ON A SPECTRAL PROPERTY OF JACOBI MATRICES

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ABSTRACT. Let J be a Jacobi matrix with elements b_k on the main diagonal and elements a_k on the auxiliary ones. We suppose that J is a compact perturbation of the free Jacobi matrix. In this case the essential spectrum of J coincides with $[-2,2]$, and its discrete spectrum is a union of two sequences $\{x_j^\pm\}, x_j^+ \ge 2, x_j^- \le -2$, tending to ± 2. We denote sequences $\{a_{k+1}-a_k\}$ and $\{a_{k+1}+a_{k-1}-2a_k\}$ by ∂a and $\partial^2 a$, respectively.

The main result of the note is the following theorem.

Theorem. Let J be a Jacobi matrix described above and σ be its spectral measure. Then $a-1, b\in l^4,\ \partial^2 a, \partial^2 b \in l^2$, if and only if

$$i)\ \int_{-2}^{2} \log \sigma'(x)(4-x^2)^{5/2}\,dx > -\infty, \qquad ii)\ \sum_j (x_j^\pm \mp 2)^{7/2} < \infty.$$

INTRODUCTION

Let us consider a Jacobi matrix

$$J = J(a,b) = \begin{bmatrix} b_0 & a_0 & 0 \\ a_0 & b_1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

defined by two sequences $a=\{a_k\},\ a_k>0$, and $b=\{b_k\},\ b_k\in\mathbb{R}$. We suppose that J is a compact perturbation of the so-called free Jacobi matrix J_0,

$$J_0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

The (scalar) spectral measure σ of J is defined by the relation

$$((J-z)^{-1}e_0, e_0) = \int_{\mathbb{R}} \frac{d\sigma(x)}{x-z},$$

with $z\in\mathbb{C}\backslash\mathbb{R}$. It is well-known that in this case the continuous part σ_c of the measure is supported by $[-2,2]$, and its discrete part σ_d lives on the union of two sequences $\{x_j^\pm\}$ such that $x_j^+\ge 2,\ x_j^-\le -2$, and $x_j^\pm \to \pm 2$, as $j\to\infty$.

Let us put $\partial a = \{a_{k+1}-a_k\}$ and $\partial^2 a = \{a_{k+1}+a_{k-1}-2a_k\}$ and denote by 1 the sequence consisting of ones. The following theorem holds.

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Theorem 1. *Let $J = J(a,b)$ and σ be its spectral measure. Then*

$$a-1, b \in l^4,\ \partial^2 a, \partial^2 b \in l^2, \tag{1}$$

if and only if

$$i)\ \int_{-2}^{2} \log \sigma'(x)(4-x^2)^{5/2}\, dx > -\infty, \quad ii)\ \sum_j (x_j^{\pm} \mp 2)^{7/2} < \infty. \tag{2}$$

The theorem has an immediate corollary.

Corollary 1. *Let $J = J(a,b)$ and σ be its spectral measure. If $a-1, b \in l^4,\ \partial a, \partial b \in l^2$, then relations* (2) *hold.*

In this direction we may also prove the following proposition.

Proposition 1. *Let $J = J(a,b)$ and σ be its spectral measure. If $a-1, b \in l^3,\ \partial a, \partial b \in l^2$, then*

$$i)\ \int_{-2}^{2} \log \sigma'(x)(4-x^2)^{3/2}\, dx > -\infty, \quad ii)\ \sum_j (x_j^{\pm} \mp 2)^{5/2} < \infty.$$

Together with Corollary 1, this proposition leads to a natural conjecture. The conjecture would yield a right discrete counterpart of results, obtained in [6] for Schrödinger operators on the half-line.

Conjecture 1. *For an integer $k \geq 2$, conditions $a-1, b \in l^k,\ \partial a, \partial b \in l^2$ imply*

$$i)\ \int_{-2}^{2} \log \sigma'(x)(4-x^2)^{k-3/2}\, dx > -\infty, \quad ii)\ \sum_j (x_j^{\pm} \mp 2)^{k-1/2} < \infty.$$

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1. Proof of Theorem 1

It is convenient to map the domain $\hat{\mathbb{C}} \backslash [-2,2]$ onto the unit disk with the help of transformation $\zeta(z) = 1/2(z - \sqrt{z^2-4})$. We construct a measure μ on the unit circle letting $d\mu(\theta) = d\sigma(x)/(2\sin\theta)$, where $x = z(e^{i\theta})$ and $z(\zeta) = \zeta + 1/\zeta$. The sequences $\{x_j^{\pm}\}$ are mapped to points $\{\zeta_j\}$, lying on the real diameter of the unit disk. In these new terms Theorem 1 says that $a-1, b \in l^4,\ \partial^2 a, \partial^2 b \in l^2$ if and only if

$$i)\ \int_{0}^{2\pi} \log \mu'(\theta) \sin^6\theta\, d\theta > -\infty, \quad ii)\ \sum_j (1-|\zeta_j|)^7 < \infty. \tag{3}$$

Let us assume first that rank $(J - J_0) < \infty$. For $j \geq 1$, we define Chebyshev polynomials by recurrence relations $T_{j+1}(z) = zT_j(z) - T_{j-1}(z)$, and we take $T_0(z) = 2, T_1(z) = z$. We have the so-called sum rules [1, 2, 4, 7]

$$0) \qquad \frac{1}{4\pi}\int_0^{2\pi} \log\frac{2\sin\theta}{\mu'(\theta)}\,d\theta = \sum_j \log\beta_j - \sum_j \log a_j,$$

$$n) \qquad \frac{1}{2\pi}\int_0^{2\pi} \log\frac{2\sin\theta}{\mu'(\theta)}\cos n\theta\,d\theta = \frac{1}{n}\sum_j(\beta_j^n - \beta_j^{-n}) - \frac{1}{n}\mathrm{tr}\,\{T_n(J) - T_n(J_0)\},$$

where we let $\beta_j = 1/|\zeta_j|$. Taking into account that $\sin^6\theta = 1/2^5(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta)$, we get

$$\frac{1}{2\pi}\int_0^{2\pi} \log\frac{2\sin\theta}{\mu'(\theta)}\sin^6\theta\,d\theta + \frac{1}{6}\sum_j F(\beta_j) = \frac{1}{6}\mathrm{tr}\,\{G(J) - G(J_0) - 120\log A\},$$

where $A = \mathrm{diag}\,\{a_k\}$, and

$$F(\beta) = \frac{1}{2^5}\left((\beta^6 - \beta^{-6}) - 9(\beta^4 - \beta^{-4}) + 45(\beta^2 - \beta^{-2}) - 120\log\beta\right),$$

$$(4)\quad G(J) = \frac{1}{2^5}\left(T_6(J) - 9T_4(J) + 45T_2(J)\right) = \frac{1}{2^5}(J^6 - 15J^4 + 90J^2 - 110).$$

Notice that $F(\beta) = C_0(\beta-1)^7 + O((\beta-1)^8)$, where $\beta \geq 1$ and C_0 is a positive constant.

Let

$$(5) \qquad \Psi(a,b) = \Psi(J(a,b)) = \mathrm{tr}\,\{G(J) - G(J_0) - 120\log A\}.$$

The following lemma is a cornerstone of the proof of the theorem.

Main Lemma. *Let $a = \{a_k\}$, $b = \{b_k\}$ and $a_k \neq 1, b_k \neq 0$, for finitely many indices only. Let also norms $||a-1||_\infty$, $||b||_\infty$ be small enough. Then*

$$(6) \qquad \Psi(a,b) \asymp (||a-1||_4^4 + ||b||_4^4 + ||\partial^2 a||_2^2 + ||\partial^2 b||_2^2),$$

In particular, $\Psi(a,b) \geq 0$ for these a and b.

The norms $||.||_p$ refer to the standard l^p-space norms. The sign "$\asymp$" means a two-sided estimate with positive constants depending on $||a-1||_\infty$, $||b||_\infty$. The lemma will be proved in the next section.

With the exception of this lemma, the proof of Theorem 1 follows a well-known scheme [4, 5]. We give only a sketch of its proof.

Proof of Theorem 1. Since the essential spectrum of $J = J(a,b)$ is the interval $[-2,2]$, we have (see [3]) that $a_k \to 1$ and $b_k \to 0$. Consequently, we assume without loss of generality that norms $||a-1||_\infty$ and $||b||_\infty$ are small enough to apply the Main Lemma.

First, we show that condition (1) implies (3), which is equivalent to showing (2). Denote

$$\Phi(J)=\Phi(\mu)=\Phi_1(\mu)+\Phi_2(\mu)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{2\sin\theta}{\mu'(\theta)}\sin^6\theta\,d\theta+\frac{1}{6}\sum_j F\left(\frac{1}{|\zeta_j|}\right),$$

Let $a_N=\{(a_N)_k\}$ and $a'_N=\{(a'_N)_k\}$, where

$$(a_N)_k=\begin{cases} a_k, & k\le N,\\ 1, & k\ge N+1,\end{cases}\qquad (a'_N)_k=\begin{cases} 1, & k\le N,\\ a_k, & k\ge N+1.\end{cases}$$

We also consider b_N, b'_N, constructed in the same way (of course, with 1's replaced by 0's). Let $J_N=J(a_N,b_N)$. Obviously, we have that $a'_N-1, b'_N\to 0$ and $\partial^2 a'_N, \partial^2 b'_N\to 0$ in corresponding norms. Since

$$\Psi(a,b)=\sum_k \psi(a_k,a_{k+1},a_{k+2};b_k,b_{k+1},b_{k+2})$$

for a function ψ (see Sect. 2), we have that

$$|\Psi(J)-\Psi(J_N)|\le\Psi(a'_N,b'_N)\le C_1(||a'_N-1||_4^4+||b'_N||_4^4+||\partial^2 a'_N||_2^2+||\partial^2 b'_N||_2^2)\to 0,$$

or $\Psi(J_N)\to\Psi(J)$. From the other hand, $(J_N-z(\zeta))^{-1}\to(J-z(\zeta))^{-1}$, for ζ inside the unit disk and, consequently, $\mu_N\to\mu$ weakly. Using [4], Corollary 5.3 and Theorem 6.2, we get

$$\Phi_1(\mu)\le\liminf_N\Phi_1(\mu_N)$$

and

$$\lim_{N\to\infty}\Phi_2(\mu_N)=\Phi_2(\mu).$$

Summing up, we obtain

$$\Phi(\mu)\le\limsup_N\Phi(\mu_N)=\limsup_N\Psi(J_N)=\lim_{N\to\infty}\Psi(J_N)=\Psi(J).$$

We now show the converse, i.e. condition (3) yields relations (1). Recall ([4], Proposition 8.4) that if $\mu=\mu(J)$ satisfies inequality $\mu\ge\delta m$, then

$$\Psi(J)\le\Phi(J),$$

where $\delta>0$ and m is the Lebesgue measure on the unit circle. We take a J with the property $\Psi(J)<\infty$, and we put $\mu_\delta=(1-\delta)\mu+\delta m$ for $\mu=\mu(J)$ and $\delta\in(0,1)$. The measure μ_δ defines a Jacobi matrix $J_\delta=J(a_\delta,b_\delta)$. Theorem 8.1 of [4] shows that $\Phi(J)=\lim_{\delta\to 0}\Phi(J_\delta)$. On the other hand (see [4], Proposition 8.4 and Theorem 8.1),

$$\Psi(J)\le\liminf_\delta\Psi(J_\delta).$$

Hence,

$$0\le\Psi(J)\le\liminf_\delta\Psi(J_\delta)=\liminf_\delta\Phi(J_\delta)=\lim_{\delta\to 0}\Phi(J_\delta)=\Phi(J).$$

The theorem is proved. □

2. Proof of the main lemma

We make few simplifications before going into the proof. First, we drop the term $\operatorname{tr} G(J_0)$ in (5), since it is finite. Second, we discard factors $1/2^5$ (see (4)). Having two sequences $a = \{a_k\}$ and $b = \{b_k\}$, we denote the sequence $\{a_k b_k\}$ by ab. We also write $a^{\pm}$ for the sequence $\{a_{k\pm 1}\}$.

The starting point of the proof is computation of $\operatorname{tr} J^n$. According to formulas from [7], Section 6.1, we have $\operatorname{tr} J^j = \sum_k (g_j)_k$, where sequences g_j are constructed as

$$
\begin{aligned}
g_{j+1} &= h_j + h_j^- + b g_j, \\
h_{j+1} &= a^2 \sum_{l=0}^{j} g_{j-l} g_l^- - \sum_{l=0}^{j} h_{j-l} h_l,
\end{aligned} \tag{7}
$$

with $g_0 = 1, h_0 = 0$.

We give the proof of the lemma in cases when $J = J(a,b)$ with $b = 0$, and $J = J(a,b)$ with $a = 1$. Since the proof in the general case follows along the same lines and is long and tedious, we omit it.

Let us consider the case when $b = 0$.

Lemma 1. *Let $a = \{a_k\}$ and $a_k \neq 1$ for only finitely many indices. Then*

$$
\begin{aligned}
\Psi(a) = \Psi(J(a,0)) &= \textstyle\sum_k \{-9\lambda_k + 3\lambda_k^2 \lambda_{k+1} + 3\lambda\lambda_{k+1}^2 + 3\lambda_{k-1}\lambda_k\lambda_{k+1} \\
&+ 9\lambda_k^2 - 12\lambda\lambda_{k+1} + 3\lambda_{k-1}\lambda_{k+1} + \tfrac{15}{2}\lambda_k^4 + O(|\lambda_k|^5)\},
\end{aligned}
$$

where $\lambda = a^2 - 1$.

Proof. A straightforward computation using (7) yields $g_1 = g_3 = g_5 = 0$ and

$$
\begin{aligned}
g_2 &= a^2 + a^{-2}, \\
g_4 &= a^{-2}(a^{--2} + a^{-2} + a^2) + a^2(a^{-2} + a^2 + a^{+2}), \\
g_6 &= a^2(a^{-2}(a^{--2} + \cdots + a^{+2}) + a^2(a^{-2} + \cdots + a^{+2}) + a^{+2}(a^2 + \cdots + a^{++2})) \\
&+ a^{-2}(a^{--2}(a^{---2} + \cdots + a^2) + a^{-2}(a^{--2} + \cdots + a^2) + a^2(a^{-2} + \cdots + a^{+2})).
\end{aligned}
$$

We also have $\operatorname{tr}\log A = 1/2 \sum_k \log(1 + \lambda_k)$. Rewriting in terms of $\lambda = a^2 - 1$ and recalling that $\operatorname{tr} J^j = \sum_k (g_j)_k$, we get the conclusion of the lemma. □

In particular, we see that

$$
\Psi(a) = \sum_k \left\{ \psi_1(\lambda_k, \lambda_{k+1}, \lambda_{k+2}) + O(|\lambda_k|^5) \right\},
$$

where

$$\begin{aligned}\psi_1(x,y,z) &= \frac{1}{2}(x^3-20y^3+z^3+6xy^2+6y^2z+6xyz)\\ &+ \frac{3}{2}(x^2+4y^2+z^2-4xy-4yz+2xz)+\frac{5}{2}(x^4+4y^4+z^4).\end{aligned}$$

Note also that $\lambda_k = 2(a_k-1)+O((a_k-1)^2)$.

Lemma 2. *We have*

$$\psi_1(x,y,z) \asymp (x^4+y^4+z^4+(x+z-2y)^2).$$

Proof. We notice that

$$\begin{aligned}\psi_1(x,y,z) &= \frac{1}{2}((x+z-2y)(x^2+10y^2+z^2+2xy+2yz-zx)\\ &+ 3(x+z-2y)^2)+5(x^4+4y^4+z^4)),\end{aligned} \tag{8}$$

and the bound from above follows from the inequality $ab \le 1/2(a^2+b^2)$ for $a,b\ge 0$. To get the bound from below, we estimate the first term in (8) as

$$|(x+z-2y)(x^2+10y^2+z^2+2xy+2yz-zx)| \le \frac{1}{2}(x+z-2y)^2+\frac{3}{2}(x^4+4y^4+z^4),$$

and, consequently,

$$\psi_1(x,y,z) \ge \frac{1}{2}\left(\frac{5}{2}(x+z-2y)^2+\frac{7}{2}(x^4+4y^4+z^4)\right).$$

The lemma is proved. $\square$

Now, we turn to the second case $J=J(a,b)$ with $a=1$.

Lemma 3. *Let $b=\{b_k\}$ and $b_k\ne 0$ for only finitely many indices. Then*

$$\Psi(b)=\Psi(1,b)=\sum_k \left\{b^6+(-3b^4+6bb^{+3}+6b^+b^3+6b^2b^{+2})+(18b^2-24bb^++6bb^{++})\right\}.$$

Proof. The argument is exactly as in Lemma 1. We only quote expressions for g_5 and g_6

$$\begin{aligned}g_5 &= b^5+(b^{-3}+8b^3+b^{+3}+2b^{-2}b+3b^-b^2+3b^2b^++2bb^{+2})\\ &+ (b^{--}+6b^-+16b+6b^++b^{++}),\end{aligned}$$

$$\begin{aligned}g_6 &= b^6+(b^{-4}+10b^4+b^{+4}\\ &+ 4b^-b^3+3b^{-2}b^2+2bb^{-3}+2bb^{+3}+3b^2b^{+2}+4b^3b^+)\\ &+ (b^{--2}+8b^{-2}+30b^2+8b^{+2}+b^{++2}+2b^-b^{--}+2bb^{--}\\ &+ 16bb^-+16bb^++2bb^{++}+2b^+b^{++}+2b^+b^-)+20.\end{aligned}$$

$\square$

Hence, we obtain

$$\Psi(b) = \sum_k \{\psi_2(b_k, b_{k+1}, b_{k+2}) + O(b_k^6)\},$$

where

$$\psi_2(x,y,z) = -3y^4 + (6xy^3 + 6y^3 z + 3y^2x^2 + 3y^2z^2) + 3(x+z-2y)^2.$$

Lemma 4. *We have*

$$\psi_2(x,y,z) \asymp (y^4 + (x+z-2y)^2).$$

Proof. One more time, the bound from above follows at once from inequality $ab \le 1/p\, a^p + 1/q\, b^q$, where $a, b > 0$, and $1/p + 1/q = 1$.

Furthermore, we see that

$$\psi_2(x,y,z) = -3y^4 + y^2\left(\frac{3}{2}(x-z)^2 + 3(x+y+z)^2 - \frac{1}{2}(x+z-2y)^2\right) + 3(x+z-2y)^2.$$

Estimating moduli of negative terms like in Lemma 2 and taking into account

$$-3y^4 + 3y^2(x+y+z)^2 - \frac{1}{2}y^2(x+z-2y)^2 + 3(x+z-2y)^2 \ge \frac{1}{2}(y^4 + (x+z-2y)^2),$$

finishes the proof. □

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