PHASE TRANSITION AND CRITICAL BEHAVIOR IN A MODEL OF ORGANIZED CRITICALITY

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Abstract: We study a model of "organized" criticality, where a single avalanche propagates through an *a priori* static (i.e., organized) sandpile configuration. The latter is chosen according to an i.i.d. distribution from a Borel probability measure ρ on $[0,1]$. The avalanche dynamics is driven by a standard toppling rule, however, we simplify the geometry by placing the problem on a directed, rooted tree. As our main result, we characterize which ρ are critical in the sense that they do not admit an infinite avalanche but exhibit a power-law decay of avalanche sizes. Our analysis reveals close connections to directed site-percolation, both in the characterization of criticality and in the values of the critical exponents.

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1. Introduction

1.1 Motivation.

Since its discovery by Bak, Tang and Wisenfeld [1, 2], self-organized criticality (SOC) has received massive attention in the physics literature. Variants of the original sandpile model of [1] were studied and some of them even "exactly" solved (see [7] for a recent review of the subject). However, despite great efforts and literally thousands of published papers, the present mathematical understanding of SOC lags far behind the bold claims made by physicists. Much of that failure can be attributed to the fact that the models used to demonstrate SOC are difficult to formulate precisely and/or too difficult to study using the current techniques of probability theory and mathematical physics. From the perspective of the latter fields, the situation seems ripe for considering models which concern at least some aspects of SOC, provided there is a decent prospect of a self-contained rigorous analysis.

The general idea behind SOC models is very appealing. Consider for instance Zhang's sandpile model [10] on $\mathbb{Z}^2$, where each site has an energy variable which evolves in discrete time-steps according to a simple "toppling" rule: If a variable exceeds a threshold value, the excess is distributed equally among the neighbors. The neighboring sites may thus turn supercritical and the process continues until the excess is "thrown overboard" at the system boundary. What makes this dynamical rule intriguing is that, if the toppling is initiated from a "highly excited" state, then the terminal state (i.e., the state where the toppling stops) is *not* the most stable state, but one of many *least-stable*, stable states. Moreover, the latter state is critical in the sense that further insertion of a small excess typically leads to further large-scale events. Using the sandpile analogy, such events are referred to as *avalanches*.

In this paper, we study the scaling properties of a single avalanche caused by an overflow at some site of a critical (i.e., least-stable) state. However, as indicated above, the full problem is way too hard and we have to resort to simplifications. Our simplifications are twofold: First, we treat the energy variables of the critical state as independent and, second, we consider the model on a directed, rooted tree rather than $\mathbb{Z}^2$. The first assumption is fairly reasonable, at least on a coarse-grained scale, because numerical results [6] suggest a rather fast decay of spatial correlations in the critical states. The second assumption will allow us to treat the correlations between different branches of the avalanche as conditionally independent, which will greatly facilitate the analysis. Finally, the reduced geometry allows for the existence of a natural monotonicity not apparent in the full-fledged model.

While placing the model on a tree simplifies the underlying geometry, some complexity is retained due to the generality of the single-site energy variable distribution. In fact, the set of underlying distributions plays the role of a parameter space in our case. As our main result, we characterize the subspace of distributions for which the configurations of energy variables have exactly the behavior expected from the SOC states: *no infinite avalanches* but a *power-law decay* of avalanche sizes. As it turns out, there is a close connection to site-percolation on the underlying graph, both in the characterization of criticality and in the values of the critical exponents. However, the significance of this connection for the general SOC models has not yet been clarified.

1.2 The model.

In order to precisely define our single-avalanche model, we need to introduce some notation. Let $b > 1$ be an integer and let $\mathbb{T}_b$ be a b-nary rooted tree, with the root vertex denoted by $\varnothing$. We use $|\sigma| = k$ to denote that $\sigma \in \mathbb{T}_b$ is on the k-th layer. When $|\sigma| = k$, we represent σ as a k-component object. Each component is an integer in $\{1, \dots, b\}$; hence the site label can be used to trace the path from σ back to the root. If σ is an ℓ-th level site with $\ell > 0$, we let $m(\sigma)$ denote the "mother-site." Explicitly, if $\sigma = (\sigma_1, \dots, \sigma_\ell)$, then $m(\sigma) = (\sigma_1, \dots, \sigma_{\ell-1})$. The edges of $\mathbb{T}_b$ are the usual directed edges $\big\{(\sigma', \sigma) \in \mathbb{T}_b \times \mathbb{T}_b \colon \sigma' = m(\sigma)\big\}$.

Let $\mathcal{M}$ be the space of all probability measures on the Borel σ-algebra of $[0,1]$. Fix a $\rho \in \mathcal{M}$ and let $\mathbb{P}_\rho = \rho^{\mathbb{T}_b}$. Let $\mathbb{E}_\rho$ denote the expectation with respect to $\mathbb{P}_\rho$. The dynamical rule driving the evolution is defined as follows: Let $\boldsymbol{X} = (X_\sigma)_{\sigma \in \mathbb{T}_b}$ be the collection of i.i.d. random variables with joint probability distribution $\mathbb{P}_\rho$ and let $v \in (0, \infty)$. The process generates the sequence

$$\boldsymbol{X}^{(v)}(t) = \big(X_\sigma^{(v)}(t)\big)_{\sigma \in \mathbb{T}_b}, \qquad t = 0, 1, \dots, \tag{1.1}$$

obtained from the initial condition

$$X_\sigma^{(v)}(0) = \begin{cases} X_\varnothing + v, & \text{if } \sigma = \varnothing, \\ X_\sigma, & \text{otherwise,} \end{cases} \tag{1.2}$$

by successive applications of the deterministic (Markov) update rule

$$X_\sigma^{(v)}(t+1) = \begin{cases} X_\sigma^{(v)}(t) + \frac{1}{b} X_{m(\sigma)}^{(v)}(t), & \text{if } X_{m(\sigma)}(t) \ge 1, \\ 0, & \text{if } X_\sigma^{(v)}(t) \ge 1, \\ X_\sigma^{(v)}(t), & \text{otherwise.} \end{cases} \tag{1.3}$$

Note that, if $X_\sigma^{(v)}(t+1) = X_\sigma^{(v)}(t)$ for all $\sigma \in \mathbb{T}_b$, then $X_\sigma^{(v)}(t) \le 1$ and the process has effectively *stopped*. (However, we let $X_\sigma^{(v)}(t)$ be defined by (1.3) for all $t \ge 0$.)

Here is an informal description of the above process: Starting at the root we first check whether $X_\varnothing + v \ge 1$ or not. If not, the process stops but if so, then this value is distributed evenly among the "daughter" cells, which have their values updated to $X_\sigma^{(v)}(1) = X_\sigma + \frac{1}{b}(X_\varnothing + v)$. The value $X_\varnothing^{(v)}(1)$ is set to zero and we say that the root has "avalanched." If none of the updated "daughter" values exceed one, the process terminates; however, if there is any first-level σ with $X_\sigma^{(v)}(1) \ge 1$, then $X_\sigma^{(v)}(1)$ is set to zero, the value $X_\sigma^{(v)}(1)$ is evenly distributed among the "daughters" of σ and we say that σ has "avalanched." The process at future times is described similarly.

Obviously, the variables X_σ play the role of the "energy variables" in the description of Zhang's avalanche model in Section 1.1. In our case the critical threshold is one, but, in (1.3), we chose to distribute the entire value of an "avalanching" site rather than just the excess to the (forward) neighbors. This choice is slightly more advantageous technically.

1.3 Main questions and outline.

Let $\mathcal{A}^{(v)}(t) = \{\sigma \in \mathbb{T}_b \colon X_\sigma^{(v)}(t) = 0,\, X_\sigma^{(v)}(s) \neq 0 \text{ for some } s < t\}$ be the set of sites that have "avalanched" by time t. Similarly, let $\mathcal{A}^{(v)} = \bigcup_{t \ge 0} \mathcal{A}^{(v)}(t)$ be the set of sites that will ever avalanche. We use $|\mathcal{A}^{(v)}|$ to denote the number of sites in the avalanched set (which includes the possibility of $|\mathcal{A}^{(v)}| = \infty$). The set $\mathcal{A}^{(v)}$ and its dependence on ρ and v are the primary focus of our study.

The first question is whether the process $\boldsymbol{X}^{(v)}(t)$ lives forever, i.e., is there an infinite avalanche? More precisely, for what measures $\rho \in \mathcal{M}$ is the probability

$$A_\infty^{(v)} = \mathbb{P}_\rho\big(|\mathcal{A}^{(v)}| = \infty\big) \tag{1.4}$$

non-zero for some value of v? A related question is whether the average size of the avalanched set is finite. The relevant object is defined by

$$\chi^{(v)} = \mathbb{E}_\rho\bigl(|\mathcal{A}^{(v)}|\bigr). \tag{1.5}$$

(Notice that, due to the directed nature of the dynamical rule, both quantities $A_\infty^{(v)}$ and $\chi^{(v)}$ are monotone in the underlying measure and v.) Again, we ask: For what measures ρ we have $\chi^{(v)} = \infty$ for some v? In addition, we might ask: Is the divergence of the mean avalanche size equivalent to the onset of infinite avalanches or can there be an *intermediate phase*?

To give answers to the above questions, we will parametrize the set $\mathcal{M}$ by values of a particular functional $\mathfrak{z}\colon \mathcal{M} \to [0,1]$. Here $\mathfrak{z}(\rho)$ roughly corresponds to the conditional probability in distribution $\mathbb{P}_\rho$ that, given the avalanche has reached a site $\sigma \in \mathbb{T}_b$ far away from the root, the site σ will also avalanche. (The definition of $\mathfrak{z}$ is somewhat technical and we refer the reader to Section 2.2 for more details.) The characterization of the avalanche regime in terms of $\mathfrak{z}$ is then very transparent: There is a *critical value* $\mathfrak{z}_{\rm c} = \frac{1}{b}$, such that the quantity $\chi^{(v)}$ for measure ρ diverges if $\mathfrak{z}(\rho) > \mathfrak{z}_{\rm c}$ and v is sufficiently large, while it is finite for all v if $\mathfrak{z}(\rho) < \mathfrak{z}_{\rm c}$. Similarly we show, for a reduced class of measures, that $A_\infty^{(v)}$ for measure ρ vanishes for all v if and only if $\mathfrak{z}(\rho) \le \mathfrak{z}_{\rm c}$. These results are formulated as Theorems 2.4 and 3.1 in Sections 2.2 and 3.1, respectively. (Outside the reduced class of measures, there are some exceptions to the rule that $A_\infty^{(v)} \equiv 0$ for measures ρ with $\mathfrak{z}(\rho) = \mathfrak{z}_{\rm c}$, i.e., there are some measures which avalanche also *at* criticality, see Remarks 1 and 2 in Section 2 for more details. These examples are fairly contrived, so we exclude them from further considerations.)

Note that, for both quantities (1.4) and (1.5), the transitions happen at the same value, $\mathfrak{z}_{\rm c}$, which rules out the possibility of an intermediate phase. To elucidate the behavior of $\mathfrak{z}$ near $\mathfrak{z}_{\rm c}$, it is worthwhile to introduce appropriate *critical exponents*. In particular, we ask whether there is a critical exponent $\gamma > 0$ such that

$$\chi^{(v)} \sim \bigl(\mathfrak{z}_{\rm c} - \mathfrak{z}(\rho)\bigr)^{-\gamma}, \qquad \mathfrak{z}(\rho) \uparrow \mathfrak{z}_{\rm c}, \tag{1.6}$$

an exponent $\beta > 0$ such that

$$A_\infty^{(v)} \sim \bigl(\mathfrak{z}(\rho) - \mathfrak{z}_{\rm c}\bigr)^{\beta}, \qquad \mathfrak{z}(\rho) \downarrow \mathfrak{z}_{\rm c}, \tag{1.7}$$

and, finally, an exponent $\delta > 0$ such that if $\mathfrak{z}(\rho) = \mathfrak{z}_{\rm c}$, then

$$\mathbb{P}_\rho\bigl(|\mathcal{A}^{(v)}| \ge n\bigr) \sim n^{-1/\delta}, \qquad n \to \infty. \tag{1.8}$$

All of these three relations of course include an appropriate interpretation of the symbol "$\sim$" and, with the exception of the last relation, also an interpretation of the limit "$\mathfrak{z}(\rho)$ tends to $\mathfrak{z}_{\rm c}$."

The relations for the critical exponents are the subject of Theorem 4.1 in Section 4. The upshot is that all three exponents take the *mean-field percolation* values,

$$\gamma = 1, \quad \beta = 1, \quad \delta = 2. \tag{1.9}$$

Neither the fact that the critical value $\mathfrak{z}_{\rm c}$ equals the percolation threshold for site percolation on $\mathbb{T}_b$ is a coincidence. Indeed, the avalanche problem can be characterized in terms of a correlated-percolation problem on $\mathbb{T}_b$ (see Section 2).

We finish with a brief outline of the paper: Section 2 contains our percolation criteria for the existence of infinite avalanches leading naturally to the definition of the functional $\mathfrak{z}$. In Section 3 we show that $\mathfrak{z}_{\rm c} = \frac{1}{b}$ is the unique critical "point" of our model, thus ruling out the possibility of an intermediate phase. Section 4 proves the above relations for the critical exponents. Finally, in Section 5 we develop a coupling argument which is the core of the proofs of the aforementioned results in Sections 3 and 4. The principal results of this paper are Theorem 2.4 (Section 2.2), Theorem 3.1 (Section 3.1) and Theorem 4.1 (Section 4.1).

2. Percolation criteria

2.1 Simple percolation bounds.

We start by deriving criteria for the presence and absence of an infinite avalanche based on a comparison to site percolation on $\mathbb{T}_b$. Let $x_\star$ denote the maximum of the support of ρ, i.e.,

$$x_\star = \sup\bigl\{y \in [0,1] \colon \rho([y,1]) > 0\bigr\}, \tag{2.1}$$

and let us define θ_b by

$$\theta_b = \frac{b}{b-1} x_\star. \tag{2.2}$$

It is noted that if $X_\varnothing + v \le \theta_b$, then the largest value that $X_\sigma(t)$ for any $\sigma \in \mathbb{T}_b$ could conceivably achieve (just prior to its own avalanche) is θ_b.

Proposition 2.1 (1) *If $\rho([\frac{b-1}{b},1]) > \frac{1}{b}$, then $A_\infty^{(v)} > 0$ for all $v > 1 - x_\star$.*
(2) *If either $\theta_b < 1$ or $\theta_b > 1$ and $\rho([1 - \frac{1}{b}\theta_b, 1]) \le \frac{1}{b}$, then $A_\infty^{(v)} = 0$ for all $v \ge 0$.*

In both cases we note that the quantity $\frac{1}{b}$ on the right-hand side of the inequalities is the percolation threshold for $\mathbb{T}_b$. Obviously, this is no coincidence; indeed, the proof of part (1) is easily generalizable to any transitive infinite graph.

Proof of Proposition 2.1. Let us start with (1): A site $\sigma \ne \varnothing$ is called occupied if $X_\sigma \ge 1 - \frac{1}{b}$, while the root $\varnothing$ is called occupied if $X_\varnothing + v \ge 1$. Denoting by $\mathcal{C}^{(v)}$ the set of occupied sites containing the origin, it is not hard to see that $\mathcal{A}^{(v)} \supset \mathcal{C}^{(v)}$. Indeed, assuming $X_\varnothing + v > 1$, each daughter site of the origin receives at least $\frac{1}{b}$; those daughter sites σ with $X_\sigma \ge 1 - \frac{1}{b}$ will be triggered, which will in turn cause avalanches in the next generation of occupied sites, etc. Evidently, whenever the occupied sites percolate, there is an infinite avalanche.

Part (2) is proved in a similar fashion. Suppose first that $\theta_b > 1$ and call a site $\sigma \ne \varnothing$ occupied if $X_\sigma \ge 1 - \frac{1}{b}\theta_b$, and vacant otherwise. The definition is as before for the root. As observed previously, if $X_\varnothing + v \le \theta_b$, then no site receives more than $\frac{1}{b}\theta_b$ from its parent. Under these circumstances, a vacant site will never avalanche and, denoting the occupied cluster of the origin by $\bar{\mathcal{C}}^{(v)}$, we have $\mathcal{A}^{(v)} \subset \bar{\mathcal{C}}^{(v)}$. Since $\rho([1 - \frac{1}{b}\theta_b, 1]) \le \frac{1}{b}$ was assumed, we have that $|\bar{\mathcal{C}}^{(v)}| < \infty$ almost surely and thus $|\mathcal{A}^{(v)}| < \infty$ whenever $X_\varnothing + v \le \theta_b$. It is then easy to show, however, that $|\mathcal{A}^{(v)}| < \infty$ almost surely for all $v \ge 0$. Indeed, let $k \ge 0$ be an integer so large that

$$(x_\star + v - \theta_b) b^{-k} < \theta_b - 1. \tag{2.3}$$

If σ is a site with $|\sigma| = k$ that has been reached by an avalanche, then σ could not receive more than

$$x_\star\bigl(b^{-1} + \dots + b^{-(k-1)}\bigr) + b^{-k}(X_\varnothing + v) = b^{-1}\theta_b + b^{-k}(X_\varnothing + v - \theta_b) \tag{2.4}$$

from its parent. Now, if σ is vacant, then the maximal possible value for $X_\sigma(k)$ (i.e., prior to its own avalanche) is no larger than $1 + b^{-k}(x_\star + v - \theta_b)$. By (2.3), this amount is strictly less than θ_b, so by our previous reasoning, σ cannot give rise to an infinite avalanche. By absence of percolation, there is a "barrier" $\mathbb{S}_k$ of vacant sites above the $(k+1)$-st layer in $\mathbb{T}_b$, that every path from the root to infinity must path through. Our previous arguments show that the avalanche cannot go beyond the union of occupied connected components rooted at $\mathbb{S}_k$. Hence, $|\mathcal{A}^{(v)}| < \infty$ with probability one.

The case $\theta_b < 1$ is handled analogously. Indeed, a simple calculation reveals that the right-hand side of (2.4) plus $x_\star$ is eventually strictly less than one and the avalanche terminates within a deterministic (v-dependent) amount of time. □

The arguments in the proof immediately give us the following corollary:

Corollary 2.2 *If $\theta_b \neq 1$ and $\rho([1-\frac{1}{b}\theta_b, \frac{b-1}{b})) = 0$, then there is an infinite avalanche if and only if occupied sites, i.e., sites $\sigma \in \mathbb{T}_b$ with value $X_\sigma \geq 1-\frac{1}{b}$, percolate. In addition, if $X_\varnothing + v \leq \theta_b$, then $\mathcal{A}^{(v)}$ coincides exactly with the occupied connected component of the root.*

Remark 1. The exceptional cases, $\theta_b = 1$, can only arise from the circumstance that $x_\star = 1-\frac{1}{b}$. (Notice that the proof of Proposition 2.1(2) does not apply because the inequality (2.3) cannot be satisfied.) For $\theta_b = 1$, the situation is marginal and, in fact, slightly subtle. Indeed, if $x_\star = 1-\frac{1}{b}$ and $\mathbb{P}(X \geq x_\star) = \frac{1}{b}$, then the existence of an infinite avalanche depends on the detailed asymptotic of $\mathbb{P}(X \geq x_\star - \epsilon)$ as $\epsilon \downarrow 0$, see Remark 2 in the next section. We exclude the cases $\theta_b = 1$ from our analysis because we believe that this "pathological" behavior is in no way generic.

2.2 Phase transition.

As is seen from Corollary 2.2, in certain cases the avalanche problem reduces to the usual (independent) percolation model. The general problem can also be presented as a percolation phenomenon albeit with correlations. Indeed, let $X_1, \dots, X_n$ are i.i.d. with distribution ρ and let

$$Q_n^{(\theta)} = X_n + \frac{X_{n-1}}{b} + \cdots + \frac{X_1}{b^{n-1}} + \frac{\theta}{b^n}, \tag{2.5}$$

In the case $n = 0$, we let $Q_0^{(\theta)} = \theta$. Similarly, for $\sigma \in \mathbb{T}_b$, we define $Q_\sigma^{(\theta)}$ by (2.5) with $n = |\sigma|$ and $X_1, \dots, X_{|\sigma|}$ being the values along the unique path connecting σ to the root. Explicitly, we set $Q_\varnothing^{(\theta)} = \theta$ and define

$$Q_\sigma^{(\theta)} = X_\sigma + \frac{1}{b} Q_{m(\sigma)}^{(\theta)}, \qquad \sigma \neq \varnothing. \tag{2.6}$$

Note that here θ plays the role of the quantity $X_\varnothing + v$. Clearly, $Q_n^{(\theta)} \overset{\mathcal{D}}{=} Q_\sigma^{(\theta)}$, whenever $n = |\sigma|$. (Note, however, that these quantities live on different probability spaces.)

Proposition 2.3 *Let $v \geq 0$ and let $\theta = X_\varnothing + v$. For each $\sigma \in \mathbb{T}_b$, let us call σ* open *if $Q_\sigma^{(\theta)} \geq 1$ and* closed *otherwise. Then $\sigma \in \mathcal{A}^{(v)}$ if and only if σ belongs to the open cluster containing the root. In particular, percolation of open sites is the necessary and sufficient condition for infinite avalanches.*

Proof. By definition, $Q_\varnothing^{(\theta)} = \theta = X_\varnothing + v$. Now, if $X_\sigma(t) = Q_\sigma^{(\theta)}$ for a site $\sigma \in \mathbb{T}_b$ that avalanches at time $t = |\sigma|$, then any daughter site σ' of σ will have its value updated to

$$X_{\sigma'}(t+1) = X_{\sigma'} + \frac{1}{b} Q_\sigma^{(\theta)} = Q_{\sigma'}^{(\theta)}. \tag{2.7}$$

Hence, if the site $\sigma \in \mathbb{T}_b$ avalanches at time $t = |\sigma|$, then $Q_\sigma^{(\theta)} = X_\sigma(t) \geq 1$. It follows that $\mathcal{A}^{(v)}$, with $v = \theta - X_\varnothing$, is the set of sites that are open and connected to the root by a path of open sites. □

Remark 2. Let us indicate what makes the case $x_\star = 1 - \frac{1}{b}$ so subtle. Given a sequence (c_k) of positive numbers, let us call $\sigma \in \mathbb{T}_b$ open if $X_\sigma \ge x_\star - c_{|\sigma|} b^{-|\sigma|}$ and closed otherwise. Letting $p_k = \mathbb{P}(X \ge x_\star - c_k b^{-k})$ and supposing, e.g., $bp_k = 1 + k^{-1/2}$, a general result of Lyons [9] implies that the open sites percolate. An easy argument shows that if σ is connected to $\varnothing$ by a path of open sites, then $Q_\sigma^{(\theta)} \ge 1 + b^{-k}(v - 1 - \sum_{\ell \le k} c_\ell)$ for $v = \theta - X_\varnothing$. Thus, if $v > 1 + \sum_{k \ge 0} c_k$, then, by Proposition 2.3, there is an infinite avalanche with a non-zero probability.

On a similar basis, we can write down the necessary and sufficient conditions for divergence of the expected size of avalanches. The criterion will be based on the asymptotic growth of the quantity

$$Z_n(\theta) = \mathbb{P}\big(Q_k^{(\theta)} \ge 1,\, k = 0, \dots, n\big), \qquad n \ge 0. \tag{2.8}$$

Notice that $Z_n(\theta) = 0$ whenever $\theta < 1$.

Theorem 2.4 *(1) For all $\theta \ge 1$, the limit*

$$\mathfrak{z} = \mathfrak{z}(\rho) = \lim_{n \to \infty} Z_n(\theta)^{1/n} \tag{2.9}$$

exists and is independent of θ.
(2) For all $\rho, \rho' \in \mathcal{M}$, the function $\alpha \mapsto \mathfrak{z}(\alpha\rho + (1-\alpha)\rho')$ is continuous in $\alpha \in [0,1]$.
(3) Let $\rho \in \mathcal{M}$ and let $x_\star$ correspond to ρ via (2.1). *Define $\mathfrak{z}_{\mathrm{c}} = \frac{1}{b}$. If $\mathfrak{z}(\rho) < \mathfrak{z}_{\mathrm{c}}$, then $\mathbb{E}_\rho(|\mathcal{A}^{(v)}|) < \infty$ for all $v \in (0, \infty)$, while if $\mathfrak{z}(\rho) > \mathfrak{z}_{\mathrm{c}}$, then $\mathbb{E}_\rho(|\mathcal{A}^{(v)}|) = \infty$ for all $v > 1 - x_\star$.*

Theorem 2.4 defines a free-energy like functional $\mathfrak{z}$ and gives the characterization of the divergence of $\chi^{(v)}$, as already discussed in Section 1.3. The continuity statement in part (2) indicates that the sets of "avalanching" and "non-avalanching" measures $\rho \in \mathcal{M}$ are separated by a "surface" (i.e., set of codimension one) of phase transitions. We will not try to make the latter more precise; our main reason for including part (2) is to have an interpretation of the limit $\mathfrak{z}(\rho) \to \mathfrak{z}_{\mathrm{c}}$, which will be needed in the discussion of the critical behavior. Under additional mild restrictions on ρ, it will be shown in Section 4 that $\mathbb{E}_\rho(|\mathcal{A}^{(v)}|) = \infty$ even for the critical measures ρ, i.e., those satisfying $\mathfrak{z}(\rho) = \mathfrak{z}_{\mathrm{c}}$.

Proof of Theorem 2.4(1). We will start with the cases $\theta = 1$ and $\theta \ge \theta_b$ which are amenable to subadditive-type arguments. Examining $Z_{n+m}(\theta)$, we may write (by conditioning on $X_1, \dots, X_m$)

$$Z_{n+m}(\theta) = \mathbb{E}\Big(Z_n(Q_m^{(\theta)}) \prod_{j=0}^{m} \mathsf{1}_{\{Q_j^{(\theta)} \ge 1\}}\Big). \tag{2.10}$$

Since $\theta \mapsto Q_n^{(\theta)}$ is manifestly increasing in θ, so is the event on the right-hand side of (2.8) and also $Z_n(\theta)$ itself. Notice that if $\theta \ge \theta_b$, then $Q_k^{(\theta)} \le \theta$ for any $k \ge 0$, while if $\theta = 1$, then the conditions in (2.8) force $Q_k^{(\theta)} \ge 1$. Thus, for $\theta = 1$ we obtain the submultiplicative bound

$$Z_{n+m}(1) \ge Z_n(1) Z_m(1), \tag{2.11}$$

while for any $\theta \ge \theta_b$ we get the supermultiplicative bound

$$Z_{n+m}(\theta) \le Z_n(\theta) Z_m(\theta). \tag{2.12}$$

By standard theorems, $Z_n(1)^{1/n}$ tends to a limit, $\mathfrak{z}_1$, while $Z_n(\theta)^{1/n}$ for $\theta \ge \theta_b$ tends to a (possibly θ-dependent) limit $\mathfrak{z}_\theta$. Moreover, (2.10) in fact implies that $Z_{n+m}(\theta) \le Z_n(\theta_b + \theta b^{-m})$ and $\mathfrak{z}_\theta$ is thus constant for all $\theta > \theta_b$. We will use $\mathfrak{z}_\star$ to denote the common value of $\mathfrak{z}_\theta$ for $\theta > \theta_b$. Note that $Z_n(1)^{1/n} \le \mathfrak{z}_1$ while $Z_n(\theta)^{1/n} \ge \mathfrak{z}_\star$ for all $n \ge 1$ and all $\theta > \theta_b$.

Since $\theta \mapsto Z_n(\theta)$ is non-decreasing, to prove (2.9), we just need to show that $\mathfrak{z}_\star$ equals $\mathfrak{z}_1$. If $x_\star < 1 - \frac{1}{b}$, then $\mathfrak{z}_\star = 0$ and there is nothing to prove, so let us suppose that $x_\star \geq 1 - \frac{1}{b}$ for the rest of the proof. First we will examine the cases with $x_\star > 1 - \frac{1}{b}$ and $\theta \in [1, \theta_b)$. We claim that for any $\theta \in [1, \theta_b)$, there is an $H(\theta) < \infty$ such that

$$Z_n(\theta) \leq H(\theta)\mathfrak{z}_1^n, \qquad n \geq 1. \tag{2.13}$$

Indeed, let $\epsilon > 0$ be such that $\theta_b - \theta > \epsilon\frac{b}{b-1}$ and $x_\star - \epsilon \geq 1 - \frac{1}{b}$ and pick m so that

$$(x_\star - \epsilon)\Bigl[1 + \frac{1}{b} + \cdots + \frac{1}{b^{m-1}}\Bigr] + \frac{1}{b^m} \geq \theta. \tag{2.14}$$

Define $\kappa_\epsilon = \rho([x_\star - \epsilon, x_\star]) > 0$. Consider the formula (2.8) for $Z_{n+m}(1)$ but with the first m coordinates restricted to the event $\mathcal{E} = \{X_1, \dots, X_m \geq x_\star - \epsilon\}$. Notice that on $\mathcal{E}$, the conditions involving $Q_1^{(1)}, \dots, Q_m^{(1)}$ are automatically satisfied. By a derivation similar to (2.10-2.11) we have

$$Z_{n+m}(1) \geq \kappa_\epsilon^m Z_n(\theta). \tag{2.15}$$

Along with the upper bound $Z_{n+m}(1) \leq \mathfrak{z}_1^{n+m}$, this implies (2.13) with $H(\theta) = (\mathfrak{z}_1/\kappa_\epsilon)^m$.

Now we can show that $\mathfrak{z}_\star = \mathfrak{z}_1$ whenever $x_\star > 1 - \frac{1}{b}$ and $\kappa_\epsilon = \rho([x_\star - \epsilon, 1]) < \mathfrak{z}_1$ for some $\epsilon > 0$ with $x_\star - \epsilon > 1 - \frac{1}{b}$. Let $\theta > \theta_b$ be small enough that $\theta_\epsilon = x_\star - \epsilon + \frac{\theta}{b} < \theta_b$. Then

$$Z_n(\theta) \leq \kappa_\epsilon Z_{n-1}(\theta) + (1-\kappa_\epsilon)Z_{n-1}(\theta_\epsilon) = Z_{n-1}(\theta)\biggl[\kappa_\epsilon + (1-\kappa_\epsilon)\frac{Z_{n-1}(\theta_\epsilon)}{Z_{n-1}(\theta)}\biggr]. \tag{2.16}$$

Using (2.13) and the bound $Z_{n-1}(\theta) \geq \mathfrak{z}_\star^{n-1}$ we obtain

$$Z_n(\theta) \leq Z_{n-1}(\theta)\biggl[\kappa_\epsilon + (1-\kappa_\epsilon)H(\theta_\epsilon)\Bigl(\frac{\mathfrak{z}_1}{\mathfrak{z}_\star}\Bigr)^{n-1}\biggr]. \tag{2.17}$$

Let $\kappa_\epsilon(n)$ denote the quantity in the square brackets, and let us set $n = 2m$ in (2.17) and iterate the bound m times. This gives $Z_{2m}(\theta) \leq \kappa_\epsilon(m)^m Z_m(\theta)$. If we still entertain the possibility that $\mathfrak{z}_1 < \mathfrak{z}_\star$, then the $m \to \infty$ limit gives $\mathfrak{z}_\star \leq \lim_{m\to\infty}\kappa_\epsilon(m) = \kappa_\epsilon$, which contradicts the bound $\mathfrak{z}_1 \geq \kappa_\epsilon$. Therefore, once $x_\star > 1 - \frac{1}{b}$ and $\kappa_\epsilon < \mathfrak{z}_1$ for some $\epsilon > 0$ we have $\mathfrak{z}_1 = \mathfrak{z}_\star$.

Next we show that $\mathfrak{z}_\star = \mathfrak{z}_1$ whenever $x_\star > 1 - \frac{1}{b}$ but $\kappa_\epsilon = \mathfrak{z}_1$ for all $\epsilon > 0$ with $x_\star - \epsilon > 1 - \frac{1}{b}$. (Note that this means that $\mathfrak{z}_1 = \rho(\{x_\star\})$.) First we note that, using (2.15) with $n = 1$, we have $Z_{m+1}(1) \geq \kappa_\epsilon^m \rho([1 - \frac{1}{b}\theta, x_\star])$ whenever θ, ϵ and m satisfy (2.14). As a consequence of (2.11), we have

$$\frac{\mathfrak{z}_1}{\kappa_\epsilon} \geq \biggl[\frac{\rho([1-\frac{1}{b}\theta, x_\star])}{\kappa_\epsilon}\biggr]^{\frac{1}{m+1}}, \tag{2.18}$$

which implies that $\mathfrak{z}_1 > \kappa_\epsilon$ for some $\epsilon > 0$ whenever $\rho((1 - \frac{1}{b}\theta_b, \frac{b-1}{b}]) > 0$. (This fact will be important later, so we restate it as a corollary right after this proof.) Hence, we can as well assume that $\rho((1 - \frac{1}{b}\theta_b, \frac{b-1}{b}]) = 0$. To prove that $\mathfrak{z}_1 = \mathfrak{z}_\star$, let $\theta > \theta_b$ be small enough that $\theta_0 = 1 + \frac{1}{b}(\theta - \theta_b) < \theta_b$. Now either $X_k = x_\star$ for all $k = 1, \dots, n$, or there is a k such that $X_k \leq 1 - \frac{1}{b}\theta_b$. Noting that then $Q_k^{(\theta)} \leq \theta_0$, we thus have

$$Z_n(\theta) \leq \rho\bigl(\{x_\star\}\bigr)^n + \sum_{k=1}^n \rho\bigl(\{x_\star\}\bigr)^{k-1}\rho\bigl([0, 1 - \tfrac{1}{b}\theta_b]\bigr)Z_{n-k}(\theta_0), \tag{2.19}$$

Using (2.13), this gives $Z_n(\theta) \leq \mathfrak{z}_1^n + n\mathfrak{z}_1^{n-1}\rho([0, 1 - \frac{1}{b}\theta_b])H(\theta_0)$, proving that $\mathfrak{z}_\star = \mathfrak{z}_1$ also in this case.

It remains to establish the equality $\mathfrak{z}_1 = \mathfrak{z}_\star$ in the case $x_\star = 1 - \frac{1}{b}$. Note that then $\theta_b = 1$. Immediately, we have $Z_n(1) = \rho(\{x_\star\})^n$ and therefore $\mathfrak{z}_1 = \rho(\{x_\star\})$, while for any $\theta > \theta_b$ we have $\bigcap_{k=0}^n \{Q_k^{(\theta)} \ge 1\} \subset \bigcap_{k=1}^n \{X_k \ge x_\star - b^{-k}(\theta - \theta_b)\}$. Therefore,

$$Z_n(\theta) \le \prod_{k=1}^n \mathbb{P}\big(X_k \ge x_\star - b^{-k}(\theta - \theta_b)\big), \tag{2.20}$$

which implies that $\mathfrak{z}_\star \le \lim_{k\to\infty} \mathbb{P}(X_1 \ge x_\star - b^{-k}(\theta - \theta_b)) = \rho(\{x_\star\}) = \mathfrak{z}_1$. □

This completes the proof of part (1). One argument in the proof deserves special attention:

Corollary 2.5 *Let $\rho \in \mathcal{M}$ and suppose that $x_\star > 1 - \frac{1}{b}$ and $\rho((1 - \frac{1}{b}\theta_b, \frac{b-1}{b}]) > 0$. Then there is an $\epsilon > 0$ with $x_\star - \epsilon > \frac{b-1}{b}$ such that $\mathfrak{z}(\rho) > \rho([x_\star - \epsilon, x_\star])$.*

Next we will prove the continuity of $\alpha \mapsto \mathfrak{z}(\alpha\rho + (1-\alpha)\rho')$ as stated in Theorem 2.4(2):

Proof of Theorem 2.4(2). Throughout this proof we will write $Z_n^{(\rho)}(\theta)$ instead of just $Z_n(\theta)$ to emphasize the dependence on the underlying measure ρ. Let $\rho_0, \rho_1 \in \mathcal{M}$ and let $\rho_\alpha = (1-\alpha)\rho_0 + \alpha\rho_1$. Clearly, to prove (2), it suffices to show that $\alpha \mapsto \mathfrak{z}(\rho_\alpha)$ is right continuous at $\alpha = 0$.

Fix $\alpha > 0$ and let (T_k) be a sequence of $0,1$-valued i.i.d. random variables with $\text{Prob}(T_k = 0) = \alpha$. Let (X_k) and (X_k') be two independent sequences of i.i.d. random variables, both independent of (T_k), with distributions $\rho_0^{\mathbb{N}}$ and $\rho_1^{\mathbb{N}}$, respectively. Let $(X_k^{(\alpha)})$ be the sequence defined by

$$X_k^{(\alpha)} = T_k X_k + (1 - T_k) X_k', \qquad k \ge 1. \tag{2.21}$$

Clearly, $(X_k^{(\alpha)})$ are i.i.d. with joint distribution $\rho_\alpha^{\mathbb{N}}$. Let us use $\mathbb{P}_\alpha$ to denote the joint distribution of (X_k), (X_k'), and (T_k).

Let $Q_n^{(\theta,\alpha)}$ be given by (2.5) with $X_1, \dots, X_n$ replaced by $X_1^{(\alpha)}, \dots, X_n^{(\alpha)}$. Then $Z_n^{(\rho_\alpha)}(\theta)$ is given by (2.8) with $Q_n^{(\theta)}$ replaced by $Q_n^{(\theta,\alpha)}$ and $\mathbb{P}$ replaced by $\mathbb{P}_\alpha$. As will be seen shortly, the main object of interest is the conditional expectation given the values (T_k):

$$Z_{n,\alpha}\big(\,\theta\,|(T_k)\big) = \mathbb{P}_\alpha\big(Q_\ell^{(\theta,\alpha)} \ge 1,\ \ell = 0, \dots, n\big|(T_k)\big). \tag{2.22}$$

Indeed, let $\theta \in [1, \theta_b]$ and, given (T_k), let (I_i) be the connected blocks of sites $k \in \{0, \dots, 1\}$ such that $T_k = 1$ and let (J_j) be the connected sets of sites with $T_k = 0$. By (2.21), the $X_k^{(\alpha)}$ for $k \in I_i$ are distributed according to ρ_0, while those for $k \in J_j$ are distributed according to ρ_1. Then an analogue of (2.10) for the quantity in (2.22) along with the bounds $Z_n(1) \le Z_n(\theta) \le Z_n(\theta_b)$ for $\theta \in [1, \theta_b]$ allow us to conclude that

$$\prod_i Z_{|I_i|}^{(\rho_0)}(1) \prod_j Z_{|J_j|}^{(\rho_1)}(1) \le Z_{n,\alpha}\big(\,\theta\,|(T_k)\big) \le \prod_i Z_{|I_i|}^{(\rho_0)}(\theta_b) \prod_j Z_{|J_j|}^{(\rho_1)}(\theta_b). \tag{2.23}$$

In order to estimate the right hand side of (2.23), note that the existence of the limit in (2.9) implies that for all $\delta > 0$ there is $C_\delta \in [1, \infty)$, such that for both $\rho = \rho_0$ and $\rho = \rho_1$,

$$Z_n^{(\rho)}(\theta_b) \le C_\delta (1+\delta)^n \mathfrak{z}(\rho)^n, \qquad n \ge 1. \tag{2.24}$$

Let $\mathbb{E}_\alpha$ denote the expectation with respect to $\mathbb{P}_\alpha$. Using (2.24) in (2.23), observing that the total number of occurrences of C_δ is less than $2k_1(Y)$, where $k_1(Y) = \sum_j |J_j|$, and noting that $k_1(Y)$ has the binomial distribution with parameter α under $\mathbb{P}_\alpha$ allows us to write

$$Z_n^{(\rho_\alpha)}(\theta) = \mathbb{E}_\alpha Z_{n,\alpha}\big(\,\theta\,|(T_k)\big) \le (1+\delta)^n \big((1-\alpha)\mathfrak{z}(\rho_0) + \alpha C_\delta^2 \mathfrak{z}(\rho_1)\big)^n. \tag{2.25}$$

By taking $n \to \infty$, we get $\lim_{\alpha\downarrow 0} \mathfrak{z}(\rho_\alpha) \le (1+\delta)\mathfrak{z}(\rho_0)$. But δ was arbitrary, hence, $\lim_{\alpha\downarrow 0} \mathfrak{z}(\rho_\alpha) \le \mathfrak{z}(\rho_0)$. The argument for the lower bound, $\lim_{\alpha\downarrow 0} \mathfrak{z}(\rho_\alpha) \ge \mathfrak{z}(\rho_0)$, is completely analogous. □

Finally, we also need to prove part (3) of Theorem 2.4:

Proof of Theorem 2.4(3). By Proposition 2.3, $\sigma \in \mathcal{A}^{(v)}$ is exactly the event that the path between (and including) σ and $\varnothing$ consists of sites σ' with $Q_{\sigma'}^{(\theta)} \ge 1$, where $\theta = X_\varnothing + v$. But then

$$\mathbb{P}_\rho(\sigma \in \mathcal{A}^{(v)}) = \mathbb{E}_\rho\bigl(Z_{|\sigma|}(X_\varnothing + v)\bigr), \tag{2.26}$$

where the final average is over $X_\varnothing$. To get the expected size of $\mathcal{A}^{(v)}$, we sum over all σ,

$$\mathbb{E}_\rho\bigl(|\mathcal{A}^{(v)}|\bigr) = \sum_{n\ge 0} \mathbb{E}_\rho\bigl(Z_n(X_\varnothing + v)\bigr) b^n \tag{2.27}$$

The existence of the limit $Z_n^{1/n}(\theta)$ independent of θ (for $\theta \ge 1$) tells us that $\mathbb{E}_\rho(|\mathcal{A}^{(v)}|) < \infty$ whenever $\mathfrak{z}(\rho) < \mathfrak{z}_c$, while $\mathbb{E}_\rho(|\mathcal{A}^{(v)}|) = \infty$ once $\mathfrak{z}(\rho) > \mathfrak{z}_c$ and $v > 1 - x_\star$. □

3. ABSENCE OF INTERMEDIATE PHASE

3.1 Sharpness of phase transition.

The goal of this section is to show that the phase transitions defined by presence/absence of an infinite avalanche and divergence of avalanche size occur at the same "point," $\mathfrak{z}_c = \frac{1}{b}$. This rules out the possibility of an intermediate phase. Moreover, we will prove that the transition is *second order* in the sense that there is no infinite avalanche at $\mathfrak{z} = \mathfrak{z}_c$.

Unfortunately, our proof will require certain restrictions on the underlying measure ρ. To ease exposition, we will assume that the measure ρ is provided by a bounded density ϕ_ρ with respect to the Lebesgue measure on $[0,1]$. We remark that the boundedness assumption is not essential for the validity of our results: Indeed, with almost no extra work, we may relax the assumption to the case when $\phi_\rho \in L^p([0,1])$ for some $p > 1$. Moreover, the behavior of the measure in the interval $[0, 1 - \frac{1}{b}\theta_b)$ is utterly inconsequential since, with probability one, only a finite number of sites σ with X_σ in this range can participate in an avalanche. Thus, for all intents and purposes, the measure there should be chosen as whatever is easiest; the only issue of significance is the total mass in this region.

For the range $[1 - \frac{1}{b}, 1]$, singularities of the measure do not play any important role either. However, in the intermediate region, $I = [1 - \frac{1}{b}\theta_b, \frac{b-1}{b})$, some restrictions are needed to push through our analysis and to avoid the sort of counterexamples described in Remarks 1 and 2 of Section 2. With a significant amount of additional work (and spreading lots of little provisos here and there in the proofs), most of what appears in the present paper can be proved under the weaker assumption that ρ has an L^p density, for a $p > 1$, in the interval I.

Definition. *Let $\mathcal{M}^\flat$ be the set of Borel probability measures ρ on $[0,1]$ that are absolutely continuous with respect to Lebesgue measure on $[0,1]$ and the associated density ϕ_ρ is bounded in L^∞ norm on $[0,1]$, i.e., $\|\phi_\rho\|_\infty < \infty$.*

It is worth noting that $\mathcal{M}^\flat$ is a convex subset of $\mathcal{M}$. The ability to take convex combinations of elements of $\mathcal{M}^\flat$ will be crucial in the discussion of the critical behavior, see Section 4.

Our second main theorem is then as follows:

Theorem 3.1 *Suppose that $\rho \in \mathcal{M}^\flat$ and define $\mathfrak{z}_{\rm c} = \frac{1}{b}$.*
(1) If $\mathfrak{z}(\rho) \le \mathfrak{z}_{\rm c}$, then $\mathbb{P}_\rho(|\mathcal{A}^{(v)}| = \infty) = 0$ for all $v \in (0,\infty)$.
(2) If $\mathfrak{z}(\rho) > \mathfrak{z}_{\rm c}$, then $\mathbb{P}_\rho(|\mathcal{A}^{(v)}| = \infty) > 0$ for all $v \in (1-x_\star,\infty)$.

The proof of Theorem 3.1 requires introducing two auxiliary random variables V_∞ and Q_∞. These will be defined in next two subsections, the proof is therefore deferred to Section 3.4. The random variable Q_∞ will be a cornerstone of our analysis of the critical process, see Section 4. The underlying significance of both V_∞ and Q_∞ is the distributional identity that each of them satisfies.

3.2 Definition of V_∞.

In this section we define a random variable V_∞ which is, roughly speaking, the minimal value of v that needs to be added to the root in order to trigger an infinite avalanche. For $n \ge 1$ let

$$V_n = \inf\bigl\{v \in (0,\infty) \colon \boldsymbol{X}^{(v)}(t+1) \ne \boldsymbol{X}^{(v)}(t),\, t = 0,\dots,n-1\bigr\}. \tag{3.1}$$

(A logical extension of this definition to $n=0$ is $V_0 \equiv 0$.) In plain words, if $v \ge V_n$, then the avalanche process will propagate to at least the n-th level. Clearly, V_n is an increasing sequence; we let V_∞ denote the $n\to\infty$ limit of V_n. Formally, V_∞ can be infinite; in fact, since the event $\{V_\infty < \infty\}$ is clearly a tail event, $\mathbb{P}_\rho(V_\infty<\infty)$ is either one or zero.

Let us use Ψ_n to denote the distribution function of V_n, i.e.,

$$\Psi_n(\vartheta) = \mathbb{P}_\rho(V_n \le \vartheta). \tag{3.2}$$

The aforementioned properties of V_n lead us to a few immediate observations about Ψ_n: First, Ψ_n is a decreasing sequence of non-decreasing functions. Second, the limit

$$\Psi(\vartheta) = \lim_{n\to\infty} \Psi_n(\vartheta), \tag{3.3}$$

exists for all $\vartheta \in (0,\infty)$ and $\Psi(\vartheta) = \mathbb{P}_\rho(V_\infty \le \vartheta)$. Third, $\Psi \not\equiv 0$ if and only if $\mathbb{P}_\rho(V_\infty < \infty) = 1$. Moreover, each of Ψ_n is in principle computable:

Lemma 3.2 *The sequence (Ψ_n) satisfies the recurrence equation*

$$\Psi_{n+1}(\vartheta) = \mathbb{E}_\rho\biggl(\Phi_b\Bigl(\Psi_n\bigl(\tfrac{X_\varnothing+\vartheta}{b}\bigr)\Bigr)\mathsf{1}_{\{X_\varnothing \ge 1-\vartheta\}}\biggr), \qquad n \ge 0, \tag{3.4}$$

where $\Psi_0(\vartheta) = \mathsf{1}_{\{\vartheta\ge 0\}}$ and

$$\Phi_b(y) = 1-(1-y)^b, \qquad 0 \le y \le 1. \tag{3.5}$$

Proof. Let $\mathbb{T}_b^{(\sigma)}$ denote the subtree of $\mathbb{T}_b$ rooted at σ and let $V_n^{(\sigma)}$ denote the random variable defined in the same way as V_n but here for the tree $\mathbb{T}_b^{(\sigma)}$. Then we have

$$\{V_{n+1} \le \vartheta\} = \{X_\varnothing \ge 1-\vartheta\} \cap \Bigl\{\min_{\sigma\in\{1,\dots,b\}} V_n^{(\sigma)} \le \frac{X_\varnothing+\vartheta}{b}\Bigr\}. \tag{3.6}$$

But for all $\sigma \in \{1,\dots,b\}$, the $V_n^{(\sigma)}$'s are i.i.d. with common distribution function Ψ_n, so we have

$$\mathbb{P}_\rho\bigl(\min_{\sigma\in\{1,\dots,b\}} V_n^{(\sigma)} \le \vartheta\bigr) = \Phi_b\bigl(\Psi_n(\vartheta)\bigr). \tag{3.7}$$

From here the claim follows by noting that $V_n^{(\sigma)}$ are independent of $X_\varnothing$. □

Corollary 3.3 *The distribution function of V_∞ satisfies the equation*

$$\Psi(\vartheta) = \mathbb{E}_\rho\biggl(\Phi_b\Bigl(\Psi\bigl(\tfrac{X_\varnothing+\vartheta}{b}\bigr)\Bigr)\mathsf{1}_{\{X_\varnothing\ge 1-\vartheta\}}\biggr), \qquad \vartheta\ge 0. \tag{3.8}$$

Proof. This is an easy consequence of (3.4) and the Bounded Convergence Theorem. □

On the basis of (3.8) and some percolation arguments, the answer to the important question whether $\Psi\equiv 0$ or not can be given by checking whether $\Psi(\vartheta)=0$ for reasonable values of ϑ:

Proposition 3.4 *Suppose that $\Psi\not\equiv 0$. Then*

$$\inf\bigl\{\vartheta\ge 0\colon \Psi(\vartheta)>0\bigr\} = 1-x_\star. \tag{3.9}$$

Proof. Let $\vartheta_\star$ denote the infimum on the left-hand side of (3.9). First note that if $x_\star < 1-\frac1b$, then Proposition 2.1(2) guarantees that there is no infinite avalanche so we can as well suppose that $x_\star \ge 1-\frac1b$. Now, since $\{V_\infty\le\vartheta\}\subset\{X_\varnothing\ge 1-\vartheta\}$, we have $\mathbb{P}_\rho(V_\infty<1-x_\star)=0$ and thus $\vartheta_\star\ge 1-x_\star$. Next we can use (3.8) to conclude that

$$\vartheta>\vartheta_\star \quad\Rightarrow\quad \frac{x_\star+\vartheta}{b}\ge\vartheta_\star, \tag{3.10}$$

which in turn implies that $\vartheta_\star\le\frac{x_\star}{b-1}$. Consequently, $\vartheta_\star\in[1-x_\star,\frac{x_\star}{b-1}]$. But if $x_\star=1-\frac1b$, then $\frac{x_\star}{b-1}=1-x_\star$ and we are done, so we can assume $x_\star>1-\frac1b$ for the rest of this proof.

Since ρ is absolutely continuous with respect to the Lebesgue measure on $[0,1]$, there is an $\eta>0$ such that $x_\star-\eta>1-\frac1b$ and $\rho([x_\star-\eta,x_\star])<\frac1b$. Now $\frac1b$ is the threshold for the site percolation on $\mathbb{T}_b$, so the sites with $X_\sigma>x_\star-\eta$ do not percolate. Let $\mathbb{G}_n=\{\sigma\in\mathbb{T}_b\colon|\sigma|=n\}$ be the n-th generation of $\mathbb{T}_b$. Pick two integers N,N' such that $N'\ge N$ and let $\mathcal{H}_{N,N'}$ be the event that $\mathbb{G}_N$ and $\mathbb{G}_{N'}$ are separated by a "barrier" of sites σ with $X_\sigma\le x_\star-\eta$. By taking $N'\gg N\gg 1$, the probability of $\mathcal{H}_{N,N'}$ can be made as close to one as desired.

Let $\vartheta>\vartheta_\star$ and pick N_0 so large that ϑb^{-N_0} is less than $\frac\eta2$. Find $N,N'\ge N_0$ such that $1-\mathbb{P}_\rho(\mathcal{H}_{N,N'})$ is strictly smaller than $\mathbb{P}_\rho(|\mathcal{A}^{(\vartheta)}|=\infty)$, i.e., we have $\mathbb{P}_\rho(\{|\mathcal{A}^{(\vartheta)}|=\infty\}\cap\mathcal{H}_{N,N'})>0$. Now for any $\epsilon\in(0,\frac\eta2)$, we will produce a configuration with an infinite avalanche that has a starting value $v=1-x_\star+\epsilon$. Draw a configuration $(\bar X_\sigma)$ subject to the constraint that $\bar X_\sigma\ge x_\star-\epsilon$ for all $\sigma\in\mathbb{T}_b$ with $|\sigma|\le N'$. Let (X_σ) belong to the set $\{|\mathcal{A}^{(\vartheta)}|=\infty\}\cap\mathcal{H}_{N,N'}$ and define X'_σ by putting

$$X'_\sigma=\begin{cases}\bar X_\sigma\vee X_\sigma, & \text{if } |\sigma|\le N',\\ X_\sigma, & \text{otherwise.}\end{cases} \tag{3.11}$$

Let $X_\sigma^{\prime,(v)}(t)$ denote the process corresponding to the initial configuration (X'_σ) and initial value $v>0$, and let $X_\sigma^{(\vartheta)}(t)$ be the corresponding process for (X_σ) and ϑ. Let $\mathcal{A}^{\prime,(v)}$ and $\mathcal{A}^{(\vartheta)}$ be the corresponding avalanche sets.

The configuration (X_σ) exhibits an infinite avalanche, so there is a site σ on one of the aforementioned "barriers" separating $\mathbb{G}_N$ and $\mathbb{G}_{N'}$, which belongs to an infinite oriented path inside $\mathcal{A}^{(\vartheta)}$. By the assumption that $x_\star-\eta>1-\frac1b$ it is clear that, if $v>1-x_\star+\epsilon$ and $t=|\sigma|$, then $\mathcal{A}^{\prime,(v)}$ will reach σ. But $X'_{\sigma'}\ge X_\sigma$ for all sites on the path from $\varnothing$ to σ, so we have

$$X_\sigma^{\prime,(v)}(t)-X_\sigma^{(\vartheta)}(t)\ge\eta-\epsilon-\frac{\vartheta-v}{b^N}>0, \tag{3.12}$$

where we used that $b^N\vartheta \le \frac{\eta}{2}$ and $\epsilon < \frac{\eta}{2}$ to derive the last inequality. Now the set $\mathcal{A}^{(\vartheta)}$ contains a path from σ to infinity and, by (3.12) and $X'_{\sigma'} \ge X_{\sigma'}$ for σ' "beyond" σ, this path will also be contained in $\mathcal{A}^{\prime,(v)}$. Consequently, an infinite avalanche will occur in configuration (X'_σ) starting from a value $v > 1 - x_\star + \epsilon$ whenever it did in configuration (X_σ) starting from ϑ. This establishes $\vartheta_\star = 1 - x_\star$, as claimed. □

3.3 Definition of Q_∞.

The second random variable, denoted by Q_∞ is a limiting version of the objects $Q_n^{(\theta)}$ defined in (2.5). Let $Y = (Y_1, Y_2, \dots)$ be a sequence of i.i.d. random variables with joint distribution $\mathbb{P} = \rho^{\mathbb{N}}$. These are, in a certain sense, the same quantities as the X's discussed earlier, however, the Y's will be ordered in the opposite way. Similarly to (2.5), let

$$Q_{n,k}^{(\theta)} = Y_k + \frac{1}{b}Y_{k+1} + \dots + \frac{1}{b^{n-k}}Y_n + \frac{\theta}{b^{n-k+1}}, \qquad 1 \le k \le n. \tag{3.13}$$

For completeness, we also let $Q_{0,1}^{(\theta)} = \theta$.

Let $\mathcal{B}$ be the Borel σ-algebra on $[0,1]^{\mathbb{N}}$ equipped with the standard product topology. Suppose that $\rho([\frac{b-1}{b}, 1]) > 0$. For any $n \ge 1$ and $\theta \ge 1$, let $\mathbb{P}_n^{(\theta)}$ be the conditional law on $\mathcal{B}$ defined by

$$\mathbb{P}_n^{(\theta)}(\,\cdot\,) = \mathbb{P}\big(\,\cdot\,\big|\, Q_{n,\ell}^{(\theta)} \ge 1,\ \ell = 2, \dots, n\big), \tag{3.14}$$

The latter is well defined because $\mathbb{P}(Q_{n,\ell}^{(\theta)} \ge 1) > 0$ for all $\ell = 2, \dots, n$, $\{Q_{n,\ell}^{(\theta)} \ge 1\}$ are increasing and $\mathbb{P}(\cdot)$ is FKG. Intentionally, the variable Y_1 is not constrained by the conditioning in (3.14).

Next we give conditions for the existence of the limiting law $\lim_{n\to\infty} \mathbb{P}_n^{(\theta)}$:

Proposition 3.5 *Let $\rho \in \mathcal{M}^\flat$ and let $\theta_0 > \theta_b$. Then there are $A = A(\rho, \theta_0) < \infty$ and $\zeta = \zeta(\rho) > 0$ such that for all bounded measurable functions $f = f(Y_1, \dots, Y_k)$ and all $\theta, \theta' \in [1, \theta_0]$,*

$$\big|\mathbb{E}_{n+1}^{(\theta)}(f) - \mathbb{E}_n^{(\theta')}(f)\big| \le A e^{-\zeta(n-k)} \|f\|_\infty, \qquad n \ge k. \tag{3.15}$$

In particular, whenever $\theta \ge 1$, the limit law

$$\widehat{\mathbb{P}}(\cdot) = \lim_{n\to\infty} \mathbb{P}_n^{(\theta)}(\cdot) \tag{3.16}$$

exists and is independent of θ. Moreover, the bounds $A(\rho, \theta_0) < \infty$ and $\zeta(\rho) > 0$ are uniform in any convex set $\mathcal{N} \subset \mathcal{M}^\flat$ with finitely many extreme points.

The proof of Proposition 3.5 uses a coupling argument, which requires some rather extensive preparations and is therefore deferred to Section 5. (The actual proof appears at the end of Section 5.3.)

We will use $\widehat{\mathbb{E}}$ to denote the expectation with respect to $\widehat{\mathbb{P}}$ whenever the latter is well defined. Let us define a random variable Q_∞ on $([0,1]^{\mathbb{N}}, \mathcal{B}, \widehat{\mathbb{P}})$ by the formula

$$Q_\infty = \sum_{k\ge1} \frac{Y_k}{b^{k-1}}. \tag{3.17}$$

Notice that Q_∞ is supported in $[\frac{1}{b}, \theta_b]$, because Y_1 is not constrained by the conditioning in (3.14).

Corollary 3.6 *Let $\rho \in \mathcal{M}^\flat$ and let $\theta \ge 1$. Let $Q_{n,1}^{(\theta)}$ be as in* (3.13)*, where the variables $Y_1, \dots, Y_n$ are distributed according to $\mathbb{P}_n^{(\theta)}$. Then $Q_{n,1}^{(\theta)}$ tends to Q_∞ in distribution as $n \to \infty$. Moreover, for each*

$\theta_0 > \theta_b$ and each $C < \infty$ there are constants $D = D(\rho, \theta_0) < \infty$ and $\varsigma = \varsigma(\rho) > 0$ such that if $f(\theta)$ is a function obeying the Lipschitz bound on $[0, \theta_0]$,

$$\big|f(\theta) - f(\theta')\big| \le C\|f\|_\infty\, |\theta - \theta'|, \qquad \theta, \theta' \in [0, \theta_0], \tag{3.18}$$

where $\|f\|_\infty = \sup_{\theta \le \theta_0} |f(\theta)|$, then

$$\Big|\mathbb{E}_n^{(\theta)}\big(f(Q_{n,1}^{(\theta)})\big) - \widehat{\mathbb{E}}\big(f(Q_\infty)\big)\Big| \le D\,\|f\|_\infty e^{-\varsigma n} \tag{3.19}$$

holds for all $\theta \in [1, \theta_0]$. The bounds $D(\rho, \theta_0) < \infty$ and $\varsigma(\rho) > 0$ are uniform in any convex set $\mathcal{N} \subset \mathcal{M}^\flat$ with finitely many extreme points.

The proof of Corollary 3.6 is given in Section 5.4. As already mentioned, a principal tool for our later investigations will be the distributional identity for Q_∞ stated below.

Proposition 3.7 *Let $\rho \in \mathcal{M}^\flat$. If X is a random variable with law $\mathbb{P} = \rho$, independent of Q_∞, then*

$$\mathbb{P} \otimes \widehat{\mathbb{P}}\biggl(X + \frac{Q_\infty}{b} \in \cdot \,\bigg|\, Q_\infty \ge 1\biggr) = \widehat{\mathbb{P}}(Q_\infty \in \cdot\,). \tag{3.20}$$

The proof of Proposition 3.7 will also be given in Section 5. Proposition 3.5 and the proof of Proposition 3.7 immediately yield an extension of Theorem 2.4(1), stated as Corollary 3.8, which will also be useful in subsequent developments. The proof of Corollary 3.8 is given in Section 5.4.

Corollary 3.8 *Suppose that $\rho \in \mathcal{M}^\flat$. Then $\mathfrak{z}(\rho) = \widehat{\mathbb{P}}(Q_\infty \ge 1)$. Moreover, if $\mathfrak{z}(\rho) > 0$, then the limit*

$$\psi_\rho(\theta) = \lim_{n\to\infty} Z_n(\theta)\mathfrak{z}(\rho)^{-n} \tag{3.21}$$

exists for all $\theta \ge 0$ and, for all $\theta_0 > \theta_b$, there are $A' = A'(\rho, \theta_0) < \infty$ and $\zeta' = \zeta'(\rho) > 0$ such that

$$\big|Z_n(\theta)\mathfrak{z}(\rho)^{-n} - \psi_\rho(\theta)\big| \le A' e^{-\zeta' n} \tag{3.22}$$

holds for all $\theta \in [0, \theta_0]$ and all $n \ge 1$. Furthermore, the function ψ_ρ has the following properties:

1. *$\psi_\rho(\theta) \in (0, \infty)$ for all $\theta \ge 1$ while $\psi_\rho(\theta) = 0$ for $\theta < 1$.*
2. *$\theta \mapsto \psi_\rho(\theta)$ is non-decreasing and Lipschitz continuous for all $\theta \ge 1$. More precisely, there is a $C = C(\rho, \theta_0) < \infty$ such that $|\psi_\rho(\theta) - \psi_\rho(\theta')| \le C\psi_\rho(\theta_0)|\theta - \theta'|$ for all $\theta, \theta' \in [1, \theta_0]$.*
3. *If $\rho, \rho' \in \mathcal{M}^\flat$ and $\rho_\alpha = (1-\alpha)\rho + \alpha\rho'$ for each $\alpha \in [0,1]$, then $\alpha \mapsto \psi_{\rho_\alpha}(\theta)$ is continuous in $\alpha \in [0,1]$ for all $\theta \ge 0$.*

The bounds $A'(\rho, \theta_0) < \infty$, $\zeta'(\rho) > 0$ and $C(\rho, \theta_0) < \infty$ are uniform in any convex set $\mathcal{N} \subset \mathcal{M}^\flat$ with finitely many extreme points.

Remark 3. The Lipschitz continuity of $\theta \mapsto \psi_\rho(\theta)$ is a direct consequence of our assumption that ρ has a *bounded* density ϕ_ρ with respect to the Lebesgue measure on $[0, 1]$. If ϕ_ρ is only in $L^p([0, 1])$ for some $p > 1$, then the appropriate concept will be Hölder continuity with a p-dependent exponent. The same will be true for various other Lipschitz continuous quantities later in this paper.

3.4 Proof of Theorem 3.1.

With random variable Q_∞ at our disposal, the sharpness of the phase transition in our avalanche model is almost immediate.

Proof of Theorem 3.1. Let $\rho \in \mathcal{M}^\flat$ and abbreviate $\mathfrak{z} = \mathfrak{z}(\rho)$. Let $x_\star$ be as in (2.1). We begin by introducing the quantity

$$G_n = \widehat{\mathbb{E}}\Big(\Psi_n\big(\tfrac{Q_\infty}{b}\big)1_{\{Q_\infty \geq 1\}}\Big). \tag{3.23}$$

The recursive equation (3.4) and Proposition 3.7 then give

$$\begin{aligned} G_{n+1} &= \widehat{\mathbb{P}}(Q_\infty \geq 1)\,\mathbb{E}_\rho \otimes \widehat{\mathbb{E}}\bigg(\Psi_n\Big(\frac{X_\varnothing + \frac{1}{b}Q_\infty}{b}\Big)1_{\{X_\varnothing + \frac{1}{b}Q_\infty \geq 1\}}\,\bigg|\,Q_\infty \geq 1\bigg) \\ &= \mathfrak{z}\,\widehat{\mathbb{E}}\Big(\Phi_b\Big(\Psi_n\big(\tfrac{Q_\infty}{b}\big)1_{\{Q_\infty \geq 1\}}\Big)\Big), \end{aligned} \tag{3.24}$$

where we have used the fact that $\mathfrak{z} = \widehat{\mathbb{P}}(Q_\infty \geq 1)$ from Corollary 3.8.

Let us first analyze the cases $b\mathfrak{z} \leq 1$. By using Jensen's inequality in (3.24) we get that

$$G_{n+1} \leq \mathfrak{z}\,\Phi_b(G_n) \leq \frac{1}{b}\Phi_b(G_n). \tag{3.25}$$

An inspection of the graph of $y \mapsto \Phi_b(y)$ reveals that if (3.25) holds, then $G_n \to 0$. By Proposition 3.4, this is compatible with $\Psi \not\equiv 0$ only if

$$\frac{Q_\infty}{b} \leq 1 - x_\star \qquad \widehat{\mathbb{P}}\text{-almost surely.} \tag{3.26}$$

However, a simple argument shows that $\operatorname{esssup} Q_\infty = x_\star \frac{b}{b-1}$ whenever $\mathfrak{z} > 0$. This contradicts (3.26), because $x_\star > \frac{b-1}{b}$ implies $1 - x_\star < x_\star \frac{b}{b-1}$. Thus, if $b\mathfrak{z} \leq 1$, then Ψ must be identically zero.

Next we will attend to the cases $b\mathfrak{z} > 1$. We will suppose that $\Psi_n \to 0$ and work to derive a contradiction. Since $n \mapsto \Psi_n$ is a monotone sequence of monotone functions, the convergence to Ψ is uniform on $[0,1]$ and, in particular, on the range of values that $\frac{1}{b}Q_\infty$ takes. Using that $\Phi_b(y) \geq by - \frac{1}{2}b(b-1)y^2$ for all $y \in [0,1]$ and invoking (3.24), we can write

$$G_{n+1} \geq b\mathfrak{z}(1 - \epsilon_n)G_n, \tag{3.27}$$

where $\epsilon_n = \frac{1}{2}(b-1)\Psi_n(1)$. Since $b\mathfrak{z} > 1$ and $\epsilon_n \to 0$, we have $G_{n+1} \geq G_n$ for n large enough. An inspection of (3.4) shows that, since $x_\star > 1 - \frac{1}{b}$, we have $\Psi_n(\vartheta) > 0$ for all $\vartheta > \frac{1}{b}$. Hence $G_n > 0$ for all $n \geq 0$. But then (3.27) forces G_n to stay uniformly bounded away from zero, in contradiction with our assumption that $G_n \to 0$. Therefore, once $b\mathfrak{z} > 1$, we must have $\Psi \not\equiv 0$. □

4. Critical behavior

4.1 Critical exponents.

In this section we establish, under certain conditions on ρ, the essential behavior of the model at the critical point $\mathfrak{z}_c = \frac{1}{b}$. In particular, we describe the asymptotics for the critical distribution of avalanche sizes, the power law behavior for the probability of an infinite avalanche as $\mathfrak{z} \downarrow \mathfrak{z}_c$ and, finally, the exponent for the divergence of $\chi^{(v)}$ as $\mathfrak{z} \uparrow \mathfrak{z}_c$.

Theorem 4.1 *Let $\rho \in \mathcal{M}^\flat$ and let $x_\star$ be as in* (2.1)*. Suppose $\mathfrak{z}(\rho) = \mathfrak{z}_c$, where $\mathfrak{z}_c = \frac{1}{b}$. Then there are functions $\tau, \mathcal{T} \colon (1 - x_\star, \infty) \to (0, \infty)$ and $\Theta \colon [0, \infty) \to [0, \infty)$ such that the following holds:*

(1) If $\rho' \in \mathcal{M}^\flat$ and $\rho_\alpha = \alpha\rho' + (1-\alpha)\rho$ satisfies $\mathfrak{z}(\rho_\alpha) < \mathfrak{z}_c$ for all $\alpha \in (0,1]$, then for all $v > 1 - x_\star$,

$$\mathbb{E}_{\rho_\alpha}\bigl(|\mathcal{A}^{(v)}|\bigr) = \frac{\tau(v)}{\mathfrak{z}_c - \mathfrak{z}(\rho_\alpha)}\bigl[1+o(1)\bigr], \qquad \alpha \downarrow 0. \tag{4.1}$$

(2) For all $v \ge 0$,

$$\mathbb{P}_\rho\bigl(|\mathcal{A}^{(v)}| \ge n\bigr) = \frac{\Theta(v)}{n^{1/2}}\bigl[1+o(1)\bigr], \qquad n \to \infty, \tag{4.2}$$

where $\Theta(v) > 0$ for $v > 1 - x_\star$.
(3) If $\rho' \in \mathcal{M}^\flat$ and $\rho_\alpha = \alpha\rho' + (1-\alpha)\rho$ satisfies $\mathfrak{z}(\rho_\alpha) > \mathfrak{z}_c$ for all $\alpha \in (0,1]$, then for all $v > 1 - x_\star$,

$$\mathbb{P}_{\rho_\alpha}\bigl(|\mathcal{A}^{(v)}| = \infty\bigr) = \bigl(\mathfrak{z}(\rho_\alpha) - \mathfrak{z}_c\bigr)\,\mathcal{T}(v)\bigl[1+o(1)\bigr], \qquad \alpha \downarrow 0. \tag{4.3}$$

Remark 4. The proof of Theorem 4.1 makes frequent use of the properties of the random variable Q_∞ defined in Section 3.2. The relevant statements are Propositions 3.5 and 3.7 and Corollaries 3.6 and 3.8, whose proofs come only in Section 5. Modulo these claims, Section 4 is essentially self-contained and can be read without a reference to Section 5.

Part (1) of Theorem 4.1 can be proved based on the already-available information; the other parts will require some preparations and their proofs are postponed to the next section.

Proof of Theorem 4.1(1). Let $\rho, \rho' \in \mathcal{M}^\flat$ be such that $\mathfrak{z}(\rho_\alpha) < \mathfrak{z}_c = \mathfrak{z}(\rho)$ for $\rho_\alpha = (1-\alpha)\rho + \alpha\rho'$ and all $\alpha \in (0,1]$. Let $\chi^{(v)}(\alpha) = \mathbb{E}_{\rho_\alpha}(|\mathcal{A}^{(v)}|)$. By (2.27),

$$\chi^{(v)}(\alpha) = \sum_{n\ge 0} \mathbb{E}_{\rho_\alpha}\bigl(Z_n^{(\rho_\alpha)}(X_\varnothing + v)\bigr)\, b^n, \tag{4.4}$$

where $\mathbb{E}_{\rho_\alpha}$ is the expectation with respect to $X_\varnothing$ in ρ_α and $Z_n^{(\rho_\alpha)}$ is defined by (2.8) using ρ_α.

In order to estimate the sum we will use A' and ζ' to denote the worst case scenarios for the quantities $A'(\rho_\alpha)$ and $\zeta'(\rho_\alpha)$ from Corollary 3.8: $A' = \inf_{0\le\alpha\le1} A'(\rho_\alpha)$ and $\zeta' = \inf_{0\le\alpha\le1} \zeta'(\rho_\alpha)$. Then we have, for all $n \ge 1$ and all $\theta \ge 1$,

$$b^n Z_n^{(\rho_\alpha)}(\theta) = b^n \mathfrak{z}(\rho_\alpha)^n \psi_{\rho_\alpha}(\theta) + b^n \mathfrak{z}(\rho_\alpha)^n E_n(\theta), \tag{4.5}$$

where $\psi_{\rho_\alpha}(\theta)$ is as in (3.21) while $E_n(\theta)$ is the "error term." Using the bounds from Corollary 3.8, $E_n(\theta)$ is estimated by $|E_n(\theta)| \le A' e^{-\zeta' n}$. By continuity of $\alpha \mapsto \psi_{\rho_\alpha}(\theta)$, we get

$$\sum_{n\ge 0} b^n Z_n^{(\rho_\alpha)}(\theta) = \frac{\psi_\rho(\theta) + o(1)}{1 - b\mathfrak{z}(\rho_\alpha)}, \tag{4.6}$$

where $o(1)$ tends to zero as $\alpha \downarrow 0$ uniformly on compact sets of θ.

Let $\tau(v) = b^{-1}\mathbb{E}_\rho(\psi_\rho(X_\varnothing + v))$ and note that $\tau(v) > 0$ for all $v > 1 - x_\star$. Then we have

$$\chi^{(v)}(\alpha) = \frac{\tau(v)}{\mathfrak{z}_c - \mathfrak{z}(\rho_\alpha)}\bigl[1+o(1)\bigr], \tag{4.7}$$

where $o(1)$ tends to zero as $\alpha \downarrow 0$, for all $v \ge 1 - x_\star$. □

It remains to establish parts (2) and (3) of Theorem 4.1. To ease derivations, instead of looking at the asymptotic size of $\mathcal{A}^{(v)}$, we will focus on a slightly different set:

$$\mathcal{B}^{(\theta)} = \begin{cases} \{\varnothing\}, & \text{if } \mathcal{A}^{(\theta - X_\varnothing)} = \emptyset, \\ \bigl\{\sigma \in \mathbb{T}_b \colon m(\sigma) \in \mathcal{A}^{(\theta - X_\varnothing)}\bigr\}, & \text{otherwise.} \end{cases} \tag{4.8}$$

(Here we take $\mathcal{A}^{(\theta')} = \emptyset$ whenever $\theta' < 1$.) Clearly, $\mathcal{B}^{(\theta)}$ is the original avalanche set together with its boundary (i.e., the set of sites in $\mathbb{T}_b$, where the avalanche has "spilled" some material). Since both sets are connected and both contain the root (with the exception of the case $\mathcal{A}^{(\theta - X_\varnothing)} = \emptyset$), their sizes satisfy the relation:

$$|\mathcal{B}^{(\theta)}| = (b-1)|\mathcal{A}^{(\theta - X_\varnothing)}| + 1. \tag{4.9}$$

(This relation holds even if $\mathcal{A}^{(\theta - X_\varnothing)} = \emptyset$.) The asymptotic probability of the events $\{|\mathcal{A}^{(v)}| \geq n\}$ as $n \to \infty$ is thus basically equivalent to that of $\{|\mathcal{B}^{(\theta)}| \geq (b-1)n\}$.

4.2 Avalanches in an external field.

Following a route which is often used in the analysis of critical systems, our proof of Theorem 4.1 will be accomplished by the addition of extra degrees of freedom that play the role of an *external field*. Let $\lambda \in [0,1]$ be fixed and let us color each site of $\mathbb{T}_b$ "green" with probability λ. Given a realization of this process, let $\mathcal{G}$ denote the random set of "green" sites in $\mathbb{T}_b$. Let $\mathbb{P}_{\rho,\lambda}(\cdot)$ be the joint probability distribution of the "green" sites and (X_σ). The principal quantity of interest is then

$$B_\infty(\theta,\lambda) = \mathbb{P}_{\rho,\lambda}\bigl(\mathcal{B}^{(\theta)} \cap \mathcal{G} \neq \emptyset\bigr). \tag{4.10}$$

It is easy to check that, as $\lambda \downarrow 0$, the number $B_\infty(\theta,\lambda)$ tends to the probability $\mathbb{P}_\rho(|\mathcal{B}^{(\theta)}| = \infty)$. In particular, Theorem 3.1 guarantees that $B_\infty(\theta,\lambda) \to 0$ as $\lambda \downarrow 0$ if $\mathfrak{z}(\rho) \leq \mathfrak{z}_{\mathrm{c}}$, while $B_\infty(\theta,\lambda)$ stays uniformly positive as $\lambda \downarrow 0$ when $\mathfrak{z}(\rho) > \mathfrak{z}_{\mathrm{c}}$ and $\theta \geq 1$.

Let $\psi_\rho(\theta)$ be as in Corollary 3.8 and let $c_\rho \in (0,\infty)$ be the quantity defined by

$$\frac{1}{c_\rho^2} = \frac{b-1}{2}\,\widehat{\mathbb{E}}\Bigl(\bigl[\mathbb{E}\bigl(\psi_\rho\bigl(X + \tfrac{Q_\infty}{b}\bigr)\bigr)\bigr]^2\,\Big|\,Q_\infty \geq 1\Bigr). \tag{4.11}$$

Here X and Q_∞ are independent with distributions $\mathbb{P} = \rho$ and $\widehat{\mathbb{P}}$, respectively. It turns out that the asymptotics of $B_\infty(\theta,\lambda)$ for critical ρ can be described very precisely:

Proposition 4.2 *Let $\rho \in \mathcal{M}^\flat$ satisfy $\mathfrak{z}(\rho) = \mathfrak{z}_{\mathrm{c}}$. For each $\theta \in (0,\infty)$,*

$$\lim_{\lambda \downarrow 0} \frac{B_\infty(\theta,\lambda)}{\sqrt{\lambda}} = c_\rho \psi_\rho(\theta). \tag{4.12}$$

Proposition 4.2 is proved in Section 4.4. Now we are ready to prove Theorem 4.1(2):

Proof of Theorem 4.1(2). We begin by noting the identity

$$\frac{B_\infty(\theta,\lambda)}{\lambda} = \sum_{n \geq 1} \mathbb{P}_\rho\bigl(|\mathcal{B}^{(\theta)}| \geq n\bigr)(1-\lambda)^{n-1}, \qquad \lambda \in (0,1]. \tag{4.13}$$

Since $B_\infty(\theta,\lambda) = \sqrt{\lambda}\,(c_\rho\psi_\rho(\theta) + o(1))$ as $\lambda \downarrow 0$ and since $n \mapsto \mathbb{P}_\rho(|\mathcal{B}^{(\theta)}| \geq n)$ is a decreasing sequence, standard Tauberian theorems (e.g., Karamata's Tauberian Theorem for Power Series, see Corollary 1.7.3 in [3]) guarantee that

$$\mathbb{P}_\rho(|\mathcal{B}^{(\theta)}| \geq n) = c_\rho \frac{\psi_\rho(\theta)}{\Gamma(\frac{1}{2})} \frac{1}{\sqrt{n}}\bigl[1 + o(1)\bigr], \qquad n \to \infty, \tag{4.14}$$

(Strictly speaking, the above Tauberian theorem applies only when $\psi_\rho(\theta) > 0$; in the opposite case, i.e., when $\theta < 1$, we have $\mathcal{B}^{(\theta)} = \{\varnothing\}$ and there is nothing to prove.) In order to obtain the corresponding asymptotics for $\mathbb{P}_\rho(|\mathcal{A}^{(v)}|\ge n)$, we first note that, by (4.9),

$$\mathbb{P}_\rho\bigl(|\mathcal{A}^{(v)}|\ge n\bigr)=\mathbb{P}_\rho\bigl(|\mathcal{B}^{(X_\varnothing+v)}|\ge (b-1)n+1\bigr). \tag{4.15}$$

By applying (4.14) on the right-hand side and invoking the Bounded Convergence Theorem, we immediately get the desired formula (4.2) with

$$\Theta(v)=\frac{c_\rho}{(b-1)^{1/2}\Gamma(\frac12)}\,\mathbb{E}_\rho\bigl(\psi_\rho(X_\varnothing+v)\bigr), \tag{4.16}$$

where $\mathbb{E}_\rho$ is the expectation over $X_\varnothing$. Clearly, $v\mapsto\Theta(v)$ is non-decreasing because $\theta\mapsto\psi_\rho(\theta)$ is non-decreasing, while $\Theta(v)>0$ for $v>1-x_\star$ because $\psi_\rho(\theta)>0$ for $\theta\ge 1$. □

Similarly we can also describe the asymptotics of $\mathbb{P}_\rho(|\mathcal{B}^{(\theta)}|=\infty)$ as $\mathfrak{z}(\rho)\downarrow\mathfrak{z}_{\rm c}$:

Proposition 4.3 *Let $\rho,\rho'\in\mathcal{M}^\flat$ and define $\rho_\alpha=(1-\alpha)\rho+\alpha\rho'$. Suppose that $\mathfrak{z}(\rho)=\mathfrak{z}_{\rm c}$ and $\mathfrak{z}(\rho_\alpha)>\mathfrak{z}_{\rm c}$ for all $\alpha\in(0,1]$. Then for all $\theta\in(0,\infty)$,*

$$\frac{\mathbb{P}_\rho(|\mathcal{B}^{(\theta)}|=\infty)}{\mathfrak{z}(\rho_\alpha)-\mathfrak{z}_{\rm c}}=bc_\rho^2\psi_\rho(\theta)+o(1),\qquad \alpha\downarrow 0, \tag{4.17}$$

where $\psi_\rho(\theta)$ is as in Corollary 3.8 and c_ρ is as in (4.11)*.*

Proposition 4.3 is proved in Section 4.5. Now we are ready finish the proof of Theorem 4.1(3):

Proof of Theorem 4.1(3). By (4.9) we clearly have that

$$\mathbb{P}_\rho\bigl(|\mathcal{A}^{(v)}|=\infty\bigr)=\mathbb{P}_\rho\bigl(|\mathcal{B}^{(X_\varnothing+v)}|=\infty\bigr). \tag{4.18}$$

By conditioning on $X_\varnothing+v=\theta$ and invoking (4.17), we can easily derive that the asymptotic formula (4.3) holds with $\mathcal{T}$ given by $\mathcal{T}(v)=bc_\rho^2\mathbb{E}_\rho(\psi_\rho(X_\varnothing+v))$. □

As we have seen, Propositions 4.2 and 4.3 have been instrumental in the proof of Theorem 4.1(2) and (3). The following three sections are devoted to the proofs of the two propositions. After some preliminary estimates, which constitute a substantial part of Section 4.3, we will proceed to establish the critical asymptotics (Section 4.4). The supercritical cases can then be handled along very much the same lines of argument, the necessary changes are listed in Section 4.5.

4.3 Preliminaries.

This section collects some facts about the quantity $B_\infty(\theta,\lambda)$ and its θ and λ dependence. We begin by proving a simple identity for $B_\infty(\theta,\lambda)$:

Lemma 4.4 *Let $\rho\in\mathcal{M}$ and let Φ_b be as in* (3.5)*. Then*

$$B_\infty(\theta,\lambda)=\lambda+(1-\lambda)\mathbb{1}_{\{\theta\ge1\}}\,\Phi_b\bigl(\mathbb{E}_\rho B_\infty\bigl(X_\varnothing+\tfrac1b\theta,\lambda\bigr)\bigr), \tag{4.19}$$

Proof. If $\theta<1$, then $B_\infty(\theta,\lambda)=\lambda$ and (4.19) clearly holds true. Let us therefore suppose that $\theta\ge1$. Let $\mathcal{B}_\sigma^{(\theta)}$ denote the object $\mathcal{B}^{(\theta)}$ for the subtree of $\mathbb{T}_b$ rooted at σ. Then

$$\bigl\{\mathcal{B}^{(\theta)}\cap\mathcal{G}\ne\emptyset\bigr\}=\{\varnothing\in\mathcal{G}\}\cup\Bigl(\{\varnothing\notin\mathcal{G}\}\cap\bigcup_{\sigma=1}^{b}\bigl\{\mathcal{B}_\sigma^{(X_\sigma+\frac1b\theta)}\cap\mathcal{G}\ne\emptyset\bigr\}\Bigr). \tag{4.20}$$

The claim then follows by using the independence of the sets in the large parentheses on the right hand side of (4.20) under the measure $\mathbb{P}_{\rho,\lambda}(\cdot)$. $\square$

Our next claim concerns continuity properties of $B_\infty(\theta,\lambda)$ as a function of θ:

Lemma 4.5 *For each $\rho \in \mathcal{M}^\flat$ satisfying $\mathfrak{z}(\rho) < \mathfrak{z}_c e$ and each $\theta_0 > \theta_b$ there is a $C = C(\rho,\theta_0) < \infty$ such that*

$$\bigl|B_\infty(\theta,\lambda) - B_\infty(\theta',\lambda)\bigr| \le C B_\infty(\theta_0,\lambda)|\theta-\theta'| \tag{4.21}$$

for all $\lambda \ge 0$ and all $\theta,\theta' \in [1,\theta_0]$. The bound $C(\rho,\theta_0) < \infty$ is uniform in any convex set $\mathcal{N} \subset \{\rho \in \mathcal{M}^\flat \colon \mathfrak{z}(\rho) < \mathfrak{z}_c e\}$ with finitely many extreme points.

Proof. Let us assume that $\theta \ge \theta'$. To derive (4.21), we will regard $B_\infty(\theta,\lambda)$ and $B_\infty(\theta',\lambda)$ as originating from the same realization of (X_σ) and the "green" sites. Then $\Delta = B_\infty(\theta,\lambda) - B_\infty(\theta',\lambda)$ is dominated by the probability (under $\mathbb{P}_{\rho,\lambda}$) that there is a site $\sigma \in \mathbb{T}_b$, $\sigma \ne \varnothing$, with the properties:

(1) $Q_{\sigma'}^{(\theta')} \ge 1$ for all $\sigma' = m^k(\sigma)$ with $k = 1,\dots,|\sigma|$.
(2) $Q_\sigma^{(\theta')} < 1$ but $Q_\sigma^{(\theta)} \ge 1$.
(3) $\mathcal{B}_\sigma^{(\theta_0)} \cap \mathcal{G} \ne \emptyset$, where $\mathcal{B}_\sigma^{(\theta_0)}$ is the set $\mathcal{B}^{(\theta_0)}$ for the subtree $\mathbb{T}_b^{(\sigma)}$ rooted at σ.

Indeed, any realization of (X_σ) and the "green" sites contributing to Δ obeys $\mathcal{B}^{(\theta')} \cap \mathcal{G} = \emptyset$ and $\mathcal{B}^{(\theta)} \cap \mathcal{G} \ne \emptyset$. But then there must be a site σ on the inner boundary of $\mathcal{B}^{(\theta')}$ where the avalanche corresponding to θ' stops but that corresponding to θ goes on. (Since $\theta,\theta' \ge 1$, we must have $\sigma \ne \varnothing$.) Consequently, $Q_{\sigma'}^{(\theta')} \ge 1$ for any σ' on the path connecting σ to the root, but $Q_\sigma^{(\theta')} < 1 \le Q_\sigma^{(\theta)}$, justifying conditions (1) and (2) above. Since $Q_\sigma^{(\theta)} \le \theta_0$, and since the θ-avalanche continuing on from σ must eventually reach a "green" site, we see that also condition (3) above must hold.

Let $\rho \in \mathcal{M}^\flat$ be such that $\mathfrak{z}(\rho) < \mathfrak{z}_c e$. Using the independence of the events described in (1), (2) and (3) above, and recalling the definitions (2.8) and (3.14), we can thus estimate

$$\Delta \le B_\infty(\theta_0,\lambda) \sum_{\sigma \in \mathbb{T}_b \smallsetminus \{\varnothing\}} Z_{|\sigma|-1}(\theta')\, \mathbb{P}_{|\sigma|}^{(\theta')}\bigl(Q_{|\sigma|,1}^{(\theta)} \ge 1 > Q_{|\sigma|,1}^{(\theta')}\bigr). \tag{4.22}$$

Abbreviate $K_n(\theta,\theta') = \mathbb{P}_n^{(\theta')}(Q_{n,1}^{(\theta)} \ge 1 > Q_{n,1}^{(\theta')})$. Since Y_1 is independent of all the other Y's in the measure $\mathbb{P}_n^{(\theta')}$, we have

$$K_n(\theta,\theta') = \bigl\{\rho([1-\tfrac{\vartheta'}{b}, 1-\tfrac{\vartheta}{b}))\colon \vartheta'-\vartheta \le |\theta-\theta'|b^{-n+1}\bigr\}. \tag{4.23}$$

Here ϑ, resp., ϑ' play the role of $Q_{n,2}^{(\theta)}$, resp., $Q_{n,2}^{(\theta')}$ and the interval in the argument of ρ exactly corresponds to the inequalities $Q_{n,1}^{(\theta)} = Y_1 + \frac{1}{b}\vartheta \ge 1 > Y_1 + \frac{1}{b}\vartheta' = Q_{n,1}^{(\theta')}$.

To estimate the supremum, we recall that $\rho(\mathrm{d}x) = \phi_\rho(x)\mathrm{d}x$ where ϕ_ρ is bounded. Then

$$K_n(\theta,\theta') \le \|\phi_\rho\|_\infty\, |\theta-\theta'|\, b^{-n+1}, \qquad n \in \mathbb{N}. \tag{4.24}$$

Now, by Corollary 3.8, $Z_n(\theta) \le C\mathfrak{z}(\rho)^n$ for some $C < \infty$ uniformly in ρ on convex sets $\mathcal{N} \subset \{\rho \in \mathcal{M}^\flat \colon \mathfrak{z}(\rho) < \mathfrak{z}_{\rm c} e\}$ with finitely many extreme points and uniformly in $\theta \le \theta_0$. Therefore, the right-hand side of (4.22) is bounded by $B_\infty(\theta_0,\lambda)|\theta-\theta'|$ times a sum that converges whenever $\mathfrak{z}(\rho) < \mathfrak{z}_{\rm c} e$, uniformly in $\rho \in \mathcal{N}$, where $\mathcal{N}$ is as above. This proves the desired claim. □

Let $\rho \in \mathcal{M}^\flat$ and let Q_∞ be the random variable defined in Section 3.3, independent of both the green sites and X_σ. Let us introduce the quantity

$$B^\star_\infty(\lambda) = \widehat{\mathbb{E}}\bigl(B_\infty(Q_\infty,\lambda)\bigr), \tag{4.25}$$

The significance of $B^\star_\infty(\lambda)$ is that it represents a stationary form of $B_\infty(\cdot,\lambda)$, i.e., $B^\star_\infty(\lambda)$ is a very good approximation of the probability $\mathbb{P}_{\rho,\lambda}(\mathcal{B}^{(\theta')}_\sigma \cap \mathcal{G} = \emptyset \,|\, \sigma \in \mathcal{A}^{(v)})$, where $\theta' = Q^{(X_\varnothing + v)}_\sigma$ and where $\mathcal{B}^{(\theta')}_\sigma$ is the quantity $\mathcal{B}^{(\theta')}$ for trees rooted at σ very far from $\varnothing$. Let

$$\varkappa_\rho(\lambda) = \widehat{\mathbb{E}}\Bigl(\bigl[\mathbb{E}\bigl(B_\infty\bigl(X + \tfrac{Q_\infty}{b},\lambda\bigr)\bigr)\bigr]^2 \,\Big|\, Q_\infty \ge 1\Bigr), \tag{4.26}$$

where X and Q_∞ are independent with distributions $\mathbb{P} = \rho$ and $\widehat{\mathbb{P}}$, respectively. For critical distributions, $B^\star_\infty(\lambda)$ and $\varkappa_\rho(\lambda)$ are related as follows:

Lemma 4.6 *Let $\rho \in \mathcal{M}^\flat$ be such that $\mathfrak{z}(\rho) = \mathfrak{z}_{\rm c}$. Then*

$$B^\star_\infty(\lambda) = 1 - \frac{b-1}{2\lambda}\,\varkappa_\rho(\lambda)\bigl[1+o(1)\bigr], \qquad \lambda \downarrow 0. \tag{4.27}$$

Proof. Since $B_\infty(\theta,\lambda) \to 0$ as $\lambda \downarrow 0$, we can expand Φ_b on the right hand side of (4.19) to the second order of Taylor expansion, use that $\mathfrak{z}(\rho) = \widehat{\mathbb{P}}(Q_\infty \ge 1)$ and apply $b\mathfrak{z}(\rho) = 1$ with the result

$$B^\star_\infty(\lambda) = \lambda + (1-\lambda)B^\star_\infty(\lambda) - \frac{b-1}{2}\,\varkappa_\rho(\lambda)\bigl[1+o(1)\bigr], \qquad \lambda \downarrow 0. \tag{4.28}$$

(Here we noted that $B_\infty(X + \frac{1}{b}Q_\infty) \le B_\infty(\theta_b)$ allows us to estimate the error in the Taylor expansion by $\varkappa_\rho(\lambda)B_\infty(\theta_b)\mathcal{O}(1)$, which is $\varkappa_\rho(\lambda)o(1)$ as $\lambda \downarrow 0$.) Subtracting $(1-\lambda)B^\star_\infty(\lambda)$ from both sides and dividing by λ, (4.27) follows. □

Note that, by the resulting expression (4.27), $\varkappa_\rho(\lambda)/\lambda$ tends to a definite limit as $\lambda \downarrow 0$. In the supercritical cases, on the other hand, Lemma 4.6 gets replaced by the following claim:

Lemma 4.7 *Let $\rho, \rho' \in \mathcal{M}^\flat$ and define $\rho_\alpha = (1-\alpha)\rho + \alpha\rho'$. Suppose that $\mathfrak{z}(\rho) = \mathfrak{z}_{\rm c}$ and $\mathfrak{z}(\rho_\alpha) > \mathfrak{z}_{\rm c}$ for all $\alpha \in (0,1]$. Let $B^\star_\infty(0,\alpha)$ denote the quantity $B^\star_\infty(0)$ for the underlying measure ρ_α. Then*

$$B^\star_\infty(0,\alpha) = \frac{b-1}{2b}\frac{\varkappa_{\rho_\alpha}(0)}{\mathfrak{z}(\rho_\alpha) - \mathfrak{z}_{\rm c}}\bigl[1+o(1)\bigr], \qquad \alpha \downarrow 0. \tag{4.29}$$

Proof. As in Lemma 4.6, we use that $B_\infty(\theta,0,\alpha) \to 0$ as $\alpha \to 0$, where $B_\infty(\theta,0,\alpha)$ denotes the quantity $B_\infty(\theta,0)$ for the underlying measure ρ_α. However, instead of (4.28), this time we get

$$B^\star_\infty(0,\alpha)\bigl(1 - b\mathfrak{z}(\rho_\alpha)\bigr) = -\frac{b-1}{2}\,\varkappa_{\rho_\alpha}(0)\bigl[1+o(1)\bigr], \qquad \alpha \downarrow 0, \tag{4.30}$$

where we again used that the error in the Taylor approximation can be bounded by $\varkappa_{\rho_\alpha} o(1)$. Dividing by $\mathfrak{z}(\rho_\alpha) - \mathfrak{z}_{\rm c} \ne 0$, (4.29) follows. □

4.4 Critical asymptotics.

The purpose of this section is to finally give the proof of Proposition 4.2. We begin by proving an appropriate upper bound on $B_\infty(\theta,\lambda)$. Note that, despite being used only marginally, equation (4.27) is a key ingredient of the proof.

Lemma 4.8 *Let $\rho\in\mathcal{M}^\flat$ satisfy $\mathfrak{z}(\rho)=\mathfrak{z}_{\rm c}$. For each $\theta\ge1$ there is a $K(\theta)\in(0,\infty)$ such that*

$$\limsup_{\lambda\downarrow0}\frac{B_\infty(\theta,\lambda)}{\sqrt{\lambda}}\le K(\theta). \tag{4.31}$$

Proof. Let $\mathfrak{z}=\mathfrak{z}(\rho)$. We begin by proving (4.31) for $\theta=1$. Let $\iota(\rho)=\widehat{\mathbb{E}}(\rho([1-\frac1bQ_\infty,1])^2|Q_\infty\ge1)$ and recall the definition of $\varkappa_\rho(\lambda)$ in (4.26). Using that $B_\infty(\theta,\lambda)\ge B_\infty(1,\lambda)1_{\{\theta\ge1\}}$ we have $\varkappa_\rho(\lambda)\ge\iota(\rho)B_\infty(1,\lambda)^2$. Inserting this in (4.27), we have

$$B^\star_\infty(\lambda)\le1-\frac{b-1}{2\lambda}\iota(\rho)\,B_\infty(1,\lambda)^2\bigl[1+o(1)\bigr],\qquad\lambda\downarrow0. \tag{4.32}$$

Since the left-hand side is always non-negative, (4.31) for $\theta=1$ follows with $K(1)^{-2}=\frac{b-1}{2}\iota(\rho)$.

Next we will show that for any $\theta<\theta_b$, $B_\infty(\theta,\lambda)$ is bounded above by a (θ-dependent) multiple of $B_\infty(1,\lambda)$. Indeed, pick an $\epsilon>0$ such that $\theta_b-\theta>\epsilon\frac{b}{b-1}$ and let m be so large that (2.14) holds. Fix a directed path of m steps in $\mathbb{T}_b$ starting from the root. By conditioning on the event that $X_\sigma\ge x_\star-\epsilon$ for all $\sigma\ne\varnothing$ in the path, we have $B_\infty(1,\lambda)\ge\rho([x_\star-\epsilon,1])^mB_\infty(\theta,\lambda)$, i.e.,

$$B_\infty(\theta,\lambda)\le C(\theta)B_\infty(1,\lambda),\qquad\theta<\theta_b, \tag{4.33}$$

with $C(\theta)=\rho([x_\star-\epsilon,1])^{-m}<\infty$.

As the third step we prove that (4.31) holds for values θ in slight excess of θ_b. (The reader will notice slight similarities with the final stages of the proof of Theorem 2.4(1).) Let $\epsilon>0$ be such that $x_\star-\epsilon>\frac{b-1}{b}$. By Corollary 2.5 and the fact that $\rho\in\mathcal{M}^\flat$, we can assume that $\kappa_\epsilon=\rho([x_\star-\epsilon,x_\star])<\mathfrak{z}$. If $\theta>\theta_b$ is such that $\theta_\epsilon=x_\star-\epsilon+\frac1b\theta<\theta_b$, then (4.19) and the bound $\Phi_b(y)\le by$ imply

$$B_\infty(\theta,\lambda)\le\lambda+(1-\lambda)b\Bigl[\kappa_\epsilon B_\infty(\theta,\lambda)+(1-\kappa_\epsilon)B_\infty(\theta_\epsilon,\lambda)\Bigr], \tag{4.34}$$

because $X+\frac1b\theta\le\theta$ for all X in the support of ρ. Since $(1-\lambda)b\kappa_\epsilon<b\kappa_\epsilon<1$, we have

$$B_\infty(\theta,\lambda)\le\frac{\lambda+(1-\lambda)(1-\kappa_\epsilon)bC(\theta_\epsilon)B_\infty(1,\lambda)}{1-(1-\lambda)b\kappa_\epsilon}. \tag{4.35}$$

Dividing by $\sqrt{\lambda}$ and taking $\lambda\downarrow0$, (4.31) follows with $K(\theta)=b(1-\kappa_\epsilon)C(\theta_\epsilon)K(1)/(1-b\kappa_\epsilon)$.

Finally, it remains to prove (4.31) for general $\theta\ge\theta_b$. But for that we just need to observe that

$$B_\infty(\theta,\lambda)\le[1-(1-\lambda)^{b^{k+1}}]+(1-\lambda)^{b^{k+1}}B_\infty(\theta_b+\theta b^{-k},\lambda) \tag{4.36}$$

as follows by conditioning on the first k layers of $\mathbb{T}_b$ to be green-free. By taking k large enough, $\theta_b+\theta b^{-k}$ is arbitrary close to θ_b, so the result follows by the preceding arguments. □

Lemma 4.8 allows us to write the following expression for $B_\infty(\theta,\lambda)$:

Lemma 4.9 *Let $\rho\in\mathcal{M}^\flat$ satisfy $\mathfrak{z}(\rho)=\mathfrak{z}_{\rm c}$. Let $\epsilon(\lambda,\theta)$ be defined by*

$$B_\infty(\theta,\lambda)=\psi_\rho(\theta)B^\star_\infty(\lambda)+\epsilon(\lambda,\theta), \tag{4.37}$$

where $\psi_\rho(\theta)$ is as in (3.21). *Then* $\lim_{\lambda\downarrow 0}\epsilon(\lambda,\theta)\lambda^{-1/2}=0$ *uniformly on compact sets of* θ.

Proof. Recall the notation $Q_{n,1}^{(\theta)}$ from (3.13), and let $\mathbb{E}_n^{(\theta)}$ denote the expectation with respect to the measure $\mathbb{P}_n^{(\theta)}$ in (3.14). We will first show that

$$B_\infty(\theta,\lambda)=Z_n(\theta)b^n\,\mathbb{E}_n^{(\theta)}\bigl(B_\infty(Q_{n,1}^{(\theta)},\lambda)\bigr)+\tilde\epsilon_n(\lambda) \tag{4.38}$$

holds with an $\tilde\epsilon_n(\lambda)$ satisfying $\lim_{\lambda\downarrow 0}\tilde\epsilon_n(\lambda)\lambda^{-1/2}=0$ for all $n\ge 1$. Let $\mathbb{G}_n$ denote the n-th generation of $\mathbb{T}_b$, i.e., $\mathbb{G}_n=\{\sigma\in\mathbb{T}_b\colon|\sigma|=n\}$, and let $\mathbb{H}_n=\bigcup_{m<n}\mathbb{G}_m$. Recall the notation $\mathcal{B}_\sigma^{(\theta)}$ for the object $\mathcal{B}^{(\theta)}$ on the subtree $\mathbb{T}_b^{(\sigma)}$ of $\mathbb{T}_b$ rooted at σ and let $Q_\sigma^{(\theta)}$ be as described in (2.6). Given a $\sigma\in\mathbb{G}_n$, let $\pi(\sigma)=\{m^k(\sigma)\colon k=1,\dots,n\}$ be the path of connecting σ to the root.

A moment's thought reveals that, if $\mathcal{G}\cap\mathbb{H}_n=\emptyset$ (i.e., if there are no green sites in the first $n-1$ generations of $\mathbb{T}_b$), then in order for $\mathcal{B}^{(\theta)}\cap\mathcal{G}\ne\emptyset$ to occur, the following must hold: First, there is a $\sigma\in\mathbb{G}_n$, such that $Q_{\sigma'}^{(\theta)}\ge 1$ for all $\sigma'\in\pi(\sigma)$. Second, the avalanche starting from this σ with an initial amount $Q_\sigma^{(\theta)}$ reaches $\mathcal{G}$. Introducing the event

$$\mathcal{U}_n=\bigcup_{\sigma\in\mathbb{G}_n}\Bigl(\bigl\{\mathcal{B}_\sigma^{(Q_\sigma^{(\theta)})}\cap\mathcal{G}\ne\emptyset\bigr\}\cap\bigcap_{\sigma'\in\pi(\sigma)}\{Q_{\sigma'}^{(\theta)}\ge 1\}\Bigr), \tag{4.39}$$

we thus have

$$\mathbb{P}_{\rho,\lambda}(\mathcal{U}_n)\le B_\infty(\theta,\lambda)\le\mathbb{P}_{\rho,\lambda}(\mathcal{U}_n)+\mathbb{P}_{\rho,\lambda}\bigl(\{\mathcal{G}\cap\mathbb{H}_n\ne\emptyset\}\bigr). \tag{4.40}$$

Since $\mathbb{P}_\rho(\mathcal{G}\cap\mathbb{H}_n\ne\emptyset)=\mathcal{O}(\lambda)$, it clearly suffices to show that $\mathbb{P}_{\lambda,\rho}(\mathcal{U}_n)$ has the same asymptotics as claimed on the right-hand side of (4.38).

Since $\mathcal{U}_n$ is the union of b^n events with the same probability, the upper bound

$$\mathbb{P}_{\rho,\lambda}(\mathcal{U}_n)\le b^nZ_n(\theta)\mathbb{E}_n^{(\theta)}\bigl(B_\infty(Q_{n,1}^{(\theta)},\lambda)\bigr) \tag{4.41}$$

directly follows using the identity

$$\mathbb{E}_\rho\Bigl(B_\infty(Q_\sigma^{(\theta)},\lambda)\prod_{\sigma'\in\pi(\sigma)}\mathsf{1}_{\{Q_{\sigma'}^{(\theta)}\ge 1\}}\Bigr)=Z_n(\theta)\mathbb{E}_n^{(\theta)}\bigl(B_\infty(Q_{n,1}^{(\theta)},\lambda)\bigr). \tag{4.42}$$

To derive the lower bound, we use the inclusion-exclusion formula. The exclusion term (i.e., the sum over intersections of pairs of events from the union in (4.39)) is estimated, using the bound in Lemma 4.8, to be less than $K(\bar\theta)^2b^{2n}\lambda$, where $\bar\theta=\theta\vee\theta_b$. This proves (4.38).

Since $\mathfrak{z}(\rho)b=1$, Corollary 3.8 tells us that $Z_n(\theta)b^n=\psi_\rho(\theta)+o(1)$. The final task is to show that $\mathbb{E}_n^{(\theta)}(B_\infty(Q_{n,1}^{(\theta)},\lambda))$ can safely be replaced by its limiting version, $B_\infty^\star(\lambda)$. We cannot use Corollary 3.6 directly, because $\theta\mapsto B_\infty(\theta,\lambda)$ is known to be Lipschitz continuous only for $\theta\ge 1$. However, by Lemma 4.4 we know that $B_\infty(\theta,\lambda)=\lambda$ for $\theta<1$, which means that we can write

$$B_\infty(\theta,\lambda)=B_\infty\bigl(\theta\vee 1,\lambda\bigr)+\bigl[\lambda-B_\infty(1,\lambda)\bigr]\mathsf{1}_{\{\theta<1\}}. \tag{4.43}$$

Now, $B_\infty^1(\theta,\lambda)=B_\infty(\theta\vee 1,\lambda)$ is Lipschitz continuous in θ for all $\theta\ge 0$, so by (4.21) and (3.19),

$$\Bigl|\mathbb{E}_n^{(\theta)}\bigl(B_\infty^1(Q_{n,1}^{(\theta)},\lambda)\bigr)-\widehat{\mathbb{E}}\bigl(B_\infty^1(Q_\infty,\lambda)\bigr)\Bigr|\le DB_\infty(\bar\theta,\lambda)e^{-\varsigma n} \tag{4.44}$$

where $\varsigma>0$ and $D=D(\bar\theta)<\infty$. To estimate the contribution of the second term in (4.43), we first note that $\lambda-B_\infty(1,\lambda)$ is a constant bounded between $-B_\infty(\bar\theta,\lambda)$ and zero. Hence, we thus need to estimate the difference $\mathbb{P}_n^{(\theta)}(Q_{n,1}^{(\theta)} < 1) - \widehat{\mathbb{P}}(Q_\infty < 1)$. But that can be done using Proposition 3.5: Let $k = \lfloor \frac{n}{2} \rfloor$ and use the monotonicity of $\theta \mapsto Q_{k,1}^{(\theta)}$ and (3.15) to estimate

$$\left|\mathbb{P}_n^{(\theta)}(Q_{n,1}^{(\theta)} < 1) - \widehat{\mathbb{P}}(Q_\infty < 1)\right| \le \mathbb{P}_n^{(\theta)}(Q_{k,1}^{(\bar\theta)} \ge 1) - \widehat{\mathbb{P}}(Q_{k,1}^{(1)} \ge 1) \le A'' e^{-\zeta(n-k)}, \tag{4.45}$$

where $A'' = A/(1-e^{-\zeta})$. By combining all the previous estimates and invoking (4.31), we find that the difference $\mathbb{E}_n^{(\theta)}(B_\infty(Q_{n,1}^{(\theta)}, \lambda)) - B_\infty^\star(\lambda)$ is proportional to $e^{-\varsigma' n}\sqrt{\lambda}$, where $\varsigma' > 0$. Using this back in (4.38) the claim follows by taking the limits $\lambda \downarrow 0$ and $n \to \infty$. $\square$

Lemmas 4.8 and 4.9 finally allow us to prove Proposition 4.2:

Proof of Proposition 4.2. Note that, by using (4.37) in (4.26) and the definition of c_ρ in (4.11), we have

$$\frac{b-1}{2} \varkappa_\rho(\lambda) = B_\infty^\star(\lambda)^2 c_\rho^{-2} + o(\lambda), \qquad \lambda \downarrow 0. \tag{4.46}$$

Then the fact that $B_\infty^\star(\lambda)$ tends to zero as $\lambda \downarrow 0$ forces, in light of (4.27), that $\frac{b-1}{2\lambda}\varkappa_\rho(\lambda) \to 1$ as $\lambda \downarrow 0$. This in turn gives that

$$B_\infty^\star(\lambda) = \sqrt{\lambda}\big(c_\rho + o(1)\big), \qquad \lambda \downarrow 0. \tag{4.47}$$

Plugging this back in (4.37) proves the desired claim. $\square$

4.5 Supercritical case.

Here we will indicate the changes to the arguments from the previous two sections that are needed to prove Proposition 4.3. We begin with an analogue of Lemma 4.8:

Lemma 4.10 *Let $\rho, \rho' \in \mathcal{M}^\flat$ and define $\rho_\alpha = (1-\alpha)\rho + \alpha\rho'$. Suppose that $\mathfrak{z}(\rho) = \mathfrak{z}_\mathrm{c}$ and $\mathfrak{z}(\rho_\alpha) > \mathfrak{z}_\mathrm{c}$ for all $\alpha \in (0,1]$. Then for each $\theta \ge 1$, there is a constant $K'(\theta) \in (0,\infty)$ such that*

$$\limsup_{\alpha \downarrow 0} \frac{\mathbb{P}_{\rho_\alpha}(|\mathcal{B}^{(\theta)}| = \infty)}{\mathfrak{z}(\rho_\alpha) - \mathfrak{z}_\mathrm{c}} \le K'(\theta). \tag{4.48}$$

Proof. The only important change compared to the proof of Lemma 4.8 is the derivation of the bound for $\theta = 1$. Indeed, in this case we use that $\varkappa_{\rho_\alpha}(0) \ge B_\infty^\star(0,\alpha) B_\infty(1,0,\alpha)$ in (4.29), where $B_\infty(1,0,\alpha)$ is the quantity $B_\infty(\theta,\lambda)$ for $\lambda = 0$, $\theta = 1$ and $\rho = \rho_\alpha$. Applying $B_\infty^\star(0,\alpha) > 0$ for all $\alpha \in (0,1]$, as follows by Theorem 3.1(2), we find that (4.48) holds with $K'(1) = \frac{2b}{b-1}$. Once we set $\lambda = 0$, the rest of the proof can literally be copied. $\square$

Next we need to state the appropriate version of Lemma 4.9:

Lemma 4.11 *Let $\rho, \rho' \in \mathcal{M}^\flat$ and define $\rho_\alpha = (1-\alpha)\rho + \alpha\rho'$. Suppose that $\mathfrak{z}(\rho) = \mathfrak{z}_\mathrm{c}$ and $\mathfrak{z}(\rho_\alpha) > \mathfrak{z}_\mathrm{c}$ for all $\alpha \in (0,1]$. Then*

$$\frac{\mathbb{P}_{\rho_\alpha}(|\mathcal{B}^{(\theta)}| = \infty)}{\mathfrak{z}(\rho_\alpha) - \mathfrak{z}_\mathrm{c}} = \psi_\rho(\theta)\, \frac{\widehat{\mathbb{E}}_\alpha(\mathbb{P}_{\rho_\alpha}(|\mathcal{B}^{(Q_\infty)}| = \infty))}{\mathfrak{z}(\rho_\alpha) - \mathfrak{z}_\mathrm{c}} + o(1), \qquad \alpha \downarrow 0, \tag{4.49}$$

where $\widehat{\mathbb{E}}_\alpha$ is the expectation corresponding to $\widehat{\mathbb{P}}$ for measure ρ_α.

Proof. Also in this case the required changes are only minuscule. First, we have an analogue of (4.38),

$$\mathbb{P}_{\rho_\alpha}\big(|\mathcal{B}^{(\theta)}| = \infty\big) = b^n Z_n^{(\rho_\alpha)}(\theta)\, \mathbb{E}_{n,\alpha}^{(\theta)}\Big(\mathbb{P}_{\rho_\alpha}\big(|\mathcal{B}^{(Q_{n,1}^{(\theta)})}| = \infty\big)\Big) + \tilde\epsilon_n'(\alpha), \qquad \alpha \downarrow 0, \tag{4.50}$$

where $\mathbb{E}_{n,\alpha}^{(\theta)}$ is the expectation $\mathbb{E}_n^{(\theta)}$ and $Z_n^{(\rho_\alpha)}$ the object $Z_n(\theta)$ for the underlying measure ρ_α and where $\tilde{\epsilon}_n'(\alpha)$ is the quantity in (4.38) for $\lambda = 0$ and $\rho = \rho_\alpha$. We claim that

$$\lim_{\alpha\downarrow 0} \frac{\tilde{\epsilon}_n'(\alpha)}{\mathfrak{z}(\rho_\alpha) - \mathfrak{z}_{\text{c}}} = 0 \tag{4.51}$$

for all finite $n \ge 1$. Indeed, the entire derivation (4.39-4.44) carries over, provided we set $\lambda = 0$. The role of the "small parameter" is now taken over by $\mathfrak{z}(\rho_\alpha) - \mathfrak{z}_{\text{c}}$. A computation shows that $\tilde{\epsilon}_n(\alpha) = O((\mathfrak{z}(\rho_\alpha) - \mathfrak{z}_{\text{c}})^2)$ as $\alpha \downarrow 0$, proving (4.51).

To finish the proof, it now remains to note that $b^n Z_n^{(\rho_\alpha)}(\theta) \to b^n Z_n^{(\rho)}(\theta)$ as $\alpha \downarrow 0$ and that, by Corollary 3.8 and the fact that $\mathfrak{z}(\rho) = \mathfrak{z}_{\text{c}}$, we have $b^n Z_n^{(\rho)}(\theta) = \psi_\rho(\theta) + o(1)$ as $n \to \infty$. □

Recall the definition of c_ρ in (4.11). To prove Proposition 4.3, we will need to know some basic continuity properties of c_ρ in ρ. Note that these do not follow simply from the continuity of $\alpha \mapsto \psi_{\rho_\alpha}(\theta)$, because also the expectation $\widehat{\mathbb{E}}$ in (4.11) depends on the underlying measure.

Lemma 4.12 *Let $\rho, \rho' \in \mathcal{M}^\flat$ be such that $\rho_\alpha = (1-\alpha)\rho + \alpha\rho'$ satisfies $\mathfrak{z}(\rho_\alpha) > 0$ for all $\alpha \in [0,1]$. Let c_ρ be as in* (4.11). *Then $\lim_{\alpha\downarrow 0} c_{\rho_\alpha} = c_\rho$.*

Proof. Let $\psi_{\rho_\alpha}^*(\theta) = \mathbb{E}_{\rho_\alpha}(\psi_{\rho_\alpha}(X_\varnothing + \frac{1}{b}\theta))$. In general, $\psi_{\rho_\alpha}(\theta)$ is Lipschitz continuous for $\theta \ge 1$. Thus, ψ_{ρ_α} converges uniformly to ψ_ρ on compact sets of θ. Hence, we just need to show

$$\lim_{\alpha\downarrow 0} \widehat{\mathbb{E}}_\alpha\bigl(\psi_\rho^*(Q_\infty)^2 \big| Q_\infty \ge 1\bigr) = \widehat{\mathbb{E}}\bigl(\psi_\rho^*(Q_\infty)^2 \big| Q_\infty \ge 1\bigr). \tag{4.52}$$

Choose $n \ge 1$ and replace $\widehat{\mathbb{E}}_\alpha$, $\widehat{\mathbb{E}}$ and Q_∞ by their finite-n versions. By Corollary 3.6, the error thus incurred is uniformly small in $\alpha \in [0,1]$. Hence, it is enough to show that

$$\lim_{\alpha\downarrow 0} \mathbb{E}_{n,\rho_\alpha}^{(\theta)}\bigl(\psi_\rho^*(Q_{n,1}^{(\theta)})^2 \big| Q_{n,1}^{(\theta)} \ge 1\bigr) = \mathbb{E}_{n,\rho}^{(\theta)}\bigl(\psi_\rho^*(Q_{n,1}^{(\theta)})^2 \big| Q_{n,1}^{(\theta)} \ge 1\bigr), \tag{4.53}$$

for some $\theta \in [1, \theta_b]$, where $\mathbb{E}_{n,\rho}^{(\theta)}$ denotes the expectation with respect to $\mathbb{P}_n^{(\theta)}$ for measure ρ. However, in (4.53) only a finite number of coordinates are involved and the result follows easily. □

With Lemmas 4.10, 4.12 and 4.11, we can finish the proof of Proposition 4.3:

Proof of Proposition 4.3. From (4.49) we have

$$\frac{b-1}{2} \varkappa_{\rho_\alpha}(0) = B_\infty^\star(0,\alpha)^2 c_{\rho_\alpha}^{-2} + o\bigl(\mathfrak{z}(\rho_\alpha) - \mathfrak{z}_{\text{c}}\bigr), \qquad \alpha \downarrow 0. \tag{4.54}$$

Using this in (4.29) and invoking Lemma 4.12, we have

$$\frac{B_\infty^\star(0,\alpha)}{\mathfrak{z}(\rho_\alpha) - \mathfrak{z}_{\text{c}}} = b c_\rho^2 + o(1), \qquad \alpha \downarrow 0. \tag{4.55}$$

The proof is finished by plugging this back into (4.49) and invoking the continuity of $\alpha \mapsto \psi_{\rho_\alpha}(\theta)$. $\square$

5. Coupling argument

5.1 Coupling measure.

The goal of this section is to define a coupling of the measures $\mathbb{P}_n^{(\theta)}$ and $\mathbb{P}_n^{(\theta')}$ that appear in (3.15). As the first step, we will write $\mathbb{P}_n^{(\theta)}(\cdot)$ as the distribution of a time-inhomogeneous process. To have the process running in forward time direction, we will need to express all quantities in terms of the original variables (X_k), which relate to the Y's through

$$X_k = Y_{n-k+1} \quad \text{or} \quad Y_k = X_{n-k+1}, \qquad 1 \le k \le n, \tag{5.1}$$

see Section 3.3. Abusing the notation slightly, $\mathbb{P}_n^{(\theta)}(\cdot)$ will temporarily be used to denote the distribution of the $X_1, \dots, X_n$ as well. We will return to the Y's in the proofs of Propositions 3.5 and 3.7.

Let $Z_n(\theta)$ be as in (2.8) and note that, since $\rho \in \mathcal{M}^\flat$, we have $Z_n(\theta) > 0$ for all $n \ge 0$ and all $\theta \ge 1$. Given $1 \le k \le n-1$ and, for $k > 1$, a sequence $(X_1, \dots, X_{k-1}) \in [0,1]^{k-1}$, let $t_{n,k}^{(\theta)}(\cdot) = t_{n,k}^{(\theta)}(\,\cdot\,|X_1, \dots, X_{k-1})$ be given by

$$t_{n,k}^{(\theta)}(x) = \frac{Z_{n-k-1}(x + \frac{1}{b}Q_{k-1}^{(\theta)})}{Z_{n-k}(Q_{k-1}^{(\theta)})} \mathsf{1}_{\{Q_{k-1}^{(\theta)} \ge 1\}}, \qquad 0 \le x \le 1, \tag{5.2}$$

where the indicator ensures that we are not dividing by zero. We will often leave the $(X_1, \dots, X_{k-1})$-dependence of $t_{n,k}^{(\theta)}$ implicit.

To interpret these objects, let us consider the case $k = 1$. Suppose that we wish to elucidate the distribution of X_1 knowing that the process *will* survive long enough to produce an X_{n-1}. (The variable X_n corresponds to Y_1, which will be uncorrelated with the other Y's.) The only prior history we know is the value of θ; obviously we are only interested in the case $\theta \ge 1$. The total weight of all configurations is just $Z_{n-1}(\theta)$; hence the denominator of (5.2). Now, if X_1 takes value x, the weight of configurations in which the process survives is like the weight of a string of length $n-2$ with an effective "θ" given by $x + \frac{1}{b}\theta$. Hence $Z_{n-2}(x + \frac{1}{b}\theta)$ in the numerator. (Notice that if $x + \frac{1}{b}\theta < 1$, this automatically vanishes.) We conclude that $\mathbb{P}_n^{(\theta)}(X_1 \in \mathrm{d}x) = t_{n,1}^{(\theta)}(x)\rho(\mathrm{d}x)$.

A similar reasoning shows that the probability of $\{X_k \in \mathrm{d}x\}$ given the values of $X_1, \dots, X_{k-1}$ equals $t_{n,k}^{(\theta)}(x)\rho(\mathrm{d}x)$. This allows us to view $\mathbb{P}_n^{(\theta)}$ as the distribution of an inhomogeneous process:

Lemma 5.1 *For all $\theta \ge 1$, all $n \ge 1$ and all Borel-measurable sets $A \subset [0,1]^n$,*

$$\mathbb{P}_n^{(\theta)}(A) = \mathbb{E}\Big(\mathsf{1}_A \prod_{k=1}^{n-1} t_{n,k}^{(\theta)}(X_k|X_1, \dots, X_{k-1})\Big). \tag{5.3}$$

Proof. The result immediately follows from the formula

$$\prod_{k=1}^{n-1} t_{n,k}^{(\theta)}(X_k|X_1, \dots, X_{k-1}) = \frac{1}{Z_{n-1}(\theta)} \Big\{ \prod_{k=1}^{n-1} \mathsf{1}_{\{X_k + \frac{1}{b}Q_{k-1}^{(\theta)} \ge 1\}} \Big\}, \tag{5.4}$$

the identity $Q_k^{(\theta)} = X_k + \frac{1}{b} Q_{k-1}^{(\theta)}$ and the definition of $\mathbb{P}_n^{(\theta)}(\cdot)$, see (3.14). □

Next we will define the coupled measure. The idea is to use the so-called Vasershtein coupling, see [8], which generates new (coupled) pairs from the "maximal overlap" of the individual distributions. Let $\theta, \theta' \ge 1$, and suppose that the corresponding sequences $X = (X_1, \dots, X_{k-1}) \in [0,1]^{k-1}$ and $X' = (X'_1, \dots, X'_{k-1}) \in [0,1]^{k-1}$ have been generated. Assume also that a sequence $(\omega_1, \dots, \omega_{k-1}) \in \{0,1\}^{k-1}$ satisfying $\omega_\ell \le \mathbb{1}_{\{X_\ell = X'_\ell\}}$ for all $1 \le \ell \le k-1$ has been generated. (This sequence marks down when X_ℓ was coupled with X'_ℓ. Note that we could have that $X_\ell = X'_\ell$ even when X_ℓ and X'_ℓ are not coupled.) Let t be the quantity $t_{n,k}^{(\theta)}$ for the sequence X and let t' be the corresponding quantity for the sequence X'. Let

$$R(\,\cdot\,) = R_{n,k}^{(\theta,\theta')}(\,\cdot\,| X_1, \dots, X_{k-1}; X'_1, \dots, X'_{k-1}; \omega_1, \dots, \omega_{k-1}) \tag{5.5}$$

be the transition kernel of the joint process, which is a probability measure on $[0,1] \times [0,1] \times \{0,1\}$ defined by the expression

$$R\bigl(\mathrm{d}x \times \mathrm{d}x' \times \{\omega\}\bigr) = \begin{cases} t(x) \wedge t'(x)\, \rho(\mathrm{d}x) \delta_x(\mathrm{d}x'), & \text{if } \omega = 1, \\ \frac{1}{1-q}\, [t(x) - t'(x)]_+ \, [t'(x') - t(x')]_+ \, \rho(\mathrm{d}x) \rho(\mathrm{d}x'), & \text{if } \omega = 0. \end{cases} \tag{5.6}$$

Here $t(x) \wedge t'(x)$ denotes the minimum of $t(x)$ and $t'(x)$ and $[t(x) - t'(x)]_+$ denotes the positive part of $t(x) - t'(x)$. The quantity $q = q_{n,k;X,X'}^{(\theta,\theta')}$ is given by

$$q = \int t(x) \wedge t'(x)\, \rho(\mathrm{d}x) = 1 - \int \bigl[t(x) - t'(x)\bigr]_+ \rho(\mathrm{d}x). \tag{5.7}$$

The interpretation of (5.6) is simple: In order to sample a new triple (X_k, X'_k, ω_k), we first choose $\omega_k \in \{0,1\}$ with $\mathrm{Prob}(\omega_k = 1) = q$. If $\omega_k = 1$, the pair (X_k, X'_k) is sampled from distribution $\frac{1}{q} t(x) \wedge t'(x)\, \rho(\mathrm{d}x) \delta_x(\mathrm{d}x')$—and, in particular, X_k gets glued together with X'_k—while for the case $\omega_k = 0$ we use the distribution in the second line of (5.6).

Remark 5. It turns out that whenever the above processes X and X' have glued together, they have a tendency to stay glued. However, the above coupling is *not* monotone, because the processes may come apart no matter how long they have been glued together. Our strategy lies in showing that q tends to one rapidly enough so that the number of "unglueing" instances is finite almost surely.

Let $\mathbb{P}_n^{(\theta,\theta')}(\cdot)$ be the probability measure on $[0,1]^n \times [0,1]^n \times \{0,1\}^n$ assigning mass

$$\mathbb{P}_n^{(\theta,\theta')}(B) = \sum_{(\omega_k)} \int_B \rho(\mathrm{d}x_n) \rho(\mathrm{d}x'_n) \mathbb{1}_{\{\omega_n = 1\}} \prod_{k=1}^{n-1} R_{n,k;x,x',\omega}^{(\theta,\theta')}\bigl(\mathrm{d}x_k \times \mathrm{d}x'_k \times \{\omega_k\}\bigr) \tag{5.8}$$

to any Borel-measurable set $B \subset [0,1]^n \times [0,1]^n \times \{0,1\}^n$. Here $R_{n,k;x,x',\omega}^{(\theta,\theta')}(\mathrm{d}x_k \times \mathrm{d}x'_k \times \{\omega_k\}) = R_{n,k}^{(\theta,\theta')}(\mathrm{d}x_k \times \mathrm{d}x'_k \times \{\omega_k\} | x_1, \dots, x_{k-1}; x'_1, \dots, x'_{k-1}; \omega_1, \dots, \omega_{k-1})$. As can be expected from the construction, $\mathbb{P}_n^{(\theta)}(\cdot)$ and $\mathbb{P}_n^{(\theta')}(\cdot)$ are the first and second marginals of $\mathbb{P}_n^{(\theta,\theta')}(\cdot)$, respectively:

Lemma 5.2 *Let $\theta, \theta' \ge 1$. Then*

$$\mathbb{P}_n^{(\theta,\theta')}\big(A \times [0,1]^n \times \{0,1\}^n\big) = \mathbb{P}_n^{(\theta)}(A) \tag{5.9}$$

and

$$\mathbb{P}_n^{(\theta,\theta')}\big([0,1]^n \times A \times \{0,1\}^n\big) = \mathbb{P}_n^{(\theta')}(A), \tag{5.10}$$

for all Borel-measurable $A \subset [0,1]^n$.

Proof. To prove formula (5.9), let $X = (X_1, \dots, X_{k-1})$ and $X' = (X'_1, \dots, X'_{k-1})$ be two sequences from $[0,1]^{k-1}$. If $Q_{k-1}^{(\theta)} \ge 1$ and the same holds for the corresponding quantity for the sequence X', let $t(\cdot) = t_{n,k}^{(\theta)}(\cdot)$, $t'(\cdot) = t_{n,k}^{(\theta')}(\cdot)$, and let $R(\cdot)$ and q be as in (5.6) and (5.7), respectively. Using (5.7) we have, for all Borel sets $C \subset [0,1]$,

$$\sum_{\omega\in\{0,1\}} \int_{C\times[0,1]} R(\mathrm{d}x \times \mathrm{d}x' \times \{\omega\}) = \int_C \big(t(x) \wedge t'(x) + [t(x) - t'(x)]_+\big)\rho(\mathrm{d}x) = \int_C t(x)\rho(\mathrm{d}x). \tag{5.11}$$

In other words, Hence, the first marginal of the coupled process is a process on $[0,1]$ with the transition kernel $t(\cdot)\rho(\cdot)$, which, as shown in Lemma 5.1, generates $\mathbb{P}_n^{(\theta)}$. This proves (5.9); the proof of (5.10) is analogous. □

Clearly, the number q represents the probability that the two processes get coupled. The following lemma provides a bound that will be useful in controlling q:

Lemma 5.3 *Let $\theta, \theta' \ge 1$, $1 \le k \le n-1$ and $X = (X_1, \dots, X_{k-1}) \in [0,1]^{k-1}$ and $X' = (X'_1, \dots, X'_{k-1}) \in [0,1]^{k-1}$. Let Q be the quantity $Q_{k-1}^{(\theta)}$ corresponding to X and let Q' be the quantity $Q_{k-1}^{(\theta')}$ corresponding to X'. If $Q \wedge Q' \ge 1$, then*

$$q_{n,k;X,X'}^{(\theta,\theta')} \ge \frac{Z_{n-k}(Q\wedge Q')}{Z_{n-k}(Q \vee Q')}. \tag{5.12}$$

Proof. Let t be the quantity $t_{n,k}^{(\theta)}$ for the sequence X and let t' be the corresponding quantity for the sequence X'. By inspection of (5.2) and monotonicity of $\theta \mapsto Z_n(\theta)$,

$$t(x) \ge \frac{Z_{n-k-1}(x + \frac{1}{b}(Q\wedge Q'))}{Z_{n-k}(Q \vee Q')}, \tag{5.13}$$

and similarly for $t'(x)$. From here the claim follows by integrating with respect to $\rho(\mathrm{d}x)$. □

5.2 Domination by a discrete process.

The goal of this section is to show that the coupled measure defined in the previous section has the desirable property that, after a finite number of steps, the processes X and X' get stuck forever. Since the information about coalescence of X and X' is encoded into the sequence ω, we just need to show that, eventually, $\omega_k = 1$. For technical reasons, we will concentrate from the start on infinite sequences $(\omega_k)_{k\in\mathbb{N}}$: Let $P_n^{(\theta,\theta')}(\cdot)$ be the law of $(\omega_k)_{k\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ induced by the distribution $\mathbb{P}_n^{(\theta,\theta')}(\cdot)$ and the requirement $P_n^{(\theta,\theta')}(\omega_k = 1,\, k \ge n) = 1$.

The coalescence of X and X' will be shown by a comparison with a simpler stochastic process on $\{0,1\}^{\mathbb N}$ whose law will be distributionally lower than $P_n^{(\theta,\theta')}(\cdot)$, i.e., in the FKG sense. Let $\preccurlyeq$ be the partial order on $\omega,\omega'\in\{0,1\}^{\mathbb N}$ defined by

$$\omega\preccurlyeq\omega'\quad\Leftrightarrow\quad\omega_k\le\omega'_k,\qquad k\ge1. \tag{5.14}$$

Next, note that, by $x_\star>\frac{b-1}{b}$, we have $1-b(1-x_\star)>\theta_b-1$. Choose a number $\delta_\rho\in(\theta_b-1,1-b(1-x_\star))$ and, noting that $\rho([1-\frac{1-\delta_\rho}{b},x_\star])>0$, define a collection of weights $(\lambda_\rho(s))$ by

$$\frac{1-\lambda_\rho(s)}{\lambda_\rho(s)}=\sum_{k\ge s}\;\sup_{\theta-\theta'\le\delta_\rho b^{-k}}\frac{\rho\bigl([1-\frac{\theta}{b},1-\frac{\theta'}{b})\bigr)}{\rho\bigl([1-\frac{1-\delta_\rho}{b},x_\star]\bigr)},\qquad s\in\mathbb N\cup\{0\}. \tag{5.15}$$

Note that $s\mapsto\lambda(s)$ is increasing. It is also easy to verify that $\lambda_\rho(\cdot)\in(0,1]$, so any of these weights can be interpreted as a probability. This allows us to define a process on $(\omega'_k)_{k\in\mathbb N}\in\{0,1\}^{\mathbb N}$, with the transition kernel

$$p_\rho(\,\omega'_k=1\,|\,\omega'_1,\dots,\omega'_{k-1})=\lambda_\rho\bigl(\min\{0\le j\le k-1\colon\omega'_{k-j-1}=0\}\bigr), \tag{5.16}$$

where, for definiteness, we set $\omega'_0=0$. Let $\widetilde P_\rho(\cdot)$ denote the law of the entire process with transition probabilities $p_\rho(\,\cdot\,|\,\cdot\,)$ and "initial" value $\omega'_0=0$.

Proposition 5.4 *Let $\rho\in\mathcal M^\flat$ and let δ_ρ be as above. For all $n\ge1$ and all θ,θ' with $1\le\theta,\theta'\le\theta_b$, the measure $P_n^{(\theta,\theta')}(\cdot)$ stochastically dominates $\widetilde P_\rho(\cdot)$ in partial order $\preccurlyeq$.*

Let δ_ρ be fixed for the rest of this Subsection. In order to give a proof of Proposition 5.4, we first establish a few simple bounds.

Lemma 5.5 *Let $\rho\in\mathcal M^\flat$ and let δ_ρ be as above. Let $n\ge0$ and suppose $\theta,\theta'\ge1$ satisfy $0\le\theta-\theta'\le\delta_\rho b^{-k}$ for some $k\ge0$. Then*

$$\frac{Z_n(\theta')}{Z_n(\theta)}\ge\lambda_\rho(k). \tag{5.17}$$

Proof. Consider a configuration $X_1,\dots,X_n$ which contributes to $Z_n(\theta)$ but *not* to $Z_n(\theta')$. This implies that there is an $\ell\in\{1,\dots,n\}$ where $Q_\ell^{(\theta)}\ge1$ but $Q_\ell^{(\theta')}<1$. With this in mind, we claim the identity

$$\prod_{m=1}^n\mathsf 1_{\{Q_m^{(\theta)}\ge1\}}-\prod_{m=1}^n\mathsf 1_{\{Q_m^{(\theta')}\ge1\}}=\sum_{\ell=1}^n\Bigl[\prod_{m=1}^{\ell-1}\mathsf 1_{\{Q_m^{(\theta')}\ge1\}}\Bigr]\mathsf 1_{\{Q_\ell^{(\theta')}<1\le Q_\ell^{(\theta)}\}}\Bigl[\prod_{m=\ell+1}^n\mathsf 1_{\{Q_m^{(\theta)}\ge1\}}\Bigr]. \tag{5.18}$$

Thence,

$$Z_n(\theta)-Z_n(\theta')=\sum_{\ell=1}^n\mathbb E\biggl(Z_{n-\ell}\bigl(Q_\ell^{(\theta)}\bigr)\,\mathsf 1_{\{Q_\ell^{(\theta')}<1\le Q_\ell^{(\theta)}\}}\prod_{m=1}^{\ell-1}\mathsf 1_{\{Q_m^{(\theta')}\ge1\}}\biggr). \tag{5.19}$$

Since $\theta-\theta'\le\delta_\rho b^{-k}$, we have $Q_\ell^{(\theta)}-1\le Q_\ell^{(\theta)}-Q_\ell^{(\theta')}\le\delta_\rho b^{-k-\ell}$ for any ℓ contributing on the right-hand side. In particular, we have $Q_\ell^{(\theta)}\le1+\frac{\delta_\rho}{b}$, which implies $Z_{n-\ell}(Q_\ell^{(\theta)})\le Z_{n-\ell}(1+\frac{\delta_\rho}{b})$. Then

$$Z_n(\theta)-Z_n(\theta')\le\sum_{\ell=1}^nZ_{n-\ell}\bigl(1+\tfrac{\delta_\rho}{b}\bigr)\mathbb E\biggl(\rho\bigl([1-\tfrac1bQ_{\ell-1}^{(\theta)},1-\tfrac1bQ_{\ell-1}^{(\theta')})\bigr)\prod_{m=1}^{\ell-1}\mathsf 1_{\{Q_m^{(\theta')}\ge1\}}\biggr), \tag{5.20}$$

or, replacing $\rho([1-\frac{1}{b}Q_{\ell-1}^{(\theta)}, 1-\frac{1}{b}Q_{\ell-1}^{(\theta')}))$ by its maximal value,

$$Z_n(\theta)-Z_n(\theta') \le \sum_{\ell=1}^{n} Z_{n-\ell}\bigl(1+\tfrac{\delta_\rho}{b}\bigr)\, Z_{\ell-1}(\theta') \sup_{\vartheta-\vartheta'\le\delta_\rho b^{-k-\ell+1}} \rho\bigl([1-\tfrac{\vartheta}{b}, 1-\tfrac{\vartheta'}{b})\bigr). \tag{5.21}$$

On the other hand, by simply demanding that $X_\ell \ge 1-\frac{1-\delta_\rho}{b}$ (which implies $Q_\ell^{(\theta)} \ge 1+\frac{\delta_\rho}{b}$) in (2.10) we have for all $1\le \ell\le n$ that

$$Z_n(\theta') \ge Z_{n-\ell}\bigl(1+\tfrac{\delta_\rho}{b}\bigr)\rho\bigl([1-\tfrac{1-\delta_\rho}{b}, x_\star]\bigr) Z_{\ell-1}(\theta'). \tag{5.22}$$

Using (5.22) in (5.21), and applying (5.15), we have

$$Z_n(\theta)-Z_n(\theta') \le \frac{1-\lambda_\rho(k)}{\lambda_\rho(k)} Z_n(\theta'), \tag{5.23}$$

whereby the claim directly follows. □

Next we prove a bound between kernels (5.6) and (5.16):

Lemma 5.6 *Let $1\le k\le n-1$ and let $\omega'=(\omega'_1,\dots,\omega'_{k-1})\in\{0,1\}^{k-1}$, $X=(X_1,\dots,X_{k-1})\in[0,1]^{k-1}$, $X'=(X'_1,\dots,X'_{k-1})\in[0,1]^{k-1}$ and $\omega=(\omega_1,\dots,\omega_{k-1})\in\{0,1\}^{k-1}$. For all $\theta,\theta'\ge 1$ and all $\ell=1,\dots,k-1$, let $Q_\ell^{(\theta)}$ correspond to X via* (2.5)*, and let $Q_\ell^{(\theta')}$ correspond to X'. Suppose that*

$$Q_j^{(\theta)}\ge 1,\quad Q_j^{(\theta')}\ge 1 \quad\textit{and}\quad \omega'_j\le\omega_j\le 1_{\{X_j=X'_j\}},\qquad j=1,\dots,k-1. \tag{5.24}$$

If $R_{n,k;X,X',\omega}^{(\theta,\theta')}(\cdot)$ is the quantity defined in (5.8)*, then*

$$R_{n,k;X,X',\omega}^{(\theta,\theta')}\bigl(\{\omega_k=1\}\bigr)\ge p_\rho(\,\omega'_k=1\,|\,\omega'_1,\dots,\omega'_{k-1}), \tag{5.25}$$

for all θ,θ' with $1\le\theta,\theta'\le\theta_b$.

Proof. Note that, since $1\le\theta,\theta'\le\theta_b$ and $1+\delta_\rho\ge\theta_b$, we have $1\le Q_\ell^{(\theta)},Q_\ell^{(\theta')}\le 1+\delta_\rho$ and thus $|Q_\ell^{(\theta)}-Q_\ell^{(\theta')}|\le\delta_\rho$ for all $\ell=1,\dots,k-1$. This allows us to define the quantity

$$s=\max\bigl\{\ell\colon 0\le\ell\le k,\ |Q_{k-1}^{(\theta)}-Q_{k-1}^{(\theta')}|\le\delta_\rho b^{-\ell}\bigr\}. \tag{5.26}$$

By Lemmas 5.3 and 5.5, we have $R(\{\omega_k=1\})\ge\lambda_\rho(s)$, where $R(\cdot)$ stands for the quantity on the left-hand side of (5.25). Recall our convention $\omega'_0=0$ and let

$$s'=\min\bigl\{0\le j\le k-1\colon \omega'_{k-j-1}=0\bigr\}. \tag{5.27}$$

In other words, s' is the length of the largest contingent block of 1's in ω' directly preceding ω'_k. We claim that $s\ge s'$. Indeed, by our previous reasoning, $|Q_{k-s'-1}^{(\theta)}-Q_{k-s'-1}^{(\theta')}|\le\delta_\rho$. By our assumptions, $1=\omega'_j\le 1_{\{X_j=X'_j\}}$ and, therefore, $X_j=X'_j$ for all $j=k-s',\dots,k-1$. This implies

$$\bigl|Q_{k-1}^{(\theta)}-Q_{k-1}^{(\theta')}\bigr|\le\delta_\rho b^{-s'} \tag{5.28}$$

and hence $s\ge s'$. Using that s' is the argument of λ in (5.16) we have $R(\{\omega_k=1\})\ge\lambda_\rho(s)\ge\lambda_\rho(s')=p_\rho(\,\omega'_k=1\,|\,\omega'_1,\dots,\omega'_{k-1})$. This proves the claim. □

Now we are ready to prove Proposition 5.4:

Proof of Proposition 5.4. The inequality (5.25) is a sufficient condition for the existence of so-called Strassen's coupling, see [8]. In particular, the inhomogeneous-time process generating the triples

(X_k, X_k', ω_k) can be coupled with the process generating ω_k' in such a way that (5.24) holds at all times less than n. The (ω, ω') marginal of this process will be, by definition, concentrated on $\{\omega \succcurlyeq \omega'\}$. Since $\omega_k = 1$ for $k > n$, $P_n^{(\theta,\theta')}$-almost surely, the required stochastic domination follows. □

5.3 Existence of the limiting measure.

The goal of this section is to show that, under proper conditions, the process ω' with distribution $\widetilde{P}_\rho(\cdot)$ equals one except at a finite number of sites. Then we will give the proof of Proposition 3.5. Let

$$p_n = \begin{cases} \bigl(1-\lambda_\rho(n)\bigr)\prod_{k=0}^{n-1}\lambda_\rho(k), & \text{if } n \in \mathbb{N}\cup\{0\}, \\ \prod_{k=0}^{\infty}\lambda_\rho(k), & \text{if } n = \infty, \end{cases} \tag{5.29}$$

and observe that p_n is the probability of seeing a block of 1's of length n in the prime configuration. We begin with an estimate of $\lambda(k)$:

Lemma 5.7 *For each $\rho \in \mathcal{M}^\flat$, there is $C(\rho) < \infty$ and $\varpi > 0$ such that*

$$1 - \lambda_\rho(k) \le C(\rho) e^{-\varpi k}. \tag{5.30}$$

Moreover, the bound $C(\rho) < \infty$ is uniform in any subset $\mathcal{N} \subset \mathcal{M}^\flat$ with finitely many extreme points.

Proof. Let ϕ_ρ be the density of ρ with respect to the Lebesgue measure on $[0,1]$. Then

$$\sup_{\theta-\theta' \le \delta_\rho b^{-n}} \rho\bigl([1-\tfrac{\theta}{b}, 1-\tfrac{\theta'}{b})\bigr) \le \delta_\rho b^{-n} \|\phi_\rho\|_\infty. \tag{5.31}$$

The claim then follows by inspection of (5.15) with $\varpi = \log b$ and an appropriate choice of $C(\rho)$. The bound on $C(\rho)$ is uniform in any $\mathcal{N}$ with the above properties, because the bound $\|\phi_\rho\|_p < \infty$ is itself uniform. □

The preceding estimate demonstrates that the discrete process locks, and in fact does so fairly rapidly. Indeed, we now have $p_\infty > 0$, which ensures that eventually the configuration is all ones, and further that the p_n tend to zero exponentially. It remains to show that the waiting times till locking are themselves exponential.

Lemma 5.8 *Let $\rho \in \mathcal{M}^\flat$ and, for $n \ge 1$, let $\mathcal{E}(n) = \{\omega' \in \{0,1\}^{\mathbb{N}} \colon \omega_j' = 1,\ j \ge n\}$. Let $\alpha_0 > 0$ be such that $\varphi(\alpha) = \sum_{0\le k<\infty} e^{\alpha(k+1)} p_k < \infty$ for all $\alpha \in (0, \alpha_0)$. Then*

$$\widetilde{P}_\rho\bigl(\mathcal{E}(n)^{\mathrm{c}}\bigr) \le n\, e^{-\mu(\rho) n}, \qquad n \ge 1, \tag{5.32}$$

where

$$\mu(\rho) = \sup\bigl\{\alpha \ge 0 \colon \varphi(\alpha) \le 1\bigr\}. \tag{5.33}$$

We note that both quantities α_0 and $\mu(\rho)$ are nontrivial. Indeed, $\alpha_0 \ge \varpi > 0$ and, since p_∞ can be written as $p_\infty = 1 - \sum_{n\ge0} p_n > 0$, we have that $\mu(\rho) > 0$.

Proof. An inspection of (5.16) shows that "blocks of 1's" form a renewal process. Indeed, suppose ξ_ℓ for $\ell = 1, \dots, k-1$ mark down the lengths of first $k-1$ "blocks of 1's" including the terminating zero (i.e., $\xi_\ell = n$ refers to a block of $n-1$ ones and followed by a zero). Denoting $N_{k-1} = \sum_{j=1}^{k-1} \xi_j$, the k-th block's length is then

$$\xi_k = \min\{j > 0 \colon \omega'_{j+N_{k-1}} = 0\}. \tag{5.34}$$

As is seen from (5.16), (ξ_ℓ) can be continued into an infinite sequence of i.i.d. random variables on $\mathbb{N}\cup\{\infty\}$ with distribution $\text{Prob}(\xi_k = n+1) = p_n$, where p_n is as in (5.29). The physical sequence terminates after the first $\xi_k=\infty$ is encountered. Let $\mathcal{G}_n(k)$ be the event that $\xi_1,\dots,\xi_k$ are all finite and $\sum_{i=1}^k \xi_i > n$. Then, clearly, $\mathcal{E}(n)^{\text{c}} = \bigcup_{k=1}^n \mathcal{G}_n(k)$.

The probability of $\mathcal{G}_n(k)$ is easily bounded using the exponential Chebyshev inequality:

$$\text{Prob}\big(\mathcal{G}_n(k)\big) \le \varphi(\alpha)^k e^{-\alpha n}, \qquad 0\le \alpha<\alpha_0. \tag{5.35}$$

Noting that $\sum_{k=1}^n \varphi(\alpha)^k \le n$ for $\alpha\le\mu(\rho)$, the claim follows. □

Now we are finally ready to prove Proposition 3.5:

Proof of Proposition 3.5. Let $\rho\in\mathcal{M}^\flat$ and n be fixed. Let $k\le n$ and suppose that f is a function that depends only on the first k of the Y-coordinates. Let $\theta_0>\theta_b$ and let $\theta,\theta'\in[1,\theta_0]$. Noting that $\mathbb{P}_n^{(\theta)}(\cdot|Q_{n,m}^{(\theta)}\in \text{d}Q) = \mathbb{P}_{n-m}^{(Q)}(\cdot)$, we have

$$\big|\mathbb{E}_{n+1}^{(\theta)}(f) - \mathbb{E}_n^{(\theta')}(f)\big| \le \mathbb{E}_{n+1}^{(\theta)}\Big(\big|\mathbb{E}_n^{(Q_{n+1,n}^{(\theta)})}(f) - \mathbb{E}_n^{(\theta')}(f)\big|\Big). \tag{5.36}$$

Since $Q_{n+1,n}^{(\theta)}\in[1,\theta_0]$ by our choice of θ, we just need to estimate $|\mathbb{E}_n^{(\theta)}(f)-\mathbb{E}_n^{(\theta')}(f)|$ by the right-hand side of (3.15) for all $\theta,\theta'\in[1,\theta_0]$.

Introduce the quantity

$$D_n(f) = \sup\big\{|\mathbb{E}_n^{(\theta)}(f) - \mathbb{E}_n^{(\theta')}(f)|\colon \theta,\theta'\in[1,\theta_0]\big\}. \tag{5.37}$$

We need to show $D_n(f)$ is exponentially small in n. By Lemmas 5.1, 5.2, and Proposition 5.4, the probability that $X_i\ne X_i'$ for some $n-k\le i\le n$ under the coupling measure $\mathbb{P}_n^{(\theta,\theta')}(\cdot)$ is dominated by the probability that $\omega_i'=0$ for some $n-k\le i\le n$ under $\widetilde{P}_\rho(\cdot)$. Since f depends only on the first k of the Y variables (i.e., the *last* k of the X variables), the coupling inequality gives us

$$\big|\mathbb{E}_n^{(\theta)}(f) - \mathbb{E}_n^{(\theta')}(f)\big| \le 2\|f\|_\infty\, \widetilde{P}_\rho\big(\mathcal{E}(n-k)^{\text{c}}\big), \tag{5.38}$$

where $\mathcal{E}(n-k)$ is as in Lemma 5.8.

Let $\mu=\mu(\rho)$ be as in Lemma 5.8. Then (5.32) and (5.38) give

$$D_n(f) \le 2\|f\|_\infty (n-k)\, e^{-\mu(n-k)} \le 4(\mu e)^{-1}\|f\|_\infty e^{-\frac12\mu(n-k)}, \tag{5.39}$$

This proves (3.15) with $\zeta=\frac12\mu$ and $A=4(\mu e)^{-1}$. The bounds $\zeta>0$ and $A<\infty$ are uniform in sets $\mathcal{N}\subset\mathcal{M}^\flat$ with finitely-many extreme points, because the bound $\mu(\rho)>0$ is itself uniform. The existence of the limit (3.16) and its independence of θ is then a direct consequence of (3.15). □

5.4 Distributional identity.

Here we will show the validity of the distributional identity (3.20). The proof we follow requires establishing that the distribution of Q_∞ has no atom at $Q_\infty=1$:

Lemma 5.9 *Let $\rho\in\mathcal{M}^\flat$. Then $\widehat{\mathbb{P}}(Q_\infty=1)=0$.*

Proof. Notice that the almost-sure bound $Q_{n,1}^{(1)}\le Q_\infty\le Q_{n,1}^{(\theta_b)}$ holds for all $n\ge1$, with $Q_{n,1}^{(1)}\uparrow Q_\infty$ and $Q_{n,1}^{(\theta_b)}\downarrow Q_\infty$ as $n\to\infty$. Therefore,

$$\widehat{\mathbb{P}}(Q_\infty=1) = \lim_{n\to\infty}\widehat{\mathbb{P}}\big(Q_{n,1}^{(1)}<1,\, Q_{n,1}^{(\theta_b)}\ge1\big). \tag{5.40}$$

But Y_1 is unconstrained under $\widehat{\mathbb{P}}(\cdot)$ which by $0 \le Q_{n,1}^{(\theta_b)} - Q_{n,1}^{(1)} \le (\theta_b - 1)b^{-n}$ allows us to write

$$\widehat{\mathbb{P}}\bigl(Q_{n,1}^{(1)} < 1,\, Q_{n,1}^{(\theta_b)} \ge 1\bigr) \le \text{ l.h.s. of (5.31)}. \tag{5.41}$$

Hence, $\widehat{\mathbb{P}}(Q_{n,1}^{(1)} < 1,\, Q_{n,1}^{(\theta_b)} \ge 1) \to 0$ as $n \to \infty$ and we have $\widehat{\mathbb{P}}(Q_\infty = 1) = 0$, as claimed. □

Proof of Proposition 3.7. Let X be a random variable with distribution $\mathbb{P}(\cdot) = \rho(\cdot)$, independent of $Y_1, Y_2, \dots$, and let $\theta \ge 1$. For all $a \in \mathbb{R}$, define the (distribution) functions

$$F_n^{(\theta)}(a) = \mathbb{P}_n^{(\theta)}\bigl(Q_{n,1}^{(\theta)} \ge a\bigr). \tag{5.42}$$

and

$$\widetilde{F}_n^{(\theta)}(a) = \mathbb{P} \otimes \mathbb{P}_n^{(\theta)}\biggl(X + \frac{Q_{n,1}^{(\theta)}}{b} \ge a,\, Q_{n,1}^{(\theta)} \ge 1\biggr). \tag{5.43}$$

Since $Q_{n,1}^{(\theta)} \overset{\mathcal{D}}{=} Q_{n+1,2}^{(\theta)}$, $X \overset{\mathcal{D}}{=} Y_1$ and $Y_1 + \frac{1}{b}Q_{n+1,2}^{(\theta)} = Q_{n+1,1}^{(\theta)}$, these functions obey the relation

$$\widetilde{F}_n^{(\theta)}(a) = F_n^{(\theta)}(1)\, F_{n+1}^{(\theta)}(a), \qquad n \ge 1,\, a \in \mathbb{R}. \tag{5.44}$$

Let $F(a) = \widehat{\mathbb{P}}(Q_\infty \ge a)$ and let

$$\widetilde{F}(a) = \mathbb{P} \otimes \widehat{\mathbb{P}}\biggl(X + \frac{Q_\infty}{b} \ge a,\, Q_\infty \ge 1\biggr). \tag{5.45}$$

Both $F(\cdot)$ and $\widetilde{F}(\cdot)$ are non-increasing, left-continuous and they both have a right-limit at every $a \in \mathbb{R}$. In particular, both functions are determined by their restriction to any dense subset of $\mathbb{R}$. The proof then boils down to showing that there is a set $A \subset \mathbb{R}$ dense in $\mathbb{R}$ such that

$$\lim_{n\to\infty} F_n^{(\theta)}(a) = F(a) \qquad a \in A \cup \{1\}, \tag{5.46}$$

and

$$\lim_{n\to\infty} \widetilde{F}_n^{(\theta)}(a) = \widetilde{F}(a), \qquad a \in A. \tag{5.47}$$

Indeed, then (5.44) implies $\widetilde{F}(a) = F(1)F(a)$ for all $a \in A$, which by continuity extends to all $a \in \mathbb{R}$, proving (3.20).

Let A be the set of continuity points of both $F(\cdot)$ and $\widetilde{F}(\cdot)$. Clearly, A^{c} is countable and hence A is dense in $\mathbb{R}$. The limits in (5.46) will be taken in too stages; first we take the limit of the distribution and then that of the event. Since, $Q_{m,1}^{(1)} \le Q_{n,1}^{(\theta)} \le Q_{m,1}^{(\theta_b)}$ for any $m \le n$, we have, by (3.16),

$$\widehat{\mathbb{P}}(Q_{m,1}^{(1)} \ge a) \le \liminf_{n\to\infty} F_n^{(\theta)}(a) \le \limsup_{n\to\infty} F_n^{(\theta)}(a) \le \widehat{\mathbb{P}}(Q_{m,1}^{(\theta_b)} \ge a) \tag{5.48}$$

for all $\theta \ge 1$ and all $m \ge 1$. The $m \to \infty$ of the extremes exists by monotonicity. Since $Q_{m,1}^{(\theta_b)} \ge Q_\infty$, the right-hand side converges to $F(a)$. As for the left-hand side, it is clear that the event $\{Q_\infty > a\}$ implies that, *eventually*, $\{Q_{m,1}^{(1)} \ge a\}$ occurs. Thus the limit of the extreme left is at least as big as $\widehat{\mathbb{P}}(Q_\infty > a)$. However, the latter equals $F(a)$ because, by assumption, a is a continuity point of F. This proves (5.46). The argument for the limit (5.47) is fairly similar; the right-hand side will directly converge to $\widetilde{F}(a)$, while the limit of the left hand side will be no smaller than $\mathbb{P} \otimes \widehat{\mathbb{P}}(X + \frac{1}{b}Q_\infty >$

$a,\, Q_\infty > 1)$. However, by Lemma 5.9 we have that $\widehat{\mathbb{P}}(Q_\infty = 1) = 0$ and thus the limit equals $\widetilde{F}(a)$, because $a \in A$. $\square$

Proof of Corollary 3.6. The proof of $Q_{n,1}^{(\theta)} \xrightarrow{\mathcal{D}} Q_\infty$ is immediate from (5.46). To prove (3.19), we note that (3.18) and (2.5) imply the deterministic bounds

$$\big|f(Q_{2n,1}^{(\theta)}) - f(Q_{n,1}^{(\theta_b)})\big| \le C\|f\|_\infty\, b^{-n}\theta_0, \tag{5.49}$$

and

$$\big|f(Q_\infty) - f(Q_{n,1}^{(\theta_b)})\big| \le C\|f\|_\infty\, b^{-n}\theta_0, \tag{5.50}$$

where we used that $Q_{2n,1}^{(\theta)} \le \theta_0$ for $\theta \le \theta_0$. The bound (5.49) implies that

$$\big|\mathbb{E}_{2n}^{(\theta)}(f(Q_{2n,1}^{(\theta)})) - \mathbb{E}_{2n}^{(\theta)}(f(Q_{n,1}^{(\theta_b)}))\big| \le C'\|f\|_\infty e^{-\eta n}, \tag{5.51}$$

where $C' < \infty$ and $\eta > 0$, while the bound (5.50) guarantees that $\widehat{\mathbb{E}}(f(Q_\infty))$ can be replaced by $\widehat{\mathbb{E}}(f(Q_{n,1}^{(\theta_b)}))$ with a similar error. Then (3.19) with $2n$ replacing n boils down to the estimate of

$$\Big|\mathbb{E}_{2n}^{(\theta)}\big(f(Q_{n,1}^{(\theta)})\big) - \widehat{\mathbb{E}}\big(f(Q_{n,1}^{(\theta)})\big)\Big|. \tag{5.52}$$

But, by Proposition 3.5, the latter is bounded by $A\|f\|_\infty e^{-\zeta n}$. Combining all of the previous estimates, the claim follows. $\square$

Proof of Corollary 3.8. We begin by showing that $\mathfrak{z}(\rho) = \widehat{\mathbb{P}}(Q_\infty \ge 1)$. Indeed, we can use that $Z_n(\theta) = 0$ for $\theta < 1$ to compute

$$\begin{aligned}\widehat{\mathbb{E}}\big(Z_n(Q_\infty)\big) &= \widehat{\mathbb{P}}(Q_\infty \ge 1)\, \mathbb{E}\otimes\widehat{\mathbb{E}}\Big(Z_{n-1}\big(X + \tfrac{1}{b}Q_\infty\big)\,\Big|\, Q_\infty \ge 1\Big)\\ &= \widehat{\mathbb{P}}(Q_\infty \ge 1)\, \widehat{\mathbb{E}}\big(Z_{n-1}(Q_\infty)\big) = \dots = \widehat{\mathbb{P}}(Q_\infty \ge 1)^{n+1},\end{aligned} \tag{5.53}$$

where we used Proposition 3.7 to derive the second equality. From here $\mathfrak{z}(\rho) = \widehat{\mathbb{P}}(Q_\infty \ge 1)$ follows by noting that $\widehat{\mathbb{P}}(Q_\infty \ge 1) Z_n(1) \le \widehat{\mathbb{E}}(Z_n(Q_\infty)) \le Z_n(\theta_b)$ and applying Theorem 2.4(1).

In order to prove the existence of the limit (3.21), we first notice that

$$\frac{Z_{n+1}(\theta)}{Z_n(\theta)} = \mathbb{P}_{n+1}^{(\theta)}\big(Q_{n+1,1}^{(\theta)} \ge 1\big). \tag{5.54}$$

Next we claim that $\mathbb{P}_{n+1}^{(\theta)}(Q_{n+1,1}^{(\theta)} \ge 1) - \mathfrak{z}(\rho)$, for $\theta \ge 1$, decays exponentially with n. Indeed, let $\theta_0 > \theta_b$ and $\theta \in [1, \theta_0]$, pick $k = \lfloor \frac{n}{2} \rfloor$, use $Q_{k,1}^{(1)} \le Q_{n+1,1}^{(\theta)} \le Q_{k,1}^{(\theta_b)}$ and apply Proposition 3.5, to get

$$\widehat{\mathbb{P}}\big(Q_{k,1}^{(1)} \ge 1\big) - \bar{A}e^{-\zeta k} \le \mathbb{P}_{n+1}^{(\theta)}\big(Q_{n+1,1}^{(\theta)} \ge 1\big) \le \widehat{\mathbb{P}}\big(Q_{k,1}^{(\theta_0)} \ge 1\big) + \bar{A}e^{-\zeta k}, \tag{5.55}$$

where $\bar{A} < \infty$ is proportional to $A(\rho, \theta_0)$ from (3.15). On the other hand, we clearly have

$$\widehat{\mathbb{P}}\big(Q_{k,1}^{(1)} \ge 1\big) \le \widehat{\mathbb{P}}(Q_\infty \ge 1) \le \widehat{\mathbb{P}}\big(Q_{k,1}^{(\theta_0)} \ge 1\big). \tag{5.56}$$

But the right and left-hand sides of this inequality differ only by $\widehat{\mathbb{P}}(Q_{k,1}^{(1)} < 1,\, Q_{k,1}^{(\theta_0)} \ge 1)$, which can be estimated as in (5.41) by a number tending to zero exponentially fast as $k \to \infty$. From here we have

$$\left|\frac{Z_{n+1}(\theta)}{Z_n(\theta)\mathfrak{z}(\rho)} - 1\right| \le A' e^{\zeta' n}, \qquad \theta \in [1, \theta_0], \tag{5.57}$$

where $A' = A'(\rho, \theta_0) < \infty$ and $\zeta' = \zeta'(\rho) > 0$. The uniformity of these estimates is a consequence of the uniformity of the bounds $A < \infty$ and $\zeta > 0$ and that as in (5.41).

The existence of the limit (3.21) for $\theta \in [1, \theta_0]$ is a direct consequence of (5.57) and the identity

$$\psi_\rho(\theta) = \lim_{n\to\infty} Z_n(\theta)\mathfrak{z}(\rho)^{-n} = \lim_{n\to\infty} \prod_{k=0}^{n-1} \frac{Z_{k+1}(\theta)}{\mathfrak{z}(\rho) Z_k(\theta)} = \prod_{k=0}^{\infty} \frac{Z_{k+1}(\theta)}{\mathfrak{z}(\rho) Z_k(\theta)}, \tag{5.58}$$

and the fact that the corresponding infinite product converges. For $\theta < 1$ we have $Z_n(\theta) = 0$ and the limit exists trivially. To prove that $\theta \mapsto \psi_\rho(\theta)$ is Lipschitz continuous for $\theta \ge 1$, we first note that, by (5.23) and the result of Lemma 5.7,

$$\bigl|Z_n(\theta) - Z_n(\theta')\bigr| \le C|\theta - \theta'|\psi_\rho(\theta_0)\mathfrak{z}(\rho)^{-n}, \qquad \theta, \theta' \in [1, \theta_0], \tag{5.59}$$

where $C = C(\rho, \theta_0) < \infty$ is on sets $\mathcal{N} \subset \mathcal{M}^\flat$ with finitely many extreme points. From here the bound in part (2) directly follows.

Let $Z_n^{(\rho)}(\theta)$ denote explicitly that $Z_n(\theta)$ is computed using the underlying measure ρ. The continuity of $\alpha \mapsto \psi_{\rho_\alpha}(\theta)$ then follows using three facts: First, $\alpha \mapsto Z_n^{(\rho_\alpha)}(\theta)$, being an expectation with respect to ρ_α^n, is continuous. Second, by Theorem 2.4(2), $\alpha \mapsto \mathfrak{z}(\rho_\alpha)$ is also continuous. Third, the infinite product (5.58) converges uniformly in α. □

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