Superfluidity in Dilute Trapped Bose Gases

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A commonly used theoretical definition of superfluidity in the ground state of a Bose gas is based on the response of the system to an imposed velocity field or, equivalently, to twisted boundary conditions in a box. We are able to carry out this program in the case of a dilute interacting Bose gas in a trap, and we prove that a gas with repulsive interactions is 100% superfluid in the dilute limit in which the Gross-Pitaevskii equation is exact. This is the first example in an experimentally realistic continuum model in which superfluidity is rigorously verified.

I. INTRODUCTION

The phenomenological two-fluid model of superfluidity (see, e.g.,[1]) is based on the idea that the particle density ρ is composed of two parts, the density ρ_s of the inviscid superfluid and the normal fluid density ρ_n. If an external velocity field is imposed on the fluid (for instance by moving the walls of the container) only the viscous normal component responds to the velocity field, while the superfluid component stays at rest. In accord with these ideas the superfluid density in the ground state is often defined as follows[2]: Let E_0 denote the ground state energy of the system in the rest frame and E_0' the ground state energy, measured in the moving frame, when a velocity field $\mathbf{v}$ is imposed. Then for small $\mathbf{v}$

$$\frac{E_0'}{N} = \frac{E_0}{N} + (\rho_s/\rho)\tfrac{1}{2}m\mathbf{v}^2 + O(|\mathbf{v}|^4) \qquad (1)$$

where N is the particle number and m the particle mass. At positive temperatures the ground state energy should be replaced by the free energy. (Remark: It is important here that (1) holds uniformly for all large N; i.e., that the error term $O(|\mathbf{v}|^4)$ can be bounded independently of N. For fixed N and a finite box, Eq. (1) with $\rho_s/\rho = 1$ always holds for a Bose gas with an arbitrary interaction if $\mathbf{v}$ is small enough, owing to the discreteness of the energy spectrum[3].) There are other definitions of the superfluid density that may lead to slightly different results[4], but this is the one we shall use in this paper. We shall not dwell on this issue here, since it is not clear that there is a "one-size-fits-all" definition of superfluidity. For instance, in the definition we use here the ideal Bose gas is a perfect superfluid in its ground state, whereas the definition of Landau in terms of a linear dispersion relation of elementary excitations would indicate otherwise. We emphasize that we are not advocating any particular approach to the superfluidity question; our contribution here consists in taking one standard definition and making its consequences explicit.

One of the unresolved issues in the theory of superfluidity is its relation to Bose-Einstein condensation (BEC). It has been argued that in general neither condition is necessary for the other (c.f., e.g.,[5–7]). A simple example illustrating the fact that BEC is not necessary for superfluidity is the 1D hard-core Bose gas. This system is well known to have a spectrum like that of an ideal Fermi gas[8], and it is easy to see that it is superfluid in its ground state in the sense of (1). On the other hand, it has no BEC[9,10]. The definition of the superfluid velocity as the gradient of the phase of the condensate wave function[2,11] is clearly not applicable in such cases.

We do not give a historical overview of superfluidity because excellent review articles are available[11,12]. While the early investigations of superfluidity and Bose-Einstein condensation were mostly concerned with liquid Helium 4, it has become possible in recent years to study these phenomena in dilute trapped gases of alkali atoms[13]. The experimental success in realizing BEC in such gases has led to a large number of theoretical papers on this subject. Most of these works take BEC for granted and start off with the Gross-Pitaevskii (GP) equation to describe the condensate wave function. A rigorous justification of these assumptions is however a difficult task, and only very recently BEC has been rigorously proved for a physically realistic many-body Hamiltonian[14]. It is clearly of interest to show that superfluidity also holds in this model and this is what we accomplish here. We prove that the ground state of a Bose gas with short range, repulsive interaction is 100% superfluid in the dilute limit in which the Gross-Pitaevskii description of the gas is exact. This is the limit in which the particle number tends to infinity, but the ratio Na/L, where a is the scattering length of the interaction potential and L the box size, is kept fixed. (The significance of the parameter Na/L is that it is the ratio of the ground state energy per particle, $\sim Na/L^3$, to the lowest excitation energy in the box, $\sim 1/L^2$.) In addition we show that the gas remains 100% Bose-Einstein condensed in this limit, also for a finite velocity $\mathbf{v}$. Both results can be generalized from periodic boxes to (non-constant) velocity fields in a cylindrical geometry.

The results of this paper have been conjectured for many years, and it is gratifying that they can be proved from first principles. They represent the first example of a rigorous verification of superfluidity in an experimentally realistic continuum model.

We wish to emphasize that in this GP limit the fact that there is 100% condensation does not mean that no significant interactions occur. The kinetic and potential energies can differ markedly from that obtained with a simple variational function that is an N-fold product of one-body condensate wave functions. This assertion might seem paradoxical, and the explanation is that near the GP limit the region in which the wave function differs from the condensate function has a tiny volume that goes to zero as $N \to \infty$. Nevertheless, the interaction energy, which is proportional to N, resides in this tiny volume.

II. SETTING AND MAIN RESULTS

We consider a Bose gas with the Hamiltonian

$$H_N = -\mu \sum_{j=1}^{N} \nabla_j^2 + \sum_{1 \le i < j \le N} v(|\mathbf{r}_i - \mathbf{r}_j|) \ , \qquad (2)$$

where $\mu = \hbar^2/(2m)$ and the interaction potential v is nonnegative and of finite range. The two-body scattering length of v is denoted by a. The Hamiltonian acts on totally symmetric functions Ψ of N variables $\mathbf{r}_i = (x_i, y_i, z_i) \in \mathcal{K} \subset \mathbb{R}^3$, where $\mathcal{K}$ denotes the cube $[0, L]^3$ of side length L. (We could easily use a cuboid of sides L_1, L_2, L_3 instead.) We assume periodic boundary conditions in all three coordinate directions.

Imposing an external velocity field $\mathbf{v} = (0, 0, \pm|\mathbf{v}|)$ means that the momentum operator $\mathbf{p} = -\mathrm{i}\hbar\nabla$ is replaced by by $\mathbf{p} - m\mathbf{v}$, retaining the periodic boundary conditions. The Hamiltonian in the moving frame is thus

$$H'_N = -\mu \sum_{j=1}^{N} \left(\nabla_j + \mathrm{i}\boldsymbol{\varphi}/L\right)^2 + \sum_{1 \le i < j \le N} v(|\mathbf{r}_i - \mathbf{r}_j|) \ , \qquad (3)$$

where $\boldsymbol{\varphi} = (0, 0, \varphi)$ and the dimensionless phase φ is connected to the velocity $\mathbf{v}$ by

$$\varphi = \frac{\pm|\mathbf{v}|Lm}{\hbar} \ . \qquad (4)$$

Let $E_0(N, a, \varphi)$ denote the ground state energy of (3) with periodic boundary conditions. Obviously it is no restriction to consider only the case $-\pi \le \varphi \le \pi$, since E_0 is periodic in φ with period 2π. For Ψ_0 the ground state of H'_N, let γ_N be its one-particle reduced density matrix

$$\gamma_N(\mathbf{r}, \mathbf{r}') = N \int_{\mathcal{K}^{N-1}} \Psi_0(\mathbf{r}, \mathbf{r}_2, \ldots, \mathbf{r}_N) \times \Psi_0^*(\mathbf{r}', \mathbf{r}_2, \ldots, \mathbf{r}_N)\, d\mathbf{r}_2 \cdots d\mathbf{r}_N \ . \qquad (5)$$

We are interested in the *Gross-Pitaevskii* (GP) limit $N \to \infty$ with Na/L fixed. We also fix the box size L. This means that a should vary like $1/N$ which can be achieved by writing $v(\mathbf{r}) = a^{-2} v_1(\mathbf{r}/a)$, where v_1 is a fixed potential with scattering length 1, while a changes with N.

Theorem 1 (Superfluidity of homogeneous gas). *For* $|\varphi| \le \pi$

$$\lim_{N \to \infty} \frac{E_0(N, a, \varphi)}{N} = 4\pi\mu a\rho + \mu \frac{\varphi^2}{L^2} \qquad (6)$$

in the limit $N \to \infty$ *with* Na/L *and* L *fixed. Here* $\rho = N/L^3$*, so* $a\rho$ *is fixed too. In the same limit, for* $|\varphi| < \pi$*,*

$$\lim_{N \to \infty} \frac{1}{N} \gamma_N(\mathbf{r}, \mathbf{r}') = \frac{1}{L^3} \qquad (7)$$

in trace class norm, i.e., $\lim_{N\to\infty} \mathrm{Trace}\left[\,\left|\gamma_N/N - |L^{-3/2}\rangle\langle L^{-3/2}|\right|\,\right] = 0$.

Note that, by the definition (1) of ρ_s and Eq. (4), Eq. (6) means that $\rho_s = \rho$, i.e., there is 100% superfluidity. For $\varphi = 0$, Eq. (6) was first proved in[15]. Eq. (7) for $\varphi = 0$ is the BEC proved in[14].

Remarks. 1. By a unitary gauge transformation,

$$\big(U\Psi\big)(\mathbf{r}_1, \ldots, \mathbf{r}_N) = e^{\mathrm{i}\varphi(\sum_i z_i)/L}\, \Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) \ , \qquad (8)$$

the passage from (2) to (3) is equivalent to replacing periodic boundary conditions in a box by the *twisted boundary condition*

$$\Psi(\mathbf{r}_1 + (0, 0, L), \mathbf{r}_2, \ldots, \mathbf{r}_N) = e^{\mathrm{i}\varphi}\Psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) \quad (9)$$

in the direction of the velocity field, while retaining the original Hamiltionian (2).

2. The criterion $|\varphi| \le \pi$ means that $|\mathbf{v}| \le \pi\hbar/(mL)$. The corresponding energy $\frac{1}{2}m(\pi\hbar/(mL))^2$ is the gap in the excitation spectrum of the one-particle Hamiltonian in the finite-size system.

3. The reason that we have to restrict ourselves to $|\varphi| < \pi$ in the second part of Theorem 1 is that for $|\varphi| = \pi$ there are two ground states of the operator $(\nabla + \mathrm{i}\boldsymbol{\varphi}/L)^2$ with periodic boundary conditions. All we can say in this case is that there is a subsequence of γ_N that converges to a density matrix of rank ≤ 2, whose range is spanned by these two functions.

Theorem 1 can be generalized in various ways to a physically more realistic setting. As an example, let $\mathcal{C}$ be a finite cylinder based on an annulus centered at the origin. Given a bounded, real function $a(r, z)$ let A be the vector field (in polar coordinates) $A(r, \theta, z) = \varphi a(r, z)\widehat{e}_\theta$, where $\widehat{e}_\theta$ is the unit vector in the θ direction. We also allow for a bounded external potential $V(r, z)$ that does not depend on θ.

Using the methods of Appendix A in[16], it is not difficult to see that there exists a $\varphi_0 > 0$, depending only on

$\mathcal{C}$ and $a(r,z)$, such that for all $|\varphi| < \varphi_0$ there is a unique minimizer $\phi^{\rm GP}$ of the Gross-Pitaevskii functional

$$\mathcal{E}^{\rm GP}[\phi] = \int_{\mathcal{K}} \Big(\mu\big|\big(\nabla + {\rm i}A(\mathbf{r})\big)\phi(\mathbf{r})\big|^2 + V(\mathbf{r})|\phi(\mathbf{r})|^2 + 4\pi\mu N a|\phi(\mathbf{r})|^4\Big) d^3\mathbf{r} \quad (10)$$

under the normalization condition $\int|\phi|^2 = 1$. This minimizer does not depend on θ, and can be chosen to be positive, for the following reason: The relevant term in the kinetic energy is $T = -r^{-2}[\partial/\partial\theta + {\rm i}\varphi\, r\, a(r,z)]^2$. If $|\varphi\, r\, a(r,z)| < 1/2$, it is easy to see that $T \geq \varphi^2 a(r,z)^2$, in which case, without raising the energy, we can replace ϕ by the square root of the θ-average of $|\phi|^2$. This can only lower the kinetic energy[17] and, by convexity of $x \to x^2$, this also lowers the ϕ^4 term.

We denote the ground state energy of $\mathcal{E}^{\rm GP}$ by $E^{\rm GP}$, depending on Na and φ. The following Theorem 2 concerns the ground state energy E_0 of

$$H_N^A = \sum_{j=1}^N \Big[-\mu\big(\nabla_j + {\rm i}A(\mathbf{r}_j)\big)^2 + V(\mathbf{r}_j)\Big] + \sum_{1\leq i<j\leq N} v(|\mathbf{r}_i - \mathbf{r}_j|)\ , \quad (11)$$

with Neumann boundary conditions on $\mathcal{C}$, and the one-particle reduced density matrix γ_N of the ground state, respectively. Different boundary conditions can be treated in the same manner, if they are also used in (10).

Remark. As a special case, consider a uniformly rotating system. In this case $A(\mathbf{r}) = \varphi r \widehat{e}_\theta$, where 2φ is the angular velocity. H_N^A is the Hamiltonian in the rotating frame, but with external potential $V(\mathbf{r}) + \mu A(\mathbf{r})^2$ (see e.g.[11] (p. 131)).

Theorem 2 (Superfluidity in a cylinder). *For* $|\varphi| < \varphi_0$

$$\lim_{N\to\infty} \frac{E_0(N,a,\varphi)}{N} = E^{\rm GP}(Na,\varphi) \quad (12)$$

in the limit $N\to\infty$ with Na fixed. In the same limit,

$$\lim_{N\to\infty} \frac{1}{N}\gamma_N(\mathbf{r},\mathbf{r}') = \phi^{\rm GP}(\mathbf{r})\phi^{\rm GP}(\mathbf{r}') \quad (13)$$

in trace class norm, i.e., $\lim_{N\to\infty} {\rm Trace}\big[\,\big|\gamma_N/N - |\phi^{\rm GP}\rangle\langle\phi^{\rm GP}|\,\big|\,\big] = 0$.

In the case of a uniformly rotating system, where 2φ is the angular velocity, the condition $|\varphi| < \varphi_0$ in particular means that the angular velocity is smaller than the critical velocity for creating vortices[18].

Remark. In the special case of the curl-free vector potential $A(r,\theta) = \varphi r^{-1}\widehat{e}_\theta$, i.e., $a(r,z) = r^{-1}$, one can say more about the role of φ_0. In this case, there is a unique GP minimizer for all $\varphi \notin \mathbb{Z} + \frac{1}{2}$, whereas there are two minimizers for $\varphi \in \mathbb{Z} + \frac{1}{2}$. Part two of Theorem 2 holds in this special case for all $\varphi \notin \mathbb{Z} + \frac{1}{2}$, and (12) is true even for all φ.

III. PROOFS

In the following, we will present only a proof of Theorem 1 for simplicity. Theorem 2 can be proved using the same methods, and additionally the methods of[14] to deal with the inhomogeneity of the system.

Before giving the formal proofs, we outline the main ideas. The strategy is related to the one in[14], but requires substantial generalizations of the techniques. A crucial element of the proof, stated in Lemma 1 below, is the fact that the interaction energy can be localized in small balls around each particle. This part uses a Lemma of Dyson[19], and its generalization in[15], which converts a strong short range potential into a soft potential. This Lemma can be also be applied to the case of an external velocity field, i.e., a $U(1)$ gauge field in the kinetic term of the Hamiltonian, owing to the "diamagnetic inequaltity"[17]. This inequaltity says that the additional gauge field increases the kinetic energy density.

The second main part of the proof is the generalized Poincaré inequality given in Lemma 2. We recall that an essential ingredient of the proof of Bose-Einstein condensation in[14] was showing that the fact that the kinetic energy density is small in most of the configuration space implies that the one-body reduced density matrix is essentially constant. The difficulty comes from the fact that the region in which the kinetic energy is small can, in principle, be broken up into disjoint subregions, thereby permitting different constants in different subregions. The fact that this does not happen is the content of the generalized Poincaré inequality. In the present case we have an additional complication coming from the imposed gauge field. The old Poincaré inequality does not suffice; one now has to measure the kinetic energy density relative to the lowest energy of a free particle in the gauge field rather than to zero. This is an essential complication. While the previous (generalized) Poincaré inequality could, after some argumentation, be related to the standard Poincaré inequality[17], this new one, with the gauge field, requires a totally new proof.

Proof of Theorem 1. As in[15] we define $Y = (4\pi/3)\rho a^3$. Note that in the limit considered, $Y \sim N^{-2}$. We first consider the upper bound to E_0. Using the ground state Ψ_0 for $\varphi = 0$ as a trial function, we immediately get

$$E_0(N,a,\varphi) \leq \langle\Psi_0, H_N'\Psi_0\rangle = E_0(N,a,0) + N\mu\frac{\varphi^2}{L^2}\ , \quad (14)$$

since $\langle\Psi_0, \nabla_i\Psi_0\rangle = 0$. From[16] we know that $E_0(N,a,0) \leq 4\pi\mu N\rho a(1 + {\rm const.}\, Y^{1/3})$, which has the right form as $N\to\infty$.

For the lower bound to the ground state energy we need the following Lemma.

Lemma 1 (Localization of energy). *For all symmetric, normalized wave functions $\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_N)$ with* *periodic boundary conditions on $\mathcal{K}$, and for $N \geq Y^{-1/17}$,*

$$\frac{1}{N}\langle\Psi, H_N'\Psi\rangle \geq \left(1 - \mathrm{const.}\, Y^{1/17}\right)\Big(4\pi\mu\rho a + \mu\int_{\mathcal{K}^{N-1}} d\mathbf{X}\int_{\Omega_\mathbf{X}} d\mathbf{r}_1 \left|(\nabla_1 + \mathrm{i}\varphi/L)\Psi(\mathbf{r}_1,\mathbf{X})\right|^2\Big)\ , \quad (15)$$

where $\mathbf{X} = (\mathbf{r}_2, \ldots, \mathbf{r}_N)$, $d\mathbf{X} = \prod_{j=2}^N d\mathbf{r}_j$, and

$$\Omega_\mathbf{X} = \left\{\mathbf{r}_1 :\ \min_{j\geq 2}|\mathbf{r}_1 - \mathbf{r}_j| \geq R\right\} \quad (16)$$

with $R = aY^{-5/17}$.

Proof. Since Ψ is symmetric, the left side of (15) can be written as

$$\int_{\mathcal{K}^{N-1}} d\mathbf{X}\int_{\mathcal{K}} d\mathbf{r}_1 \Big[\mu\left|(\nabla_1 + \mathrm{i}\varphi/L)\Psi(\mathbf{r}_1,\mathbf{X})\right|^2 + \tfrac{1}{2}\sum_{j\geq 2} v(|\mathbf{r}_1 - \mathbf{r}_j|)|\Psi(\mathbf{r}_1,\mathbf{X})|^2\Big]\ . \quad (17)$$

For any $\varepsilon > 0$ and $R > 0$ this is

$$\geq \varepsilon T + (1-\varepsilon)(T^{\mathrm{in}} + I) + (1-\varepsilon)T_\varphi^{\mathrm{out}}\ , \quad (18)$$

with

$$T = \mu\int_{\mathcal{K}^{N-1}} d\mathbf{X}\int_{\mathcal{K}} d\mathbf{r}_1 \left|\nabla_1|\Psi(\mathbf{r}_1,\mathbf{X})|\right|^2\ , \quad (19)$$

$$T^{\mathrm{in}} = \mu\int_{\mathcal{K}^{N-1}} d\mathbf{X}\int_{\Omega_\mathbf{X}^c} d\mathbf{r}_1 \left|\nabla_1|\Psi(\mathbf{r}_1,\mathbf{X})|\right|^2\ , \quad (20)$$

$$T_\varphi^{\mathrm{out}} = \mu\int_{\mathcal{K}^{N-1}} d\mathbf{X}\int_{\Omega_\mathbf{X}} d\mathbf{r}_1 \left|(\nabla_1 + \mathrm{i}\varphi/L)\Psi(\mathbf{r}_1,\mathbf{X})\right|^2\ , \quad (21)$$

and

$$I = \tfrac{1}{2}\int_{\mathcal{K}^{N-1}} d\mathbf{X}\int_{\mathcal{K}} d\mathbf{r}_1 \sum_{j\geq 2} v(|\mathbf{r}_1 - \mathbf{r}_j|)|\Psi(\mathbf{r}_1,\mathbf{X})|^2\ . \quad (22)$$

Here

$$\Omega_\mathbf{X}^c = \{\mathbf{r}_1 :\ |\mathbf{r}_1 - \mathbf{r}_j| < R \text{ for some } j \geq 2\} \quad (23)$$

is the complement of $\Omega_\mathbf{X}$, and the diamagnetic inequality $|(\nabla + \mathrm{i}\varphi/L)f(\mathbf{r})|^2 \geq \left|\nabla|f(\mathbf{r})|\right|^2$ has been used. The proof is completed by using the results of[15] and[14] (see also[20]) which tell us that for $\varepsilon = Y^{1/17}$ and $R = aY^{-5/17}$

$$\varepsilon T + (1-\varepsilon)(T^{\mathrm{in}} + I) \geq \left(1 - \mathrm{const.}\, Y^{1/17}\right)4\pi\mu\rho a \quad (24)$$

as long as $N \geq Y^{-1/17}$. (This estimate is highly non-trivial. Among several other things it uses a generalization of Dyson's lemma[19].) □

The following Lemma 2 is needed for a lower bound on the second term in (15). It is stated for $\mathcal{K}$ the $L \times L \times L$-cube with periodic boundary conditions, but it can be generalized to arbitrary connected sets $\mathcal{K}$ that are sufficiently nice so that the Rellich-Kondrashov Theorem (see[17] (Thm. 8.9)) holds on $\mathcal{K}$. In particular, this is the case if $\mathcal{K}$ has the 'cone property'[17]. Another possible generalization is to include general bounded vector fields replacing $\boldsymbol{\varphi}$, see[21].

If Ω is any subset of $\mathcal{K}$ we shall denote $\int_\Omega f^*(\mathbf{r})g(\mathbf{r})d\mathbf{r}$ by $\langle f, g\rangle_\Omega$ and $\langle f, f\rangle_\Omega^{1/2}$ by $\|f\|_{L^2(\Omega)}$. We also denote $\nabla + \mathrm{i}\boldsymbol{\varphi}$ by ∇_φ for short.

Lemma 2 (Generalized Poincaré inequality). *For any $|\varphi| < \pi$ there are constants $c > 0$ and $C < \infty$ such that for all subsets $\Omega \subset \mathcal{K}$ and all functions f on the torus $\mathcal{K}$ the following estimate holds:*

$$\|\nabla_\varphi f\|_{L^2(\Omega)}^2 \geq \frac{\varphi^2}{L^2}\|f\|_{L^2(\mathcal{K})}^2 + \frac{c}{L^2}\|f - L^{-3}\langle 1, f\rangle_\mathcal{K}\|_{L^2(\mathcal{K})}^2 - C\left(\|\nabla_\varphi f\|_{L^2(\mathcal{K})}^2 + \frac{1}{L^2}\|f\|_{L^2(\mathcal{K})}^2\right)\left(\frac{|\Omega|^c}{|\mathcal{K}|}\right)^{1/2}\ . \quad (25)$$

Here $|\Omega^c|$ is the volume of $\Omega^c = \mathcal{K}\setminus\Omega$, the complement of Ω in $\mathcal{K}$.

Proof. We shall derive (25) from a special form of this inequality that holds for all functions that are orthogonal to the constant function. Namely, for any positive $\alpha < 2/3$ and some constants $c > 0$ and $\widetilde{C} < \infty$ (depending only on α and $|\varphi| < \pi$) we claim that

$$\|\nabla_\varphi h\|_{L^2(\Omega)}^2 \geq \frac{\varphi^2 + c}{L^2}\|h\|_{L^2(\mathcal{K})}^2 - \widetilde{C}\left(\frac{|\Omega^c|}{|\mathcal{K}|}\right)^\alpha \|\nabla_\varphi h\|_{L^2(\mathcal{K})}^2\ , \quad (26)$$

provided $\langle 1, h\rangle_\mathcal{K} = 0$. (Remark: Eq. (26) holds also for $\alpha = 2/3$, but the proof is slightly more complicated in that case. See[21].) If (26) is known the derivation of (25) is easy: For any f, the function $h = f - L^{-3}\langle 1, f\rangle_\mathcal{K}$ is orthogonal to 1. Moreover,

$$\begin{aligned}
&\|\nabla_\varphi h\|_{L^2(\Omega)}^2 \\
&= \|\nabla_\varphi h\|_{L^2(\mathcal{K})}^2 - \|\nabla_\varphi h\|_{L^2(\Omega^c)}^2 \\
&= \|\nabla_\varphi f\|_{L^2(\Omega)}^2 - \frac{\varphi^2}{L^2}|\langle L^{-3/2}, f\rangle_\mathcal{K}|^2\left(1 + \frac{|\Omega^c|}{|\mathcal{K}|}\right) \\
&\quad + 2\frac{\varphi}{L}\mathrm{Re}\,\langle L^{-3/2}, f\rangle_\mathcal{K}\langle\nabla_\varphi f, L^{-3/2}\rangle_{\Omega^c} \\
&\leq \|\nabla_\varphi f\|_{L^2(\Omega)}^2 - \frac{\varphi^2}{L^2}|\langle L^{-3/2}, f\rangle_\mathcal{K}|^2 \\
&\quad + \frac{|\varphi|}{L}\left(L\|\nabla_\varphi f\|_{L^2(\mathcal{K})}^2 + \frac{1}{L}\|f\|_{L^2(\mathcal{K})}^2\right)\left(\frac{|\Omega^c|}{|\mathcal{K}|}\right)^{1/2}
\end{aligned} \quad (27)$$

and

$$\frac{\varphi^2+c}{L^2}\|h\|^2_{L^2(\mathcal{K})} = \frac{\varphi^2}{L^2}\left(\|f\|^2_{L^2(\mathcal{K})} - |\langle L^{-3/2}, f\rangle_{\mathcal{K}}|^2\right) + \frac{c}{L^2}\|f - L^{-3}\langle 1, f\rangle_{\mathcal{K}}\|^2_{L^2(\mathcal{K})} . \quad (28)$$

Setting $\alpha = \frac{1}{2}$, using $\|\nabla_\varphi h\|_{L^2(\mathcal{K})} \leq \|\nabla_\varphi f\|_{L^2(\mathcal{K})}$ in the last term in (26) and combining (26), (27) and (28) gives (25) with $C = |\varphi| + \widetilde{C}$.

We now turn to the proof of (26). For simplicity we set $L = 1$. The general case follows by scaling. Assume that (26) is false. Then there exist sequences of constants $C_n \to \infty$, functions h_n with $\|h_n\|_{L^2(\mathcal{K})} = 1$ and $\langle 1, h_n\rangle_{\mathcal{K}} = 0$, and domains $\Omega_n \subset \mathcal{K}$ such that

$$\lim_{n\to\infty}\left\{\|\nabla_\varphi h_n\|^2_{L^2(\Omega_n)} + C_n|\Omega_n^c|^\alpha\|\nabla_\varphi h_n\|^2_{L^2(\mathcal{K})}\right\} \leq \varphi^2 . \quad (29)$$

We shall show that this leads to a contradiction.

Since the sequence h_n is bounded in $L^2(\mathcal{K})$ it has a subsequence, denoted again by h_n, that converges weakly to some $h \in L^2(\mathcal{K})$ (i.e., $\langle g, h_n\rangle_{\mathcal{K}} \to \langle g, h\rangle_{\mathcal{K}}$ for all $g \in L^2(\mathcal{K})$). Moreover, by Hölder's inequality the $L^p(\Omega_n^c)$ norm $\|\nabla_\varphi h_n\|_{L^p(\Omega_n^c)} = (\int_{\Omega_n^c}|h(\mathbf{r})|^p d\mathbf{r})^{1/p}$ is bounded by $|\Omega_n^c|^{\alpha/2}\|\nabla_\varphi h_n\|_{L^2(\mathcal{K})}$ for $p = 2/(\alpha+1)$. From (29) we conclude that $\|\nabla_\varphi h_n\|_{L^p(\Omega_n^c)}$ is bounded and also that $\|\nabla_\varphi h_n\|_{L^p(\Omega_n)} \leq \|\nabla_\varphi h_n\|_{L^2(\Omega_n)}$ is bounded. Altogether, $\nabla_\varphi h_n$ is bounded in $L^p(\mathcal{K})$, and by passing to a further subsequence if necessary, we can therefore assume that $\nabla_\varphi h_n$ converges weakly in $L^p(\mathcal{K})$. The same applies to ∇h_n. Since $p = 2/(\alpha+1)$ with $\alpha < 2/3$ the hypotheses of the Rellich-Kondrashov Theorem[17] (Thm 8.9) are fulfilled and consequently h_n converges *strongly* in $L^2(\mathcal{K})$ to h (i.e., $\|h - h_n\|_{L^2(\mathcal{K})} \to 0$). We shall now show that

$$\liminf_{n\to\infty}\|\nabla_\varphi h_n\|^2_{L^2(\Omega_n)} \geq \|\nabla_\varphi h\|^2_{L^2(\mathcal{K})} . \quad (30)$$

This will complete the proof because the h_n are normalized and orthogonal to 1 and the same holds for h by strong convergence. Hence the right side of (30) is necessarily $> \varphi^2$, since for $|\varphi| < \pi$ the lowest eigenvalue of $-\nabla_\varphi^2$, with constant eigenfunction, is non-degenerate. This contradicts (29).

Eq. (30) is essentially a consequence of the weak lower semicontinuity of the L^2 norm, but the dependence on Ω_n leads to a slight complication. First, Eq. (29) and $C_n \to \infty$ clearly imply that $|\Omega_n^c| \to 0$, because $\|\nabla_\varphi h_n\|^2_{L^2(\mathcal{K})} > \varphi^2$. By choosing a subsequence we may assume that $\sum_n|\Omega_n^c| < \infty$. For some fixed N let $\widetilde{\Omega}_N = \mathcal{K}\setminus\cup_{n\geq N}\Omega_n^c$. Then $\widetilde{\Omega}_N \subset \Omega_n$ for $n \geq N$. Since $\|\nabla_\varphi h_n\|^2_{L^2(\Omega_n)}$ is bounded, $\nabla_\varphi h_n$ is also bounded in $L^2(\widetilde{\Omega}_N)$ and a subsequence of it converges weakly in $L^2(\widetilde{\Omega}_N)$ to $\nabla_\varphi h$. Hence

$$\liminf_{n\to\infty}\|\nabla_\varphi h_n\|^2_{L^2(\Omega_n)} \geq \liminf_{n\to\infty}\|\nabla_\varphi h_n\|^2_{L^2(\widetilde{\Omega}_N)} \geq \|\nabla_\varphi h\|^2_{L^2(\widetilde{\Omega}_N)} . \quad (31)$$

Since $\widetilde{\Omega}_N \subset \widetilde{\Omega}_{N+1}$ and $\cup_N\widetilde{\Omega}_N = \mathcal{K}$ (up to a set of measure zero), we can now let $N \to \infty$ on the right side of (31). By monotone convergence this converges to $\|\nabla_\varphi h\|^2_{L^2(\mathcal{K})}$. This proves (30) which, as remarked above, contradicts (29). □

We now are able to finish the proof of Theorem 1. From Lemmas 1 and 2 we infer that, for any symmetric Ψ with $\langle\Psi,\Psi\rangle = 1$ and for N large enough,

$$\begin{aligned}&\frac{1}{N}\langle\Psi, H_N'\Psi\rangle\left(1 - \text{const.}\,Y^{1/17}\right)^{-1}\\ &\geq 4\pi\mu\rho a + \mu\frac{\varphi^2}{L^2}\\ &\quad -CY^{1/17}\Big(\frac{1}{L^2} + \frac{1}{N}\big\langle\Psi, \textstyle\sum_j(\nabla_j + \mathrm{i}\boldsymbol{\varphi})\Psi\big\rangle\Big)\\ &\quad +\frac{c}{L^2}\int_{\mathcal{K}^{N-1}}d\mathbf{X}\int_{\mathcal{K}}d\mathbf{r}_1\Big|\Psi(\mathbf{r}_1,\mathbf{X})\\ &\qquad -L^{-3}\big[\textstyle\int_{\mathcal{K}}d\mathbf{r}\Psi(\mathbf{r},\mathbf{X})\big]\Big|^2 ,\end{aligned} \quad (32)$$

where we used that $|\Omega^c| \leq \frac{4\pi}{3}NR^3 = \text{const.}\,L^3Y^{2/17}$. From this we can infer two things. First, since the kinetic energy, divided by N, is certainly bounded independent of N, as the upper bound shows, we get that

$$\liminf_{N\to\infty}\frac{E_0(N,a,\varphi)}{N} \geq 4\pi\mu\rho a + \mu\frac{\varphi^2}{L^2} \quad (33)$$

for any $|\varphi| < \pi$. By continuity this holds also for $|\varphi| = \pi$, proving (6). (To be precise, $E_0/N - \mu\varphi^2L^{-2}$ is concave in φ, and therefore stays concave, and in particular continuous, in the limit $N \to \infty$.) Secondly, since the upper and the lower bound to E_0 agree in the limit considered, the positive last term in (32) has to vanish in the limit. I.e., we get that for the ground state wave function Ψ_0 of H_N'

$$\lim_{N\to\infty}\int_{\mathcal{K}^{N-1}}d\mathbf{X}\int_{\mathcal{K}}d\mathbf{r}_1\Big|\Psi_0(\mathbf{r}_1,\mathbf{X}) - L^{-3}\big[\textstyle\int_{\mathcal{K}}d\mathbf{r}\Psi_0(\mathbf{r},\mathbf{X})\big]\Big|^2 = 0 . \quad (34)$$

This proves (7), since

$$\int_{\mathcal{K}^{N-1}}d\mathbf{X}\int_{\mathcal{K}}d\mathbf{r}_1\Big|\Psi_0(\mathbf{r}_1,\mathbf{X}) - L^{-3}\big[\textstyle\int_{\mathcal{K}}d\mathbf{r}\Psi_0(\mathbf{r},\mathbf{X})\big]\Big|^2 = 1 - \frac{1}{NL^3}\int_{\mathcal{K}\times\mathcal{K}}\gamma(\mathbf{r},\mathbf{r}')d\mathbf{r}d\mathbf{r}' , \quad (35)$$

and therefore $N^{-1}\langle L^{-3/2}|\gamma_N|L^{-3/2}\rangle \to 1$. As explained in[14,20] this suffices for the convergence $N^{-1}\gamma_N \to |L^{-3/2}\rangle\langle L^{-3/2}|$ in trace class norm. □

IV. CONCLUSIONS

We have shown that a Bose gas with short range, repulsive interactions is both a 100% superfluid and also

100% Bose-Einstein condensed in its ground state in the Gross-Pitaevskii limit where the parameter Na/L is kept fixed as $N \to \infty$. This is a simultaneous large N and low density limit, because the dimensionless density parameter ρa^3 is here proportional to $1/N^2$. If ρa^3 is not zero, but small, a depletion of the Bose-Einstein condensate of the order $(\rho a^3)^{1/2}$ is expected (see, e.g.,[22]). Nevertheless, complete superfluidity in the ground state, e.g. of Helium 4, is experimentally observed. It is an interesting open problem to deduce this property rigorously from first principles. In the case of a one-dimensional hard-core Bose gas superfluidity in the ground state is easy to show, but nevertheless there is no Bose-Einstein condensation at all, not even in the ground state[9,10].

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3 The ground state with $\mathbf{v} = 0$ remains an eigenstate of the Hamiltonian with arbitrary $\mathbf{v}$ since its total momentum is zero. Its energy is $\frac{1}{2}mN\mathbf{v}^2$ above the ground state energy for $\mathbf{v} = 0$. Since in a finite box the spectrum of the Hamiltonian for arbitrary $\mathbf{v}$ is discrete and the energy gap above the ground state is bounded away from zero for $\mathbf{v}$ small, the ground state for $\mathbf{v} = 0$ is at the same time the ground state of the Hamiltonian with $\mathbf{v}$ if $\frac{1}{2}mN\mathbf{v}^2$ is smaller than the gap.

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