.4) in Theorem 4.1 that

$$s_j R(E + i0; H_a) X^{\pm}_{am} R(E + i0; K_{am}) s_j = \tilde{r}_L$$

for any $L \gg 1$. We approximate $R(E + i0; H_a) X^{\infty}_{am}$ in the form (3.24). Then (4.4) again implies

$$s_j R(E + i0; H_a) X^{\infty}_{am} R(E + i0; K_{am}) s_j = \tilde{r}_L$$

by use of calculus of pseudodifferential operators. Recall that the coefficients of $X_{am} = g_m V_{am}$ have support in $\{|x - d_m| < 2Md^\sigma\}$ and are uniformly bounded by (3.10). By elliptic estimate, it follows from (4.7) that

$$\|s_j R(E + i0; H_a) X_{am} R(E + i0; K_{am}) s_j\| = O(d^{-1+c\sigma})$$

for some $c > 0$. According to (3.30), the operator $R(E + i0; K_{am})$ is written as

$$R(E + i0; K_{am}) = e^{iq_m} R(E + i0; H_b) e^{-iq_m}, \quad b = a \setminus \{m\},$$

and $q_m = q_m(x; \theta)$ defined by (3.2) with $\theta = \hat{d}_{jm}$ behaves like

$$q_m = \alpha_m \gamma(d_j - d_m; \theta) + O(d^{-1+\sigma}) = \alpha_m \gamma(-\theta; \theta) + O(d^{-1+\sigma}) = \alpha_m \pi + O(d^{-1+\sigma})$$

on $S_j = \operatorname{supp} s_j$. Thus we obtain

$$\|s_j \left(R(E + i0; H_a) - R(E + i0; H_b)\right) s_j\| = O(d^{-1+c\sigma})$$

and (4.13) is verified by making repeated use of the above argument.

We end the section by proving a lemma similar to Lemma 4.2. In later application, the lemma is used when centers are all on segment (3.27) for some direction $\theta \in S^1$.

Lemma 4.3 *Assume that a subset $a \subset \{1, \dots, N\}$ satisfy* (3.27) *for some $\theta \in S^1$. Let d_m, $m \in a$, be an interior point of segment* (3.27) *and let d_l, $l \notin a$, be also on the segment.*

(1) *Assume that H_a satisfy* (4.4). *Then*

$$b_{l1} R(E + i0; H_a) Z_{am}^+ = \tilde{r}_L, \qquad Z_{am}^- R(E + i0; H_a) b_{l1} = \tilde{r}_L$$

for $Z_{am}^\pm = Z_{am}^\pm(\theta)$ defined by (3.28).

(2) *Let $b = a \setminus \{m\}$. Assume that H_b satisfy* (4.4). *Then* (1) *is true for $K_{am}(\theta)$.*

Proof. The lemma can be shown in almost the same way as Lemma 4.2. As is seen in section 3,

$$|x - y - 2t\xi| \geq c\left(|t| + |y| + d\right)^{1/2+\sigma}$$

for $\pm t > 0$, when $x \in B_l$ and $(y, \xi) \in \operatorname{supp} Z_{am}^\pm$. This implies the desired results. $\square$

Lemmas 4.1, 4.2 and 4.3 are often used in proving Theorem 4.1 in the next section. We should note that Theorem 4.1 is not used in the above proof of these lemmas.

5. Resolvent estimates

The present section is devoted to proving Theorem 4.1. We denote by $|a|$ the number of elements in a. We prove the theorem by induction on $|a|$, $0 \le |a| \le N$. The proof is rather lengthy and it is divided into several steps. Throughout the proof, we use the notation $\|Q\| \simeq O(d^\nu)$ when a bounded operator $Q : L^2 \to L^2$ obeys the bound $\|Q\| = O(d^{\nu+\varepsilon})$ for any $\varepsilon > 0$ small enough.

Proof of Theorem 4.1. (1) We start by the case $H_a = H_0 = -\Delta$ with $|a| = 0$. We know by the principle of limiting absorption that

$$\langle x\rangle^{-\rho} R(E \pm i0; H_0)\langle x\rangle^{-\rho} : L^2 \to L^2$$

is bounded for $\rho > 1/2$. Hence (4.3) and (4.4) follow at once. The resolvent $R(E + i0; H_0)$ has the kernel

$$G_0(x, y; E) = (i/4)H_0^{(1)}(\sqrt{E}|x - y|)$$

and the Hankel function $H_0^{(1)}(z)$ of first kind and order zero is known to behave like

$$H_0^{(1)}(z) = (2/\pi)^{1/2} \exp(i(z - \pi/4))z^{-1/2} \left(1 + O(|z|^{-1})\right) \tag{5.1}$$

at infinity. If $j \ne k$, then

$$\|b_{j1}R(E + i0; H_0)b_{k1}\| = O(d^\sigma) \tag{5.2}$$

by evaluating the Hilbert–Schmidt norm. This is still valid for $j = k$ by the principle of limiting absorption, so that (4.5) is obtained. Similarly (4.6) and (4.7) can be shown to hold true for $c = 1$.

(2) Next we consider the case $|a| = 1$. In this case, $H_a = H_m = H(A_m)$ for some m, $1 \le m \le N$. We know again by the principle of limiting absorption that

$$\langle x - d_m\rangle^{-\rho} R(E \pm i0; H_a)\langle x - d_m\rangle^{-\rho} : L^2 \to L^2$$

is bounded for $\rho > 1/2$. Hence (4.3) and (4.4) follow and we have

$$\|b_{m1}R(E + i0; H_a)b_{m1}\| \simeq O(1). \tag{5.3}$$

This implies (4.5) with $j = k = m$.

We shall prove (4.5) in the case when at least one of j and k is not m. Set $A_{jk} = b_{j1}R(E + i0; H_a)b_{k1}$. We may assume that $k \ne m$. If $k = m$, then a similar argument applies to A^*_{jm} with $j \ne m$. Let ψ_m be defined by (3.8). Then $\psi_m b_k = b_k$ and we have

$$A_{jk} = b_{j1}\psi_m R(E + i0; K_{am})b_{k1} + b_{j1}R(E + i0; H_a)V_{am}R(E + i0; K_{am})b_{k1}$$

by Lemma 3.1, where $K_{am} = K_{am}(\theta) = e^{iq_m}H_0e^{-iq_m}$ is defined by (3.6) with $a = \{m\}$. The direction $\theta \in S^1$ is specified later. The first operator on the right side obeys the bound (5.2), and hence it satisfies (4.5). We estimate the second operator. First we deal with the case that $j = m$ and $k \neq m$. We take $\theta = \widehat{d}_{km}$. Then it is obvious that m and $\theta = \widehat{d}_{km}$ satisfy (3.12). We decompose $V_{am} = V_{am}(\theta)$ as in (3.25) and use Lemma 4.2. We have already shown that H_a satisfies (4.4). The symbol $Y_{am}^{\pm}(x, \xi)$ has support in

$$\{|x - d_m| > Md^{1/2+\sigma},\ |(\widehat{x - d_m}) - \theta| < \delta,\ E/3 < |\xi|^2 < 3E,\ \pm\hat{\xi} \cdot \theta > -\delta\}.$$

We can take $M \gg 1$ so large that the free particles starting from $\operatorname{supp} Y_{am}^{+}$ and $\operatorname{supp} Y_{am}^{-}$ fulfill the assumption in the lemma for $l = j\ (= m)$ and $l = k$ respectively. Hence we can show

$$b_{m1} R(E + i0; H_a) Y_{am}^{\pm} R(E + i0; K_{am}) b_{k1} = \tilde{r}_L$$

by making use of an argument similar to that in the proof of Lemma 4.2. We can also show

$$b_{m1} R(E + i0; H_a) Y_{am}^{\infty} R(E + i0; K_{am}) b_{k1} \ = \tilde{r}_L$$

by calculus of pseudodifferential operators. The coefficients of $Y_{am} = h_m V_{am}$ have support in $\tilde{B}_m = \{|x - d_m| < 2Md^{1/2+\sigma}\}$ and satisfy (3.10). Set $\tilde{b}_{m1}(x) = \tilde{b}_m(x)\langle x - d_m\rangle^{-1/2}$ for the characteristic function $\tilde{b}_m$ of $\tilde{B}_m$. Then it follows from (5.2) that $\|\tilde{b}_{m1} R(E + i0; K_{am}) b_{k1}\| = O(d^{\sigma})$ and hence we have

$$\|\langle x - d_m\rangle^{1/2} Y_{am} R(E + i0; K_{am}) b_{k1}\| = O(d^{\sigma})$$

by elliptic estimate. The estimate (5.3) remains true for $\tilde{b}_{m1}$, so that

$$\|b_{m1} R(E + i0; H_a) Y_{am} R(E + i0; K_{am}) b_{k1}\| \simeq O(d^{\sigma}).$$

Thus we have proved

$$\|A_{mk}\| = \|b_{m1} R(E + i0; H_a) b_{k1}\| \simeq O(d^{\sigma}). \tag{5.4}$$

Next we consider the case that $j \neq m$ and $k \neq m$. The argument is divided into two cases (a) and (b) according to the location of three centers $d_j,\ d_k$ and d_m : (a) three centers are on an even segment with d_m as an interior point ; (b) otherwise. Assume case (b). We take the direction perpendicular to $\hat{d}_{jk}$ as θ if three centers are not on an even segment, and $\theta = \hat{d}_{km}$ if three centers are on an even segment with d_m as an endpoint. Then m and θ satisfy (3.12). We decompose $V_{am}(\theta)$ as in (3.25). By (5.2) and (5.4), we have

$$\|b_{j1} R(E + i0; H_a) Y_{am} R(E + i0; K_{am}) b_{k1}\| \simeq O(d^{2\sigma}).$$

We use Lemma 4.2 and repeat the same argument as above for the other operators with $Y_{am}^{\pm}$ and Y_{am}^{∞}. Then we obtain

$$\|A_{jk}\| = \|b_{j1} R(E + i0; H_a) b_{k1}\| \simeq O(|d|^{2\sigma}). \tag{5.5}$$

We note that this is true even for $j = k$. If three centers are on the segment (3.27) with d_m as an interior point for some direction θ, then we can show a similar bound by decomposing $V_{am}(\theta)$ as in (3.28) and by making use of Lemma 4.3. Thus we have proved (4.5) with $c > 2$.

We show that (4.5) implies (4.6). We set $B_{jk} = b_{j1}R(E + i0; H_a)s_{k1}$ for $j \neq k$. If $j = m$ and $k \neq m$, then we take $\theta = \widehat{d}_{km}$ and we have

$$B_{mk} = b_{m1}\psi_m R(E + i0; K_{am})s_{k1} + b_{m1}R(E + i0; H_a)V_{am}R(E + i0; K_{am})s_{k1}$$

with $V_{am} = V_{am}(\theta)$. We decompose V_{am} as in (3.25) and use Lemma 4.2 and (5.3) to obtain that $\|B_{mk}\| \simeq O(d^{-1/4+\sigma})$. If $j \neq m$ and $k = m$, then we take $\theta = \widehat{d}_{jm}$ and we apply a similar argument to

$$B^*_{jm} = s_{m1}\psi_m R(E - i0; K_{am})b_{j1} + s_{m1}R(E - i0; H_a)V_{am}R(E - i0; K_{am})b_{j1}.$$

We have $\|B_{jm}\| \simeq O(d^{-1/4+\sigma})$ by decomposing V_{am} as in (3.14). The same bound is also obtained in the case that $j \neq m$ and $k \neq m$. We divide the argument to the two cases (a) and (b) above and we take θ as before. In case (b), we decompose $V_{am}(\theta)$ as in (3.14) and use Lemma 4.2, while in case (a), we decompose $V_{am}(\theta)$ as in (3.28) and use Lemma 4.3. In either case, we have by (5.4) that

$$\|B_{jk}\| = O(d^{-1/4+c\sigma}) \tag{5.6}$$

with $c > 2$. Thus (4.6) is obtained. Similarly we can show that (4.6) implies (4.7). Set $C_{jk} = s_{j1}R(E + i0; H_a)s_{k1}$ for $j \neq k$. If either j or k is m, then we get $\|C_{jk}\| \simeq O(d^{-1/2+\sigma})$ by decomposing V_{am} as in (3.14) and by making use of Lemma 4.2 and (5.3). If $j \neq m$ and $k \neq m$, then we decompose V_{am} as in (3.28) or (3.14) according to case (a) or (b). It follows from (5.6) that $\|C_{jk}\| = O(d^{-1/2+c\sigma})$ with $c > 4$. Thus we have proved (4.3) $\sim$ (4.7) in the case $|a| = 1$.

(3) We proceed to the case $|a| = n \geq 2$, assuming the case $0 \leq |a| \leq n - 1$ as inductive assumption. We assume that (4.3) $\sim$ (4.7) are true for some $c = c_{n-1}$. We begin by proving (4.3) and (4.4). This step occupies the core of the proof. We set

$$Q_\nu = r_\nu R(E + i0; H_a)r_\nu, \qquad P_j = r_\nu R(E + i0; H_a)b_{j1}$$

for $\nu \gg 1$, where $r_{\pm\nu}$ is the multiplication operator defined by (4.2). Let j, $1 \leq j \leq N$, be fixed arbitrarily through the step. We take $m \in a$ such that $|d_{jm}| = \max\{|d_{jk}| : k \in a\}$ and we write Q_ν as the sum of two operators

$$Q_{1\nu} = r_\nu R(E + i0; H_a)\psi_m r_\nu, \quad Q_{2\nu} = r_\nu R(E + i0; H_a)(1 - \psi_m)r_\nu,$$

where ψ_m is defined by (3.8). Define the auxiliary operator $K_{am} = K_{am}(\theta_m)$ by (3.6) with $\theta_m = \hat{d}_{jm}$. Then Lemma 3.1 yields

$$Q_{1\nu} = r_\nu\psi_m R(E + i0; K_{am})r_\nu + r_\nu R(E + i0; H_a)V_{am}R(E + i0; K_{am})r_\nu.$$

By inductive assumption, the first operator on the right side obeys the bound $O(d^{-\nu})$. To estimate the second operator, we decompose $V_{am} = V_{am}(\theta_m)$ as in (3.14). We have

$$\|r_\nu R(E+i0;H_a)X_{am}R(E+i0;K_{am})r_\nu\| = O(d^{-\nu/2})\|P_m\|$$

again by inductive assumption. Recall that $R(E+i0;H_a)X^\infty_{am}$ is approximated in the form (3.24). Hence a calculus of pseudodifferential operators yields

$$\|r_\nu R(E+i0;H_a)X^\infty_{am}R(E+i0;K_{am})r_\nu\| = O(d^{-\nu}) + O(d^{-L})\|Q_\nu\|$$

for any $L \gg 1$. We prove the following two estimates

$$\begin{aligned} \|r_\nu R(E+i0;H_a)X^+_{am}r_{-\nu/3}\| &= O(d^{-\nu/2}) + O(d^{-L})\|Q_\nu\|, && (5.7)\\ \|r_{-\nu/3}X^-_{am}R(E+i0;K_{am})r_{\nu/2}\| &= O(1) && (5.8)\end{aligned}$$

after completing the proof. Let $\Pi_j(\theta)$ be defined by (3.5). We denote by $\pi_m(x)$ the characteristic function of $\Pi_m(\theta_m)$ and we set $R_m = r_\nu R(E+i0;H_a)\pi_m r_{\nu/3}$. Since

$$\|r_{\nu/3}R(E+i0;K_{am})r_\nu\| = O(d^{-2\nu/3})\|r_{\nu/3}R(E+i0;K_{am})r_{\nu/3}\| = O(d^{-\nu}),$$

we have

$$\|r_\nu R(E+i0;H_a)X^+_{am}R(E+i0;K_{am})r_\nu\| = O(d^{-3\nu/2}) + O(d^{-L})\|Q_\nu\|$$

by (5.7), and since $\pi_m X^-_{am} = X^-_{am}$, it follows from (5.8) that

$$\|r_\nu R(E+i0;H_a)X^-_{am}R(E+i0;K_{am})r_\nu\| = O(d^{-\nu/2})\|R_m\|.$$

We combine the above estimates to obtain

$$\|Q_{1\nu}\| = O(d^{-\nu}) + O(d^{-\nu/2})\left(\|P_m\| + \|R_m\|\right) + O(d^{-L})\|Q_\nu\|. \qquad (5.9)$$

We evaluate the other operator $Q_{2\nu}$. Let $m \in a$ be as above. We take $l \in a$ such that $|d_{ml}| = \max\{|d_{mk}| : k \in a\}$. Since $|a| \ge 2$, $l \ne m$. We define $K_{al} = K_{al}(\theta_l)$ by (3.6) with $\theta_l = \widehat{d}_{ml}$. Since $1-\psi_m$ vanishes around d_l, we have

$$Q_{2\nu} = r_\nu(1-\psi_m)R(E+i0;K_{al})r_\nu + r_\nu R(E+i0;H_a)W_{al}R(E+i0;K_{al})r_\nu$$

by Lemma 3.1, where

$$W_{al} = [1-\psi_m, K_{al}] - (H_a - K_{al})(1-\psi_m).$$

As is easily seen, $(H_a - K_{al})(1-\psi_m) = 0$ and hence the coefficients of W_{al} have support in $\{d^\sigma/2 < |x - d_m| < d^\sigma\}$ only. Thus we have

$$\|Q_{2\nu}\| = O(d^{-\nu}) + O(d^{-\nu/2})\|P_m\|.$$

This, together with (5.9), implies that

$$\|Q_\nu\| = O(d^{-\nu}) + O(d^{-\nu/2}) \left(\|P_m\| + \|R_m\|\right). \tag{5.10}$$

The next task is to evaluate P_m and R_m on the right side of (5.10). Since $\psi_l b_m = b_m$ for $l \neq m$, we have

$$P_m = r_\nu \psi_l R(E + i0; K_{al}) b_{m1} + r_\nu R(E + i0; H_a) V_{al} R(E + i0; K_{al}) b_{m1} \tag{5.11}$$

with $V_{al} = V_{al}(\theta_l)$. The first operator obeys the bound $O(d^{-\nu/2})$ by inductive assumption. We estimate the second operator. Note that $l \in a$ and $\theta_l = \widehat{d}_{ml}$ satisfy (3.12). We decompose V_{al} as in (3.14). The coefficients of X_{al} have support in $\tilde{S}_l = \{|x - d_l| < 2Md^\sigma\}$, and hence we have

$$\|\tilde{s}_{l1} R(E + i0; K_{al}) b_{m1}\| = O(d^{-1/4+c\sigma})$$

by making use of (4.6) as inductive assumption for K_{al}, where $\tilde{s}_{l1}(x) = \tilde{s}_l(x)\langle x - d_l\rangle^{-1/2}$ with characteristic function $\tilde{s}_l$ of $\tilde{S}_l$. This implies that

$$\|r_\nu R(E + i0; H_a) X_{al} R(E + i0; K_{al}) b_{m1}\| = o(1)\|P_l\|$$

for $0 < \sigma \ll 1$ small enough. We claim that

$$\|r_\nu R(E + i0; H_a) X_{al}^+ r_{-\nu/3}\| = O(d^{-\nu/2}) + O(d^{-L})\|Q_\nu\|. \tag{5.12}$$

This is also verified after completing the proof. Since

$$\|r_{\nu/3} R(E + i0; K_{al}) b_{m1}\| = O(d^{-\nu/6})$$

by inductive assumption, it follows from (5.12) that

$$\|r_\nu R(E + i0; H_a) X_{al}^+ R(E + i0; K_{al}) b_{m1}\| = O(d^{-\nu/2}) + O(d^{-L})\|Q_\nu\|.$$

By Lemma 4.2, we have $\|r_{-L} X_{al}^- R(E + i0; K_{al}) b_{m1}\| = O(1)$ for any $L \gg 1$, so that

$$\|r_\nu R(E + i0; H_a) X_{al}^- R(E + i0; K_{al}) b_{m1}\| = O(d^{-L})\|Q_\nu\|.$$

A similar bound is true for X_{al}^∞. Thus we have shown that P_m obeys

$$\|P_m\| = O(d^{-\nu/2}) + o(1)\|P_l\| + O(d^{-L})\|Q_\nu\|. \tag{5.13}$$

A similar argument applies to R_m. We represent R_m as

$$R_m = r_\nu \psi_l R(E + i0; K_{al}) \pi_m r_{\nu/3} + r_\nu R(E + i0; H_a) V_{al} R(E + i0; K_{al}) \pi_m r_{\nu/3}.$$

The first operator obeys

$$\|r_\nu \psi_l R(E + i0; K_{al}) \pi_m r_{\nu/3}\| = O(d^{-2\nu/3})\|r_{\nu/3} R(E + i0; K_{al}) r_{\nu/3}\| = O(d^{-\nu}).$$

We estimate the second operator by decomposing $V_{al} = V_{al}(\theta_l)$ as in (3.14). Since $\|\tilde{s}_{l1}R(E+i0;K_{al})\pi_m r_{\nu/3}\| = O(d^{-\nu/6})$, we have

$$\|r_\nu R(E+i0;H_a)X_{al}R(E+i0;K_{al})\pi_m r_{\nu/3}\| = o(1)\|P_l\|$$

and it follows from (5.12) that

$$\|r_\nu R(E+i0;H_a)X_{al}^+R(E+i0;K_{al})\pi_m r_{\nu/3}\| = O(d^{-\nu/2}) + O(d^{-L})\|Q_\nu\|.$$

The free particle starting from $\operatorname{supp} X_{al}^-$ never passes over $\Pi_m(\theta_m)$ for $t<0$. This enables us to use the same argument as in the proof of Lemma 4.2 and we can show that

$$\|r_{-L}X_{al}^-R(E+i0;K_{al})\pi_m r_{\nu/3}\| = O(1)$$

for any $L \gg 1$. The operator X_{al}^∞ also satisfies the same bound. Thus we get

$$\|R_m\| = O(d^{-\nu/2}) + o(1)\|P_l\| + O(d^{-L})\|Q_\nu\|. \tag{5.14}$$

We have to estimate P_l on the right side of (5.13) and (5.14). To do this, we represent P_l as

$$P_l = r_\nu\psi_m R(E+i0;K_{am})b_{l1} + r_\nu R(E+i0;H_a)V_{am}R(E+i0;K_{am})b_{l1}$$

with $V_{am} = V_{am}(\theta_m),\ \theta_m = \hat{d}_{jm}$. Then we have

$$\|P_l\| = O(d^{-\nu/2}) + o(1)\|P_m\| + O(d^{-L})\|Q_\nu\| \tag{5.15}$$

by the same argument as used to prove (5.13). This, together with (5.10), (5.13) and (5.14), forms a closed system of inequalities on $Q_\nu,\ P_m,\ R_m$ and P_l. Hence we can obtain $\|Q_\nu\| = O(d^{-\nu})$. This proves (4.3). We can also obtain $\|P_m\| = O(d^{-\nu/2})$ simultaneously. To prove (4.4), we represent P_j as

$$P_j = r_\nu\psi_m R(E+i0;K_{am})b_{j1} + r_\nu R(E+i0;H_a)V_{am}R(E+i0;K_{am})b_{j1}$$

for $j \neq m$. Then we can show

$$\|P_j\| = O(d^{-\nu/2}) + o(1)\|P_m\| + O(d^{-L})\|Q_\nu\|$$

again in the same way as used to prove (5.13). Hence $\|P_j\| = O(d^{-\nu/2})$ and (4.4) is proved.

(4) Let m and l, $m \neq l$, be as above. Both m and l are in a. The aim of this step is to prove that

$$\|F_1\| = \|b_{m1}R(E+i0;H_a)\tilde{b}_{m1}\| = O(d^{c\sigma}) \tag{5.16}$$

for $c = c_{n-1}$. To prove this, we set $F_2 = b_{m1}R(E+i0;H_a)\tilde{s}_{l1}$. Then

$$\begin{aligned}
F_1 &= b_{m1}\psi_l R(E+i0;K_{al})\tilde{b}_{m1} + b_{m1}R(E+i0;H_a)V_{al}R(E+i0;K_{al})\tilde{b}_{m1},\\
F_2 &= b_{m1}\psi_m R(E+i0;K_{am})\tilde{s}_{l1} + b_{m1}R(E+i0;H_a)V_{am}R(E+i0;K_{am})\tilde{s}_{l1}
\end{aligned}$$

by Lemma 3.1. We decompose V_{al} as in (3.14) and V_{am} as in (3.25). In the previous step, we have already proved (4.4) for H_a, and hence H_a satisfies the assumption in Lemma 4.2. If we use inductive assumptions for K_{am} and K_{al}, then we have

$$\|F_1\| = O(d^{c\sigma}) + O(d^{-1/4+c\sigma})\|F_2\|, \quad \|F_2\| = O(d^{-1/4+c\sigma}) + O(d^{-1/4+c\sigma})\|F_1\|$$

for $c = c_{n-1}$. This implies (5.16).

(5) We proceed to proving (4.5), (4.6) and (4.7). Once (5.16) has been established, (4.5) is proved in almost the same way as in the case $|a| = 1$. We give only a sketch. We set $A_{jk} = b_{j1}R(E + i0; H_a)b_{k1}$. We first deal with the case that j or k is in a. Assume for brevity that $k \in a$. Then we take $m \in a$ as before for j. If $k = m$, we have

$$A^*_{jm} = b_{m1}\psi_m R(E - i0; K_{am})b_{j1} + b_{m1}R(E - i0; H_a)V_{am}R(E - i0; K_{am})b_{j1}$$

for $V_{am} = V_{am}(\theta)$ with $\theta = \theta_m = \hat{d}_{jm}$. We decompose V_{am} as in (3.25) and use Lemma 4.2. Then it follows from (5.16) that

$$\|A_{jm}\| = O(d^{2c\sigma}) \tag{5.17}$$

for $c = c_{n-1}$. If $k \neq m$, a similar argument applies to

$$A_{jk} = b_{j1}\psi_m R(E + i0; K_{am})b_{k1} + b_{j1}R(E + i0; H_a)V_{am}R(E + i0; K_{am})b_{k1}.$$

We use (5.17) and the inductive assumption for K_{am} to obtain

$$\|A_{jk}\| = O(d^{3c\sigma}). \tag{5.18}$$

This is true even for $j = k \in a$. If neither j nor k is in a, then the argument is divided into two cases : (a) $d_j,\ d_k$ and $d_l,\ l \in a$, are all on an even segment with d_l as interior points ; (b) otherwise. We take $m \in a$ and θ as in the case $|a| = 1$, and we decompose $V_{am}(\theta)$ as in (3.28) or (3.25) according to case (a) or (b). In either case, we can show $\|A_{jk}\| = O(d^{4c\sigma})$ by (5.18). Thus we have proved (4.5) with $c = 4c_{n-1}$. We can also show in almost the same way as in the case $|a| = 1$ that (4.5) implies (4.6) and that (4.6) implies (4.7). We skip the details. Thus (4.3) $\sim$ (4.7) have been proved in the case $|a| = n$ and the proof is now complete. □

We prove (5.7), (5.8) and (5.12) which remain unproved. The proof uses Lemma 4.1. The operator $R(E + i0; H_a)X^+_{am}$ is approximated by (3.19). This, together with Lemma 4.1, implies (5.7), and (5.12) is shown in exactly the same way. The approximation for $X^-_{am}R(E + i0; K_{am})$ takes the form (3.22), and

$$\|r_{\nu/2}R(E + i0; K_{am})r_{\nu/2}\| = O(d^{-\nu/2})$$

by inductive assumption. Hence (5.8) again follows from Lemma 4.1.

6. Scattering by a chain of fields : Proof of Theorem 1.2

The remaining two sections are devoted to proving the second main theorem. We recall our situation. We are studying the magnetic scattering by a chain of three point–like fields on an even line. The three centers are along the direction $\omega_1 = (1, 0)$ in the order of d_1, d_2 and d_3. If we identify these centers with coordinates $(d_j, 0)$, then $d_1 < d_2 < d_3$. We derive the asymptotic formula as $d \to \infty$ for the backward scattering amplitude $f_d = f_d(\omega_1 \to -\omega_1; E)$ only. A similar argument applies to $f_d(\omega_1 \to \tilde{\omega}; E)$ with $\tilde{\omega} \neq -\omega_1$, and the derivation is easier. For brevity, we further assume that the three centers are at even intervals, so that $|d_{12}| = |d_{23}| = d$ and $\delta_0 = 1$ in (1.9).

Throughout the proof, we use the notation $f \sim g$ to denote $f = g + o(1)$ as $d \to \infty$. We follow almost the same argument as in the proof of Theorem 1.1 to represent the amplitude under consideration. Let $e_0(x) \in C^\infty(\boldsymbol{R}^2)$ be again a smooth function such that $\operatorname{supp} e_0 \subset \Sigma(d^\sigma, \omega_1, \delta)$ and $e_0 = 1$ on $\Sigma(2d^\sigma, \omega_1, 2\delta)$. We set $e_j(x) = e_0(x - d_j)$, $1 \le j \le 3$, and define $\theta_d(x) = \sum_{j=1}^{3} \alpha_j \gamma(x - d_j; \omega_1)$. Then

$$(H_d - E)e^{i\theta_d} e_1 = e^{i\theta_d}(H_0 - E)e_1$$

and we obtain

$$f_d \sim (ic(E)/4\pi)(R(E + i0; H_d)e^{i\theta_d}[H_0, e_1]\varphi_0, e^{i\theta_d}[H_0, e_1]\overline{\varphi}_0), \tag{6.1}$$

where $\varphi_0 = \varphi_0(x; \omega_1, E) = \exp(i\sqrt{E}x_1)$ (see (2.14) and (4.8)). We introduce a partition of unity over the x_2 axis. We set $\chi_{0j}(x_2) = \chi(|x_2|/3^j d^\sigma)$ and

$$\chi_{1j}(x_2) = (1 - \chi_{0j}(x_2))\, \chi(|x_2|/3^j d^{1/2+\sigma}), \qquad \chi_{2j}(x_2) = 1 - \chi(|x_2|/3^j d^{1/2+\sigma})$$

for $1 \le j \le 3$, where $\chi \in C_0^\infty[0, \infty)$ is a cut–off function with property (2.3). Then $\chi_{0j} + \chi_{1j} + \chi_{2j} = 1$ for each j. We decompose the term on the right side of (6.1) into the sum of

$$f_{lm} = (ic(E)/4\pi)(R(E + i0; H_d)e^{i\theta_d}\chi_{l1}[H_0, e_1]\varphi_0, e^{i\theta_d}\chi_{m1}[H_0, e_1]\overline{\varphi}_0)$$

with $0 \le l, m \le 2$. The coefficients of differential operator $\chi_{01}[H_0, e_1]$ have support in $\{|x - d_1| < c\, d^\sigma\}$ for some $c > 1$. If we use (4.13) and Lemma 2.2, then we can prove

$$f_{00} \sim f_1(\omega_1 \to -\omega_1; E)$$

in the same way as in the proof of Theorem 1.1. We assert that $f_{01} = o(1)$ and $f_{10} = o(1)$. Note that

$$\|\chi_{11}\langle x - d_1\rangle^{1/2}[H_0, e_1]\varphi_0\|_{L^2} = O(d^{1/4+\sigma/2}),$$

where $\|\ \|_{L^2}$ denotes the L^2 norm in $L^2(\boldsymbol{R}^2)$. We define

$$\beta_1(\xi) = \chi(|\xi - \sqrt{E}\omega_1|/\delta^{1/2}), \qquad \tilde{\beta}_1(\xi) = \chi(|\xi + \sqrt{E}\omega_1|/\delta^{1/2})$$

and write β_1 and $\tilde{\beta}_1$ for $\beta_1(D_x)$ and $\tilde{\beta}_1(D_x)$. The assertion is obtained as a consequence of the following lemma.

Lemma 6.1 *Let $s_j(x)$ be the characteristic function of S_j defined by* (1.12) *and let $f_1(x)$ be defined by $f_1 = f(x)\langle x - d_1\rangle^{-1/2}$ with characteristic function $f(x)$ of set*

$$F = \{x = (x_1, x_2) : d^\sigma/C_1 < |x_2| < C_1 d^{1/2+\sigma},\ \ \delta < |(\widehat{x - d_1}) - \omega_1| < 2\delta\}$$

for $C_1 > 1$. Then one has

$$\|s_1\tilde{\beta}_1 R(E + i0; H_d)\beta_1 f_1\| + \|s_1\beta_1 R(E - i0; H_d)\tilde{\beta}_1 f_1\| = O(d^{-3/4+c\sigma})$$

for some $c > 0$.

Proof. We consider the first operator only. If $x \in F$, then $c_1 d^\sigma < x_1 - d_1 < c_2 d^{1/2+\sigma}$ for some $c_1,\ c_2 \gg 1$. Hence we can prove that

$$\|s_1\tilde{\beta}_1 R(E + i0; H_1)\beta_1 f_1\| = O(d^{-L})$$

for any $L \gg 1$, because the classical particle starting from $(x, \xi) \in F \times \operatorname{supp}\beta_1$ at time $t = 0$ moves like the free particle for $t > 0$. Define

$$K_{a2} = K_{a2}(\omega_1) = e^{iq_2} H_1 e^{-iq_2} = H(A_a - A_2 + \nabla q_2) = H(A_1 + \nabla q_2)$$

by (3.6) with $a = \{1, 2\}$. Then

$$R(E + i0; H_a)\psi_2 = \psi_2 R(E + i0; K_{a2}) + R(E + i0; H_a)V_{a2}R(E + i0; K_{a2})$$

by Lemma 3.1, where $\psi_2(x)$ and $V_{a2} = V_{a2}(\omega_1)$ are defined by (3.8) and (3.9) respectively. We evaluate the second operator on the right side by decomposing V_{a2} as in (3.14). By Theorem 4.1,

$$\|s_1 R(E + i0; H_a)s_2\| = O(d^{-1/2+c\sigma}), \quad \|s_2 R(E + i0; K_{a2})f_1\| = O(d^{-1/4+c\sigma})$$

and hence it follows that

$$\|s_1 R(E + i0; H_a)X_{a2}R(E + i0; K_{a2})f_1\| = O(d^{-3/4+c\sigma})$$

for another $c > 0$. The other operators with $X_{a2}^{\pm}$ and X_{a2}^{∞} are evaluated by making use of the argument as in the proof of Theorem 4.1. We can show that these operators obey the bound $O(d^{-L})$. Since $\psi_2 = 1$ on the support of f_1 by (3.8), we get $\psi_2\beta_1 f_1 = \beta_1 f_1 + \tilde{r}_L$ and hence

$$\|s_1\tilde{\beta}_1 R(E + i0; H_a)\beta_1 f_1\| = O(d^{-3/4+c\sigma})$$

for H_a with $a = \{1, 2\}$. Thus the desired bound for $H_d = H_a,\ a = \{1, 2, 3\}$, is obtained by repeating the same argument as above. □

We fix b as the subset $b = \{2, 3\} \subset \{1, 2, 3\}$, and define the auxiliary operator

$$K_b = e^{iq_1} H_b e^{-iq_1} = H(A_2 + A_3 + \nabla q_1) = H(A_d - A_1 + \nabla q_1) = K_{a1}(-\omega_1) \qquad (6.2)$$

by (3.6) with $a = \{1, 2, 3\}$, where $q_1(x) = q_1(x; -\omega_1) = \alpha_1\gamma(x - d_1; -\omega_1)$ on

$$\{x : |x - d_1| > d^\sigma,\ \ |(\widehat{x - d_1}) + \omega_1| > \delta\},$$

so that $H_d = K_b$ there (see (3.7)).

Lemma 6.2 *Let $b = \{2,3\}$ be as above and let f_1 be as in Lemma* 6.1*. Denote by $g(x)$ the characteristic function of set*

$$G = \{x : x \in \Pi_1(-\omega_1),\ \ |x - d_1| < Cd^\sigma\}$$

with $C \gg 1$, where $\Pi_j(\theta)$ is defined by (3.5)*. Then*

$$\|gR(E+i0;H_b)\beta_1 f_1\| + \|f_1\tilde{\beta}_1 R(E+i0;H_b)g\| = O(d^{-3/4+c\sigma})$$

and

$$\|f_1\tilde{\beta}_1 R(E+i0;H_b)\beta_1 f_1\| = O(d^{-1/2+c\sigma}).$$

Proof. We first consider the operator $gR(E+i0;H_b)\beta_1 f_1$. The free particle starting from $(x,\xi) \in F \times \operatorname{supp}\beta_1$ at time $t=0$ never passes over G for $t>0$, and hence we have $\|gR(E+i0;H_0)\beta_1 f_1\| = O(d^{-L})$ for any $L \gg 1$. Let

$$K_{a2} = K_{a2}(\omega_1) = e^{iq_2}H_0 e^{-iq_2} = H(\nabla q_2)$$

be defined by (3.6) with $a = \{2\}$. We apply Lemma 3.1 to H_2 and K_{a2}, and we use the same argument as in the proof of Lemma 6.1. By Theorem 4.1, we have

$$\|gR(E+i0;H_2)s_2\| = O(d^{-1/2+c\sigma}), \quad \|s_2R(E+i0;K_{a2})f_1\| = O(d^{-1/4+c\sigma}).$$

This implies that $\|gR(E+i0;H_2)\beta_1 f_1\| = O(d^{-3/4+c\sigma})$. We repeat this argument to obtain the desired bound for $gR(E+i0;H_b)\beta_1 f_1$, and a similar argument applies to $f_1\tilde{\beta}_1 R(E+i0;H_b)g$. The second bound can be also shown in a similar way. We give only a sketch for a proof. It is obvious that $\tilde{\beta}_1 R(E+i0;H_0)\beta_1 = 0$. By Theorem 4.1, we have

$$\|f_1R(E+i0;H_2)s_2\| + \|s_2R(E+i0;K_{a2})f_1\| = O(d^{-1/4+c\sigma})$$

for K_{a2} as above, and hence it follows that

$$\|f_1\tilde{\beta}_1 R(E+i0;H_2)\beta_1 f_1\| = O(d^{-1/2+c\sigma}).$$

The desired bound for H_b is obtained by repeating the argument again. □

Lemma 6.3 *Let f_1 be again as in Lemma* 6.1 *and let K_b be defined by* (6.2)*. Then*

$$\|f_1\tilde{\beta}_1\left(R(E+i0;H_d) - R(E+i0;K_b)\right)\beta_1 f_1\| = O(d^{-3/2+c\sigma})$$

and

$$\|f_1\tilde{\beta}_1 R(E+i0;H_d)\beta_1 f_1\| = O(d^{-1/2+c\sigma}).$$

Proof. This lemma is also proved in almost the same way as Lemma 6.1. We use Lemma 3.1 for H_d and K_b. Then

$$R(E \pm i0; H_d)\psi_1 - \psi_1 R(E \pm i0; K_b) = R(E \pm i0; H_d)V_1 R(E \pm i0; K_b)$$

and hence

$$\psi_1 R(E + i0; H_d) - R(E + i0; K_b)\psi_1 = R(E + i0; K_b)V_1^* R(E + i0; H_d),$$

where $V_1 = V_{a1}(-\omega_1)$ is defined by (3.9) with $a = \{1, 2, 3\}$. We first consider the operator $f_1\tilde{\beta}_1 R(E + i0; H_d)g$ with g in Lemma 6.2. We decompose V_1 as in (3.14). We see by Theorem 4.1 and Lemma 6.2 that

$$\|f_1\tilde{\beta}_1 R(E + i0; K_b)V_1^* R(E + i0; H_d)g\| = O(d^{-3/4+c\sigma})$$

and hence $\|f_1\tilde{\beta}_1 R(E + i0; H_d)g\| = O(d^{-3/4+c\sigma})$. This, together with Lemma 6.2 again, implies that

$$\|f_1\tilde{\beta}_1 R(E + i0; H_d)V_1 R(E + i0; K_b)\beta_1 f_1\| = O(d^{-3/2+c\sigma})$$

for another $c > 0$. Thus the first bound follows and the second one is obtained from Lemma 6.2 at once. □

We now analyze f_{11} by use of Lemma 6.3. It follows that

$$f_{11} \sim (ic(E)/4\pi)(R(E + i0; K_b)e^{i\theta_d}\chi_{11}[H_0, e_1]\varphi_0, e^{i\theta_d}\chi_{11}[H_0, e_1]\overline{\varphi}_0).$$

Let Z be the set consisting of two components $+$ and $-$. We write $\chi_{11} = \chi_{11+}(x_2) + \chi_{11-}(x_2)$, where $\chi_{11\pm}$ has support in $\{3d^\sigma < \pm x_2 < 6d^{1/2+\sigma}\}$. Then f_{11} admits the decomposition

$$f_{11} \sim (ic(E)/4\pi) \sum_{\varepsilon,\varepsilon' \in Z} (e^{-i\theta_d}R(E + i0; K_b)e^{i\theta_d}\chi_{11\varepsilon}[H_0, e_1]\varphi_0, \chi_{11\varepsilon'}[H_0, e_1]\overline{\varphi}_0).$$

The operator on the right side is represented as

$$e^{-i\theta_d}R(E + i0; K_b)e^{i\theta_d} = e^{-i(\theta_d - q_1)}R(E + i0; H_b)e^{i(\theta_d - q_1)}$$

with q_1 as in (6.2). We calculate the phase factor as

$$\theta_d(x) - q_1(x) = \theta_b(x) + \alpha_1\left(\gamma(x - d_1; \omega_1) - \gamma(x - d_1; -\omega_1)\right)$$

on $\operatorname{supp}\nabla e_1$, where $\theta_b(x) = \sum_{j=2}^3 \alpha_j\gamma(x - d_j; \omega_1)$. If, in addition, $x \in \operatorname{supp}\chi_{11\pm}$, then

$$\alpha_1\left(\gamma(x - d_1; \omega_1) - \gamma(x - d_1; -\omega_1)\right) = \mp\alpha_1\pi.$$

On the other hand, $[H_0, e_1]\varphi_0$ behaves like

$$[H_0, e_1]\varphi_0 = -2i\sqrt{E}\,(\partial_1 e_1)\,\varphi_0 + O(|x - d_1|^{-2})$$

and similarly for $[H_0, e_1]\overline{\varphi}_0$. If we set

$$u_{1\varepsilon}(x) = e^{-i\varepsilon\alpha_1\pi}\left(\partial_1 e_1\right)(x), \qquad \eta_{1\varepsilon}(x_2) = 2i\sqrt{E}\chi_{11\varepsilon}(x_2)$$

for $\varepsilon \in Z$, then it follows from Lemma 6.2 that

$$f_{11} \sim (ic(E)/4\pi) \sum_{\varepsilon,\varepsilon' \in Z} (R(E+i0; H_b)e^{i\theta_b}\eta_{1\varepsilon}u_{1\varepsilon}\varphi_0, e^{i\theta_b}\overline{\eta}_{1\varepsilon'}u_{1\varepsilon'}\overline{\varphi}_0). \tag{6.3}$$

We consider f_{lm} with $l = 2$ or $m = 2$. We shall show that these terms are all negligible. To see this, we write $\chi_{21} = \chi_{21+}(x_2) + \chi_{21-}(x_2)$, and we define

$$T_\pm(x, D_x) = \chi_{21\pm}[H_0, e_1]\beta_1\gamma_0, \qquad \tilde{T}_\pm(x, D_x) = \chi_{21\pm}[H_0, e_1]\tilde{\beta}_1\gamma_0,$$

where $\chi_{21\pm}$ has support in $\{\pm x_2 > 3d^{1/2+\sigma}\}$ and γ_0 is defined by $\gamma_0 = \gamma_0(D_{x_2})$ with $\gamma_0(\xi_2) = \chi(d^{1/2}|\xi_2|)$. The symbol $T_\pm(x, \xi)$ obeys

$$|\partial_x^\beta \partial_\xi^\alpha T_\pm(x, \xi)| \le C_{\alpha\beta}\left(|x - d_1| + d^{1/2+\sigma}\right)^{-|\beta|} d^{|\alpha|/2}$$

and similarly for $\tilde{T}_\pm(x, \xi)$. The classical particle starting from $\operatorname{supp} T_\pm$ at $t = 0$ passes far away from centers d_2 and d_3 for $t > 0$ and it moves like the free particle. This enables us to construct an approximation to $R(E+i0; H_d)T_\pm$ in a form similar to (3.29). We can construct a similar approximation for $\tilde{T}_\pm^* R(E+i0; H_d)$ also, and we can show

$$\tilde{T}_\pm^* R(E+i0; H_d)T_+ = \tilde{r}_L, \qquad \tilde{T}_\pm^* R(E+i0; H_d)T_- = \tilde{r}_L$$

in the same way as used to prove (4.10). This implies that $f_{lm} = o(1)$ for $l = 2$ or $m = 2$.

We end the section by summing up the results obtained above. Let $\theta_b(x)$, $u_{1\varepsilon}$ and $\eta_{1\varepsilon}$ be as in (6.3). Then f_d behaves like

$$\begin{aligned} f_d \ &\sim \ f_1(\omega_1 \to -\omega_1; E) \\ &+ \ (ic(E)/4\pi) \sum_{\varepsilon,\varepsilon' \in Z} (R(E+i0; H_b)e^{i\theta_b}\eta_{1\varepsilon}u_{1\varepsilon}\varphi_0, e^{i\theta_b}\overline{\eta}_{1\varepsilon'}u_{1\varepsilon'}\overline{\varphi}_0). \end{aligned} \tag{6.4}$$

7. Analysis on free resolvent : Completion of proof

In this section we complete the proof by calculating the coefficients of amplitudes $f_j(\omega_1 \to -\omega_1; E)$, $2 \le j \le 3$. Let $e_2(x) = e_0(x - d_2)$ be as in section 6. By definition, $e_2 = 1$ on $\operatorname{supp} \partial_1 e_1 \cap \operatorname{supp} \chi_{11}$, so that $e_2\eta_{1\varepsilon}u_{1\varepsilon} = \eta_{1\varepsilon}u_{1\varepsilon}$. If we make use of the relation

$$(H_b - E)e^{i\theta_b}e_2 = e^{i\theta_b}(H_0 - E)e_2,$$

then

$$(H_b - E)e^{i\theta_b}e_2 R(E+i0; H_0) = e^{i\theta_b}e_2 + e^{i\theta_b}[H_0, e_2]R(E+i0; H_0)$$

and hence we obtain the rather formal representation

$$R(E+i0;H_b)e^{i\theta_b}\eta_{1\varepsilon}u_{1\varepsilon}\varphi_0 = e^{i\theta_b}e_2R(E+i0;H_0)\eta_{1\varepsilon}u_{1\varepsilon}\varphi_0 - R(E+i0;H_b)v_\varepsilon$$

for the term on the right side of (6.4), where

$$v_\varepsilon = e^{i\theta_b}[H_0, e_2]R(E+i0;H_0)\eta_{1\varepsilon}u_{1\varepsilon}\varphi_0.$$

The argument below shows that $R(E+i0;H_b)v_\varepsilon$ makes sense. Similarly we define

$$\tilde{v}_\varepsilon = e^{i\theta_b}[H_0, e_2]R(E-i0;H_0)\overline{\eta}_{1\varepsilon}u_{1\varepsilon}\overline{\varphi}_0.$$

Then we have

$$R(E-i0;H_b)e^{i\theta_b}\overline{\eta}_{1\varepsilon}u_{1\varepsilon}\overline{\varphi}_0 = e^{i\theta_b}e_2R(E-i0;H_0)\overline{\eta}_{1\varepsilon}u_{1\varepsilon}\overline{\varphi}_0 - R(E-i0;H_b)\tilde{v}_\varepsilon.$$

Note that $\beta_1\varphi_0 = \varphi_0$ and $\tilde{\beta}_1\overline{\varphi}_0 = \overline{\varphi}_0$. Since $\tilde{\beta}_1R(E+i0;H_0)\beta_1 = 0$, a calculus of pseudodifferential operators shows that

$$\begin{aligned}(R(E+i0;H_0)\eta_{1\varepsilon}u_{1\varepsilon}\varphi_0, \overline{\eta}_{1\varepsilon'}u_{1\varepsilon'}\overline{\varphi}_0) &= O(d^{-L}) \\ (v_\varepsilon, e^{i\theta_b}e_2R(E-i0;H_0)\overline{\eta}_{1\varepsilon'}u_{1\varepsilon'}\overline{\varphi}_0) &= O(d^{-L})\end{aligned}$$

for any $L \gg 1$. Thus it follows from (6.4) that

$$f_d \sim f_1 + (ic(E)/4\pi)\sum_{\varepsilon,\varepsilon'\in Z}(R(E+i0;H_b)v_\varepsilon, \tilde{v}_{\varepsilon'}). \tag{7.1}$$

We decompose the second term on the right side into the sum of

$$f_{lm,\varepsilon\varepsilon'} = (ic(E)/4\pi)(R(E+i0;H_b)\chi_{l2}v_\varepsilon, \chi_{m2}\tilde{v}_{\varepsilon'}), \quad 0 \le l,\ m \le 2,$$

for each $\varepsilon,\ \varepsilon' \in Z$. We study the behavior as $d \to \infty$ of $f_{lm,\varepsilon\varepsilon'}$. The analysis in the present section is based on the following lemma.

Lemma 7.1 *Let B_j be defined by* (4.1) *and let $b_{j1}(x) = b_j(x)\langle x - d_j\rangle^{-1/2}$ be as in Theorem* 4.1. *Assume that $x = (x_1, x_2) \in B_2$ and $y = (y_1, y_2) \in B_1$, and denote by $K_\pm$ the integral operator with kernel*

$$K_\pm(x,y) = c_\pm(E)d^{-1/2}e^{\pm i\sqrt{E}x_1}\exp(\pm id^{-1}\sqrt{E}(x_2-y_2)^2/2)e^{\mp i\sqrt{E}y_1},$$

where $c_\pm(E) = \left(8\pi\sqrt{E}\right)^{-1/2}e^{\pm i\pi/4}$. Then

$$\|b_{21}\left(R(E\pm i0;H_0) - K_\pm\right)b_{11}\| = O(d^{-1/2+4\sigma}).$$

Proof. By (5.1), the Green kernel $G_0(x,y;E)$ of $R(E+i0;H_0)$ behaves like

$$G_0(x,y;E) = c_+(E)\exp(i\sqrt{E}|x-y|)|x-y|^{-1/2}\left(1+O(|x-y|^{-1})\right) \tag{7.2}$$

for $|x-y| \gg 1$. We calculate $|x-y|$ for (x,y) as in the lemma. By assumption, $|d_{12}| = |d_2 - d_1| = d$. We set $\tilde{x} = x_1 - d_2$ and $\tilde{y} = y_1 - d_1$. Then we have

$$\begin{aligned} |x-y| &= \left((d_{12}+\tilde{x}-\tilde{y})^2 + (x_2-y_2)^2\right)^{1/2} \\ &= d\left(1+2d^{-1}(\tilde{x}-\tilde{y}) + d^{-2}\left((\tilde{x}-\tilde{y})^2+(x_2-y_2)^2\right)\right)^{1/2}. \end{aligned}$$

Note that $|\tilde{x}|+|\tilde{y}|+|x_2|+|y_2| = O(d^{1/2+\sigma})$. Since

$$(1+t)^{1/2} = 1+t/2-t^2/8+O(t^3)$$

as $t \to 0$, a simple computation yields

$$|x-y| = (x_1-y_1) + d^{-1}(x_2-y_2)^2/2 + O(d^{-1/2+3\sigma}).$$

Thus the lemma is verified by evaluating the Hilbert–Schmidt norm. □

We apply the above lemma to $f_{lm,\varepsilon\varepsilon'}$ with $0 \le l,\ m \le 1$. Since

$$\int (\partial_1 e_1)(x_1,x_2)\,dx_1 = -1$$

for $x_2 \in \operatorname{supp}\chi_{11}$, we have

$$(K_+\eta_{1\varepsilon}u_{1\varepsilon}\varphi_0)(x) = -e^{-i\varepsilon\alpha_1\pi}\rho_{1\varepsilon}(x_2)\varphi_0,$$

where $\rho_{1\varepsilon}(x_2) = (Y\eta_{1\varepsilon})(x_2)$, and Y acts on $\eta(x_2)$ as

$$(Y\eta)(x_2) = c_+(E)d^{-1/2}\int \exp(id^{-1}\sqrt{E}(x_2-y_2)^2/2)\eta(y_2)\,dy_2. \tag{7.3}$$

Similarly we have

$$(K_-\overline{\eta}_{1\varepsilon}u_{1\varepsilon}\overline{\varphi}_0)(x) = -e^{-i\varepsilon\alpha_1\pi}\overline{\rho}_{1\varepsilon}(x_2)\overline{\varphi}_0.$$

The estimates obtained in Lemmas 6.1 and 6.3 remain true for H_b under natural modification. For example, we have

$$\|\langle x-d_2\rangle^{1/2}[H_0,e_2]\chi_{12}\tilde{\beta}_1 R(E+i0;H_b)\beta_1\chi_{12}[H_0,e_2]\langle x-d_2\rangle^{1/2}\| = O(d^{-1/2+c\sigma}),$$

which follows from the second bound in Lemma 6.3. If we take account of these estimates, then Lemma 7.1 shows that $f_{lm,\varepsilon\varepsilon'}$ behaves like

$$e^{-i\varepsilon\alpha_1\pi}e^{i\varepsilon'\alpha_1\pi}(ic(E)/4\pi)(R(E+i0;H_b)e^{i\theta_b}\chi_{l2}[H_0,e_2]\rho_{1\varepsilon}\varphi_0, e^{i\theta_b}\chi_{m2}[H_0,e_2]\overline{\rho}_{1\varepsilon'}\overline{\varphi}_0)$$

as $d \to \infty$, when $0 \le l,\ m \le 1$. We evaluate these terms. We use the integral formula

$$\int e^{\pm i\gamma t^2}\, dt = (\pi/\gamma)^{1/2}\, e^{\pm i\pi/4}, \quad \gamma > 0, \tag{7.4}$$

to analyze the behavior of the term with $l = m = 0$. Recall that $\eta_{1\varepsilon} = 2i\sqrt{E}\chi_{11\varepsilon}(x_2)$. We have

$$\rho_{1\varepsilon}(x_2) = (Y\eta_{1\varepsilon})\,(x_2) = 2i\sqrt{E}c_+(E)d^{-1/2}\int \exp(id^{-1}\sqrt{E}(x_2-y_2)^2/2)\chi_{11\varepsilon}(y_2)\,dy_2$$

by definition. If $|x_2| < cd^\sigma$ for some $c > 0$, then $\partial_2\rho_{1\varepsilon}(x_2) = O(d^{-1/2+2\sigma})$. A change of variable shows that

$$\rho_{1+}(x_2) = 2i\sqrt{E}c_+(E)\int_0^\infty \exp(i\sqrt{E}t^2/2)\,dt + O(d^{-1/2+\sigma})$$

and hence it follows from formula (7.4) that $\rho_{1+}(x_2) = -1/2 + O(d^{-1/2+\sigma})$ uniformly in x_2 as above. A similar relation is true for $\rho_{1-}(x_2)$. Thus we see that $f_{00,\varepsilon\varepsilon'}$ behaves like

$$\left(e^{-i\varepsilon\alpha_1\pi}e^{i\varepsilon'\alpha_1\pi}/4\right)(ic(E)/4\pi)(R(E+i0;H_b)e^{i\theta_b}\chi_{02}[H_0,e_2]\varphi_0, e^{i\theta_b}\chi_{02}[H_0,e_2]\overline{\varphi}_0)$$

as $d \to \infty$. We now follow the argument in section 6. If we use Lemma 2.2, then the coefficient of $f_2(\omega_1 \to -\omega_1; E)$ is calculated by summing

$$\sum_{\varepsilon,\varepsilon'\in Z} f_{00,\varepsilon\varepsilon'} \sim (\cos\alpha_1\pi)^2\, f_2(\omega_1 \to -\omega_1; E).$$

We write $\chi_{12} = \chi_{12+}(x_2) + \chi_{12-}(x_2)$, where $\chi_{12\pm}$ has support in $\{9d^\sigma < \pm x_2 < 18d^{1/2+\sigma}\}$. Then we can prove that $f_{10,\varepsilon\varepsilon'} = o(1)$ and $f_{01,\varepsilon\varepsilon'} = o(1)$ and that

$$f_{11,\varepsilon\varepsilon'} \sim (ic(E)/4\pi)\sum_{\kappa,\kappa'\in Z}(R(E+i0;H_3)e^{i\theta_3}\eta_{2\varepsilon\kappa}u_{2\varepsilon\kappa}\varphi_0, e^{i\theta_3}\overline{\eta}_{2\varepsilon'\kappa'}u_{2\varepsilon'\kappa'}\overline{\varphi}_0)$$

for each $\varepsilon,\ \varepsilon' \in Z$, where $\theta_3(x) = \alpha_3\gamma(x - d_3; \omega_1)$ and

$$u_{2\varepsilon\kappa}(x) = e^{-i\varepsilon\alpha_1\pi}e^{-i\kappa\alpha_2\pi}\,(\partial_1 e_2)\,(x), \qquad \eta_{2\varepsilon\kappa}(x_2) = 2i\sqrt{E}\chi_{12\kappa}(x_2)\rho_{1\varepsilon}(x_2).$$

We show that $f_{lm,\varepsilon\varepsilon'}$ with $l = 2$ or $m = 2$ obeys $f_{lm,\varepsilon\varepsilon'} = o(1)$. To do this, we write $\chi_{22}v_\varepsilon = \chi_{22+}v_\varepsilon + \chi_{22-}v_\varepsilon$ as in the previous section, and similarly for $\tilde{v}_\varepsilon$. We consider

$$\chi_{22+}v_\varepsilon = \chi_{22+}e^{i\theta_b}[H_0,e_2]R(E+i0;H_0)\eta_{1\varepsilon}u_{1\varepsilon}\varphi_0$$

which has support in

$$F_+ = \{x : x_1 - d_2 > cd^{1/2+\sigma}/\delta,\ x_2 > 9d^{1/2+\sigma},\ \delta < |\widehat{(x-d_2)} - \omega_1| < 2\delta\}$$

for some $c > 0$. The Green kernel $G_0(x,y;E)$ obeys (7.2) for $|x-y| \gg 1$, and it has $i\sqrt{E}|x-y|$ as a phase factor. We cover $\operatorname{supp}\partial_1 e_1 \cap \operatorname{supp}\chi_{11}$ with balls of radius $d^{1/2-\sigma}$. If $|y - a_0| < d^{1/2-\sigma}$, then

$$|x-y| = |x-a_0| - ((x-a_0)\cdot(y-a_0))\,/|x-a_0| + O(d^{-2\sigma})$$

for $x \in F_+$, and we have

$$|\partial_x^\beta \exp\left(i\sqrt{E}(|x-y|-|x-a_0|)\right)| \le C_\beta \left(|x-d_2|+d\right)^{-(1/2+\sigma)|\beta|}.$$

Thus the function $\chi_{22+}v_\varepsilon$ under consideration takes the form

$$\chi_{22+}v_\varepsilon = \sum_k \exp(i\sqrt{E}|x-a_k|)h_k(x),$$

where $h_k(x)$ has support in F_+ and obeys

$$|\partial_x^\beta h_k(x)| \le C_\beta \left(|x-d_2|+d\right)^{-(1/2+\sigma)|\beta|}$$

uniformly in k. The summation above ranges over a finite number of k and the total number is at most of order $O(d^{4\sigma})$. Let $\{\gamma_0, \gamma_\pm\}$ be a partition of unity normalized by $\gamma_0(\xi_2)+\gamma_+(\xi_2)+\gamma_-(\xi_2)=1$, where $\gamma_0 = \chi(d^{1/2}|\xi_2|)$ again and $\gamma_\pm$ has support in $\{\pm\xi_2 > d^{-1/2}\}$. Then the identity operator is decomposed into the sum of pseudodifferential operators

$$1 = \beta_1\left(\gamma_0+\gamma_++\gamma_-\right)+(1-\beta_1)$$

and similarly for $\tilde{\beta}_1$. If $y=(y_1,y_2)\in \operatorname{supp}\chi_{11}$, then $|y_2| < 6d^{1/2+\sigma}$. Since $x_2 > 9d^{1/2+\sigma}$ for $x \in F_+$, we have

$$\sqrt{E}\nabla_x|x-a_k| \in \operatorname{supp}\beta_1 \cap \operatorname{supp}\gamma_+$$

for $x \in F_+$, and we can show that

$$\|r_{-L}\beta_1\gamma_0\chi_{22+}v_\varepsilon\|_{L^2} + \|r_{-L}\beta_1\gamma_-\chi_{22+}v_\varepsilon\|_{L^2} + \|r_{-L}\left(1-\beta_1\right)\chi_{22+}v_\varepsilon\|_{L^2} = O(1)$$

for any $L \gg 1$. A similar bound is true for $\chi_{22+}\tilde{v}_\varepsilon$ with β_1 replaced by $\tilde{\beta}_1$. On the other hand, the classical particle starting from $(x,\xi) \in F_+ \times (\operatorname{supp}\beta_1 \cap \operatorname{supp}\gamma_+)$ at time $t=0$ passes far away from center d_3 for $t>0$ and it moves like the free particle. Hence the particle does not take momentum around $-\sqrt{E}\omega_1$. If we take account of these facts, then we have

$$(R(E+i0;H_b)\chi_{22+}v_\varepsilon, \chi_{22+}\tilde{v}_{\varepsilon'}) = o(1).$$

In fact, this can be shown in almost the same way as used to prove (4.10) and (4.11), and the argument is based on the construction of approximation to resolvent $R(E \pm i0; H_b)$ (see (3.29) for example). Thus it follows that the terms $f_{lm,\varepsilon\varepsilon'}$ with $l=2$ or $m=2$ are all negligible.

We here sum up the results which have been obtained so far. We use the new notation

$$u_{2\varepsilon} = e^{-i\varepsilon_1\alpha_1\pi}e^{-i\varepsilon_2\alpha_2\pi}\left(\partial_1 e_2\right)(x), \qquad \eta_{2\varepsilon} = 2i\sqrt{E}\chi_{12\varepsilon_2}(x_2)\rho_{1\varepsilon_1}(x_2)$$

for $\varepsilon = (\varepsilon_1, \varepsilon_2) \in Z \times Z$. Then f_d behaves like

$$
\begin{aligned}
f_d &\sim f_1(\omega_1 \to -\omega_1; E) + (\cos\alpha_1\pi)^2 f_2(\omega_1 \to -\omega_1; E) \\
&+ (ic(E)/4\pi) \sum_{\varepsilon, \varepsilon' \in Z \times Z} (R(E+i0; H_3) e^{i\theta_3} \eta_{2\varepsilon} u_{2\varepsilon} \varphi_0, e^{i\theta_3} \overline{\eta}_{2\varepsilon'} u_{2\varepsilon'} \overline{\varphi}_0). \qquad (7.5)
\end{aligned}
$$

We proceed to the final step. We calculate the coefficient of $f_3(\omega_1 \to -\omega_1; E)$. If we make use of the relation

$$
(H_3 - E) e^{i\theta_3} e_3 = e^{i\theta_3} (H_0 - E) e_3
$$

and if we repeat the same argument as above, then we obtain from (7.5) that

$$
f_d \sim f_1 + (\cos\alpha_1\pi)^2 f_2 + (ic(E)/4\pi) \sum_{\varepsilon, \varepsilon' \in Z \times Z} (R(E+i0; H_3) w_\varepsilon, \tilde{w}_{\varepsilon'}),
$$

where w_ε and $\tilde{w}_\varepsilon$ are given by

$$
\begin{aligned}
w_\varepsilon &= e^{i\theta_3} [H_0, e_3] R(E+i0; H_0) \eta_{2\varepsilon} u_{2\varepsilon} \varphi_0, \\
\tilde{w}_\varepsilon &= e^{i\theta_3} [H_0, e_3] R(E-i0; H_0) \overline{\eta}_{2\varepsilon} u_{2\varepsilon} \overline{\varphi}_0.
\end{aligned}
$$

We decompose the third term on the right side into the sum of

$$
g_{lm,\varepsilon\varepsilon'} = (ic(E)/4\pi)(R(E+i0; H_3) \chi_{l3} w_\varepsilon, \chi_{m3} \tilde{w}_{\varepsilon'}), \quad 0 \le l, \ m \le 2,
$$

for each ε, $\varepsilon' \in Z \times Z$. Then the leading term comes from $g_{00,\varepsilon\varepsilon'}$ and the other terms are shown to be negligible. In particular, $g_{11,\varepsilon\varepsilon'} = o(1)$ follows from the fact that there is no center in the direction ω_1 from d_3. We calculate the coefficient in question. We have assumed that the three centers are at even intervals, so that $|d_{12}| = |d_{23}| = d$. Then $g_{00,\varepsilon\varepsilon'}$ behaves like

$$
e^{-i\varepsilon\alpha\pi} e^{i\varepsilon'\alpha\pi} (ic(E)/4\pi)(R(E+i0; H_3) e^{i\theta_3} \chi_0 [H_0, e_3] \rho_{2\varepsilon} \varphi_0, e^{i\theta_3} \chi_0 [H_0, e_3] \overline{\rho}_{2\varepsilon'} \overline{\varphi}_0)
$$

as $d \to \infty$, where $e^{i\varepsilon\alpha\pi} = e^{i\varepsilon_1\alpha_1\pi} e^{i\varepsilon_2\alpha_2\pi}$ for $\varepsilon = (\varepsilon_1, \varepsilon_2) \in Z \times Z$ and $\alpha = (\alpha_1, \alpha_2)$, and $\rho_{2\varepsilon}(x_2) = (Y\eta_{2\varepsilon})(x_2)$, Y being defined by (7.3). By definition, $\rho_{2\varepsilon}$ is written as

$$
\rho_{2\varepsilon}(x_2) = 2i\sqrt{E}\,(Y\chi_{12\varepsilon_2}\rho_{1\varepsilon_1})(x_2) = (2i\sqrt{E})^2\,(Y\,(\chi_{12\varepsilon_2} Y \chi_{11\varepsilon_1}))(x_2)
$$

for $\varepsilon = (\varepsilon_1, \varepsilon_2) \in Z \times Z$. Assume that $|x_2| < cd^\sigma$ for some $c > 0$. If $\varepsilon = (+,+)$, then $\rho_{2\varepsilon}(x_2)$ behaves like

$$
(2i\sqrt{E})^2 c_+(E)^2 d^{-1} \int_0^\infty e^{id^{-1}\sqrt{E} y_2^2/2} \left(\int_0^\infty e^{id^{-1}\sqrt{E}(z_2 - y_2)^2/2}\, dz_2 \right) dy_2 + O(d^{-1/2+2\sigma}).
$$

Hence we have

$$
\rho_{2\varepsilon}(x_2) = (2i\sqrt{E})^2 c_+(E)^2 \int_0^\infty e^{i\sqrt{E}t^2/2} \left(\int_{-t}^\infty e^{i\sqrt{E}s^2/2}\, ds \right) dt + O(d^{-1/2+2\sigma})
$$

by making change of variables, so that $\rho_{2\varepsilon} = 3/8 + O(d^{-1/2+2\sigma})$ by use of formula (7.4). Similarly $\rho_{2\varepsilon} = 3/8 + O(d^{-1/2+2\sigma})$ for $\varepsilon = (-,-)$ and $\rho_{2\varepsilon} = 1/8 + O(d^{-1/2+2\sigma})$ for $\varepsilon = (+,-)$ or $\varepsilon = (-,+)$. Thus we obtain that

$$\sum_{\varepsilon,\varepsilon' \in Z} g_{00,\varepsilon\varepsilon'} \sim (\theta_+ \cos(\alpha_1 + \alpha_2)\pi + \theta_- \cos(\alpha_1 - \alpha_2)\pi)^2 f_3(\omega_1 \to -\omega_1; E)$$

with $\theta_+ = 3/4$ and $\theta_- = 1/4$. This yields the desired coefficient and the proof of the theorem is now completed.

We conclude the paper by stating that the argument used in the special case $N = 3$ extends to the general case without any serious difficulty, although we can not represent the coefficients of amplitudes $f_j(\omega_1 \to -\omega_1; E)$, $1 \le j \le N$, in a neat form.

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Appendix : Figures of differential cross sections

The differential cross section is a quantity observable through actual experiments and it is one of the most important quantities in the scattering theory. We here figure the approximate values for $|f_d(\omega_1 \to \tilde{\omega}; E)|^2,\ \omega_1 = (1, 0)$, to see how the intensity of scattering changes with three flux parameters $\alpha_1,\ \alpha_2$ and α_3, as stated in Remark 1.1. We consider the case that the three centers $d_1,\ d_2$ and d_3 are at even intervals along direction ω_1, and we fix $|d_{12}| = |d_{23}| = d = 100$ and $\alpha_3 = 1/2$. The three figures are drawn for $|f_d(\omega_1 \to \tilde{\omega}; E)|^2$ with $E = 1$ and $4\pi/9 < \gamma(\tilde{\omega}; \omega_1) < 5\pi/9$. For example, the scattering angle $\gamma(\tilde{\omega}; \omega_1) = \pi/2$ corresponds to the value $\pi/2 = 1.57....$ on the horizontal axis in the figures below.

Figure 1 $(\alpha_1 = 1/4,\ \alpha_2 = 1/4)$: three coefficients do not vanish.
Figure 2 $(\alpha_1 = 1/2,\ \alpha_2 = 1/2)$: coefficient of $f_2(\omega_1 \to \tilde{\omega}; E)$ vanishes.
Figure 3 $(\alpha_1 = 1/4,\ \alpha_2 = (\arctan 2)/\pi)$: coefficient of $f_3(\omega_1 \to \tilde{\omega}; E)$ vanishes.

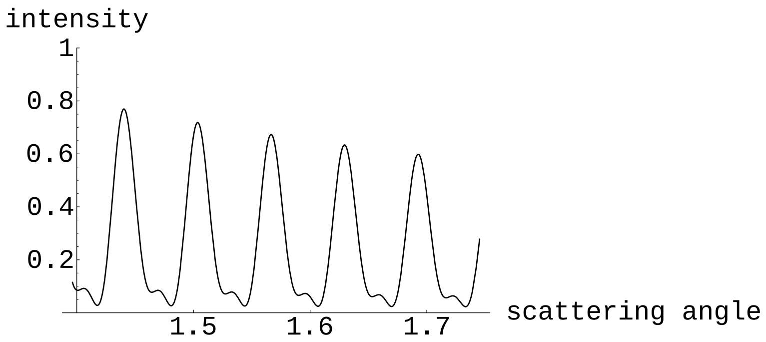

Figure 1 $\alpha_1 = 1/4,\ \alpha_2 = 1/4,\ \alpha_3 = 1/2$

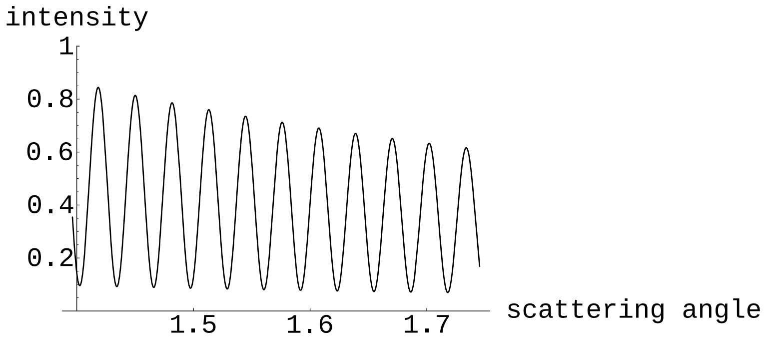

Figure 2 $\alpha_1 = 1/2,\ \alpha_2 = 1/2,\ \alpha_3 = 1/2$

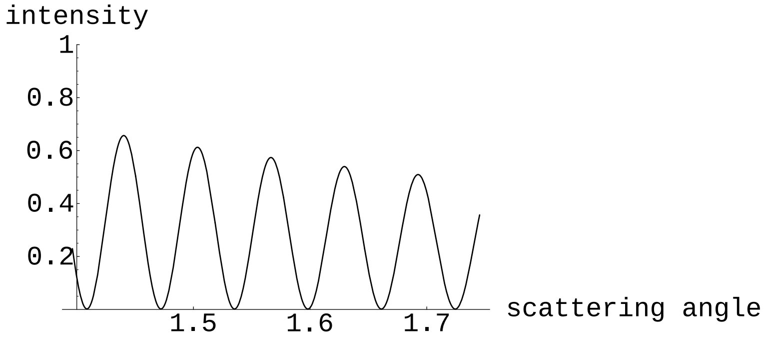

Figure 3 $\alpha_1 = 1/4,\ \alpha_2 = \pi^{-1} \arctan 2,\ \alpha_3 = 1/2$