# MULTIPLICITY OF RESONANCES EQUALS MULTIPLICITY OF THE SCATTERING MATRIX

L. NEDELEC*

Abstract. We present a easy proof of a result of Petkov and Zworski using dilation. We proove that the multiplicity of a resonance is equal to the multiplicity of the scattering matrix.

## 1. Notation

Let $r_0 > 0$. We denote the closed ball in $\mathbb{R}^n$ by $B(x, r_0) = \{y \in \mathbb{R}^n, |x-y| \leq r_0\}$. Let $\mathcal{H}$ be a complex Hilbert space with an orthogonal decomposition :

$$\mathcal{H} = \mathcal{H}_{r_0} \oplus L^2(\mathbb{R}^n \setminus B(0, r_0)).$$

We assume that $P_0 : \mathcal{D} \to \mathcal{H}$ is a self adjoint operator with domain $\mathcal{D} \subset \mathcal{H}$ satisfying :

$$1_{\mathbb{R}^n \setminus B(0,r_0)} \mathcal{D} = H^2(\mathbb{R}^n \setminus B(0, r_0)),$$

$$1_{\mathbb{R}^n \setminus B(0,r_0)} P_0(h) = -h^2 \Delta|_{\mathbb{R}^n \setminus B(0,r_0)}.$$

We assume that there exists $m_0 \in \mathbb{N}^\star$ such that $1_{B(0,r_0)}(P_0 + i)^{-m_0}$ is trace class, and that there exists $C \geq 0$ such that $P_0 \geq -C$. Let $r > r_0$. Let $\varepsilon > 0$, $a > 0$. Then there exists $b > r + a$ and a function $f_\theta$ on $\mathbb{R}^+$ such that

$$f_\theta(t) = t, \quad t \leq r + a,$$
$$f_\theta(t) = e^{i\theta} t, \quad t \geq b.$$

$$\forall t \in \mathbb{R}^+, \, 0 \leq \mathrm{Arg}(f_\theta(t)) \leq \theta,$$
$$\partial_t f_\theta(t) \neq 0,$$
$$\mathrm{Arg}(f_\theta(t)) \leq \mathrm{Arg}(\partial_t f_\theta(t)) \leq \mathrm{Arg}(f_\theta(t)) + \varepsilon.$$

Let $\Gamma_\theta = v(\mathbb{R}^n) \in \mathbb{C}^n$ where $v(x) = f_\theta(|x|)\frac{x}{|x|}$.
Let $\chi \in \mathcal{C}_0^\infty(B(0, r))$, $\tilde{\chi} \in \mathcal{C}_0^\infty(B(0, r))$ be such that

$$\forall x \in B(0, r_0), \, \chi(x) = \tilde{\chi}(x) = 1, \, \tilde{\chi}\chi = \chi.$$

*Supported in part by ACI : Méthodes mathématiques en physique de la matiére condensée.

Denote by $P_1 = -h^2\Delta$. We denote by $P_{0,\theta}$ (res. $P_{1,\theta}$) the operator $P_0$ (res. $P_1$) on the space $\Gamma_\theta$ (see [6] for the definition). A bullet will denote either 0 or 1, so $P_\bullet$ is either $P_0$ or $P_1$. It is known that the resolvents $R_\bullet(z) = (P_\bullet - z)^{-1} : \mathcal{H} \to \mathcal{D}$ has a meromorphic extension from the upper half plane $\{z \in \mathbb{C}, \mathrm{Im}(z) > 0\}$ through $]0, \infty[$ to the double cover of $\mathbb{C}$ when $n$ is odd, and to the logarithmic plane $\Lambda$ when $n$ is even. This continuation is an operator from $\mathcal{H}_{comp}$ to $\mathcal{D}_{loc}$ [4] and the poles are of finite rank. The poles of $R_0$ are called resonances of $P_0$. Denote by $R_{\bullet,\theta} = (P_{\bullet,\theta} - z)^{-1}$. We will denote the set of resonances of $P_0$ by $\mathrm{Res}(P_0)$ and will always include them according to their multiplicity $m_{R_0}(z)$, which for $z \neq 0$ are defined as

$$m_{R_0}(z) = \mathrm{Rank}\Big(\int_{\gamma_\varepsilon(z)} R_0(u)\,du\Big) = \mathrm{Tr}\left(-\frac{1}{2i\pi}\int_{\gamma_\varepsilon(z)} R_0(u)\,du\right),$$

where $\gamma_\varepsilon(z) = \{z + \varepsilon e^{it}, 0 \leq t \leq 2\pi\}$ for $\varepsilon$ sufficiently small [6]. It is known that for $\theta > 0$,

$$m_{R_0}(z) = \mathrm{Rank}\Big(\int_{\gamma_\varepsilon(z)} R_{0,\theta}(u)\,du\Big) := m_{R_{0,\theta}(z)}.$$

It is known that the resonances of $P_0$ that belong to the real axe are eigenvalues of $P_0$.
The scattering matrix is defined just as in the usual potential scattering [4], [3]. We denote by $S(\lambda)$ the standard (relative) scattering matrix. The standard scattering matrix $S(\lambda)$ continues meromorphically in $\lambda$ from $\{\lambda, \mathrm{Im}(\lambda) > 0\}$ through $]0, \infty[$ to the double cover of $\mathbb{C}$ when $n$ is odd, and to the logarithmic plane $\Lambda$ when $n$ is even. The poles of its continuation correspond to resonances of $P_0$. They are of finite rank. We define the multiplicity of a pole of $\det S(z)$ as

$$m_S(z) = -\frac{1}{2i\pi}\mathrm{Tr}\int_{\gamma_\varepsilon(z)} \Big(S(u)^{-1}\frac{d}{du}S(u)\Big)\,du.$$

## 2. RESULT

This paper is devotes to the proove of the following theorem that we could find in [4] proposition 2-2

**Theorem 2.1.** *For $z$ such that* $\mathrm{Re}(z) > 0$,

$$m_S(z) = m_{R_0}(z) - m_{R_0}(\overline{z}). \tag{2.1}$$

The proof is just based on basic computation on the dilate operator, unless the author use the generic symplicity of resonances and continuity propreties of the multiplicities.

## 3. Proof of 2.1

Assume $\mathrm{Re}(\lambda_0) > 0$ and $\lambda_0 \notin \mathrm{Res}(P_0)$. Let $f \in \mathcal{C}_0^\infty(\mathbb{R})$. We write $g(\lambda) = (\lambda - \lambda_0)^m f(\lambda)$. Let $\tilde{g}$ be an almost analytic extension of $g$ of compact support [2]. Using the Birman Krein formula [1] we get

$$\begin{aligned}
&\mathrm{Tr}[f(P_0) - (1-\chi)f(P_1)(1-\tilde{\chi})] - \mathrm{Tr}[\chi f(P_1)] \\
=&\frac{1}{2i\pi}\int_{\mathbb{R}} \Big(\frac{df}{d\lambda}\log(\det S(\lambda))\Big)\,d\lambda + \sum_{\lambda\in\mathrm{Res}(P_0)\cap\mathbb{R}} g(\lambda)(\lambda-\lambda_0)^{-m_0} \\
=&-\frac{1}{2i\pi}\int g(\lambda)(\lambda-\lambda_0)^{-m_0}\mathrm{Tr}(S'S^{-1}(\lambda))\,d\lambda \\
&+\sum_{\lambda\in\mathrm{Res}(P_0)\cap\mathbb{R}} g(\lambda)(\lambda-\lambda_0)^{-m_0}.
\end{aligned} \tag{3.1}$$

The expression of the functionnal calculus is [2]

$$f(P_\bullet) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|\mathrm{Im}(z)|>\varepsilon} \overline{\partial}\tilde{g}(z)(P_\bullet - \lambda_0)^{-m_0}R_\bullet(z)\,L(dz).$$

Denote by

$$G(z) = (P_0-\lambda_0)^{-m_0}R_0(z) - (1-\chi)(P_1-\lambda_0)^{-m_0}R_1(z)(1-\tilde{\chi}),$$
$$G_\theta(z) = (P_{0,\theta}-\lambda_0)^{-m_0}R_{0,\theta}(z) - (1-\chi)(P_{1,\theta}-\lambda_0)^{-m_0}R_{1,\theta}(z)(1-\tilde{\chi}),$$

$\langle z\rangle = (1+z^2)^{-\frac{1}{2}}$, and $\|A\|_{\mathrm{Tr}}$ the norm trace of $A$. Denote by

$$f_\varepsilon = \frac{1}{\pi}\int_{|\mathrm{Im}(z)|>\varepsilon} \overline{\partial}\tilde{g}(z)G(z)\,L(dz).$$

The estimates

$$\|\overline{\partial}\tilde{g}(z)\| \le C|\mathrm{Im}(z)|^3,$$

and [1]

$$\|G(z)\|_{\mathrm{Tr}} \le \frac{C\langle z\rangle}{|\mathrm{Im}(z)|^2},$$

give

$$\|f_\varepsilon - f_{\varepsilon'}\|_{\mathrm{Tr}} \le C\sup(\varepsilon,\varepsilon').$$

Using the completeness of the trace class we get

$$\begin{aligned}
&\mathrm{Tr}[f(P_0) - (1-\chi)f(P_1)(1-\tilde{\chi})] \\
=&\frac{1}{\pi}\lim_{\varepsilon\to 0}\mathrm{Tr}\Big[\int_{|\mathrm{Im}(z)|>\varepsilon}\overline{\partial}\tilde{g}(z)G(z)L(dz)\Big].
\end{aligned}$$

So

$$\begin{aligned}&\mathrm{Tr}[f(P_0)-(1-\chi)f(P_1)(1-\tilde{\chi})]\\&=-\frac{1}{2i\pi}\lim_{\varepsilon\to 0}\mathrm{Tr}\Big[\int_{\mathbb{R}}\tilde{g}(\lambda+i\varepsilon)G(\lambda+i\varepsilon)-\tilde{g}(\lambda-i\varepsilon)G(\lambda-i\varepsilon)\,d\lambda\Big].\end{aligned}$$

Lebesgue's theorem of dominated convergence gives

$$\begin{aligned}&\mathrm{Tr}[f(P_0)-(1-\chi)f(P_1)(1-\tilde{\chi})]\\&=-\lim_{\varepsilon\to 0}\frac{1}{2i\pi}\int_{\mathbb{R}}\mathrm{Tr}\Big[\tilde{g}(\lambda+i\varepsilon)G(\lambda+i\varepsilon)-\tilde{g}(\lambda-i\varepsilon)G(\lambda-i\varepsilon)\Big]\,d\lambda.\end{aligned}$$

For $\mathrm{Im}(z)>0$, the function $\theta\to\mathrm{Tr}\big(G_\theta(z)\big)$ is independant of $\theta$ for $\theta\geq 0$ [5]. This implies for $\theta\geq 0$

$$\begin{aligned}\mathrm{Tr}[f(P_0)-(1-\chi)f(P_1)(1-\tilde{\chi})]=&-\frac{1}{2i\pi}\lim_{\varepsilon\to 0}\int_{\mathrm{Im}(z)=\varepsilon}\mathrm{Tr}\big(\tilde{g}(z)G_\theta(z)\big)\,dz\\&+\frac{1}{2i\pi}\lim_{\varepsilon\to 0}\int_{\mathrm{Im}(z)=-\varepsilon}\mathrm{Tr}\big(\tilde{g}(z)G_{-\theta}(z)\big)\,dz.\end{aligned}$$

The function $z\to\mathrm{Tr}\big(\tilde{g}(z)G_\theta(z)\big)$ is analytic in the domain where $R_{0,\theta}(z)$ is defined, and is locally independant of $\theta$. Denote by $D(x,\varepsilon)=\{z\in\mathbb{C},|z-x|=\varepsilon\}$, $d_x^+(\varepsilon)=D(x,\varepsilon)\cap\{z,\mathrm{Im}(z)\geq 0\}$ and $d_x^-(\varepsilon)=D(x,\varepsilon)\cap\{z,\mathrm{Im}(z)\leq 0\}$. We obtain for $\varepsilon$ sufficiently small

(3.2)

$$\begin{aligned}&-\sum_{\{1,-1\}}(\pm 1)\frac{1}{2i\pi}\int_{\mathrm{Im}(z)=\pm\varepsilon}\mathrm{Tr}\big(\tilde{g}(z)G_{\pm\theta}(z)\big)\,dz\\=&-\sum_{\{1,-1\}}(\pm 1)\frac{1}{2i\pi}\int_{\mathbb{R}\setminus\cup_{z_j\in\mathbb{R}\cap\mathrm{Res}(P_0)}\{z\in\mathbb{R},|z-z_j|\leq\varepsilon\}}\tilde{g}(z)\mathrm{Tr}(G_{\pm\theta}(z))\,dz\\&+\sum_{\{1,-1\}}(\pm 1)\frac{1}{\pi}\int_{\{z,0\leq\pm\mathrm{Im}(z)\leq\varepsilon\}\setminus\cup_{z_j\in\mathbb{R}\cap\mathrm{Res}(P_0)}B(z_j,\varepsilon)}\overline{\partial}\tilde{g}(z)\mathrm{Tr}(G_{\pm\theta}(z))\,L(dz)\\&+\sum_{\{1,-1\}}\sum_{z_j\in\mathrm{Res}(P_0)}\frac{1}{2i\pi}\int_{d^{\pm}_{z_j}(\varepsilon)}\tilde{g}(z)\mathrm{Tr}(G_{\pm\theta}(z))\,dz.\end{aligned}$$

We get

$$\begin{aligned}&\left\|\sum_{\{1,-1\}}(\pm 1)\frac{1}{\pi}\int_{0\leq\pm\mathrm{Im}(z)\leq\varepsilon\setminus\cup_{z_j\in\mathbb{R}\cap\mathrm{Res}(P_0)}B(z_j,\varepsilon)}\overline{\partial}\tilde{g}(z)\mathrm{Tr}(G_{\pm\theta}(z))\,L(dz)\right\|\\&\leq 2\frac{1}{\pi}\int_{0\leq|\mathrm{Im}(z)|\leq\varepsilon\setminus\cup_{z_j\in\mathbb{R}\cap\mathrm{Res}(P_0)}\{z,|z-z_j|\leq\varepsilon\}}\big\|\overline{\partial}\tilde{g}(z)\big\|\;\|\mathrm{Tr}(G(z))\|\;L(dz)\\&\leq C|\varepsilon|.\end{aligned}$$

For $z_j \in \mathbb{R} \cap \mathrm{Res}(P_0)$, the function $z \to (z - z_j)G_\theta(z)$ is analytic. With a control on the trace norme we obtain that the function $z \to (z - z_j)\mathrm{Tr}\big(G_\theta(z)\big)$ is analytic. Therefore

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\Big(\sum_{z_j\in\mathrm{Res}(P_0)}\sum_{\{1,-1\}}\frac{1}{2i\pi}\int_{d^{\pm}_{z_j}(\varepsilon)}\tilde{g}(z)\mathrm{Tr}(G_{\pm\theta}(z))\,dz\Big)\\
&=\lim_{\varepsilon\to 0}\Big(\sum_{z_j\in\mathrm{Res}(P_0)}\sum_{\{1,-1\}}\frac{1}{2i\pi}g(z_j)\int_{d^{\pm}_{z_j}(\varepsilon)}\mathrm{Tr}(G_{\pm\theta}(z))\,dz\Big)\\
&=:\sum_{z_j\in\mathrm{Res}(P_0)}g(z_j)H_j(z_j-\lambda_0)^{-m_0}.
\end{aligned}$$

The limit $\varepsilon \to 0$ gives

$$\begin{aligned}
&\mathrm{Tr}\big(f(P_0)-(1-\chi)f(P_1)(1-\tilde{\chi})\big)-\mathrm{Tr}[\chi f(P_1)]\\
&=-\sum_{\{1,-1\}}(\pm 1)\frac{1}{2i\pi}\int_{\mathbb{R}}g(z)\mathrm{Tr}(G_{\pm\theta}(z))\,dz\\
&\quad+\sum_{z_j\in\mathrm{Res}(P_0)\cap\mathbb{R}}g(z_j)H_j(z_j-\lambda_0)^{-m_0}\\
&\quad+\sum_{\{1,-1\}}(\pm 1)\frac{1}{2i\pi}\int_{\mathbb{R}}g(z)\mathrm{Tr}[\chi(P_{1,\pm\theta}-\lambda_0)^{-m_0}R_{1,\pm\theta}(z)]\,dz.
\end{aligned}$$

This, and (3.1) implies the following equality of measures

$$\begin{aligned}
&-\frac{1}{2i\pi}(\lambda-\lambda_0)^{-m_0}\mathrm{Tr}\big(S'S^{-1}(\lambda)\big)\,d\lambda+\sum_{\lambda\in\mathrm{Res}(P_0)\cap\mathbb{R}}\delta_\lambda(\lambda-\lambda_0)^{-m_0}\\
&=-\sum_{\{1,-1\}}(\pm 1)\frac{1}{2i\pi}\mathrm{Tr}(G_{\pm\theta}(\lambda))\,d\lambda\\
&\quad+\sum_{\{1,-1\}}(\pm 1)\frac{1}{2i\pi}\mathrm{Tr}\big(\chi(P_{1,\pm\theta}-\lambda_0)^{-m_0}R_{1,\pm\theta}(\lambda)\big)\,d\lambda\\
&\quad+\sum_{\lambda\in\mathrm{Res}(P_0)\cap\mathbb{R}}\delta_\lambda H_\lambda(\lambda-\lambda_0)^{-m_0}.
\end{aligned}$$

Taking the continuous part of the measures we get

$$\begin{aligned}
(\lambda-\lambda_0)^{-m_0}\mathrm{Tr}(S'S^{-1}(\lambda))\,d\lambda&=\mathrm{Tr}(G_\theta(\lambda))d\lambda-\mathrm{Tr}(G_{-\theta}(\lambda))\,d\lambda\\
&\quad+\sum_{\{1,-1\}}(\pm 1)\frac{1}{2i\pi}\mathrm{Tr}[\chi(P_{1,\pm\theta}-\lambda_0)^{-m_0}R_{1,\pm\theta}(\lambda)]\,d\lambda.
\end{aligned}\tag{3.3}$$

Due to the analyticity of the functions we get, for all $z \in \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$,

$$\mathrm{Tr}(S'S^{-1}(z)) = (z-\lambda_0)^{m_0}\mathrm{Tr}(G_\theta(z)) - (z-\lambda_0)^{m_0}\mathrm{Tr}(G_{-\theta}(z))$$
$$+\frac{1}{2i\pi}\mathrm{Tr}[\chi(P_{1,\theta}-\lambda_0)^{-m_0}R_{1,\theta}(z)] - \frac{1}{2i\pi}\mathrm{Tr}\big(\chi(P_{1,-\theta}-\lambda_0)^{-m_0}R_{1,-\theta}(z)\big).$$

A computation of $m_{R_1}(z)$ gives

$$\begin{aligned}
0 = m_{R_1}(z) &= -\frac{1}{2i\pi}\mathrm{Tr}\int_{\gamma_\varepsilon(z)} R_{1,\theta}(u)\,du \\
&= -\frac{1}{2i\pi}\mathrm{Tr}\Big((P_{1,\theta}-\lambda_0)^{-m_0}\int_{\gamma_\varepsilon(z)}(P_{1,\theta}-\lambda_0)^{m_0}R_{1,\theta}(u)\,du\Big) \\
&= -\frac{1}{2i\pi}\mathrm{Tr}\Big((P_{1,\theta}-\lambda_0)^{-m_0}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}R_{1,\theta}(u)\,du\Big) \\
&= -\frac{1}{2i\pi}\mathrm{Tr}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}(P_{1,\theta}-\lambda_0)^{-m_0}R_{1,\theta}(u)\,du.
\end{aligned}$$

Similarly, for $m_{R_{0,\theta}}(z)$ we get

(3.4)
$$\begin{aligned}
m_{R_{0,\theta}}(z) =& -\frac{1}{2i\pi}\mathrm{Tr}\int_{\gamma_\varepsilon(z)} R_{0,\theta}(u)\,du \\
=& -\frac{1}{2i\pi}\mathrm{Tr}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}(P_{0,\theta}-\lambda_0)^{-m_0}R_{0,\theta}(u)\,du \\
=& -\frac{1}{2i\pi}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}\mathrm{Tr}\big(G_\theta(u)\big)\,du \\
&+\frac{1}{2i\pi}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}\mathrm{Tr}\big(\chi(P_{1,\theta}-\lambda_0)^{-m_0}R_{1,\theta}(u)\big)\,du.
\end{aligned}$$

By integrating (3.3), using (3.4) we obtain

(3.5)
$$\begin{aligned}
m_S(z) =& -\frac{1}{2i\pi}\int_{\gamma_\varepsilon(z)}\mathrm{Tr}S'S(u)^{-1}\,du \\
=& -\sum_{\{1,-1\}}(\pm 1)\frac{1}{2i\pi}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}\mathrm{Tr}(G_{\pm\theta})(u)\,du \\
&+\sum_{\{1,-1\}}(\pm 1)\frac{1}{2i\pi}\int_{\gamma_\varepsilon(z)}(u-\lambda_0)^{m_0}\mathrm{Tr}\big(\chi(P_{1,\pm\theta}-\lambda_0)^{-m_0}R_{1,\pm\theta}(u)\big)\,du \\
=& m_{R_{0,\theta}}(z) - m_{R_{0,-\theta}}(z).
\end{aligned}$$

Denote by $\mathcal{C}$ the operator $u \to \overline{u}$. For any operator $A$ of trace class we obtain $\mathrm{Tr}(\mathcal{C}A\mathcal{C}) = \overline{\mathrm{Tr}A}$. We get $\mathcal{C}P_0\mathcal{C} = P_0$, so $\mathcal{C}(R_{0,-\theta}(z))\mathcal{C} = R_{0,\theta}(\overline{z})$, and therefore

$$\begin{aligned} m_{R_{0,-\theta}}(z) &= \frac{1}{2i\pi}\mathrm{Tr}[\int_{\gamma_\varepsilon(z)} R_{0,-\theta}(u)\,du] \\ &= \overline{\Big(-\frac{1}{2i\pi}\mathrm{Tr}\big(\mathcal{C}(\int_{\gamma_\varepsilon(z)} R_{0,-\theta}(u)\,du)\mathcal{C}\big)\Big)} \\ &= \overline{\Big(\frac{1}{2i\pi}\mathrm{Tr}\big(\int_{\gamma_\varepsilon(\overline{z})} R_{0,\theta}(u)\,du\big)\Big)} \\ &= \overline{m_{R_{0,\theta}}(\overline{z})} = m_{R_{0,\theta}}(\overline{z}). \end{aligned}$$

The equation (3.5) implies

$$m_S(z) = m_{R_{0,\theta}}(z) - m_{R_{0,\theta}}(\overline{z}).$$

This proves Theorem 2.1. I like to thank M. Zworski for suggested me this problem.

## References


[1] T. Christiansen. Spectral asymptotics for compactly supported perturbations of the laplacian on $\mathbb{R}^n$. *Comm. Partial Differential Equations*, 23 n 5-6:933–948, 1998.

[2] B. Helffer and J. Sjöstrand. Analyse semi-classique pour l'équation de Harper. *Memoire de la S.M.F.*, 34, 1988.

[3] R. Melrose. *Geometric Scattering Theory, Stanford Lectures.* Cambridge University Press, 1995.

[4] V. Petkov and M. Zworski. Semi-classical estimates on the scattering determinant. *Ann. Henri Poincare*, 2:675–711, 2001.

[5] J. Sjöstrand. Resonances for bottles and trace formulae. *Math. Nachrichten*, 221:95–149, 2001.

[6] J. Sjöstrand and M. Zworski. Complex scaling and the distribution of scattering poles. *J. Amer. Math. Soc.*, 4:4–729–769, 1991.



L.A.G.A., Institut Galilée, Université de Paris Nord, av. J.B. Clement, F-93430 Villetaneuse, France, IUFM de l'academie de Rouen, France

*E-mail address*: `nedelec@math.univ-paris13.fr`