Macroscopic evolution of particle systems with random field Kac interactions *

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Abstract. We consider a lattice gas interacting via a Kac interaction $J_\gamma(|x-y|)$ of range γ^{-1}, $\gamma > 0$, $x, y \in \mathbb{Z}^d$ and under the influence of an external random field given by independent bounded random variables with a translation invariant distribution. We study the evolution of the system through a conservative dynamics, i.e. particles jump to nearest neighbor empty sites, with rates satisfying a detailed balance condition with respect to the equilibrium measure. We prove that rescaling space as γ^{-1} and time as γ^{-2}, in the limit $\gamma \to 0$, for dimension $d \geq 3$, the macroscopic density profile ρ satisfies, a.s. with respect to the random field, a nonlinear integral differential equation, with a diffusion matrix determined by the statistical properties of the external random field. The result holds for all values of the density, also in the presence of phase segregation, and the equation is in the form of the flux gradient for the energy functional.

1 Introduction

We consider a d-dimensional particle system interacting via a two-body Kac potential and an external random field given by independent bounded random variables with a translation invariant distribution, and look for its macroscopic behavior. Problems where a stochastic contribution is added to the energy of the system naturally arise in condensed matter physics, where the presence of impurities causes the microscopic structure to vary from point to point. Some of the vast literature on these topics may be seen by consulting the references [BMT], [OS], [FGK]. Equilibrium statistical properties of spin systems with random magnetic field have been intensively investigated, e.g. [AW], [BK], [FFS] and [I]. A review of some developments in the theory of the Ising model in random field is given in [N].

Kac potentials J_γ are two-body interactions with range γ^{-1} and strength γ^d, where γ is a dimensionless scaling parameter. When $\gamma \to 0$, i.e. very long range compared with the interparticle distance, the strength of the interaction becomes very weak, but in such a way that the total interaction between one particle and all the others is finite. Kac potentials were introduced in [KUH], and then generalized in [LP], to present a rigorous derivation of the van der Waals theory of a gas-liquid phase transition. Since then several papers appeared on the subject, and recent ones studied spin systems with Kac potential for γ small but finite. We mention only [COP], [Bo], [CP], [LMP]; [COP1], [COPV] are devoted to random field Kac models, in dimension $d = 1$. Time evolution of the macroscopic profile in particle systems interacting via long range Kac potentials were investigated either for conservative dynamics [LOP], [GL], [GLM], [MM] or for non conservative ones [DOPT]. For reviews concerning these models, see [Be], [GLP], [P].

Given β and θ positive parameters we consider the formal Hamiltonian

$$H_\gamma^{\beta,\alpha}(\eta) = -\frac{\beta}{2} \sum_{x,y \in \mathbb{Z}^d} J_\gamma(x-y)\eta(x)\eta(y) - \theta \sum_{x \in \mathbb{Z}^d} \alpha(x)\eta(x), \tag{1.1}$$

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where $\eta = (\eta(x), x \in \mathbb{Z}^d)$, $\eta(x) \in \{0,1\}$; $\eta(x) = 1$ if there is a particle at site x and $\eta(x) = 0$ if site x is empty. The external random field $\alpha = (\alpha(x), x \in \mathbb{Z}^d)$ is a collection of independent bounded random variables with translation invariant distribution.

Given the Hamiltonian (1.1), one can construct in a standard way, see [Li], [Sp], an evolution conserving the total number of particles, the so-called Kawasaki dynamics, which can be described as follows. Particles attempt to jump to nearest neighbor sites at rates depending on the energy difference before and after the exchange, provided the nearest neighbor target sites are empty; attempted jumps to occupied sites are suppressed. The rates are chosen in such a way that the system satisfies a detailed balance condition with respect to a family of Gibbs measures, parametrized by the so-called chemical potential $\lambda \in \mathbb{R}$, for some fixed temperature T, with β and θ fixed, and for γ finite. We are interested in the influence of the random field and the Kac type interaction on the transport properties of such a system, in particular on the rate of bulk diffusion. The relevant features of the system are the absence of translation invariance, for a given disorder configuration, and the non validity of the gradient condition. We will come back to this point later.

Transport properties for Kawasaki dynamics with $\beta = 0$ and $\theta = 1$ in (1.1) have been studied by A. Faggionato in her thesis [F] and by A. Faggionato and F. Martinelli in [FM]. They proved that under diffusive scaling and in dimension $d \geq 3$, the system has a hydrodynamic limit and gave a variational formula for the bulk diffusion. The diffusion matrix turns out to be a nonlinear function of the density, continuous in the open interval $(0,1)$; it is determined by the temperature T, always assumed constant and therefore omitted in the notation, and by the statistical properties of the external random field. Given the presence of both the random field and the exclusion rule, to obtain an hydrodynamic limit was not at all obvious. In the absence of the exclusion rule the rate of bulk diffusion is the result of diffusive scaling of a single particle moving with reversible rates in the random field. If the exclusion rules are present but the field is constant the bulk diffusion turns out to be independent of the density of particles. However when the random field and the exclusion rules are present, one obtains a nonlinear dependence of the bulk diffusion on the density. The case of a periodic field in one dimension had been solved exactly [W], in this case the bulk diffusion is constant, it does not depend on the density.

Same model as in [F] and [FM] was considered in [Q] in which results about a hydrodynamic behavior of the model in all dimensions were announced but not proved.

The equilibrium statistical picture of systems with Hamiltonian (1.1) with both β and θ different from zero and J_γ positive presents an interplay between the ferromagnetic properties of the interaction and the randomness of the external field, [AW], [BK]. To understand whether some of these aspects could be detected in studying the particles evolution, in this paper we look for the macroscopic behavior of such a system. It turns out that in the diffusive scaling regime we are analyzing the system ($\gamma \to 0$ with space and time scaling γ^{-1} and γ^{-2}), the equilibrium statistical properties of the full model described by the Hamiltonian (1.1) are not relevant. There, the only relevant properties are those of the local part of the Hamiltonian, i.e the one corresponding to $\beta = 0$ in (1.1). Namely the non linear partial integral differential equation we obtain for the density profile of the particle system, see (2.24), depends only through the diffusion matrix on the equilibrium statistical properties of the local part of the Hamiltonian (1.1). This has been already observed in some other special cases, see [BL], [GL], [GLM], [MM]. A key ingredient in our analysis is that the dynamics with $\beta \neq 0$ is in fact a weak perturbation of the one with $\beta \equiv 0$. This can be seen both at the levels of rates and current, see (3.8) and (4.6).

We note that if we consider a hydrodynamic space scale ϵ^{-1} much larger than γ^{-1}, say $\epsilon = \gamma^\delta$, $\delta > 1$, then the diffusion matrix will depend on the equilibrium statistical properties of the full Hamiltonian (1.1). Results in this direction, for a particular system, have been shown in [R] for all choices of initial densities and for systems under only Kac type interaction, i.e corresponding to $\theta = 0$ in (1.1), in [LOP] and [G] for initial densities restricted to a special region of phase diagram, the so called *spinodal region.*

To be more precise, we study the Kawasaki dynamics in a torus of diameter γ^{-1}. We rescale space as γ^{-1} and time as γ^{-2}. In the limit $\gamma \to 0$ we show that in dimension $d \geq 3$, the empirical densities converge, almost surely with respect to the random field, to the solution of a partial integral differential equation, see Section 2. The diffusion coefficient depends on the statistical properties but not on the realizations of the random field. To establish the hydrodynamic limit we need to prove some version of Fick's law, namely to replace the microscopic current (i.e the difference between the rate at which a particle jumps from site x to site y and the rate at which a particle jumps from y to x, x and y being nearest neighbors), by the gradient of the density field multiplied by the diffusion coefficient. The system turns out to be of the so-called *nongradient* type. Roughly speaking, the gradient condition says that the microscopic current is already the gradient of a function of the density field. The method developed by S.R.S. Varadhan ([V], see also [Q1], [VY], [KL]) for nongradient systems is to replace the microscopic current by a gradient plus a fluctuation term. However, in the presence of the random field, such a decomposition does not hold microscopically, because the fluctuations of the gradient of the density field arising from the random field are large. In [F] and [FM] this problem has been solved by introducing a mesoscopic scale such that the gradients could be considered over sufficiently large mesoscopic distances to reduce stochastic fluctuations by the central limit theorem.

The limit equation they obtained for the evolution of the density field is a nonlinear second order partial differential equation with diffusion coefficient continuous with respect to the density field. To derive the limit equation for the model with $\beta \neq 0$ and $\theta \neq 0$, we use the results of [F] and [FM], and take into account that the invariant measures for the unperturbed dynamics are not invariant for the dynamics with $\beta \neq 0$. This fact induces the presence of a new term, which is crucial in order to recognize that the limiting equation has the form of the gradient flux for an energy functional. The limiting equation for the density profile of our system is the non linear partial integral equation

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left(\sigma(\rho) \nabla \frac{\delta \mathcal{G}}{\delta \rho} \right), \tag{1.2}$$

where the energy functional $\mathcal{G}(\rho)$ is of the form

$$\mathcal{G}(\rho) = \int dr g_0(\rho(r)) - \frac{\beta}{2} \int \int J(r-r')\rho(r)\rho(r')drdr', \tag{1.3}$$

g_0 being the (strictly convex) free energy density and $\sigma(\rho)$ the conductivity, or mobility, (see [Sp], Part 2, Section 2) of the system with only short range interaction, i.e. corresponding to $\beta \equiv 0$ in (1.1). The free energy density g_0 is given by

$$g_0(\rho) = \rho \lambda_0(\rho) - p_0(\lambda_0(\rho)),$$

where

$$p_0(\lambda) = I\!\!E \left[\log \left(1 + e^{(\lambda + \theta\alpha(0))} \right) \right],$$

$I\!\!E$ stands for expectation with respect to the disorder, and, for any given $\rho \in [0,1]$, $\lambda_0(\rho)$ satisfies

$$\rho = \frac{dp_0}{d\lambda}(\lambda_0(\rho)) = I\!\!E \left[\frac{e^{(\lambda_0(\rho)+\theta\alpha(0))}}{1 + e^{(\lambda_0(\rho)+\theta\alpha(0))}} \right]. \tag{1.4}$$

To simplify notation we write λ_0 instead of $\lambda_0(\rho)$, if no confusion arises. We derive (1.2) under some mild regularity property on J, therefore (1.2) holds even if the functional $\mathcal{G}$ has minimizers having non constant density profiles. In this case phase segregation could occur; see [GL1], [A], [CCO] for results on existence, properties and qualitative behavior of solutions of equations of type (1.2) with functional (1.3) in the presence of phase segregation. In [GL2] solutions corresponding to interface dynamics are considered.

Since we study the evolution of the system at fixed temperature we assume that the latter has been absorbed into (1.1). To relate (1.2) with equation (2.24), recall that, in a regime of linear response, the diffusion matrix is linked to the mobility via the Einstein-relation, see [Sp], $D(\rho) = \sigma(\rho)(\chi(\rho))^{-1}$, where $\chi(\rho)$ is the static compressibility

$$\chi(\rho) = [\lambda_0'(\rho)]^{-1} = I\!\!E\left[\frac{e^{(\lambda_0(\rho)+\theta\alpha(0))}}{\left(1+e^{(\lambda_0(\rho)+\theta\alpha(0))}\right)^2}\right]. \tag{1.5}$$

The variational characterization derived for $D(\rho)$ in [F] and [FM] is reported in (2.20).

In section 2 we introduce the model and state the main result, the macroscopic behavior of the system. In section 3, after explaining the strategy of proof, we obtain some basic estimates. In section 4, we derive the hydrodynamic limit equation.

2 The model and the main results

Let the scaling parameter $\gamma \in (0,1)$ be such that $\gamma^{-1} \in I\!\!N$. We denote by Λ the d-dimensional torus of diameter 1, by $\Lambda_\gamma \equiv Z\!\!\!Z^d/\gamma^{-1}Z\!\!\!Z^d$ the discrete torus, and by $|V|$ the cardinality of any finite non empty subset $V \subset Z\!\!\!Z^d$.

For a fixed $A > 0$, let $\Omega_D = [-A, A]^{Z\!\!\!Z^d}$ be the set of disorder configurations on $Z\!\!\!Z^d$. On Ω_D we define a product, translation invariant probability measure $I\!\!P$. We denote by $I\!\!E$ the expectation with respect to $I\!\!P$, and by $\alpha \equiv (\alpha(x),\ x \in Z\!\!\!Z^d)$, $\alpha(x) \in [-A, A]$, a disorder configuration in Ω_D. They represent external magnetic fields acting on the particles. A configuration $\alpha \in \Omega_D$ induces in a natural way a disorder configuration α_γ on Λ_γ, by identifying a cube centered at the origin of side γ^{-1} (γ^{-1} odd and integer) with the torus Λ_γ. By a slight abuse of notation whenever in the following we refer to a disorder configuration either on Λ_γ or on $Z\!\!\!Z^d$ we denote it by α.

We denote by $\mathcal{S}_\gamma \equiv \{0,+1\}^{\Lambda_\gamma}$ and $\mathcal{S} \equiv \{0,+1\}^{Z\!\!\!Z^d}$ the configuration spaces, both equipped with the product topology. We denote by η a configuration, either in $\mathcal{S}_\gamma$ or in $\mathcal{S}$.

Given a realization $\alpha \in \Omega_D$ of the magnetic field, for β and θ positive parameters, the Hamiltonian will be (1.1) restricted to the torus Λ_γ, namely a real-valued function defined on $\mathcal{S}_\gamma$ as the sum of two terms

$$H_\gamma^{\beta,\alpha}(\eta) = \beta H_\gamma(\eta) + \theta H_s^\alpha(\eta)\ , \tag{2.1}$$

where H_s^α is the local, one body interaction,

$$H_s^\alpha(\eta) = -\sum_{x\in\Lambda_\gamma} \alpha(x)\eta(x) \tag{2.2}$$

and H_γ is the long range Kac interaction,

$$H_\gamma(\eta) = -\frac{1}{2}\sum_{(x,y)\in\Lambda_\gamma\times\Lambda_\gamma} J_\gamma(x-y)\eta(x)\eta(y)\ . \tag{2.3}$$

The pair interaction $J_\gamma(x-y)$, the so-called Kac potential, is such that $J_\gamma(x-y) \equiv \gamma^d J(\gamma(x-y))$ for $J \in \mathcal{C}^2(\Lambda, I\!\!R)$ with $J(r) = J(-r)$ (symmetry).

We denote by $\mu_\gamma^{\beta,\alpha,\lambda}$ the grand canonical Gibbs measure on $\mathcal{S}_\gamma$ associated to the Hamiltonian (2.1) with chemical potential $\lambda \in I\!R$

$$\mu_\gamma^{\beta,\alpha,\lambda}(\eta) = \frac{1}{Z_\gamma^{\beta,\alpha,\lambda}} \exp\{-H_\gamma^{\beta,\alpha}(\eta) + \lambda \sum_{x\in\Lambda_\gamma} \eta(x)\}\ , \qquad \eta \in \mathcal{S}_\gamma\ , \tag{2.4}$$

where $Z_\gamma^{\beta,\alpha,\lambda}$ is the normalization factor, so that $\mu_\gamma^{\beta,\alpha,\lambda}$ is a probability measure on $\mathcal{S}_\gamma$. When $\beta = 0$, $\mu_\gamma^{0,\alpha,\lambda}$ becomes a random Bernoulli product measure, that we denote by

$$\mu_\gamma^{\alpha,\lambda}(\eta) \equiv \mu_\gamma^{0,\alpha,\lambda}(\eta) = \frac{1}{Z_\gamma^{0,\alpha,\lambda}} \exp\{-\theta H_s^\alpha(\eta) + \lambda \sum_{x\in\Lambda_\gamma} \eta(x)\}\ , \qquad \eta \in \mathcal{S}_\gamma\ . \tag{2.5}$$

If $\lambda = 0$, we simply write μ_γ^α. Similarly, for $\alpha \in \Omega_D$, $\lambda \in I\!R$, $\mu^{\alpha,\lambda}$ is the corresponding random product measure on the infinite product space $\mathcal{S}$, and when $\lambda = 0$, we denote it by μ^α. Moreover, for a probability measure μ and a bounded function f, both defined on $\mathcal{S}$ or $\mathcal{S}_\gamma$, we denote by $\mathbf{E}^\mu(f)$ the expectation of f with respect to μ, and by $\mathbf{P}^\mu$ the corresponding probability.

The potential λ, sometimes called *annealed chemical potential*, can be adjusted to the average density ρ of particles as follows. For $\rho \in [0,1]$, the function $\lambda_0(\rho)$ is determined by (cf. (1.4))

$$I\!E\left[\int \eta(x) d\mu^{\alpha,\lambda_0(\rho)}(\eta)\right] = I\!E\left[\frac{e^{\theta\alpha(x)+\lambda_0(\rho)}}{1+e^{\theta\alpha(x)+\lambda_0(\rho)}}\right] = \rho\ . \tag{2.6}$$

The disordered Kawasaki dynamics with parameter $\beta \geq 0$ is the Markov process on $\mathcal{S}_\gamma$ defined through its infinitesimal generator $\mathcal{L}_\gamma^{\beta,\alpha}$, acting on functions $f : \mathcal{S}_\gamma \to I\!R$ as

$$\left(\mathcal{L}_\gamma^{\beta,\alpha} f\right)(\eta) = \sum_{e\in\mathcal{E}} \sum_{x\in\Lambda_\gamma} C_\gamma^{\beta,\alpha}(x, x+e; \eta)(\nabla_{x,x+e} f)(\eta)\ , \tag{2.7}$$

where $\mathcal{E} = \{e_1, \ldots, e_d\}$ is the canonical basis of $I\!R^d$ and e a generic element of $\mathcal{E}$. For $x, y \in \Lambda_\gamma$, $\eta \in \mathcal{S}_\gamma$,

$$(\nabla_{x,y} f)(\eta) = f(\eta^{x,y}) - f(\eta),$$

where $\eta^{x,y}$ is the configuration obtained from η by interchanging the values at x and y:

$$\eta^{x,y}(z) = \begin{cases} \eta(x) & \text{if} \quad z = y \\ \eta(y) & \text{if} \quad z = x \\ \eta(z) & \text{otherwise.} \end{cases} \tag{2.8}$$

The rate $C_\gamma^{\beta,\alpha}$ is given by

$$C_\gamma^{\beta,\alpha}(x, y; \eta) = \Phi\left\{(\nabla_{x,y} H_\gamma^{\beta,\alpha})(\eta)\right\}\ . \tag{2.9}$$

Here $\Phi \in \mathcal{C}^2(I\!R, (0,\infty))$ satisfies* $\Phi(0) = 1$ and the *detailed balance condition*

$$\Phi(E) = \exp(-E)\Phi(-E)\ . \tag{2.10}$$

This is equivalent to the existence of a function $\psi \in \mathcal{C}^2(I\!R, (0,\infty))$ such that

$$\Phi(E) = \exp(-E/2)\psi(E), \qquad \psi(E) = \psi(-E), \qquad \psi(0) = 1\ . \tag{2.11}$$

* The case $\Phi(0) \in (0,\infty)$ can be recovered by a time change.

Notice that $C_\gamma^{\beta,\alpha}(x,y;\eta)$ has the following properties:

a) *detailed balance condition*, see (2.10) or (2.11);

b) *positivity and boundedness*: there exists $a>0$ such that

$$a^{-1} \le C_\gamma^{\beta,\alpha}(x,y;\eta) \le a \tag{2.12}$$

c) *translation invariance*

$$C_\gamma^{\beta,\alpha}(x,y;\eta) = C_\gamma^{\beta,\tau_z\alpha}(x-z,y-z;\tau_z\eta) = \tau_z C_\gamma^{\beta,\alpha}(x-z,y-z;\eta) \tag{2.13}$$

where for z in $\mathbb{Z}^d$, τ_z denotes the space shift by z units on $\mathcal{S}\times\Omega_D$ defined for all $\eta\in\mathcal{S}$, $\alpha\in\Omega_D$ and $g:\mathcal{S}\times\Omega_D\to\mathbb{R}$ by

$$(\tau_z\eta)(x) = \eta(x+z),\ (\tau_z\alpha)(x) = \alpha(x+z),\ (\tau_z g)(\eta,\alpha) = g(\tau_z\eta,\tau_z\alpha)\ . \tag{2.14}$$

For each $\lambda\in\mathbb{R}$, the generator $\mathcal{L}_\gamma^{\beta,\alpha}$ is self-adjoint in $L^2(\mu_\gamma^{\beta,\alpha,\lambda})$ (cf. (2.4)). We could alternatively have fixed the number of particles, and got a density $\rho\in[0,(|\Lambda_\gamma|)^{-1},\cdots,1]$. Then the generator $\mathcal{L}_\gamma^{\beta,\alpha}$ is self-adjoint in $L^2(\nu_{\rho,\Lambda_\gamma}^{\beta,\alpha})$ for the canonical measure

$$\nu_{\rho,\Lambda_\gamma}^{\beta,\alpha}(\eta) = \frac{1}{Z_\gamma^{\beta,\alpha}}\exp\{-H^{\beta,\alpha}(\eta)\}\mathbb{1}_{\{\sum_{x\in\Lambda_\gamma}\eta(x)=\rho|\Lambda_\gamma|\}}\ , \qquad \eta\in\mathcal{S}_\gamma\ , \tag{2.15}$$

with $Z_\gamma^{\beta,\alpha}$ the corresponding normalization factor.

We shall denote the generator of the Markov process associated to the Hamiltonian θH_s^α, i.e. with $\beta=0$, see (2.2), by $\mathcal{L}_\gamma^\alpha$

$$\left(\mathcal{L}_\gamma^\alpha f\right)(\eta) = \sum_{e\in\mathcal{E}}\sum_{x\in\Lambda_\gamma} C^\alpha(x,x+e;\eta)(\nabla_{x,x+e}f)(\eta)\ , \tag{2.16}$$

where f is a function on $\mathcal{S}_\gamma$, and

$$C^\alpha(x,y;\eta) = \Phi\left\{\theta(\nabla_{x,y}H_s^\alpha)(\eta)\right\}\ . \tag{2.17}$$

The rate $C^\alpha(x,y;\eta)$ satisfies properties (2.10), (2.12) and (2.13). We call the process with generator $\mathcal{L}_\gamma^\alpha$ the Disordered Lattice Gas process (DLG) or the unperturbed process, to distinguish from the process generated by $\mathcal{L}_\gamma^{\beta,\alpha}$ to which we refer as the perturbed process. The DLG process for $\theta=1$ is the one considered in [F] and [FM]. The invariant measures for the DLG process are, for $\lambda\in\mathbb{R}$, the random product measures $\mu_\gamma^{\alpha,\lambda}$ defined in (2.5), or alternatively, for $\rho\in[0,|\Lambda_\gamma|^{-1},\cdots,1]$, the canonical measures obtained by setting $\beta=0$ in (2.15),

$$\nu_{\rho,\Lambda_\gamma}^{\alpha}(\eta) \equiv \nu_{\rho,\Lambda_\gamma}^{0,\alpha}(\eta)\ , \qquad \eta\in\mathcal{S}_\gamma\ . \tag{2.18}$$

Same way, the operator $\mathcal{L}_\gamma^\alpha$ is selfadjoint in $L^2(\mu_\gamma^{\alpha,\lambda})$, or alternatively in $L^2(\nu_{\rho,\Lambda_\gamma}^\alpha)$.

Our goal is to establish a law of large numbers for the density field, starting from a sequence of measures on $\mathcal{S}_\gamma$ associated to some initial density profile ρ_0. We show that for almost any disorder configuration $\alpha\in\Omega_D$, the density field converges, as γ decreases to 0, to the unique weak solution of a non-linear integral parabolic equation with initial condition ρ_0. In order to write this equation we introduce some notation, and define, after [F] and [FM] the diffusion coefficient of the integral parabolic equation. For $g\in\mathbb{G}$, where

$$\mathbb{G} \equiv \{g:\mathcal{S}\times\Omega_D\to\mathbb{R};\ \text{ local and bounded }\}\ , \tag{2.19}$$

$\Gamma_g(\eta)=\sum_{x\in\mathbb{Z}^d}\left(\tau_x g\right)(\eta,\alpha)$ is a formal expression, but the difference $(\nabla_{0,e}\Gamma_g)(\eta)=\Gamma_g(\eta^{0,e})-\Gamma_g(\eta)$ for $e\in\mathcal{E}$ makes sense. Recall that a function $g:\mathcal{S}\times\Omega_D\to\mathbb{R}$ is local if the support Δ_g of g, i.e. the smallest subset of $\mathbb{Z}^d$ such that g depends only on $\{(\eta(x),\alpha(x));x\in\Delta_g\}$, is finite; g is bounded if $\sup_\eta\sup_\alpha|g(\eta,\alpha)|<\infty$. For each $\rho\in(0,1)$, let $D(\rho)=\{D_{i,j}(\rho),\ 1\le i,j\le d\}$ be the symmetric matrix defined, for every $a\in\mathbb{R}^d$, by the variational formula

$$(a,D(\rho)a)=\frac{1}{2\chi(\rho)}\inf_{g\in\mathcal{G}}\sum_{i=1}^d\mathbb{E}\left[\mathbf{E}^{\mu^{\alpha,\lambda_0(\rho)}}\left(C^\alpha(0,e_i;\eta)\Big\{a_i\nabla_{0,e_i}\eta(0)+(\nabla_{0,e_i}\Gamma_g)(\eta)\Big\}^2\right)\right]\ , \tag{2.20}$$

where $\lambda_0(\rho)$ is defined in (2.6) and $\chi(\rho)$ is the static compressibility, see (1.5), given by

$$\chi(\rho)=\mathbb{E}\left[\int\eta(0)^2d\mu^{\alpha,\lambda_0(\rho)}(\eta)-\left(\int\eta(0)d\mu^{\alpha,\lambda_0(\rho)}(\eta)\right)^2\right]=\rho(1-\rho)\ . \tag{2.21}$$

In [FM], Theorem 2.1, for $d\ge3$ and $\rho\in(0,1)$, the existence of the symmetric diffusion matrix defined in (2.20) has been proved. Moreover, the coefficients $D_{i,j}(\cdot)$ are nonlinear continuous functions of $\rho\in(0,1)$, and there exists a positive constant C, depending on dimension and bounds on the random field, such that

$$C^{-1}\mathbb{I}\le D(\rho)\le C\mathbb{I}\ ,\qquad \rho\in(0,1)\ , \tag{2.22}$$

where $\mathbb{I}$ is the $d\times d$ identity matrix. The upper bound is an immediate consequence of (2.20); the lower bound is more delicate. In the sequel we assume that the diffusion matrix $D(\cdot)$ can be continuously extended to the closed interval $[0,1]$. In [FM], subsection 4.8, the derivation of the hydrodynamic limit in the case when $D(\rho)$ does not have a continuous extension is explained, requiring further hypotheses to be satisfied. Continuity of $D(\rho)$ in $d\ge2$ is not enough to prove uniqueness of weak solutions of the hydrodynamic equation. To guarantee this uniqueness, we shall *assume* below that the diffusion coefficient is Lipschitz continuous in ρ. This assumption together with the uniform ellipticity bounds, see (2.22), guaranties the existence of Lipshitz continuous solutions of (2.24), see [VY]. The Lipshitz continuous property for $D(\cdot)$ has not been proved for the DLG model. To prove regularity properties for the diffusion matrix, duality methods have been introduced; we refer to [LOV1], [LOV2] and references therein. For a discussion of the relation between the variational formula (2.20) and the classical Green-Kubo formula, see [Sp], part ll, Section 2.2.

We are now ready to state the main result. Fix a positive time T. For a measure μ_γ on $\mathcal{S}_\gamma$, denote by $\mathbf{P}^{\beta,\alpha}_{\mu_\gamma}$ the probability measure on the path space $D([0,T],\mathcal{S}_\gamma)$ corresponding to the Markov process $(\eta_t)_{t\in[0,T]}$ with generator $\gamma^{-2}\mathcal{L}^{\beta,\alpha}_\gamma$ starting from μ_γ, and by $\mathbf{E}^{\beta,\alpha}_{\mu_\gamma}$ the expectation with respect to $\mathbf{P}^{\beta,\alpha}_{\mu_\gamma}$. For $t\in[0,T],\eta\in\mathcal{S}_\gamma$, let the empirical measure π^γ_t be defined by

$$\pi^\gamma_t(r)\equiv\pi^\gamma(r;\eta_t)\ =\ \gamma^d\sum_{x\in\Lambda_\gamma}\eta_t(x)\,\delta_{\gamma x}(r)\ ,\qquad r\in\Lambda\ , \tag{2.23}$$

where $\delta_u(\cdot)$ is the Dirac measure on Λ concentrated on u. Since $\eta(x)\in\{0,1\}$, relation (2.23) induces from $\mathbf{P}^{\beta,\alpha}_{\mu_\gamma}$ a distribution $Q^{\beta,\alpha}_{\mu_\gamma}$ of $(\pi^\gamma_t(r);\ r\in\Lambda,\ t\in[0,T])$ on the Skorohod space $D([0,T],\mathcal{M}_1(\Lambda))$, where $\mathcal{M}_1(\Lambda)$ is the space of nonnegative measures on Λ with total mass bounded by 1, equipped with the weak topology induced through duality by $C(\Lambda)$, the continuous real functions from Λ to $\mathbb{R}$, according to

$$<\pi,U>=\int_\Lambda G(r)\pi(dr)\ .$$

The space $\mathcal{M}_1(\Lambda)$ is compact under the topology of weak convergence. As $\gamma\to0$ we expect the distributions $(Q^{\beta,\alpha}_{\mu_\gamma})_{\gamma\ge0}$ of the empirical measures to converge to the Dirac measure concentrated on the trajectories $\rho(t,r)$.

Theorem 2.1 *Let $d \geq 3$, $(\beta, \theta) \in I\!R_+^2$ and assume that $D(\rho)$ defined in (2.20) can be continuously extended to the closed interval $[0,1]$. Consider a sequence of probability measures μ_γ on $\mathcal{S}_\gamma$ associated to the initial profile ρ_0 in the following sense:*

$$\lim_{\gamma \to 0} \mu_\gamma \Big\{ \Big| \langle \pi_0^\gamma, G \rangle - \int G(u) \rho_0(u)\, du \Big| > \delta \Big\} \;=\; 0$$

for every continuous function $G : \Lambda \to I\!R$ and every $\delta > 0$. Then, for almost any disorder configuration $\alpha \in \Omega_D$, the sequence of probability measures $(Q^{\beta,\alpha}_{\mu_\gamma})_{\gamma \geq 0}$ is tight and all its limit points Q^* are concentrated on absolutely continuous paths whose densities $\rho \in \mathcal{C}([0,T], \mathcal{M}_1(\Lambda))$ are the weak solutions of the equation

$$\begin{cases} \partial_t \rho \;=\; \displaystyle\sum_{k,m=1}^{d} \partial_k \Big\{ D_{k,m}(\rho) \Big\{ \partial_m \rho - \beta \chi(\rho) \big(\partial_m J * \rho \big) \Big\} \Big\} \,, \\ \rho(0, \cdot) \;=\; \rho_0(\cdot) \end{cases} \tag{2.24}$$

with $\chi(\rho)$ given in (2.21), satisfying

$$\int_0^T ds \Big(\int_\Lambda \|\nabla \rho(s,u)\|^2 du \Big) \Big] < \infty \,,$$

where $\| \cdot \|$ is the euclidean norm. Moreover if the diffusion matrix $D(\cdot)$ is locally Lipschitz continuous for $\rho \in (0,1)$, then $(Q^{\beta,\alpha}_{\mu_\gamma})_{\gamma \geq 0}$ converges in the limit $\gamma \to 0$ to Q^*. This limit point is concentrated on the unique weak solution of equation (2.24).

Through the text J, β, θ, Φ, and A (the bound on $\alpha(x)$) will be kept fixed. We therefore avoid to write explicitly the dependence on these quantities. The proof of Theorem 2.1 is given through several steps in the next sections.

3. Strategy of proof and basic estimates

3.1. The steps to prove theorem 2.1

Following [GPV] we divide the proof of the hydrodynamic behavior (i.e. of Theorem 2.1) in three steps: tightness of the measures $(Q^{\beta,\alpha}_{\mu_\gamma})_{\gamma \geq 0}$, which enables to work with a weak limit Q^* of $Q^{\beta,\alpha}_{\mu_\gamma}$, as $\gamma \to 0$, an energy estimate to provide some regularity for functions in the support of Q^*, and identification of the support of Q^* as weak solution of (2.24). We then refer to [KL], Chapter IV, that presents arguments, by now standard, to deduce the hydrodynamic behavior of the empirical measures from the preceding results and the uniqueness of the weak solution to equation (2.24).

Proposition 3.1 (**Tightness**) *For almost any disorder configuration $\alpha \in \Omega_D$, the sequence $(Q^{\beta,\alpha}_{\mu_\gamma})_{\gamma \geq 0}$ is tight and all its limit points Q^* are concentrated on absolutely continuous paths $\pi(t, du) = \rho(t,u) du$ whose density ρ is positive and bounded above by 1 :*

$$Q^* \Big\{ \pi \,:\, \pi(t,du) = \rho(t,u) du \Big\} = 1 \,, \quad Q^* \Big\{ \pi \,:\, 0 \leq \rho(t,u) \leq 1 \Big\} = 1 \,. \tag{3.1}$$

This proof is sketched in subsection 3.3.

We then show that Q^* is supported on densities ρ that satisfy (2.24) in the weak sense. For $\ell \in I\!N$ denote by $\eta^\ell(x)$ the average density of η in a cube of width $2\ell+1$ centered in $x \in \Lambda_\gamma$, namely

$$\eta^\ell(x) = \frac{1}{(2\ell+1)^d} \sum_{y:|y-x|\le \ell} \eta(y). \tag{3.2}$$

For a function G on Λ, $e \in \mathcal{E}$, $\partial_e^\gamma G$ denotes the discrete (space) derivative in the direction e

$$\big(\partial_e^\gamma G\big)(\gamma x) \;=\; \gamma^{-1}[G(\gamma(x+e)) - G(\gamma x)] \tag{3.3}$$

with $\partial_k^\gamma G$ as a short notation for $\partial_{e_k}^\gamma G$ for $1 \le k \le d$, and $\star$ denotes the discrete convolution

$$(J\star\eta)(x) \equiv (J\star\pi^\gamma)(\gamma x;\eta) = \gamma^d \sum_{z\in\Lambda_\gamma} J(\gamma(x-z))\eta(z)\,. \tag{3.4}$$

Proposition 3.2 (**Identification of the limit equation**) *Let $d \ge 3$ and assume that $D(\rho)$ defined in (2.20) can be continuously extended in $[0,1]$. Then, for almost any disorder configuration $\alpha \in \Omega_D$, any function G in $\mathcal{C}^{1,2}([0,T]\times\Lambda)$ and any $\delta > 0$, we have*

$$\limsup_{c\to 0}\limsup_{a\to 0}\limsup_{\gamma\to 0} \mathbf{P}^{\beta,\alpha}_{\mu_\gamma}\left(\left|\mathcal{B}^{G,\gamma}_{a,c}\right| \ge \delta\right) = 0, \tag{3.5}$$

where

$$\begin{aligned}\mathcal{B}^{G,\gamma}_{a,c} &= \gamma^d \sum_{x\in\Lambda_\gamma} G(T,\gamma x)\eta_T(x) - \gamma^d \sum_{x\in\Lambda_\gamma} G(0,\gamma x)\eta_0(x) - \gamma^d \sum_{x\in\Lambda_\gamma}\int_0^T \partial_s G(s,\gamma x)\eta_s(x)ds \\ &+ \sum_{1\le k,m\le d}\int_0^T ds\gamma^{d-1}\sum_{x\in\Lambda_\gamma}\big(\partial_k^\gamma G\big)(s,\gamma x)\Big\{D_{k,m}\left(\eta_s^{[a\gamma^{-1}]}(x)\right) \\ &\times\Big\{(2c)^{-1}\left[\eta_s^{[a\gamma^{-1}]}(x+c\gamma^{-1}e_m) - \eta_s^{[a\gamma^{-1}]}(x-c\gamma^{-1}e_m)\right] - \beta\gamma\chi\left(\eta_s^{[a\gamma^{-1}]}(x)\right)\left((\partial_m^\gamma J)\star\pi_s^\gamma\right)(\gamma x;\eta)\Big\}\Big\}\end{aligned} \tag{3.6}$$

$\chi(\cdot)$ is defined in (2.21) and $\mathbf{P}^{\beta,\alpha}_{\mu_\gamma}$ is the probability measure on the path space $D([0,T],\mathcal{S}_\gamma)$ corresponding to the Markov process $(\eta_t)_{t\in[0,T]}$ with generator $\gamma^{-2}\mathcal{L}^{\beta,\alpha}_\gamma$ starting from μ_γ.

This result is proved in section 4.

The last statement is an energy estimate needed, together with the assumption that the diffusion matrix is locally Lipschitz continuous, to prove uniqueness of the weak solutions. It states that for almost any disorder configuration $\alpha \in \Omega_D$, every limit point Q^* of the sequence $(Q^{\beta,\alpha}_{\mu_\gamma})_{\gamma\ge 0}$ is concentrated on paths whose densities ρ satisfy the following energy estimate.

Proposition 3.3 (**Energy estimate**) *Let $d \ge 3$ and assume that $D(\rho)$ defined in (2.20) can be continuously extended in $[0,1]$. For almost any disorder configuration $\alpha \in \Omega_D$, let Q^* be a limit point of the sequence $(Q^{\beta,\alpha}_{\mu_\gamma})_{\gamma\ge 0}$. Then,*

$$E^{Q^*}\Big[\int_0^T ds\Big(\int_\Lambda \|\nabla\rho(s,u)\|^2 du\Big)\Big] < \infty\,, \tag{3.7}$$

where $\|\cdot\|$ is the euclidean norm in $I\!R^d$.

3.2. Basic estimates

In this section we derive some results we need later on. In the first lemma, we show that the jump rates of $\mathcal{L}_\gamma^{\beta,\alpha}$ are a perturbation of the ones of $\mathcal{L}_\gamma^\alpha$. This will enable us to transform non equilibrium estimates with respect to the perturbed process into equilibrium estimates for the unperturbed process (see Remark 3.8).

Lemma 3.4 *For every Φ as in (2.10), J as in (2.3), $x \in \Lambda_\gamma, e \in \mathcal{E}$, $\eta \in \mathcal{S}_\gamma$, for all $\alpha \in \Omega_D$*

$$\begin{aligned} &C_\gamma^{\beta,\alpha}(x,x+e;\eta) - C^\alpha(x,x+e;\eta) = \\ &\gamma\beta\Phi'(\theta[\nabla_{x,x+e}H_s^\alpha](\eta))[\eta(x+e)-\eta(x)]\big((\partial_e^\gamma J)\star\eta\big)(x) + \gamma^2 R_2(x,x+e)(\eta) \end{aligned} \tag{3.8}$$

with

$$\sup_{e\in\mathcal{E}}\sup_{x\in\Lambda_\gamma}\sup_{\eta\in\mathcal{S}_\gamma}|R_2(x,x+e)(\eta)| \le C \tag{3.9}$$

for a positive constant $C \equiv C(\beta,\theta,A,J,\Phi)$.

Proof. We have

$$[\nabla_{x,x+e}H_s^\alpha](\eta) = (\eta(x+e)-\eta(x))(\alpha(x+e)-\alpha(x)) \tag{3.10}$$

$$[\nabla_{x,x+e}H_\gamma](\eta) = -\gamma(\eta(x+e)-\eta(x))\big\{\big((\partial_e^\gamma J)\star\eta\big)(x) - \gamma^d(\partial_e^\gamma J)(0)\big\} \tag{3.11}$$

so that a Taylor expansion to the second order of the function Φ yields the result. We write (cf. (2.9), (2.17))

$$C_\gamma^{\beta,\alpha}(x,x+e;\eta) - C^\alpha(x,x+e;\eta) = \gamma R_1(x,x+e)(\eta) + \gamma^2 R_2(x,x+e)(\eta) \tag{3.12}$$

with

$$|R_1(x,x+e)(\eta)| = |\beta\Phi'(\theta[\nabla_{x,x+e}H_s^\alpha](\eta))[\eta(x+e)-\eta(x)]\big((\partial_e^\gamma J)\star\eta\big)(x)| \le C_1\ . \tag{3.13}$$

Similar analysis holds for the second term of (3.12) obtaining

$$|R_2(x,x+e)(\eta)| \le C \tag{3.14}$$

for C_1 and C positive constants depending on A (the bound on $\alpha(x)$), $\beta, \theta, C(J) = 2\sup_{1\le i\le d, u\in\Lambda}|\partial_i J(u)|$, and Φ (via $\sup_{u\in[-2\theta A, 2\theta A]}|\Phi'(u)|$ and $\sup_{u\in[-2\theta A-\beta C(J), 2\theta A+\beta C(J)]}|\Phi''(u)|$). $\square$

Notice that as a consequence we recover here that the jump rates are bounded.

In the following, we avoid to write explicitly the terms of order γ^2, by replacing them with $O_u(\gamma^2)$, which should be understood in the standard sense of $O(\gamma^2)$, but uniformly with respect to the disorder α, and either to configurations η or to the history of the process.

We start by adapting to our dynamics some arguments used in non gradient methods. We first recall the definitions of the entropy and the Dirichlet form associated to the generator of a Markov process. Recall from section 2 that for a realization α and a density ρ the Gibbs measure $\mu_\gamma^{\alpha,\lambda_0(\rho)}$ is Bernoulli product. For a probability measure μ on $\mathcal{S}_\gamma$, denote by $H(\mu|\mu_\gamma^{\alpha,\lambda_0(\rho)})$ the relative entropy of μ with respect to $\mu_\gamma^{\alpha,\lambda_0(\rho)}$:

$$H(\mu|\mu_\gamma^{\alpha,\lambda_0(\rho)}) \ = \ \sup_f\Big\{\int f(\eta)\mu(d\eta) - \log\int e^{f(\eta)}\mu_\gamma^{\alpha,\lambda_0(\rho)}(d\eta)\Big\}\ .$$

In this formula the supremum is carried over all bounded functions on $\mathcal{S}_\gamma$. Since $\mu_\gamma^{\alpha,\lambda_0(\rho)}$ gives a positive probability to each configuration, μ is absolutely continuous with respect to $\mu_\gamma^{\alpha,\lambda_0(\rho)}$ and we have an explicit formula for the entropy:

$$H(\mu|\mu_\gamma^{\alpha,\lambda_0(\rho)}) \;=\; \int \log\Big\{\frac{d\mu}{d\mu_\gamma^{\alpha,\lambda_0(\rho)}}\Big\}\, d\mu\ .$$

Moreover, since there is at most one particle per site, there exists a constant $C_0 \equiv C_0(\theta, A)$ such that

$$H(\mu|\mu_\gamma^{\alpha,\lambda_0(\rho)}) \;\le\; C_0\gamma^{-d} \tag{3.15}$$

for all probability measures μ on $\mathcal{S}_\gamma$ (cf. comments following Remark V.5.6 in [KL]).

For $f \in L^2(\mu_\gamma^{\alpha,\lambda})$ denote by $\mathcal{D}^\alpha(f)$ the Dirichlet form associated to the operator $\mathcal{L}_\gamma^\alpha$

$$\mathcal{D}^\alpha(f) = -\int_{\mathcal{S}_\gamma} f(\eta)\mathcal{L}_\gamma^\alpha f(\eta)d\mu_\gamma^{\alpha,\lambda}(\eta)\ . \tag{3.16}$$

Denote by $S = [\mathcal{L}_\gamma^{\beta,\alpha} + (\mathcal{L}_\gamma^{\beta,\alpha})^*]/2$ the symmetric part of the operator $\mathcal{L}_\gamma^{\beta,\alpha}$ in $L^2(\mu_\gamma^{\alpha,\lambda})$. Then (see [KL], Appendix 1),

$$-\int_{\mathcal{S}_\gamma} f(\eta)Sf(\eta)d\mu_\gamma^{\alpha,\lambda}(\eta) = -\int_{\mathcal{S}_\gamma} f(\eta)\mathcal{L}_\gamma^{\beta,\alpha} f(\eta)d\mu_\gamma^{\alpha,\lambda}(\eta) \ge 0\ .$$

By abuse of notation we thus denote

$$\mathcal{D}^{\beta,\alpha}(f) = -\int_{\mathcal{S}_\gamma} f(\eta)\mathcal{L}_\gamma^{\beta,\alpha} f(\eta)d\mu_\gamma^{\alpha,\lambda}(\eta)\ .$$

The first application of Lemma 3.4 is a bound on $\mathcal{D}^{\beta,\alpha}(f)$ by the Dirichlet form $\mathcal{D}^\alpha$ of the DLG dynamics.

Lemma 3.5 *For any $\beta > 0$ and $\alpha \in \Omega_D$, there exists a positive constant $C_0' \equiv C_0'(J,\beta,\theta,A)$ such that for any $f \in L^2\big(\mu_\gamma^{\alpha,\lambda_0}\big)$ and for any $M > 0$,*

$$-\mathcal{D}^{\beta,\alpha}\big(f\big) \le -\left(1-M\right)\mathcal{D}^\alpha\big(f\big) + C_0'\frac{\gamma^{2-d}}{M}\|f\|^2_{L^2(\mu_\gamma^{\alpha,\lambda})}\ . \tag{3.17}$$

Proof. By (3.12),

$$\mathcal{D}^\alpha(f) - \mathcal{D}^{\beta,\alpha}(f) = \sum_{e\in\mathcal{E}}\sum_{x\in\Lambda_\gamma}\int f(\eta)(\nabla_{x,x+e}f)(\eta)[\gamma R_1(x,x+e)(\eta) + \gamma^2 R_2(x,x+e)(\eta)]d\mu_\gamma^{\alpha,\lambda_0}(\eta)\,. \tag{3.18}$$

By the elementary inequality $2uv \le Bu^2 + B^{-1}v^2$ which holds for any $B > 0$ and by (3.13), we obtain

$$\begin{aligned}
&\int f(\nabla_{x,x+e}f)\gamma R_1(x,x+e)d\mu_\gamma^{\alpha,\lambda_0} \\
&\le \frac{M}{4a}\int(\nabla_{x,x+e}f)^2 d\mu_\gamma^{\alpha,\lambda_0} + \frac{a\gamma^2}{M}\int(R_1(x,x+e))^2 f^2 d\mu_\gamma^{\alpha,\lambda_0} \\
&\le \frac{M}{4}\int C_\gamma^\alpha(x,x+e;\eta)\left[f(\eta^{x,x+e}) - f(\eta)\right]^2 d\mu_\gamma^{\alpha,\lambda_0}(\eta) + \frac{a\gamma^2}{M}C_1^2\|f\|^2_{L^2(\mu_\gamma^{\alpha,\lambda_0})},
\end{aligned} \tag{3.19}$$

where we choose $B = M(2a)^{-1}$, for an arbitrary $M > 0$, and a comes from (2.12). Taking into account (3.14) we deal similarly with the second term of the integral in the r.h.s. of (3.18). Combining the previous estimates with (3.18) we obtain

$$\begin{aligned}
&\mathcal{D}^{\alpha}(f) - \mathcal{D}^{\beta,\alpha}(f) \\
&\le \frac{M}{2} \sum_{e\in\mathcal{E}} \sum_{x\in\Lambda_\gamma} \int C^{\alpha}(x, x+e; \eta) \left[f(\eta^{x,x+e}) - f(\eta)\right]^2 d\mu_\gamma^{\alpha,\lambda_0}(\eta) \\
&+ a\frac{d\gamma^{2-d}}{M}\{C_1^2 + \gamma^2 C^2\} \|f\|^2_{L^2(\mu_\gamma^{\alpha,\lambda_0})}.
\end{aligned} \tag{3.20}$$

Since $\mathcal{L}^\alpha$ is selfadjoint in $L^2(\mu_\gamma^{\alpha,\lambda})$

$$\mathcal{D}^{\alpha}(f) = \frac{1}{2} \sum_{e\in\mathcal{E}} \sum_{x\in\Lambda_\gamma} \int C^{\alpha}(x, x+e; \eta) \left[f(\eta^{x,x+e}) - f(\eta)\right]^2 d\mu_\gamma^{\alpha,\lambda_0}(\eta). \tag{3.21}$$

We then obtain from (3.20) that

$$-\mathcal{D}^{\beta,\alpha}(f) \le (M-1)\mathcal{D}^{\alpha}(f) + a\frac{d\gamma^{2-d}}{M}\{C_1^2 + \gamma^2 C^2\}\|f\|^2_{L^2(\mu_\gamma^{\alpha,\lambda_0})}$$

Since $\gamma < 1$ we obtain (3.17) by setting $C_0' = ad\{C_1^2 + C^2\}$. $\square$

The next lemma transforms nonequilibrium exponential estimates for the perturbed process into eigenvalue problems for $\mathcal{L}_\gamma^\alpha$ using Lemma 3.5 and the Feynman-Kac formula. For $\alpha \in \Omega_D$, $B > 0$ and for a bounded function $\mathbf{X}^\gamma : I\!R^+ \times \mathcal{S}_\gamma \times \Omega_D \to I\!R$ define

$$\begin{aligned}
\Gamma\big(s, \mathbf{X}^\gamma + B\gamma^{-2}\mathcal{L}_\gamma^\alpha\big) &= \sup spec_{L^2(\mu_\gamma^{\alpha,\lambda_0})} \left\{\mathbf{X}^\gamma(s,\cdot,\alpha) + B\gamma^{-2}\mathcal{L}_\gamma^\alpha\right\} \\
&\equiv \sup_{\{\|f\|_{L^1(\mu_\gamma^{\alpha,\lambda_0})}=1, f\ge 0\}} \left\{ \int \mathbf{X}^\gamma(s,\eta,\alpha) f(\eta) d\mu_\gamma^{\alpha,\lambda_0}(\eta) - B\gamma^{-2}\mathcal{D}^\alpha\big(\sqrt{f}\big) \right\} .
\end{aligned} \tag{3.22}$$

Lemma 3.6 *For any positive constants B and $M < 1$, for any bounded function $\mathbf{X}^\gamma : I\!R^+ \times \mathcal{S}_\gamma \times \Omega_D \to I\!R$ twice continuously differentiable in its first variable, such that for some $\tilde{C} > 0$, $\sup_{t,\eta,\alpha}\big(|\partial_t \mathbf{X}^\gamma(t,\eta,\alpha)| + |\partial_t^2 \mathbf{X}^\gamma(t,\eta,\alpha)|\big) < \tilde{C}$, we have for any $T > 0$, $\alpha \in \Omega_D$,*

$$\begin{aligned}
&\log \mathbf{E}^{\beta,\alpha}_{\mu_\gamma^{\alpha,\lambda_0}} \left[\exp\left\{B\gamma^{-d}\int_0^T \mathbf{X}^\gamma(s,\eta_s,\alpha)ds\right\}\right] \\
&\le \int_0^T \Gamma_{\mu_\gamma^{\alpha,\lambda_0}}\big(s, B\gamma^{-d}\mathbf{X}^\gamma + \gamma^{-2}(1-M)\mathcal{L}_\gamma^\alpha\big)\, ds + C_0'\frac{\gamma^{-d}}{M}T ,
\end{aligned}$$

where C_0' is the constant introduced in Lemma 3.5.

Proof. Fix $T > 0$. For $\eta \in \mathcal{S}_\gamma$, let

$$V(\eta, T) = \mathbf{E}^{\beta,\alpha}_\eta\left[\exp\left\{B\gamma^{-d}\int_0^T \mathbf{X}^\gamma(s,\eta_s,\alpha)ds\right\}\right] , \tag{3.23}$$

where $\mathbf{E}_\eta^{\beta,\alpha}$ denotes the expectation with respect to the process generated by $\mathcal{L}_\gamma^{\beta,\alpha}$ starting from the Dirac measure δ_η. By Feynman-Kac formula (cf. Appendix 1, section 7 in [KL]), V is the stochastic representation at time T of the solution of equation

$$\begin{cases} \partial_s u(\eta,s) \;=\; \gamma^{-2}\mathcal{L}_\gamma^{\beta,\alpha}u(\eta,s) + B\gamma^{-d}\mathbf{X}^\gamma(T-s,\eta,\alpha)u(\eta,s)\,, \\ u(\eta,0) \;\equiv\; 1. \end{cases} \tag{3.24}$$

To simplify notation we set $F(T-s,\eta) = B\gamma^{-d}\mathbf{X}^\gamma(T-s,\eta,\alpha)$. Multiplying (3.24) by u, integrating w.r.t. $\mu_\gamma^{\alpha,\lambda_0}$ and applying (3.17) we obtain (cf. [VY], lemma 3.7)

$$\begin{aligned}
&\partial_s\left(\frac{1}{2}\int u^2(\eta,s)d\mu_\gamma^{\alpha,\lambda_0}(\eta)\right)\\
&= \int u(\eta,s)\Big(\gamma^{-2}\mathcal{L}_\gamma^{\beta,\alpha}u(\eta,s) + F(s,\eta)u(\eta,s)\Big)d\mu_\gamma^{\alpha,\lambda_0}(\eta)\\
&\le \sup_{f\in L^2(\mu_\gamma^{\alpha,\lambda_0})}\left\{\frac{\int f\left[F(T-s,\cdot)+\gamma^{-2}(1-M)\mathcal{L}_\gamma^\alpha\right]fd\mu_\gamma^{\alpha,\lambda_0}}{\int f^2d\mu_\gamma^{\alpha,\lambda_0}}\right\}\int u^2(\eta,s)d\mu_\gamma^{\alpha,\lambda_0}(\eta)\\
&+ C_0'\frac{\gamma^{-d}}{M}\int u^2(\eta,s)d\mu_\gamma^{\alpha,\lambda_0}(\eta)\\
&= \left[\Gamma\Big(T-s,F+\gamma^{-2}(1-M)\mathcal{L}_\gamma^\alpha\Big) + C_0'\frac{\gamma^{-d}}{M}\right]\int u^2(\eta,s)d\mu_\gamma^{\alpha,\lambda_0}(\eta)\;,
\end{aligned} \tag{3.25}$$

This implies that

$$\partial_s\left(\frac{1}{2}\mathrm{Log}\int u^2(\eta,s)d\mu_\gamma^{\alpha,\lambda_0}(\eta)\right) \le \Gamma\big(T-s,F+\gamma^{-2}(1-M)\mathcal{L}_\gamma^\alpha\big) + C_0'\frac{\gamma^{-d}}{M}$$

and then, since $u(.,0)\equiv 1$

$$\begin{aligned}
\mathrm{Log}\int u(\eta,T)d\mu_\gamma^{\alpha,\lambda_0}(\eta) &\le \frac{1}{2}\mathrm{Log}\int u^2(\eta,T)d\mu_\gamma^{\alpha,\lambda_0}(\eta)\\
&\le \int_0^T \Gamma\big(T-s,F+\gamma^{-2}(1-M)\mathcal{L}_\gamma^\alpha\big)ds + C_0'\frac{\gamma^{-d}}{M}T\\
&= \int_0^T \Gamma\big(u,F+\gamma^{-2}(1-M)\mathcal{L}_\gamma^\alpha\big)du + C_0'\frac{\gamma^{-d}}{M}T
\end{aligned}$$

This together with the representation (3.23) concludes the proof of the Lemma. □

Next lemma is an important consequence of the previous one, and we will repeatedly use it in the following.

Lemma 3.7 *Under hypotheses of Lemma 3.6, for any positive constants B and $M<1$, we have*

$$\begin{aligned}
&\limsup_{\gamma\to 0}\mathbf{E}_{\mu_\gamma}^{\beta,\alpha}\Big[\Big|\int_0^T \mathbf{X}^\gamma(s,\eta_s,\alpha)ds\Big|\Big]\\
&\quad\le \Big(\limsup_{\gamma\to 0}\int_0^T \Gamma\big(s,\mathbf{X}^\gamma + B(1-M)\gamma^{d-2}\mathcal{L}_\gamma^\alpha\big)ds\Big)\\
&\qquad\vee\Big(\limsup_{\gamma\to 0}\int_0^T \Gamma\big(s,-\mathbf{X}^\gamma + B(1-M)\gamma^{d-2}\mathcal{L}_\gamma^\alpha\big)ds\Big)\\
&\quad + B\big(C_0 + C_0'M^{-1}T\big)\;,
\end{aligned} \tag{3.26}$$

where C_0, C_0' are the constants introduced in (3.15) and Lemma 3.5.

Proof. Using (3.15) and the entropy inequality we obtain that

$$\mathbf{E}^{\beta,\alpha}_{\mu_\gamma}\Big[\Big|\int_0^T \mathbf{X}^\gamma(s,\eta_s,\alpha)ds\Big|\Big] \le C_0 B \;+\; B\gamma^d \log \mathbf{E}^{\beta,\alpha}_{\mu_\gamma^{\alpha,\lambda_0}}\Big[\exp\Big\{B^{-1}\gamma^{-d}\Big|\int_0^T \mathbf{X}^\gamma(s,\eta_s,\alpha)ds\Big|\Big\}\Big] \tag{3.27}$$

for all $B>0$. Since $e^{|x|} \le e^x + e^{-x}$ and

$$\limsup_{\gamma\to 0}\Big\{\gamma^d \log(a_\gamma + b_\gamma)\Big\} \le \Big\{\limsup_{\gamma\to 0}(\gamma^d \log a_\gamma)\Big\} \vee \Big\{\limsup_{\gamma\to 0}(\gamma^d \log b_\gamma)\Big\}$$

applying Lemma 3.6 we obtain (3.26). □

Remark 3.8 . *Note that if for any $K>0$*

$$\limsup_{\gamma\to 0}\int_0^T \Gamma\left(s, \pm\mathbf{X}^\gamma + K\gamma^{d-2}\mathcal{L}^\alpha_\gamma\right) ds = 0 \tag{3.28}$$

then, from Lemma 3.7, letting first $\gamma\to 0$ then $B\to 0$ we obtain

$$\limsup_{\gamma\to 0}\mathbf{E}^{\beta,\alpha}_{\mu_\gamma}\Big[\Big|\int_0^T \mathbf{X}^\gamma(s,\eta_s,\alpha)ds\Big|\Big] = 0.$$

This remark allows us to state the replacement lemma for the perturbed process. For any local function $g(\eta,\alpha)$, for $\eta\in\mathcal{S}_\gamma$ and $\alpha\in\Omega_D$, for $\rho\in[0,1]$, $\lambda_0(\rho)$ chosen according to (1.4), define

$$\widetilde{g}(\rho) \equiv I\!E\left[\mathbf{E}^{\mu_\gamma^{\alpha,\lambda_0(\rho)}}[g]\right] \tag{3.29}$$

Lemma 3.9 (Replacement lemma for the $\mathcal{L}^{\beta,\alpha}$ process) *Let $g(\eta,\alpha)$ be a local function on $\mathcal{S}_\gamma\times\Omega_D$. For any fixed $b>0$ let*

$$\mathcal{B}_{b\gamma^{-1}}(\eta,\alpha) = \left|\frac{1}{(2b\gamma^{-1}+1)^d}\sum_{|y|\le b\gamma^{-1}}\left[\tau_y g(\eta,\alpha) - \tilde{g}(\eta^{[b\gamma^{-1}]}(0))\right]\right| \tag{3.30}$$

Then, for any $\delta>0$, $I\!P$ a.s.

$$\limsup_{b\to 0}\limsup_{\gamma\to 0}\mathbf{P}^{\beta,\alpha}_{\mu_\gamma}\left[\int_0^T\Big\{\gamma^d\sum_{x\in\Lambda_\gamma}\tau_x\mathcal{B}_{b\gamma^{-1}}(\eta_s,\alpha)\Big\}ds \ge \delta\right] = 0\,. \tag{3.31}$$

Proof. Taking into account Remark 3.8, it is enough to show that for any positive K

$$\limsup_{b\to 0}\limsup_{\gamma\to 0}\sup\, spec_{L^2(\mu_\gamma^{\alpha,\lambda_0})}\{\gamma^d\sum_{x\in\Lambda_\gamma}\tau_x\mathcal{B}_{b\gamma^{-1}}(\eta,\alpha) + K\mathcal{L}^\alpha_\gamma\} = 0\,.$$

This has been shown in [F], subsection 1.13, and [FM], proposition A.9; the proof relies on the one and two blocks estimates. Thanks to the ergodicity of the random field α and the subadditivity properties of $\sup spec$, the [GPV] techniques can be adapted to the random case, see also [K], [CX]. $\square$

We conclude this subsection with a computation used in the next section to identify the limit equation. Recall the definition of ψ given in (2.11). Denote by $\kappa(.)$ the function $\kappa(E) = e^{-E/2}\psi'(E)$.

Lemma 3.10 *For any $f : \mathcal{S}_\gamma \to \mathbb{R}$*

$$\int \kappa\big(\theta[\nabla_{x,y} H_s^\alpha](\eta)\big) f(\eta) d\mu_\gamma^{\alpha,\lambda_0}(\eta) = -\int \kappa\big(\theta[\nabla_{x,y} H_s^\alpha](\eta)\big) f(\eta^{x,y}) d\mu_\gamma^{\alpha,\lambda_0}(\eta) \, .$$

Proof. By the explicit formula (2.5) for the measure $\mu_\gamma^{\alpha,\lambda_0}$ and by a change of variables we obtain

$$\begin{aligned}
&\int \kappa\big(\theta[\nabla_{x,y} H_s^\alpha](\eta)\big) f(\eta) d\mu_\gamma^{\alpha,\lambda_0}(\eta)\\
&= \frac{1}{Z_\gamma^{\alpha,\lambda_0}} \sum_{\eta\in\mathcal{S}_\gamma} e^{-\frac{\theta}{2}[\nabla_{x,y}H_s^\alpha](\eta)} \psi'\big(\theta[\nabla_{x,y} H_s^\alpha](\eta)\big) f(\eta) e^{-\theta H_s^\alpha(\eta)} e^{\lambda_0 \sum_{x\in\Lambda_\gamma} \eta(x)}\\
&= \frac{1}{Z_\gamma^{\alpha,\lambda_0}} \sum_{\eta\in\mathcal{S}_\gamma} e^{-\frac{\theta}{2}[H_s^\alpha(\eta^{x,y}) + H_s^\alpha(\eta)]} \psi'\big(\theta[\nabla_{x,y} H_s^\alpha](\eta)\big) f(\eta) e^{\lambda_0 \sum_{x\in\Lambda_\gamma} \eta(x)}\\
&= \frac{1}{Z_\gamma^{\alpha,\lambda_0}} \sum_{\eta\in\mathcal{S}_\gamma} e^{-\frac{\theta}{2}[H_s^\alpha(\eta^{x,y}) + H_s^\alpha(\eta)]} \psi'\big(-\theta[\nabla_{x,y} H_s^\alpha](\eta)\big) f(\eta^{x,y}) e^{\lambda_0 \sum_{x\in\Lambda_\gamma} \eta(x)}\\
&= \frac{1}{Z_\gamma^{\alpha,\lambda_0}} \sum_{\eta\in\mathcal{S}_\gamma} e^{-\frac{\theta}{2}[\nabla_{x,y}H_s^\alpha](\eta)} \psi'\big(-\theta[\nabla_{x,y} H_s^\alpha(\eta)]\big) f(\eta^{x,y}) e^{-\theta H_s^\alpha(\eta)} e^{\lambda_0 \sum_{x\in\Lambda_\gamma} \eta(x)}\\
&= -\int \kappa\big(\theta[\nabla_{x,y} H_s^\alpha](\eta)\big) f(\eta^{x,y}) d\mu_\gamma^{\alpha,\lambda_0}(\eta) \, ,
\end{aligned}$$

where we have used that ψ is odd so that $\psi'(-E) = -\psi'(E)$. $\square$

Corollary 3.11 *For any $f, g : \mathcal{S}_\gamma \to \mathbb{R}$*

$$\int \Phi'\big(\theta[\nabla_{x,y} H_s^\alpha]\big)\big(\nabla_{x,y} f\big)\big(\nabla_{x,y} g\big) d\mu_\gamma^{\alpha,\lambda_0} = -\frac{1}{2}\int \Phi\big(\theta[\nabla_{x,y} H_s^\alpha]\big)\big(\nabla_{x,y} f\big)\big(\nabla_{x,y} g\big) d\mu_\gamma^{\alpha,\lambda_0} \, .$$

Proof. By (2.11), $\Phi'(E) = -(\Phi(E)/2) + \kappa(E)$, so that we have

$$\begin{aligned}
\int \Phi'\big(\theta[\nabla_{x,y} H_s^\alpha]\big)\big(\nabla_{x,y} f\big)\big(\nabla_{x,y} g\big) d\mu_\gamma^{\alpha,\lambda_0} &= -\frac{1}{2}\int \Phi\big(\theta[\nabla_{x,y} H_s^\alpha]\big)\big(\nabla_{x,y} f\big)\big(\nabla_{x,y} g\big) d\mu_\gamma^{\alpha,\lambda_0}\\
&\quad + \int \kappa\big(\theta[\nabla_{x,y} H_s^\alpha]\big)\big(\nabla_{x,y} f\big)\big(\nabla_{x,y} g\big) d\mu_\gamma^{\alpha,\lambda_0} \, .
\end{aligned}$$

The second term of the right hand side is equal to 0, because the function $\big(\nabla_{x,y} f\big)\big(\nabla_{x,y} g\big)$ is invariant under the transformation $\eta \to \eta^{x,y}$, and by Lemma 3.10. $\square$

3.3. Tightness and energy estimate

Proof of proposition 3.1. Tightness can be proven either by using, as in [KL], the Garcia-Rodemich-Rumsey inequality or by adapting to our model the exponential martingale argument used in [VY]. Lemma 4.1 in [VY] depends only on the uniform boundedness of the jump rates and can be easily extended to our model. Lemma 4.2 in [VY], in which the equilibrium measure plays a role, can be adapted applying first next lemma, then Lemma 3.7.

Lemma 3.12 *Let $f : \mathcal{S}_\gamma \times \Omega_D \to I\!R$. For any disorder configuration α, any $v \in I\!R$ and any $0 \le s \le t \le T$, we have*

$$\mathbf{E}^{\beta,\alpha}_{\mu}\Big[v\big(f(\eta_t,\alpha) - f(\eta_s,\alpha)\big)\Big] \le C_0 B + B\gamma^d \log \mathbf{E}^{\beta,\alpha}_{\mu^{\alpha,\lambda_0}_{\gamma}}\Big[\exp\Big\{B^{-1}\gamma^{-d} v\big(f(\eta_{t-s},\alpha) - f(\eta_0,\alpha)\big)\Big\}\Big]$$

for any initial measure μ and $B > 0$, where C_0 is the constant introduced in (3.15).

Proof. Denote by $\big(S^{\beta,\alpha}_\gamma(t)\big)_{t\in[0,T]}$ the semigroup associated to the generator $\gamma^{-2}\mathcal{L}^{\beta,\alpha}_\gamma$. We have

$$\mathbf{E}^{\beta,\alpha}_{\mu}\Big[v\big(f(\eta_t,\alpha) - f(\eta_s,\alpha)\big)\Big] = \mathbf{E}^{\beta,\alpha}_{S^{\beta,\alpha}_\gamma(s)\mu}\Big[v\big(f(\eta_{t-s},\alpha) - f(\eta_0,\alpha)\big)\Big] .$$

We conclude by the entropy inequality, as in (3.27). □

The limit points Q^* are actually concentrated on functions in $C([0,T],\mathcal{M}_1(\Lambda))$, since every jump produces a discontinuity of order $O_u(\gamma^d)$. Furthermore, since there is at most one particle per site, for any continuous function $G : \Lambda \to I\!R$, the quantity $| < \pi^\gamma_t, G > |$ is $I\!P-$a.s. bounded by $\gamma^d \sum_{x\in\Lambda_\gamma} |G(\gamma x)|$ that converges to $\int_\Lambda |G(r)|dr$ as $\gamma \to 0$. All limit points Q^* of the sequence $(Q^{\beta,\alpha}_{\mu_\gamma})_{\gamma\ge 0}$ are thus concentrated on paths such that $\sup_{t\in[0,T]} | < \pi_t, G > | \le \int_\Lambda |G(r)|dr$. The trajectories are therefore absolutely continuous with respect to the Lebesgue measure: (3.1) is satisfied. □

Proof of proposition 3.3. It is the same as in [FM], (3.2), Lemma 3.1, section 4.7. However, the first step of the latter proof requires an application of the Feynman Kac formula, for which we have to replace for our dynamics (4.49) by Lemma 3.7, Remark 3.8. □

4. Identification of the limit

We prove in this section Proposition 3.2. Let Q^* be a limit point of the sequence $(Q^{\beta,\alpha}_{\mu_\gamma})_{\gamma\ge 0}$ and assume, without loss of generality, that $Q^{\beta,\alpha}_{\mu_\gamma}$ converges to Q^*.

Fix a function G in $\mathcal{C}^{1,2}([0,T]\times\Lambda)$. For any $\alpha \in \Omega_D$ consider the $\mathbf{P}^{\beta,\alpha}_{\mu_\gamma}$ martingales with respect to the natural filtration associated with $(\eta_t)_{t\in[0,T]}$, $M^G_t \equiv M^{G,\gamma,\beta,\alpha}_t$ and $N^G_t \equiv N^{G,\gamma,\beta,\alpha}_t$, $t \in [0,T]$, defined by

$$\begin{aligned} M^G_t &= < \pi^\gamma_t, G_t > - < \pi^\gamma_0, G_0 > - \int_0^t \big(< \pi^\gamma_s, \partial_s G_s > + \gamma^{-2}\mathcal{L}^{\beta,\alpha}_\gamma < \pi^\gamma_s, G_s > \big)\, ds , \\ N^G_t &= \big(M^G_t\big)^2 \\ &\quad - \int_0^t \Big\{\gamma^{-2}\mathcal{L}^{\beta,\alpha}_\gamma\big(< \pi^\gamma_s, G_s > \big)^2 - 2 < \pi^\gamma_s, G_s > \gamma^{-2}\mathcal{L}^{\beta,\alpha}_\gamma < \pi^\gamma_s, G_s > \Big\}\, ds , \end{aligned}$$

where π^γ_s is the empirical measure at time s, see (2.23). A computation of the integral term of N^G_t shows that the expectation of the quadratic variation of M^G_t vanishes as $\gamma \downarrow 0$. Therefore, by Doob's inequality, for every $\delta > 0$,

$$\lim_{\gamma\to 0} \mathbf{P}^{\beta,\alpha}_{\mu_\gamma}\Big[\sup_{0\le t\le T} |M^G_t| > \delta\Big] = 0 . \tag{4.1}$$

Thanks to (2.13), a summation by parts permits to rewrite the integral term of M_t^G as

$$\int_0^t < \pi_s^\gamma, \partial_s G_s > ds \; + \; \int_0^t \Big\{ \gamma^{d-1} \sum_{k=1}^d \sum_{x \in \Lambda_\gamma} (\partial_k^\gamma G_s)(\gamma x) \mathbf{J}_{x,x+e_k}^{\gamma,\beta,\alpha}(\eta_s) \Big\} ds \; , \tag{4.2}$$

where $\mathbf{J}_{x,x+e_k}^{\gamma,\beta,\alpha}(\eta)$ is the current over the bond $(x, x+e_k)$:

$$\mathbf{J}_{x,x+e_k}^{\gamma,\beta,\alpha}(\eta) \equiv \mathbf{J}_{x,x+e_k}^{\beta,\alpha} = C_\gamma^{\beta,\alpha}(x, x+e_k; \eta) \big[\eta(x) - \eta(x+e_k) \big] \; . \tag{4.3}$$

We will often omit to write the dependence of $\mathbf{J}_{x,x+e_k}^{\gamma,\beta,\alpha}(\eta)$ on γ and η. We split the current as

$$\mathbf{J}_{x,x+e_k}^{\beta,\alpha} = \mathbf{J}_{x,x+e_k}^{\alpha} + \big[\mathbf{J}_{x,x+e_k}^{\beta,\alpha} - \mathbf{J}_{x,x+e_k}^{\alpha} \big] \; , \tag{4.4}$$

where

$$\mathbf{J}_{x,x+e_k}^{\alpha} = C^\alpha(x, x+e_k; \eta) \big[\eta(x) - \eta(x+e_k) \big] \tag{4.5}$$

is the current of the DLG process, i.e the one generated by $\mathcal{L}_\gamma^\alpha$. By Lemma 3.4,

$$\begin{aligned} \mathbf{J}_{x,x+e_k}^{\beta,\alpha} - \mathbf{J}_{x,x+e_k}^{\alpha} = & - \beta\gamma \Phi' \Big(\theta \big(\nabla_{x,x+e_k} H_s^\alpha \big)(\eta) \Big) \big(\eta(x) - \eta(x+e_k) \big)^2 \big((\partial_e^\gamma J) \star \eta \big)(x) \\ & + O_u(\gamma^2) \; . \end{aligned} \tag{4.6}$$

In the decomposition (4.4) the non gradient difficulties come from the first term $\mathbf{J}_{x,x+e_k}^{\alpha}$, for which we will follow the techniques developed in [F] and [FM]. For the remaining term, considerations in the spirit of [GLM] and [MM] apply. Having in mind (3.6), set, for $0 < a < 1$, $0 < c < 1$, $k = 1, \ldots, d$,

$$\begin{aligned} \mathbf{V}_k^{\gamma,c,a}(\eta, \alpha) \; = & \; \mathbf{J}_{0,e_k}^{\beta,\alpha} \\ & + \sum_{m=1}^d D_{k,m} \Big(\eta^{[a\gamma^{-1}]}(0) \Big) \Big\{ (2c)^{-1} \Big[\eta^{[a\gamma^{-1}]}(c\gamma^{-1} e_m) - \eta^{[a\gamma^{-1}]}(-c\gamma^{-1} e_m) \Big] \\ & - \beta\gamma\chi \Big(\eta^{[a\gamma^{-1}]}(0) \Big) \left((\partial_m^\gamma J) \star \pi^\gamma \right)(0; \eta) \Big\} \; . \end{aligned} \tag{4.7}$$

Next theorem is the main step in the proof of Proposition 3.2.

Theorem 4.1 *Let $d \geq 3$ and assume that $D(\rho)$ defined in (2.20) can be continuously extended in $[0,1]$. Then, for almost any disorder configuration $\alpha \in \Omega_D$, for any $G \in \mathcal{C}^{1,2}([0,T] \times \Lambda)$,*

$$\limsup_{c \to 0} \limsup_{a \to 0} \limsup_{\gamma \to 0} \mathbf{E}_{\mu_\gamma}^{\beta,\alpha} \Big[\Big| \gamma^{d-1} \int_0^T \sum_{x \in \Lambda_\gamma} G_s(\gamma x) \tau_x \mathbf{V}_k^{\gamma,c,a}(\eta_s, \alpha) \, ds \Big| \Big] = 0 \tag{4.8}$$

for $k = 1, \ldots, d$.

By a summation by parts, Theorem 4.1 allows to conclude the proof of Proposition 3.2. Details can be found in [KL], section VII.1.

Before proving Theorem 4.1 we introduce some notation and recall some tools of non gradient methods. We refer mainly to [F], [FM], see also [V], [VY] and [KL], section VII. Given $\alpha \in \Omega_D$, denote by $\mathcal{L}^\alpha \equiv \mathcal{L}^{0,\alpha}$ the pregenerator of the DLG process in infinite volume (cf. (2.16)),

$$\left(\mathcal{L}^\alpha f \right)(\eta) = \sum_{e \in \mathcal{E}} \sum_{x \in \mathbb{Z}^d} C^\alpha(x, x+e; \eta) (\nabla_{x,x+e} f)(\eta) \; , \tag{4.9}$$

where f is a local function on $\mathcal{S}$. We refer to [Li] for the construction of the process in the infinite volume setting, and we recall that for every $\lambda \in I\!R$, $\mathcal{L}^\alpha$ can be extended to a self-adjoint operator on $L^2(\mu^{\alpha,\lambda})$. For a finite non empty subset V of $Z\!\!Z^d$, $\rho \in [0, |V|^{-1}, \dots, 1]$ and $\alpha \in \Omega_D$, the canonical measure $\nu^\alpha_{\rho,V}$ is defined as in (2.18), with Λ_γ replaced by V. We denote by $\mathcal{M}^\alpha(V)$ the set of all canonical measures as ρ varies in $[0, |V|^{-1}, \dots, 1]$, and by ν^α a generic element of $\mathcal{M}^\alpha(V)$. Let $\mathcal{G} \subset \mathbb{G}$, see (2.19), be the space of bounded cylinder functions h for which there exists a finite non empty set $V \subset Z\!\!Z^d$ such that the support of $h(\cdot, \alpha)$ is contained in V and, for any given disorder configuration $\alpha \in \Omega_D$, all canonical expectations on V are null:

$$\mathcal{G} = \Big\{ h \in \mathbb{G} \text{ ; support of } \{h(\cdot,\alpha)\} \subset V \quad \text{and } \forall \alpha \in \Omega_D, \ \ \forall \nu^\alpha \in \mathcal{M}^\alpha(V), \ \ \mathbf{E}^{\nu^\alpha}[h(\cdot,\alpha)] = 0 \Big\} \ . \tag{4.10}$$

Given a positive density $0 < \rho < 1$, f and g in $\mathcal{G}$, define

$$V_\rho(h,g) = \lim_{\ell\to\infty} (2\ell)^{-d} I\!E \left[\mathbf{E}^{\mu^{\alpha,\lambda_0(\rho)}} \Big(\sum_{|x| \le \ell - \sqrt{\ell}} \tau_x h \ , \ \big(-\mathcal{L}^\alpha_{0,\ell}\big)^{-1} \sum_{|x|\le \ell-\sqrt{\ell}} \tau_x g \Big) \right] , \tag{4.11}$$

where $\mathcal{L}^\alpha_{0,\ell}$ is obtained from $\mathcal{L}^\alpha$ by restricting jumps to the cube $\Lambda_{0,\ell}$. In the extreme densities cases $\rho = 0$ or $\rho = 1$, i.e. when the measure is concentrated on configurations $\eta = 0$ or $\eta = 1$ in $\Lambda_{0,\ell}$, for any $\ell \in Z\!\!Z$, set $V_\rho(h,g) = 0$. It has been shown in [F] and [FM], Theorem 7.2, that the above limit exists and is finite. Moreover $V_\rho(\cdot,\cdot)$ defines a semi-inner product on $\mathcal{G}$. When $h = g$ we write $V_\rho(h)$ in place of $V_\rho(h,h)$.

We consider the orthogonal decomposition of $\mathcal{G}$, endowed with the semi-inner product V_ρ, along $\mathcal{L}^\alpha\,\mathbb{G} = \{\mathcal{L}^\alpha g \ : \ g \in \mathbb{G}\}$ and its orthogonal subspace $(\mathcal{L}^\alpha\,\mathbb{G})^\perp$. As explained briefly in the introduction, the general strategy for non gradient systems is to write the current, which has zero average with respect to all canonical and grandcanonical measures, as a linear combination of the density gradient and a fluctuation term

$$J^\alpha_{0,e} \simeq \nabla_{0,e}\eta(0) + \mathcal{L}^\alpha g \ .$$

We are neglecting the diffusion coefficient for simplicity. The fluctuation term belongs to $\mathcal{G}$, while the presence of the disorder induces $\nabla_{0,e}\eta(0) \notin \mathcal{G}$.

This aspect is present also in the non disordered systems considered in [VY]. In this paper S.R.S. Varadhan and H.T. Yau study the hydrodynamic behavior of a generic lattice gas with a translation invariant and finite range Hamiltonian satisfying some mixing conditions, with a stochastic dynamics reversible with respect to Gibbs measures. In this case the canonical expectations of $\nabla_{0,e}\eta(0)$ in a cube of size n decay as a power of n when $n \uparrow \infty$. Differently, in the case of the DLG process one can check that for any $n \in I\!N$,

$$\sup_{\rho\in[0,1]} \mathbf{E}^{\nu^\alpha_{\rho,\Lambda_{0,n}}} [\nabla_{0,e}\eta(0)] = O(1) \ . \tag{4.12}$$

Namely for any disorder configuration $\alpha \in \Omega_D$ and any chemical potential $\lambda \in I\!R$,

$$\mathbf{E}^{\mu^{\alpha,\lambda}} [\nabla_{0,e}\eta(0)] = \frac{e^\lambda (e^{\theta\alpha(e)} - e^{\theta\alpha(0)})}{(1+e^{\theta\alpha(e)+\lambda})(1+e^{\theta\alpha(0)+\lambda})} \ . \tag{4.13}$$

By the equivalence of ensembles, see [FM], Appendix A, for a positive constant c and for any density ρ on the cube $\Lambda_{0,n}$,

$$\Big| \mathbf{E}^{\mu^{\alpha,\lambda}} [\nabla_{0,e}\eta(0)] - \mathbf{E}^{\nu^\alpha_{\rho,\Lambda_{0,n}}} [\nabla_{0,e}\eta(0)] \Big| \le \frac{c}{n^d} \tag{4.14}$$

provided λ in (4.14) is chosen such that $\mathbf{E}^{\mu^{\alpha,\lambda}} \Big(|\Lambda_{0,n}|^{-1} \sum_{x\in\Lambda_{0,n}} \eta(x) \Big) = \rho$. For densities ρ such that λ is almost 0 one obtains (4.12). Here $\lambda = \lambda_{\Lambda_{0,n}}(\alpha,\rho)$ is the *empirical chemical potential*, to distinguish from the annealed chemical potential, see (1.4).

Exploiting that $I\!\!E\left[\mathbf{E}^{\mu^{\alpha,\lambda}}\left[\nabla_{0,e}\eta(0)\right]\right] = 0$, [F] and [FM] considered the gradient of the density in two sufficiently large regions. By the ergodicity of the random field α, the density fluctuations in these regions are small provided the dimension d is larger or equal to 3.

To be more precise, given $s = 2\ell + 1$ with $\ell \in I\!\!N$ and $e \in \mathcal{E}$, let $\Lambda^e_{1,s}$ and $\Lambda^e_{2,s}$ be a couple of adjacent cubes of diameter s, centered respectively at $-(\ell+1)e$ and at ℓe. For any given configuration η, denote by $m^{1,e}_s$, $m^{2,e}_s$, and m^e_s the densities respectively in $\Lambda^e_{1,s}$, $\Lambda^e_{2,s}$, and $\Lambda^e_{2,s} \cup \Lambda^e_{1,s}$. Given an integer s' with $s \le s'$, set

$$\phi^e_{s,s'} = \mathbf{E}^{\mu^\alpha}\left[m^{2,e}_s - m^{1,e}_s | m^e_{s'}\right] \quad \text{and} \quad \psi^e_{s,s'} = m^{2,e}_s - m^{1,e}_s - \phi^e_{s,s'} \ . \tag{4.15}$$

Note that $I\!\!E[\phi^e_{s,s'}] = 0$. The main step to obtain a generalized Fick's law, see [FM], Theorem 7.18, is to show the property

(P) for $d \ge 3$ and for any $e \in \mathcal{E}$, $\left((\psi^e_{n,n})/n\right)_{n\ge 0}$ is a Cauchy sequence in the space $\mathcal{G}$ endowed with the semi-inner product V_ρ, and its limit points $(\psi_e)_{e\in\mathcal{E}}$ form a basis of the subspace $(\mathcal{L}^\alpha\, \mathbb{G})^\perp$.

Then the current $\mathbf{J}^\alpha_{0,e_k}$ can be substituted in the limits $n \to \infty$ after $\gamma \to 0$ by some negligible fluctuation $\mathcal{L}^\alpha g$ plus

$$\sum_{m=1}^d D_{k,m}(\rho)\frac{\left[m^{1,e_k}_n - m^{2,e_k}_n\right]}{n} - \sum_{m=1}^d D_{k,m}(\rho)\frac{\phi^{e_k}_{n,n}}{n}, \tag{4.16}$$

where ρ is the density in a particular mesoscopic region centered in 0. Then, [FM], Theorem 5.3 shows that suitable spatial averages of $(\phi^e_{n,n})/n$ are indeed negligible when $d \ge 3$, by letting first $\gamma \to 0$ then $n \to \infty$. An important step in [F] and [FM] section 7.2, to show property *(P)* is the introduction of the following auxiliary functions: For the integer $s = 2\ell + 1$, $\ell \in I\!\!N$ and $e \in \mathcal{E}$, let

$$\mathbf{W}^e_s = -\frac{1}{|\Lambda^e_{1,s}|}\sum_{x\in\Lambda^e_{1,s}}\left\{\frac{1}{|\Lambda^e_{2,s}|}\sum_{y\in\Lambda^e_{2,s}}\omega^\alpha_{x,y}\right\} ,$$

where

$$\omega^\alpha_{x,y} = \left(1 + e^{-\theta(\alpha(x)-\alpha(y))(\eta(x)-\eta(y))}\right)\left(\eta(x) - \eta(y)\right)$$

and $\Lambda^e_{1,s}$ and $\Lambda^e_{2,s}$ are the cubes defined before (4.15). When x and y are nearest neighbors, $\omega^\alpha_{x,y}$ is the current associated to a particular choice of the rate $C^\alpha(x,y;\eta)$ corresponding to $\Phi(E) = 1 + e^{-E}$ in (2.17). It has the important property to have mean zero with respect to any measure ν^α in $\mathcal{M}^\alpha(V)$, $V \subset Z\!\!\!Z^d$ being any bounded set containing x and y. Furthermore it yields a simple integration by parts formula

$$\int \omega^\alpha_{x,y} f(\eta) d\nu^\alpha(\eta) = \int [\eta(x) - \eta(y)](\nabla_{x,y} f)(\eta) d\nu^\alpha(\eta) \ .$$

It is proven in [FM], Theorem 7.10, that for any $e \in \mathcal{E}$ and $0 \le \rho \le 1$,

$$\lim_{n\uparrow\infty} V_\rho\left(2\rho(1-\rho)\lambda'_0(\rho)\frac{\psi^e_{n,n}}{n} - \frac{\mathbf{W}^e_n}{n}\right) = 0 \ . \tag{4.17}$$

Moreover, if for $g \in \mathbb{G}$ and $h \in \mathcal{G}$ we define

$$\left(h, g\right)_{\rho,0} = \sum_{x\in Z\!\!\!Z^d} I\!\!E\left[\mathbf{E}^{\mu^{\alpha,\lambda_0(\rho)}}\left(h, \tau_x g\right)\right] \tag{4.18}$$

we obtain by the definition of $V_\rho(.,.)$ the following properties (cf. Lemma 7.1 of [FM]).

$$
\begin{aligned}
V_\rho\big(h, \mathcal{L}^\alpha g\big) &= -\big(h, g\big)_{\rho,0} \,, \\
V_\rho\left(\mathbf{J}^\alpha_{0,e_k}, \mathbf{J}^\alpha_{0,e_m}\right) &= \frac{\delta_{k,m}}{2} I\!E\left[\mathbf{E}^{\mu^{\alpha,\lambda_0(\rho)}}\left(C^\alpha(0,e_k;\eta)\big(\nabla_{0,e_k}\eta(0)\big)^2\right)\right] \,, \\
V_\rho\left(\mathbf{J}^\alpha_{0,e_k}, \frac{\mathbf{W}^{e_m}_n}{n}\right) &= -\delta_{k,m} 2\rho(1-\rho) \,,
\end{aligned}
\tag{4.19}
$$

where $\delta_{k,m}$ is the Kroenecker delta. Thanks to (4.17) and the last identity in (4.19) one obtains, cf. [FM], Theorem 7.18, that

$$
\lim_{n\to\infty} V_\rho\left(\mathbf{J}^\alpha_{0,e_k}, \frac{\psi^{e_m}_{n,n}}{n}\right) = V_\rho\left(\mathbf{J}^\alpha_{0,e_k}, \psi_{e_m}\right) = -\delta_{k,m}\chi(\rho) \,. \tag{4.20}
$$

We now turn to the proof of theorem 4.1.

proof of Theorem 4.1. From [FM], Theorem 7.22, for all $1 \le k \le d$,

$$
\inf_{g\in\mathbb{G}} \limsup_{n\uparrow\infty} \sup_{0\le\rho\le 1} V_\rho\Big(\mathbf{J}^\alpha_{0,e_k} + \sum_{m=1}^d D_{k,m}(\rho)\frac{\psi^{e_m}_{n,n}}{n} + \mathcal{L}^\alpha g(\eta,\alpha)\Big) = 0 \,, \tag{4.21}
$$

where J^α_{0,e_k} is the current of the DLG process, see (4.5), $D(\rho)$ is the diffusion matrix of (2.20), and $\psi^{e_m}_{n,n}$ the quantity defined in (4.15). By similar arguments as in [KL], chapter VII p. 179, for each $\delta > 0$, there exists $g^{k,\delta} : [0,1]\times\mathcal{S}\times\Omega_D \to I\!R^d$ with $g^{k,\delta}(\rho,\cdot,\cdot) \in \mathbb{G}$, $g^{k,\delta}$ smooth in the first variable, such that

$$
\limsup_{n\uparrow\infty} \sup_{0\le\rho\le 1} V_\rho\Big(\mathbf{J}^\alpha_{0,e_k} + \sum_{m=1}^d D_{k,m}(\rho)\frac{\psi^{e_m}_{n,n}}{n} + \mathcal{L}^\alpha g^{k,\delta}\Big) \le \delta \,. \tag{4.22}
$$

Notice that $g^{k,\delta}$ depends on α, η and on the local average $(\ell_\delta^d)^{-1}\sum_{|y|\le\ell_\delta}\eta(y)$ where ℓ_δ is the diameter of the support in η of $g^{k,\delta}$. To keep notation light we denote it simply by $g^{k,\delta}(\eta,\alpha)$.

We claim that $I\!P$-a.s.,

$$
\limsup_{\gamma\to 0} \mathbf{E}^{\beta,\alpha}_{\mu_\gamma}\Big[\Big|\int_0^T ds\Big(\gamma^{d-1}\sum_{x\in\Lambda_\gamma} G_s(\gamma x)\tau_x\mathcal{L}^{\beta,\alpha}_\gamma g^{k,\delta}(\eta_s,\alpha)\Big)\Big|\Big] = 0 \,, \tag{4.23}
$$

for all real smooth, bounded functions $G_s(u) = G(s,u)$ defined on $I\!R_+\times\Lambda_\gamma$. Indeed

$$
\begin{aligned}
&\int_0^T ds\Big(\gamma^{d-1}\sum_{x\in\Lambda_\gamma} G_s(\gamma x)\tau_x\mathcal{L}^{\beta,\alpha}_\gamma g^{k,\delta}(\eta_s,\alpha)\Big) \\
&= \gamma^{d+1}\sum_{x\in\Lambda_\gamma}\left[G_T(\gamma x)\tau_x g(\eta_T,\alpha) - G_0(\gamma x)\tau_x g(\alpha,\eta_0)\right] + \gamma\tilde{M}^{\alpha,G}_\gamma(T) + R^{\alpha,G}_\gamma(T) \,.
\end{aligned}
\tag{4.24}
$$

In the r.h.s. of (4.24), the first term is of order γ. The $\mathbf{P}^{\beta,\alpha}_{\mu_\gamma}$-martingale $(\tilde{M}^{\alpha,G}_\gamma(t))_{t\in[0,T]}$ has quadratic variation of order $O_u(\gamma^d)$. The error term $R^{\alpha,G}_\gamma(T)$ is coming from ignoring the action of the generator on $(\ell_\delta^d)^{-1}\sum_{|y|\le\ell_\delta}\eta(y)$; as in [KL], chapter VII, we can easily obtain that $\sup_{t\in[0,T]}|R^{\alpha,G}_\gamma(t)|$ tends to zero in probability and $I\!P$-a.s., as $\gamma\to 0$ and $\delta\to 0$. This yields (4.23).

We will therefore through the rest of the proof ignore the action of the generator on the non local variable of $g^{k,\delta}$, and consider effectively $g^{k,\delta}$ as a function only of η and α. Then it is equivalent to prove (4.8) or

$$\limsup_{\delta\to 0}\limsup_{c\to 0}\limsup_{a\to 0}\limsup_{\gamma\to 0}\mathbf{E}^{\beta,\alpha}_{\mu_\gamma}\Big[\Big|\int_0^T \gamma^{d-1}\sum_{x\in\Lambda_\gamma} G_s(\gamma x)\left[\tau_x\mathbf{V}^{\gamma,c,a}_k(\eta_s,\alpha)+\tau_x\mathcal{L}^{\beta,\alpha}_\gamma g^{k,\delta}(\eta_s,\alpha)\right]\,ds\Big|\Big]=0\,. \tag{4.25}$$

Next, split

$$\mathbf{V}^{\gamma,c,a}_k(\eta,\alpha)+\mathcal{L}^{\beta,\alpha}_\gamma g^{k,\delta}(\eta,\alpha)=Y^{\gamma,c,a,\delta}_k(\eta,\alpha)+Z^{\gamma,c,a,\delta}_k(\eta,\alpha) \tag{4.26}$$

where

$$\begin{aligned} Y^{\gamma,c,a,\delta}_k(\eta,\alpha)&=\mathbf{J}^\alpha_{0,e_k}+\mathcal{L}^\alpha_\gamma g^{k,\delta}(\eta,\alpha)\\ &\quad+\sum_{m=1}^d D_{k,m}\left(\eta^{[a\gamma^{-1}]}(0)\right)(2c)^{-1}\left[\eta^{[a\gamma^{-1}]}(c\gamma^{-1}e_m)-\eta^{[a\gamma^{-1}]}(-c\gamma^{-1}e_m)\right],\\ Z^{\gamma,c,a,\delta}_k(\eta,\alpha)&=\left(\mathbf{J}^{\beta,\alpha}_{0,e_k}-\mathbf{J}^\alpha_{0,e_k}\right)+\left(\mathcal{L}^{\beta,\alpha}_\gamma-\mathcal{L}^\alpha_\gamma\right)g^{k,\delta}(\eta,\alpha)\\ &\quad-\beta\gamma\sum_{m=1}^d D_{k,m}\left(\eta^{[a\gamma^{-1}]}(0)\right)\chi\left(\eta^{[a\gamma^{-1}]}(0)\right)\left(\left(\partial^\gamma_m J\right)\star\pi^\gamma\right)(0;\eta)\,. \end{aligned} \tag{4.27}$$

To conclude the proof of the theorem, taking into account (4.25), it is enough to show the two lemmas:

Lemma 4.2 *For almost any disorder configuration α, for any function $G\in\mathcal{C}^{1,2}([0,T]\times\Lambda)$,*

$$\limsup_{\delta\to 0}\limsup_{c\to 0}\limsup_{a\to 0}\limsup_{\gamma\to 0}\mathbf{E}^{\beta,\alpha}_{\mu_\gamma}\Big[\Big|\int_0^T \gamma^{d-1}\sum_{x\in\Lambda_\gamma} G_s(\gamma x)\tau_x Y^{\gamma,c,a,\delta}_k(\eta_s,\alpha)\,ds\Big|\Big]=0$$

for $k=1,\dots,d$.

Lemma 4.3 *For almost any disorder configuration α, for any function $G\in\mathcal{C}^{1,2}([0,T]\times\Lambda)$,*

$$\limsup_{\delta\to 0}\limsup_{c\to 0}\limsup_{a\to 0}\limsup_{\gamma\to 0}\mathbf{E}^{\beta,\alpha}_{\mu_\gamma}\Big[\Big|\int_0^T \gamma^{d-1}\sum_{x\in\Lambda_\gamma} G_s(\gamma x)\tau_x Z^{\gamma,c,a,\delta}_k(\eta_s,\alpha)\,ds\Big|\Big]=0$$

for $k=1,\dots,d$.

Proof of Lemma 4.2. Set

$$\mathbf{P}^{\gamma,c,a,\delta}_k(G,s,\eta,\alpha)=\gamma^{d-1}\sum_{x\in\Lambda_\gamma}G_s(\gamma x)\tau_x Y^{\gamma,c,a,\delta}_k(\eta,\alpha)\,. \tag{4.28}$$

By Corollary 3.7, it is enough to show that for any $C>0$, for any function G, for almost any $\alpha\in\Omega_D$,

$$\limsup_{\delta\to 0}\limsup_{c\to 0}\limsup_{a\to 0}\limsup_{\gamma\to 0}\int_0^T\left(\sup spec_{L^2(\mu^{\alpha,\lambda_0})}\left\{\mathbf{P}^{\gamma,c,a,\delta}_k(G,s,\cdot,\alpha)+C\gamma^{d-2}\mathcal{L}^\alpha_\gamma\right\}\right)ds=0\,.$$

This is the main content of [F] and [FM], Theorem 3.2, with the only observation that there the term $\mathcal{L}^\alpha g^{k,\delta}$ does not appear: in their case by stochastic calculus it is possible to show (see [FM], formula (4.4)) that this term is irrelevant. □

Proof of Lemma 4.3. We start by analyzing the first addend of $Z_k^{\gamma,c,a,\delta}$, the difference $\mathbf{J}^{\beta,\alpha}_{0,e_k} - \mathbf{J}^{\alpha}_{0,e_k}$. Set

$$F_1^k(\eta,\alpha) = -\beta\Phi'\big(\theta\big(\nabla_{0,e_k} H_s^\alpha\big)(\eta)\big)\big(\eta(e_k)-\eta(0)\big)^2 \ . \tag{4.29}$$

By (4.6) we obtain

$$\gamma^{d-1}\sum_{x\in\Lambda_\gamma} G(\gamma x)\big[\mathbf{J}^{\beta,\alpha}_{x,x+e_k} - \mathbf{J}^{\alpha}_{x,x+e_k}\big] = \gamma^d \sum_{x\in\Lambda_\gamma} G(\gamma x)\big((\partial_k^\gamma J)\star\eta\big)(x)\tau_x F_1^k(\eta,\alpha) + O_u(\gamma) \ .$$

Denote

$$\begin{aligned}\mathcal{A}_{a,\gamma}((\eta_s)_{s\in[0,T]}) &= \int_0^T \gamma^{d-1}\Big\{\sum_{x\in\Lambda_\gamma} G_s(\gamma x)\big[\mathbf{J}^{\beta,\alpha}_{x,x+e_k}(\eta_s) - \mathbf{J}^{\alpha}_{x,x+e_k}(\eta_s)\big]\Big\}ds \\ &- \int_0^T \gamma^d \sum_{x\in\Lambda_\gamma} G_s(\gamma x)\big((\partial_k^\gamma J)\star\eta_s\big)(x)\Big\{\frac{1}{(2a\gamma^{-1}+1)^d}\sum_{|y|\le a\gamma^{-1}}\tau_{x+y}F_1^k(\eta_s,\alpha)\Big\}ds \ .\end{aligned}$$

The smoothness of G and J induce that, for any $\alpha\in\Omega_D$,

$$\lim_{a\to 0}\limsup_{\gamma\to 0}\sup_{(\eta_s)_{s\in[0,T]}}\big|\mathcal{A}_{a,\gamma}((\eta_s)_{s\in[0,T]})\big| = 0 \ . \tag{4.30}$$

Recalling (3.29), write

$$\widetilde{F_1^k}(\rho) = I\!E\Big[\mathbf{E}^{\mu^{\alpha,\lambda_0(\rho)}}\Big(F_1^k\Big)\Big] \ . \tag{4.31}$$

By applying (4.30) together with Lemma 3.9 to the local function $F_1^k(\eta,\alpha)$, we have $I\!P$ a.s.

$$\lim_{a\to 0}\lim_{\gamma\to 0}\mathbf{E}^{\beta,\alpha}_{\mu_\gamma}\Big[\Big|\gamma^d\sum_{x\in\Lambda_\gamma}\int_0^T G(\gamma x)\tau_x\mathcal{B}^{\beta,1}_{k,\gamma,a}(\eta_s,\alpha)ds\Big|\Big] = 0 \ , \tag{4.32}$$

where

$$\mathcal{B}^{\beta,1}_{k,\gamma,a}(\eta,\alpha) = \gamma^{-1}\big(\mathbf{J}^{\beta,\alpha}_{0,e_k}(\eta) - \mathbf{J}^{\alpha}_{0,e_k}(\eta)\big) - \beta\big((\partial_k^\gamma J)\star\pi^\gamma\big)(0;\eta)\widetilde{F_1^k}(\eta^{[a\gamma^{-1}]}) \ .$$

Next we consider the second term of $Z_k^{\gamma,c,a,\delta}$, the difference $\big(\mathcal{L}^{\beta,\alpha}-\mathcal{L}^\alpha\big)g^{k,\delta}$, and repeat the same steps as for the first term. Recalling that $g^{k,\delta}$ is a local and bounded function, by Lemma 3.4 we have

$$\begin{aligned}&\big(\mathcal{L}^{\beta,\alpha}-\mathcal{L}^\alpha\big)g^{k,\delta}(\eta,\alpha) \\ &= \sum_{m=1}^d\sum_{y\in\Lambda_\gamma}\gamma\beta\Phi'\big(\theta(\nabla_{y,y+e_m}H_s^\alpha)(\eta)\big)\big(\eta(y+e_m)-\eta(y)\big)\big((\partial_m^\gamma J)\star\eta\big)(y)\big[\nabla_{y,y+e_m}g^{k,\delta}\big](\eta,\alpha) + O_u(\gamma^2)O_u(\ell^{-d}),\end{aligned}$$

where ℓ is the diameter of the support of $g^{k,\delta}$. By the smoothness of J, a Taylor expansion yields

$$\begin{aligned}&\gamma^{d-1}\sum_{x\in\Lambda_\gamma} G(\gamma x)\tau_x\big(\mathcal{L}^{\beta,\alpha}-\mathcal{L}^\alpha\big)g^{k,\delta}(\eta,\alpha) \\ &= \gamma^d\sum_{m=1}^d\sum_{x\in\Lambda_\gamma} G(\gamma x)\big((\partial_m^\gamma J)\star\pi^\gamma\big)(\gamma x;\eta)\tau_x F_2^{k,m,\delta}(\eta,\alpha) + O_u(\gamma)O_u(\ell^{-d})\end{aligned}$$

for the local and bounded functions $F_2^{k,m,\delta}$ given by

$$F_2^{k,m,\delta}(\eta,\alpha) = \beta \sum_{y\in\Lambda_\gamma} \Phi'\big(\theta(\nabla_{y,y+e_m} H_s^\alpha)(\eta)\big)\big(\nabla_{y,y+e_m}\eta(y)\big)\big[\nabla_{y,y+e_m} g^{k,\delta}\big](\eta,\alpha)\ . \tag{4.33}$$

Denote

$$\mathcal{B}_{k,\gamma,a}^{\beta,\delta,2}(\eta,\alpha) = \gamma^{-1}\big(\mathcal{L}^{\beta,\alpha} - \mathcal{L}^\alpha\big) g^{k,\delta}(\eta,\alpha) - \sum_{m=1}^d \big((\partial_m^\gamma J)\star\pi^\gamma\big)(0;\eta)\widetilde{F_2}^{k,m,\delta}\Big(\eta^{[a\gamma^{-1}]}(0)\Big) + O_u(\gamma)O_u(\ell^{-d})\ .$$

By Lemma 3.9, for any fixed $\delta > 0$,

$$\lim_{a\to 0}\lim_{\gamma\to 0} \mathbf{E}_{\mu_\gamma}^{\beta,\alpha}\Big[\Big|\gamma^d \sum_{x\in\Lambda_\gamma}\int_0^T G(\gamma x)\tau_x \mathcal{B}_{k,\gamma,a}^{\beta,\delta,2}(\eta_s,\alpha)ds\Big|\Big] = 0\ . \tag{4.34}$$

We conclude the proof by collecting the estimates (4.32), (4.34) and using Lemma 4.4 below. □

Lemma 4.4 For $1\le k,m\le d$,

$$\lim_{\delta\to 0}\sup_{0\le\rho\le 1}\Big|\delta_{k,m}\widetilde{F_1^k}(\rho) + \widetilde{F_2^{k,m,\delta}}(\rho) - \beta\chi(\rho)D_{k,m}(\rho)\Big| = 0\ . \tag{4.35}$$

Proof. Applying Corollary 3.11 with $f(\eta) = g(\eta) = \eta(0)$ to (4.29), we have

$$\begin{aligned}\widetilde{F_1^k}(\rho) =& I\!E\Big[\mathbf{E}^{\mu^{\alpha,\lambda_0(\rho)}}\Big(F_1^k\Big)\Big]\\ =& \frac{\beta}{2} I\!E\Big[\int \Phi\big(\theta\big(\nabla_{0,e_k}H_s^\alpha\big)(\eta)\big)\big(\eta(e_k)-\eta(0)\big)^2 d\mu^{\alpha,\lambda_0(\rho)}(\eta)\Big]\ .\end{aligned} \tag{4.36}$$

Corollary 3.11 applied to (4.33), with $f(\eta) = \eta(y)$ and $g(\eta,\alpha) = g^{k,\delta}(\eta,\alpha)$, reversibility and summation by parts imply that, for $1\le m\le d$ and $1\le k\le d$,

$$\begin{aligned}\widetilde{F_2}^{k,m,\delta}(\rho) &\equiv I\!E\Big[\mathbf{E}^{\mu^{\alpha,\lambda_0(\rho)}}\Big({F_2}^{k,m,\delta}\Big)\Big]\\ &= -\frac{\beta}{2}\sum_{y\in\Lambda_\gamma} I\!E\Big[\mathbf{E}^{\mu^{\alpha,\lambda_0(\rho)}}\Big(\Phi\big(\theta\big(\nabla_{y,y+e_m}H_s^\alpha\big)(\eta)\big)\big(\nabla_{y,y+e_m}\eta(y)\big)\big(\nabla_{y,y+e_m}g^{k,\delta}(\eta,\alpha)\big)\Big)\Big]\\ &=\beta\sum_{y\in\Lambda_\gamma} I\!E\Big[\mathbf{E}^{\mu^{\alpha,\lambda_0(\rho)}}\Big(\Phi\big(\theta\big(\nabla_{y,y+e_m}H_s^\alpha\big)(\eta)\big)\big(\eta(y+e_m)-\eta(y)\big)g^{k,\delta}(\eta,\alpha)\Big)\Big]\\ &= -\beta\Big(\mathbf{J}_{0,e_m}^\alpha, g^{k,\delta}\Big)_{\rho,0}\ ,\end{aligned} \tag{4.37}$$

where $(.,.)_{\rho,0}$ was defined in (4.18).

By (4.19) the terms $\widetilde{F_1^k}(\rho)$ in (4.36) and $\widetilde{F_2^{k,m,\delta}}(\rho)$ in (4.37) can be written as

$$\delta_{k,m}\widetilde{F_1^k}(\rho) = \beta V_\rho\left(\mathbf{J}_{0,e_k}^\alpha, \mathbf{J}_{0,e_m}^\alpha\right) \tag{4.38}$$

$$\widetilde{F_2^{k,m,\delta}}(\rho) = \beta V_\rho\Big(\mathbf{J}_{0,e_m}^\alpha, \mathcal{L}^\alpha g^{k,\delta}\Big)\ . \tag{4.39}$$

We obtain that

$$\sup_{0\le\rho\le1} \beta^{-1} \left| \delta_{k,m} \widetilde{F_1^k}(\rho) + \widetilde{F_2^{k,m,\delta}}(\rho) - \beta\chi(\rho) D_{k,m}(\rho) \right|$$
$$= \limsup_{n\uparrow\infty} \sup_{0\le\rho\le1} \left| V_\rho \left(\mathbf{J}^\alpha_{0,e_m}, \mathbf{J}^\alpha_{0,e_k} + \sum_{\ell=1}^d D_{k,\ell}(\rho) \frac{\psi^{e_\ell}_{n,n}}{n} + \mathcal{L}^\alpha g^{k,\delta} \right) \right| .$$

By Schwartz inequality the right hand side of this equality is bounded by

$$\limsup_{n\uparrow\infty} \sup_{0\le\rho\le1} \left\{ V_\rho^{\frac12} \left(\mathbf{J}^\alpha_{0,e_m} \right) V_\rho^{\frac12} \left(\mathbf{J}^\alpha_{0,e_k} + \sum_{\ell=1}^d D_{k,\ell}(\rho) \frac{\psi^{e_\ell}_{n,n}}{n} + \mathcal{L}^\alpha g^{k,\delta} \right) \right\}$$

which is bounded, see (4.22), by $C\sqrt{\delta}$ for some positive constant C. To conclude the proof of the lemma it remains to let $\delta \downarrow 0$. □

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