Asymptotic Size of Herman Rings of the Complex Standard Family by Quantitative Quasiconformal Surgery

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Abstract

In this paper we consider the complexification of the Arnold standard family of circle maps given by $\widetilde{F}_{\alpha,\varepsilon}(u) = u\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\frac{\varepsilon}{2}(u-\frac{1}{u})}$, with $\alpha = \alpha(\varepsilon)$ chosen so that $\widetilde{F}_{\alpha(\varepsilon),\varepsilon}$ restricted to the unit circle has a prefixed rotation number θ belonging to the set of Brjuno numbers. In this case, it is known that $\widetilde{F}_{\alpha(\varepsilon),\varepsilon}$ is analytically linearizable if ε is small enough, and so, it has a Herman ring $\widetilde{U}_\varepsilon$ around the unit circle. Using Yoccoz's estimates, one has that *the size* $\widetilde{R}_\varepsilon$ of $\widetilde{U}_\varepsilon$ (so that $\widetilde{U}_\varepsilon$ is conformally equivalent to $\{u \in \mathbb{C} : \ 1/\widetilde{R}_\varepsilon < |u| < \widetilde{R}_\varepsilon\}$) goes to infinity as $\varepsilon \to 0$, but one may ask for its asymptotic behavior.

We prove that $\widetilde{R}_\varepsilon = \frac{2}{\varepsilon}(R_0 + \mathcal{O}(\varepsilon \log \varepsilon))$, where R_0 is the conformal radius of the Siegel disk of the complex semistandard map $G(z) = z\mathrm{e}^{\mathrm{i}\omega}\mathrm{e}^{z}$, where $\omega = 2\pi\theta$. In the proof we use a very explicit quasiconformal surgery construction to relate $\widetilde{F}_{\alpha(\varepsilon),\varepsilon}$ and G, and hyperbolic geometry to obtain the quantitative result.

Contents

1 Introduction

The *complex standard family* of self maps of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is given by the two-parameter family

$$\widetilde{F}_{\alpha,\varepsilon}(u) = u\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\frac{\varepsilon}{2}(u-\frac{1}{u})},$$

where $\alpha \in [0, 2\pi)$ and $\varepsilon \in [0, 1)$. These maps are holomorphic in $\mathbb{C}^*$ and the points at 0 and infinity are essential singularities (see [Ba, Ko1, Mak, Ke, Ko2, F]). For ε small, these functions are perturbations of the rotation of angle α with respect to the origin. The interest of this family lies on the fact that it is the extension to the complex plane of the well-known *Arnold family* of circle maps (see [Ar, dMvS]). Indeed, the unit circle $\mathbf{C}_1$ is invariant under $\widetilde{F}_{\alpha,\varepsilon}$, and using the lift $\mathrm{e}^{2\pi \mathrm{i}x}$, $\widetilde{F}_{\alpha,\varepsilon|\mathbf{C}_1}$ becomes the Arnold family:

$$\begin{array}{rccl} \widetilde{f}_{\alpha,\varepsilon}: & \mathbb{T}^1 & \longrightarrow & \mathbb{T}^1 \\ & x & \longrightarrow & x + \frac{\alpha}{2\pi} + \frac{\varepsilon}{2\pi}\sin(2\pi x) \end{array} \tag{1}$$

where $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$.

It is clear that if $\varepsilon \in [0, 1)$, $\widetilde{f}_{\alpha,\varepsilon}$ is an orientation preserving diffeomorphism of $\mathbb{T}^1$, and thus, for each pair of parameters (α, ε) the *rotation number* of $\widetilde{f}_{\alpha,\varepsilon}$ is well defined (see Section 2.1 for the definition of the rotation number). The rotation number measures the asymptotic rate of rotation of points of the circle. For instance, a rigid rotation of $\mathbb{T}^1$ of the form $\mathcal{T}_\theta(x) = x + \theta$ has rotation number θ. All throughout this paper we shall fix an irrational rotation number θ and always choose the parameters α and ε such that the rotation number of $\widetilde{f}_{\alpha,\varepsilon}$ be θ. More precisely, we choose our rotation number θ among the *Brjuno set* of irrational numbers, which contains all *Diophantine numbers* (see [PM] for a precise definition of these sets). In the $(\alpha, \varepsilon)-$parameter space, the set of parameters with a given rotation number θ is called the *Arnold tongue* T_θ. If θ is rational, its Arnold tongue is a set with interior, while if the rotation number is an irrational number θ, then T_θ corresponds to a curve connecting $\varepsilon = 0$ and $\varepsilon = 1$ which is in fact the graph of a function $\varepsilon \mapsto \alpha(\varepsilon)$, with $\alpha(0) = 2\pi\theta$ (see Figure 1). If θ is a Brjuno number, the curve $\alpha(\varepsilon)$ is known to be analytic for ε small enough [Ri, FG].

Moreover, if the rotation number θ is a Brjuno number we have that for ε small enough, the map $\widetilde{f}_{\alpha(\varepsilon),\varepsilon}$ is *analytically linearizable* (see [Y2, PM, Ri]). That is, there is an analytic map $\widetilde{\eta}_\varepsilon : \mathbb{T}^1 \to \mathbb{T}^1$ that conjugates $\widetilde{f}_{\alpha(\varepsilon),\varepsilon}$ to $\mathcal{T}_\theta$, i. e.

$$\widetilde{f}_{\alpha(\varepsilon),\varepsilon} \circ \widetilde{\eta}_\varepsilon = \widetilde{\eta}_\varepsilon \circ \mathcal{T}_\theta. \tag{2}$$

Equivalently, to say that $\widetilde{F}_{\alpha(\varepsilon),\varepsilon}$ restricted to $\mathbf{C}_1$ is analytically linearizable means that there is an analytic map $\widetilde{\varphi}_\varepsilon : \mathbf{C}_1 \to \mathbf{C}_1$, such that

$$\widetilde{F}_{\alpha(\varepsilon),\varepsilon} \circ \widetilde{\varphi}_\varepsilon = \widetilde{\varphi}_\varepsilon \circ \mathcal{R}_\omega, \tag{3}$$

where $\mathcal{R}_\omega(u) = \mathrm{e}^{\mathrm{i}\omega}u$ and $\omega = 2\pi\theta$. Since the linearization $\widetilde{\varphi}_\varepsilon$ is analytic, it can be extended to a neighborhood of the unit circle of the form $A(1/r, r)$, where we define

$$A(r_1, r_2) = \{u \in \mathbb{C} : \ r_1 < |u| < r_2\} \tag{4}$$

as the straight annulus of radii r_1 and r_2. We denote by $\widetilde{A}_\varepsilon = A(1/\widetilde{R}_\varepsilon, \widetilde{R}_\varepsilon)$ the maximal annulus for which $\widetilde{\varphi}_\varepsilon$ can be analytically continued. Then, it is easy to check that, being $\widetilde{F}_{\alpha(\varepsilon),\varepsilon|\mathbf{C}_1}$

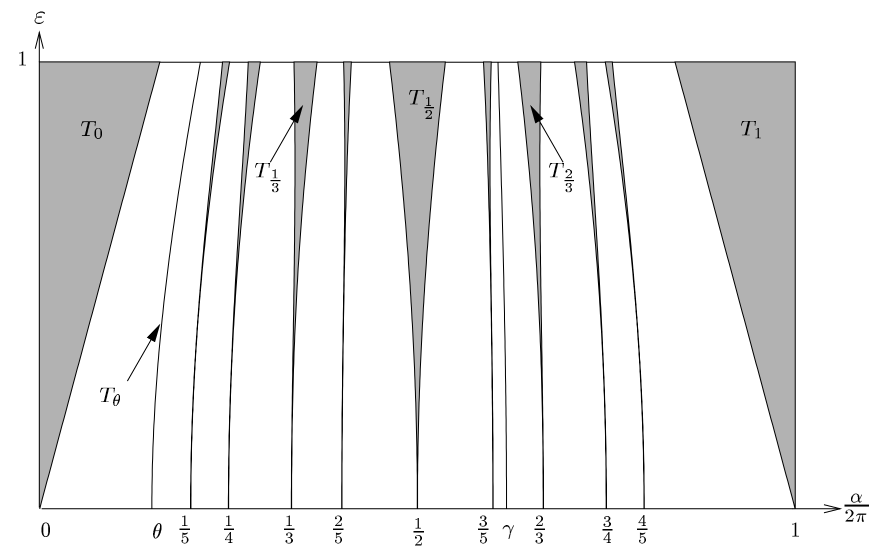

Figure 1: Rational Arnold tongues in the parameter space of the standard family up to denominator 5 (note that these are sets with interior). The curves correspond to the irrational tongues for $\gamma = \frac{\sqrt{5}-1}{2}$ and $\theta = \sqrt[5]{2} - 1$.

analytically linearizable is equivalent to the existence of a *Herman ring* $\widetilde{U}_\varepsilon$ for $\widetilde{F}_{\alpha(\varepsilon),\varepsilon}$, which is given by $\widetilde{U}_\varepsilon := \widetilde{\varphi}_\varepsilon(\widetilde{A}_\varepsilon)$. In $\widetilde{U}_\varepsilon$, every orbit under $\widetilde{F}_{\alpha(\varepsilon),\varepsilon}$ lies on an invariant closed curve which has rotation number θ. Since $\widetilde{\varphi}_\varepsilon$ is unique modulus the composition with any rotation, the constant $\widetilde{R}_\varepsilon$ is univocally defined and we will call it *the size* of the Herman ring.

The main goal of this paper is to give an asymptotic estimate for the size $\widetilde{R}_\varepsilon$ of the Herman ring $\widetilde{U}_\varepsilon$ as $\varepsilon \to 0$.

The sharpest results concerning the size of Herman rings for univalent maps on a given annulus are due to Yoccoz (see Theorems 2.1 and 2.3), who gives an estimate that can be applied to *any* analytic map F that leaves the unit circle $\mathbf{C}_1$ invariant and has rotation number θ (i.e., a lift on $\mathbb{R}$ of $F_{|\mathbf{C}_1}$ has rotation number θ), and which depends only on θ and on the size of the domain where the map is univalent. In Section 3 we will see that this general result applied to the complex standard family leads to

$$\widetilde{R}_\varepsilon \geq K\frac{\sigma(\varepsilon)}{\varepsilon},$$

where $K = \exp(-\Phi(\theta) - 2\pi C_0)$, Φ is the *Brjuno function* [MMY], C_0 is a universal constant and $\sigma(\varepsilon) = 1 + \sqrt{1-\varepsilon^2}$ is defined in such a way that $\widetilde{F}_{\alpha(\varepsilon),\varepsilon}$ is univalent in $A(\varepsilon/\sigma(\varepsilon), \sigma(\varepsilon)/\varepsilon)$.

The fact that this estimate holds for any analytic diffeomorphism having $\mathbf{C}_1$ invariant with rotation number θ and univalent at least in $A(\varepsilon/\sigma(\varepsilon), \sigma(\varepsilon)/\varepsilon)$, suggests that a better estimate can be found for the complex standard family. We shall return to this problem in a moment, but first, let us consider what is known as the *complex semistandard map* of parameter $\mathrm{e}^{\mathrm{i}\omega}$

$$G(z) = z\mathrm{e}^{\mathrm{i}\omega}\mathrm{e}^{z}.$$

Observe that $z = 0$ is a fixed point of G with derivative $\mathrm{e}^{\mathrm{i}\omega}$. Since $\omega = 2\pi\theta$, and θ is a Brjuno number, it is known [Br1, Br2] that G has a *Siegel disk* around the origin, which we denote by U. This means that if we call $\mathbb{D}_r$ the open disk of center 0 and radius r, there exists a unique maximal number $R_0 > 0$ and a unique conformal isomorphism

$$\varphi : \mathbb{D}_{R_0} \longrightarrow U, \qquad \varphi(0) = 0, \qquad \varphi'(0) = 1 \tag{5}$$

that conjugates G to the rotation $\mathcal{R}_\omega$, i.e., $G \circ \varphi = \varphi \circ \mathcal{R}_\omega$. The number R_0 is known as the *conformal radius* of the Siegel disk. Standard arguments show that R_0 is always finite. Lower bounds for R_0 (as a function of ω) could be obtained applying Yoccoz's results [Y2] to the semistandard map.

We now return to the problem of estimating the size of the Herman ring $\widetilde{U}_\varepsilon$. Our main result is the following theorem.

Theorem A. *Let θ be a Brjuno number and consider the standard map $\widetilde{F}_{\alpha,\varepsilon}(u) = u\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\frac{\varepsilon}{2}(u-\frac{1}{u})}$, with $\alpha = \alpha(\varepsilon)$ such that $\widetilde{F}_{\alpha(\varepsilon),\varepsilon}$ restricted to $\mathbf{C}_1$ has rotation number θ. Let $\widetilde{R}_\varepsilon$ be the size of its Herman ring and let R_0 be the conformal radius of the Siegel disk of the semistandard map $G(z) = z\mathrm{e}^{\mathrm{i}\omega}\mathrm{e}^z$, where $\omega = 2\pi\theta$. Then,*

$$\widetilde{R}_\varepsilon = \frac{2}{\varepsilon}(R_0 + \mathcal{O}(\varepsilon \log \varepsilon)).$$

Remark 1.1. We believe that this is the best estimate that can be obtained with our methods. However, some recent developments (work in progress) seem to indicate that a more optimal estimate could be $\widetilde{R}_\varepsilon = \frac{2}{\varepsilon}(R_0 + \mathcal{O}_2(\varepsilon))$ (with $\widetilde{R}_\varepsilon$ analytic on ε). As a vague indication, this could follow from knowing that the complexification of the Arnold tongue T_θ, with θ a Brjuno number, can be parametrized holomorphically by a complex parameter strongly related with the modulus of the ring.

An analogue of Theorem A, for *Chirikov's standard and semistandard maps* of $\mathbb{R}^2$, was proved in [SV] using KAM methods and complex matching, and therefore restricting the result to Diophantine rotation numbers. In the present paper, we prove Theorem A using quasiconformal surgery, inspired by a qualitative construction of Geyer in [G] which relates the standard and the semistandard maps. We shall modify this construction by introducing the dependence on the parameter ε and by making most of its ingredients completely explicit. These additions will give us the possibility of obtaining quantitative estimates from the geometric construction.

From all the partial results involved in the proof of Theorem A we choose the following one to be remarked here, because of its interest in itself and its possible use in other surgery constructions. Under certain conditions, this result gives an estimate for how close the Ahlfors-Bers' map (see Theorem 2.12) is to the identity.

Throughout the paper, $\delta_{\mathcal{U}}$ indicates the hyperbolic distance inside $\mathcal{U}$, where $\mathcal{U}$ is a hyperbolic set (see Section 2.3). The notation $\|\cdot\|$ denotes the infinity norm and $\mathbb{D} = \mathbb{D}_1$.

Proposition B. *Let μ be a Beltrami form on $\mathbb{C}$ (see Section 2.2) and $h : \mathbb{C} \to \mathbb{C}$ be the unique quasiconformal solution of the Beltrami equation $\frac{\partial h}{\partial \bar{z}} = \mu \frac{\partial h}{\partial z}$ fixing 0 and 1 (see Theorem 2.12). Then:*

(a) *For any $z \in \mathbb{C} \setminus \{0,1\}$, we have $\delta_{\mathbb{C}\setminus\{0,1\}}(z, h(z)) \leq \delta_{\mathbb{D}}(0, ||\mu||)$.*

(b) *There exists a universal constant $0 < \rho < 1$ such that if $\|\mu\| \leq \rho$, then for any $z \in \mathbb{D}^*_\rho = \mathbb{D}_\rho \setminus \{0\}$ verifying $\|\mu\|| \log |z|| \leq \rho$, one has:*

$$|h(z) - z| \leq C||\mu|||z|| \log |z||,$$

where $C > 0$ only depends on ρ.

The paper is organized as follows. Section 2 contains basic introductions to some of the tools and preliminary results that will be used during the proofs of Theorem A and Proposition B. The expert reader can go directly to Section 3, where the problem is scaled and restated more precisely. Section 4 and Section 5 contain the actual proofs of Theorem A and Proposition B, respectively.

2 Preliminaries

In this section we state the basic results that we need to prove Theorem A and Proposition B. In Section 2.1 we review previous results about the linearization of analytic circle maps and their translation to maps of the complex plane having an invariant circle. Section 2.2 is devoted to quasiconformal mappings and Ahlfors-Bers' theorem. Finally, in Section 2.3 we give some definitions and results in hyperbolic geometry.

2.1 Analytic linearization

Let $f : \mathbb{T}^1 \to \mathbb{T}^1$ be an orientation preserving homeomorphism of the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, with the normalization $f(0) \in [0, 1)$. To such a map one can assign a *rotation number* defined as

$$\rho(f) = \lim_{n\to\infty} \frac{f^n(x) - x}{n},$$

where x is any point in $\mathbb{T}^1$. It is well known (see e. g. [dMvS]) that f being a homeomorphism guarantees that this limit exists and is independent of the point x. With this definition, $\rho(f)$ is a rational number if and only if f has a periodic orbit. If $\rho(f)$ is irrational, all orbits are dense in the circle. We are interested in maps with an irrational rotation number.

If the rotation number of f is an irrational number θ and $f \in \mathcal{C}^2(\mathbb{T}^1)$, Denjoy's theorem (see [dMvS]) ensures that f is topologically conjugate to the rigid rotation of angle θ, $\mathcal{T}_\theta(x) = x + \theta$. That is, there exists a homeomorphism $\eta : \mathbb{T}^1 \to \mathbb{T}^1$ such that $\eta \circ \mathcal{T}_\theta = f \circ \eta$, making the following diagram commute:

$$\begin{array}{ccc} \mathbb{T}^1 & \xrightarrow{\mathcal{T}_\theta} & \mathbb{T}^1 \\ {\scriptstyle\eta}\big\downarrow & & \big\downarrow{\scriptstyle\eta} \\ \mathbb{T}^1 & \xrightarrow{f} & \mathbb{T}^1 \end{array}$$

If we require $\eta(0) = 0$ then the conjugacy is unique.

We restrict from now on to the case where f is an analytic diffeomorphism of $\mathbb{T}^1$, and therefore it can be extended to a complex strip, of certain width $\Delta > 0$, around $\mathbb{T}^1$:

$$\mathcal{A}_\Delta = \{z \in \mathbb{C}/\mathbb{Z} : \ |\mathrm{Im}(z)| < \Delta\}. \tag{6}$$

Abusing notation, we shall denote this extension again by f. If the conjugacy η is also analytic, the map f is said to be *analytically linearizable.* Then again, η can be extended to a neighbourhood of the circle, and it is easy to check (by the principle of analytic continuation) that its extension also conjugates f to $\mathcal{T}_\theta$ wherever η is defined.

We are particularly interested in the case where $F : \mathcal{U} \subset \mathbb{C} \to \mathbb{C}$ is an analytic map having the unit circle $\mathbf{C}_1$ invariant, and f is the lift to $\mathbb{T}^1$ of $F_{|\mathbf{C}_1}$. In this case, we say that $F_{|\mathbf{C}_1}$ is analytically linearizable if there exists an analytic diffeomorphism $\varphi : \mathbf{C}_1 \to \mathbf{C}_1$, such that

$\varphi \circ \mathcal{R}_\omega = F \circ \varphi$, where $\mathcal{R}_\omega(u) = \mathrm{e}^{\mathrm{i}\omega} u$ and $\omega = 2\pi\theta$. If we ask $\varphi(1) = 1$, φ is univocally defined and the relation between η and φ is given by

$$\varphi(\mathrm{e}^{2\pi \mathrm{i} x}) = \mathrm{e}^{2\pi \mathrm{i} \eta(x)}, \qquad x \in \mathbb{T}^1.$$

The image by φ of the maximal annulus where φ can be analytically continued is called the *Herman ring* of F. If R is the outer radius of this annulus (in the understanding that this annulus is symmetric with respect to the unit circle, and then it is of the form $A(1/R, R)$), the width of the strip of analyticity of f around $\mathbb{T}^1$ is $\frac{1}{2\pi}\log R$. The quantity $\frac{1}{\pi}\log R$ is called the *modulus of the ring* and we will call R the *size of the ring.*

Arnold showed in [Ar] that if θ is a Diophantine number and f is close enough to the rigid rotation $\mathcal{T}_\theta$, then f is analytically linearizable. This result was later improved by Rüssmann [Ru1, Ru2], Herman [Her1, Her2] and Yoccoz [Y2, PM]. The sharpest results are due to Yoccoz and we state them below. In the statement, $\Phi : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}^+ \cup \{\infty\}$ denotes the *Brjuno function*, a purely arithmetic $\mathbb{Z}-$periodic function. Its most important property is that $\Phi(\alpha)$ is finite if and only if α is a Brjuno number (see [MMY] for details on Φ).

The *Local Conjugacy Theorem*, due to Yoccoz, states that any analytic circle map with a Brjuno rotation number and which is univalent in a sufficiently large strip (where "large" is defined only in terms of the rotation number) is analytically linearizable. Moreover, it gives a lower bound for the linearization domain which, again, only depends on the initial domain of univalency and the rotation number.

Theorem 2.1 (Local Conjugacy Theorem). *Let θ be a Brjuno number and $\Delta > 0$ such that $\Delta > \frac{1}{2\pi}\Phi(\theta) + C_0$, where C_0 is a universal constant. Let $f : \mathbb{T}^1 \to \mathbb{T}^1$ be an analytic circle diffeomorphism, orientation preserving and with rotation number θ. We assume that f is holomorphic and univalent in the strip $\mathcal{A}_\Delta$ (see (6)). Then, f is analytically linearizable and the linearization $\eta : \mathbb{T}^1 \to \mathbb{T}^1$ is analytic in the complex strip $\mathcal{A}_d$, with*

$$d \geq \Delta - \frac{1}{2\pi}\Phi(\theta) - C_0,$$

and verifies $\eta(\mathcal{A}_d) \subset \mathcal{A}_\Delta$.

Remark 2.2. For the Arnold standard family (1) a sort of reciprocal is also true. Indeed, it was shown in [G] that if a member of the Arnold standard family is analytically linearizable, then its rotation number must be Brjuno.

If F is a holomorphic map leaving the unit circle invariant, and by applying Theorem 2.1 to the lift f of $F_{|\mathbf{C}_1}$, we can state an analogous result about the analytic linearization of F.

Theorem 2.3. *Let θ be a Brjuno number and $R > 1$ such that $R > \mathrm{e}^{\Phi(\theta) + 2\pi C_0}$. Let $F : \mathbf{C}_1 \to \mathbf{C}_1$ be an analytic diffeomorphism with rotation number θ. We assume that F is holomorphic and univalent in the annulus $A(1/R, R)$. Then, F is analytically linearizable and the linearization $\varphi : \mathbf{C}_1 \to \mathbf{C}_1$ is analytic in the annulus $A(1/r, r)$, with*

$$r \geq R\mathrm{e}^{-\Phi(\theta) - 2\pi C_0},$$

and verifies $\varphi(A(1/r, r)) \subset A(1/R, R)$.

2.2 Quasiconformal mappings and the Beltrami equation

In this section we recall shortly the relevant definitions and results to be used in the *quasiconformal surgery* procedure, which is going to be one of the main tools to prove the results of this paper. The standard reference for quasiconformal mappings is [Ah]. In this section, $\mathcal{U}, \mathcal{V} \subset \mathbb{C}$ are open sets.

Definition 2.4. Given a measurable function $\mu : \mathcal{U} \to \mathbb{C}$, we say that μ is a $k-$*Beltrami form* of $\mathcal{U}$ if $|\mu(z)| \leq k < 1$ almost everywhere in $\mathcal{U}$. Two Beltrami forms of $\mathcal{U}$ are equivalent if they coincide almost everywhere in $\mathcal{U}$.

Equivalently, a Beltrami form of $\mathcal{U}$ gives an *almost complex structure* σ, which means a measurable field of ellipses in the tangent space of $\mathcal{U}$, centered at 0 and defined up to multiplication by a non-zero real constant. The argument of the major axis of these infinitesimal ellipses, at the point $z \in \mathcal{U}$, is $\pi/2 + \arg(\mu(z))/2$, and the ratio of minor and major axes equals $(1 - |\mu(z)|)/(1 + |\mu(z)|)$.

Definition 2.5. A homeomorphism $f : \mathcal{U} \to \mathcal{V}$ is said to be $k-$*quasiconformal* if it has locally integrable weak derivatives and

$$\mu_f(z) = \frac{\frac{\partial f}{\partial \bar{z}}(z)}{\frac{\partial f}{\partial z}(z)}$$

is a $k-$Beltrami form. In this case, we say that μ_f is the *complex dilatation* or the *Beltrami coefficient* of f.

Remark 2.6. With the same definition, but skipping the hypothesis on f to be a homeomorphism, f is called a $k-$*quasiregular map*. It is easy to check that a $k-$quasiregular map is locally the composition $g \circ h$ of a holomorphic map g and a $k-$quasiconformal map h.

Definition 2.7. Given a Beltrami form μ of $\mathcal{V}$ and a quasiregular map $h : \mathcal{U} \to \mathcal{V}$, we define the *pull-back* of μ by h as the Beltrami form of $\mathcal{U}$ defined by:

$$h^*\mu = \frac{\frac{\partial h}{\partial \bar{z}} + (\mu \circ h)\overline{\frac{\partial h}{\partial z}}}{\frac{\partial h}{\partial z} + (\mu \circ h)\overline{\frac{\partial h}{\partial \bar{z}}}}.$$

Remark 2.8. Notice that if, in the previous definition, $\mu = \mu_f$ for certain quasiregular map f, then $h^*\mu_f = \mu_{f \circ h}$.

Remark 2.9. Pulling-back by holomorphic functions does not increase the maximal dilatation, k, of a $k-$Beltrami form.

Remark 2.10. The standard complex structure corresponds to $\mu_0 \equiv 0$, which is a field of circles. A quasiregular mapping f is holomorphic if and only if $f^*\mu_0 = \mu_0$.

Definition 2.11. Given a Beltrami form μ, the partial differential equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z} \tag{7}$$

is called *the Beltrami equation*. By *integration* of μ we mean the construction of a quasiconformal map f solving this equation almost everywhere, or equivalently, such that $\mu_f = \mu$.

The famous *Measurable Riemann Mapping Theorem* by Morrey, Ahlfors, Bers and Bojarski states that every almost complex structure is integrable. As we are going to use this result for Beltrami forms with $\mathcal{U} = \mathcal{V} = \mathbb{C}$, we give a statement adapted to this context.

Theorem 2.12 (Ahlfors-Bers, [Ah, BD]). *Let μ be a $k-$Beltrami form of $\mathbb{C}$. Then, there exists a unique quasiconformal map $h : \mathbb{C} \to \mathbb{C}$ such that $h(0) = 0$, $h(1) = 1$ and $\mu_h = \mu$. Furthermore, if μ_t is a family of Beltrami forms such that $\mu_t(z)$ depends analytically on t, for any $z \in \mathbb{C}$, then h_t depends analytically on t.*

Remark 2.13. The application of Ahlfors-Bers' theorem to complex dynamics is the following. Let f and μ be, respectively, a quasiregular mapping of $\mathbb{C}$ and a Beltrami form of $\mathbb{C}$, such that $f^*\mu = \mu$. If we apply Theorem 2.12 to integrate μ, and we construct a quasiconformal mapping h such that $\mu_h = \mu$, then $g = h \circ f \circ h^{-1}$ verifies $g^*\mu_0 = \mu_0$, and hence g is a holomorphic map of $\mathbb{C}$. Moreover, f and g are quasiconformally conjugate, i. e., they have the same dynamics.

2.3 Hyperbolic geometry

During this paper, besides quasiconformal surgery, we will use some results of *hyperbolic geometry* (see [Be2] for a survey). In a few words, quasiconformal surgery will be the key for the geometrical constructions we do and hyperbolic geometry will provide some of the quantitative estimates.

Definition 2.14. Given $\mathcal{U} \subseteq \mathbb{C}$ a domain (open and connected set) and given a continuous function $\lambda : \mathcal{U} \to [0, +\infty)$, with at most isolated zeros, we define the *Riemannian metric* λ on $\mathcal{U}$ as the metric having $\lambda(z)|dz|$ as a line element. More precisely, given a piecewise differentiable arc $\gamma : [a, b] \to \mathcal{U}$, the *length* of γ with respect to the metric λ is defined by

$$l_\lambda(\gamma) = \int_a^b \lambda(\gamma(t))|dz| = \int_a^b \lambda(\gamma(t))|\gamma'(t)|dt.$$

Definition 2.15. Given a Riemannian metric λ on $\mathcal{U}$ and given two points $z_1, z_2 \in \mathcal{U}$, we define the *distance* $\mathrm{d}_\lambda(z_1, z_2)$ by

$$\mathrm{d}_\lambda(z_1, z_2) = \inf\{l_\lambda(\gamma) \mid \ \gamma \subset U \text{ arc from } z_1 \text{ to } z_2\}.$$

In the case when this infimum is achieved by an arc γ^* from z_1 to z_2, this arc γ^* is called a *geodesic* between z_1 and z_2.

Any holomorphic map between two domains $\mathcal{U}$ and $\mathcal{V}$ can be used to transport a Riemannian metric on $\mathcal{V}$ to a Riemannian metric on $\mathcal{U}$.

Definition 2.16. Given an holomorphic map $f : \mathcal{U} \to \mathcal{V}$ and a Riemannian metric λ on $\mathcal{V}$, we define the *pull-back* of λ by f as the Riemannian metric on $\mathcal{U}$ given by

$$f^*\lambda = (\lambda \circ f)|f'|.$$

With this definition f is a local isometry between $(\mathcal{U}, f^*\lambda)$ and $(\mathcal{V}, \lambda)$, i.e., it preserves arc-lengths. If f is biholomorphic, then it is a global isometry.

The example that concerns us is the *hyperbolic metric*, which is a Riemannian metric defined on domains $\mathcal{U}$ that have the unit disk $\mathbb{D} := \mathbb{D}_1$ as a covering space, and which is preserved under conformal self-mappings of $\mathcal{U}$. On $\mathbb{D}$ the hyperbolic metric takes the following form.

Definition 2.17. The hyperbolic or *Poincaré* metric on $\mathbb{D}$ is the metric defined by

$$\lambda_{\mathbb{D}}(z) = \frac{2}{1-|z|^2}.$$

The Poincaré metric $\lambda_{\mathbb{D}}$ is the unique metric on $\mathbb{D}$ (up to multiplication by positive constants) invariant under conformal automorphisms of $\mathbb{D}$.

We will need an explicit expression for the distance in $\mathbb{D}$ defined by the Poincaré metric.

Proposition 2.18. *Given $w_1, w_2 \in \mathbb{D}$, we have the following formula for the hyperbolic distance $\delta_{\mathbb{D}}$ in $\mathbb{D}$:*

$$\sinh^2\left[\frac{\delta_{\mathbb{D}}(w_1,w_2)}{2}\right] = \frac{4|w_1-w_2|^2}{(1-|w_1|^2)(1-|w_2|^2)}.$$

In particular, if $0 \le r < 1$,

$$\delta_{\mathbb{D}}(0,r) = \log\left(\frac{1+r}{1-r}\right).$$

The pull-back process allows us to transport the Poincaré metric to any domain $\mathcal{U}$ that is conformally equivalent to $\mathbb{D}$. Indeed, if $\psi : \mathcal{U} \to \mathbb{D}$ is a Riemann map, then the hyperbolic metric on $\mathcal{U}$ is given by

$$\lambda_{\mathcal{U}}(z) = (\psi^*\lambda_{\mathbb{D}})(z) = \lambda_{\mathbb{D}}(\psi(z))|\psi'(z)|,$$

or equivalently, if $\varphi : \mathbb{D} \to \mathcal{U}$ is a conformal map, then

$$\lambda_{\mathcal{U}}(\varphi(z)) = \frac{\lambda_{\mathbb{D}}(z)}{|\varphi'(z)|}. \tag{8}$$

An important example of this is the upper half plane, $\mathbb{H}$, for which we can take $\psi(z) = \frac{z-i}{z+i}$, obtaining the following result.

Proposition 2.19. *The hyperbolic metric in $\mathbb{H}$ is given by $\lambda_{\mathbb{H}}(z) = \frac{1}{\mathrm{Im}(z)}$. In this case, the $\lambda_{\mathbb{H}}$-geodesics are vertical segments or arcs of circles orthogonal to the real axis.*

The hyperbolic metric can also be transported to non-simply connected domains, by means of any *universal covering map.*

Definition 2.20. A domain $\mathcal{U}$ of the Riemann sphere $\overline{\mathbb{C}}$ is called *hyperbolic* if it has at least three boundary points.

Theorem 2.21. *If $\mathcal{U}$ is a hyperbolic domain, there exists a holomorphic covering map $\varphi : \mathbb{D} \to \mathcal{U}$ (i.e., φ is a local homeomorphism at every point). Each such map is called a* universal covering map *and it is uniquely determined if we prefix $\varphi(0)$ and require $\varphi'(0) > 0$.*

Then, if $\mathcal{U}$ is a hyperbolic domain, and φ is a universal covering for $\mathcal{U}$, the hyperbolic metric $\lambda_{\mathcal{U}}|dz|$ is given as above by (8).

Using that $\varphi(z) = \exp\left(\frac{z-1}{z+1}\right)$ is a universal covering for the punctured disk $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$, we obtain the following properties for the hyperbolic metric in $\mathbb{D}^*$.

Proposition 2.22. *The hyperbolic metric in $\mathbb{D}^*$ has the form*

$$\lambda_{\mathbb{D}^*}(z) = \frac{1}{|z|\log(1/|z|)}.$$

The hyperbolic distance $\delta_{\mathbb{D}^*}$ satisfies

$$\sinh^2\left[\frac{\delta_{\mathbb{D}^*}(z_1, z_2)}{2}\right] = \frac{|\log z_1 - \log z_2|^2}{\log|z_1|\log|z_2|}, \qquad z_1, z_2 \in \mathbb{D}^*,$$

where we have chosen appropriate determinations for $\log z_1$ and $\log z_2$ (with the arguments of z_1 and z_2 differing at most π). Moreover, the geodesics in $\mathbb{D}^*$ are obtained by mapping the geodesics of $\mathbb{H}$ by the covering $\widehat{\varphi} : \mathbb{H} \to \mathbb{D}^*$ given by $\widehat{\varphi}(z) = \mathrm{e}^{\mathrm{i}z}$.

The main reason why hyperbolic geometry is very useful in complex dynamics is the fact that all holomorphic maps are contractive, when we look at them under the hyperbolic metric. This is known as the *Big Schwartz-Pick lemma* which reads as follows.

Theorem 2.23 (Big Schwartz-Pick lemma). *If $\mathcal{U}$ and $\mathcal{V}$ are hyperbolic domains and $f : \mathcal{U} \to \mathcal{V}$ is holomorphic, then*

$$\delta_{\mathcal{V}}(f(z_1), f(z_2)) \le \delta_{\mathcal{U}}(z_1, z_2)$$

for all $z_1, z_2 \in \mathcal{U}$. Moreover, for all $z \in \mathcal{U}$,

$$\frac{\lambda_{\mathcal{V}}(f(z))}{\lambda_{\mathcal{U}}(z)}|f'(z)| \le 1.$$

In this paper, we shall need to compare (locally) different hyperbolic distances. The next result, known as *Ahlfors' lemma*, gives a comparison between hyperbolic metrics. Since we are unable to provide a standard reference, we include its proof, taken from lecture notes [Pet]. The analogous comparison for hyperbolic distances (which is in fact what we really need in the proof of Proposition B) requires some work and it is therefore given in Proposition 5.1 (see Section 5.2).

Proposition 2.24 (Ahlfors' lemma). *Let $\mathcal{U} \subseteq \mathcal{V} \subset \overline{\mathbb{C}}$ be hyperbolic domains. Then for any point $z \in \mathcal{U}$,*

$$1 \le \frac{\lambda_{\mathcal{U}}(z)}{\lambda_{\mathcal{V}}(z)} \le \coth\left(\frac{1}{2}\delta_{\mathcal{V}}(z, \partial\mathcal{U})\right).$$

Proof : The left hand inequality is quite immediate if we consider the identity map $\mathrm{Id} : \mathcal{U} \to \mathcal{V}$. By the Big Schwartz-Pick lemma, $\frac{\lambda_{\mathcal{V}}(z)}{\lambda_{\mathcal{U}}(z)} \le 1$ and we are done.

For the right hand inequality, let $\varphi : \mathbb{D} \to \mathcal{V}$ be a universal covering of $\mathcal{V}$ such that $\varphi(0) = z$, and let $0 \in \mathcal{U}' \subset \mathbb{D}$ be such that $\varphi : \mathcal{U}' \to \mathcal{U}$ is conformal. See Figure 2.

Since φ is a (local) isometry between the hyperbolic metrics, we may work with $\mathbb{D}$, $\mathcal{U}'$ and 0 instead of $\mathcal{V}$, $\mathcal{U}$ and z. In particular, we have

$$\frac{\lambda_{\mathcal{U}}(z)}{\lambda_{\mathcal{V}}(z)} = \frac{\lambda_{\mathcal{U}'}(0)}{\lambda_{\mathbb{D}}(0)}.$$

Let $r = \min\{|z| : z \in \partial\mathcal{U}'\}$. If $r = 1$ then $\mathcal{U}' = \mathbb{D}$ and there is nothing to prove. Hence we suppose $r < 1$. We now apply the left hand inequality to $\mathbb{D}_r \subseteq \mathcal{U}'$ to obtain

$$1 \le \frac{\lambda_{\mathbb{D}_r}(0)}{\lambda_{\mathcal{U}'}(0)} = \frac{1}{r}\frac{\lambda_{\mathbb{D}}(0)}{\lambda_{\mathcal{U}'}(0)},$$

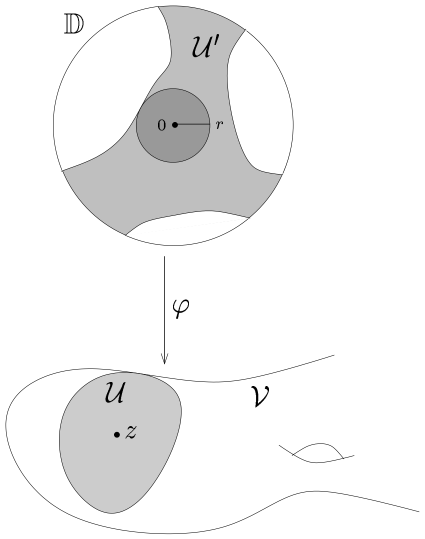

Figure 2: Sketch for the proof of Ahlfors' lemma.

and hence,

$$\frac{\lambda_{\mathcal{U}}(z)}{\lambda_{\mathcal{V}}(z)} \leq \frac{1}{r}.$$

It remains to show that $\frac{1}{r} = \coth(\frac{d}{2})$ where $d = \delta_{\mathcal{V}}(z, \partial\mathcal{U})$. To see this, we use Proposition 2.18. In particular, we observe that $\delta_{\mathbb{D}}(0, \cdot)$ has radial symmetry, and thus we have

$$d = \delta_{\mathcal{V}}(z, \partial\mathcal{U}) = \delta_{\mathbb{D}}(0, \partial\mathcal{U}') = \delta_{\mathbb{D}}(0, r) = \log\left(\frac{1+r}{1-r}\right).$$

Therefore $e^d = \frac{1+r}{1-r}$ and

$$\frac{1}{r} = \frac{e^d+1}{e^d-1} = \frac{e^{d/2}+e^{-d/2}}{e^{d/2}-e^{-d/2}} = \frac{\cosh(d/2)}{\sinh(d/2)} = \coth(d/2).$$

□

3 The complex standard family and the semistandard map

The *complex standard family*

$$\widetilde{F}_{\alpha,\varepsilon}(u) = u\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\frac{\varepsilon}{2}(u-\frac{1}{u})},$$

with $\alpha \in [0, 2\pi)$ and $\varepsilon \in [0, 1)$, is a family of holomorphic maps of $\mathbb{C}^*$ onto itself, with essential singularities at 0 and infinity. The maps of the family are symmetric with respect to the unit circle, which is invariant under $\widetilde{F}_{\alpha,\varepsilon}$. The singularities of the inverse map consist exclusively of the images of the two critical points of $\widetilde{F}_{\alpha,\varepsilon}$ (as $\widetilde{F}_{\alpha,\varepsilon}$ has no asymptotic values) which are located at

$$\widetilde{c}_{\pm}(\varepsilon) = \frac{1}{\varepsilon}(-1 \pm \sqrt{1-\varepsilon^2}) < 0.$$

Moreover, one can see that the standard map is univalent on a symmetric annulus $A(1/r_\varepsilon, r_\varepsilon)$, where

$$r_\varepsilon = -\widetilde{c}_-(\varepsilon) = \frac{1}{\varepsilon}(1 + \sqrt{1-\varepsilon^2}).$$

Notice that $r_\varepsilon \lesssim \frac{2}{\varepsilon}$ as $\varepsilon \to 0$, and so r_ε tends to infinity as ε tends to zero.

From now on, we fix a rotation number θ in the Brjuno set, and consider the analytic curve $\alpha = \alpha(\varepsilon)$ such that the rotation number of $\widetilde{F}_{\alpha(\varepsilon),\varepsilon|\mathbf{C}_1}$ is θ. Thus, for ε small enough (depending only on θ), the standard map is under the hypothesis of the Local Conjugacy Theorem of Yoccoz (Theorem 2.3) which assures the existence of a Herman ring of size

$$\widetilde{R}_\varepsilon \geq \frac{1+\sqrt{1-\varepsilon^2}}{\varepsilon} K := \frac{\sigma(\varepsilon)}{\varepsilon} K, \tag{9}$$

where $K = \exp(-\Phi(\theta) - 2\pi C_0)$, Φ is the Brjuno function and C_0 is a universal constant.

To estimate asymptotically the value of $\widetilde{R}_\varepsilon$, we start by scaling the problem hoping that the scaled value of $\widetilde{R}_\varepsilon$ has a finite limit as ε tends to zero. We perform the change of variables

$$z = \frac{\varepsilon}{2} u,$$

and we obtain a new map

$$F_{\alpha(\varepsilon),\varepsilon}(z) = z\mathrm{e}^{\mathrm{i}\alpha(\varepsilon)}\mathrm{e}^{z-\frac{\varepsilon^2}{4z}}.$$

This map shows the complex standard family as a perturbation of the semistandard map $G(z) = z\mathrm{e}^{\mathrm{i}\omega}\mathrm{e}^{z}$, with $\omega = 2\pi\theta$, as long as z is far away from zero (recall that $\alpha(0) = \omega$). Note that the limit is a singular limit at $z = 0$, since an essential singularity is converted into a fixed point. The new scaled map $F_{\alpha(\varepsilon),\varepsilon}$ leaves $\mathbf{C}_{\frac{\varepsilon}{2}}$ invariant and its critical points are now located at

$$c_\pm(\varepsilon) = \frac{1}{2}\left(-1 \pm \sqrt{1-\varepsilon^2}\right) < 0, \tag{10}$$

which approach 0 and -1 as ε tends to 0.

We also change variables on the conjugation plane so that the map

$$\varphi_\varepsilon(z) = \frac{\varepsilon}{2}\widetilde{\varphi}_\varepsilon\left(\frac{2}{\varepsilon}z\right), \tag{11}$$

where $\widetilde{\varphi}_\varepsilon$ is given in (3), is now the linearizing map of $F_{\alpha(\varepsilon),\varepsilon|\mathbf{C}_{\frac{\varepsilon}{2}}}$. The map φ_ε is defined from the annulus $A(\frac{\varepsilon^2}{4R_\varepsilon}, R_\varepsilon)$, with $R_\varepsilon := \frac{\varepsilon}{2}\widetilde{R}_\varepsilon$, to the scaled Herman ring $U_\varepsilon := \frac{\varepsilon}{2}\cdot\widetilde{U}_\varepsilon$. See Figure 3.

We will actually compare the scaled standard family, $F_{\alpha(\varepsilon),\varepsilon}(z)$, with the semistandard map, $G(z)$. The qualitative and quantitative relationship between these maps will be explained by the surgery construction in the next section.

Remark 3.1. At this point, after scaling, Theorem A is equivalent to prove

$$R_\varepsilon = R_0 + \mathcal{O}(\varepsilon \log \varepsilon), \tag{12}$$

where R_0 is the conformal radius of the Siegel disk U of the semistandard map $G(z) = z\mathrm{e}^{\mathrm{i}\omega}\mathrm{e}^{z}$ (see (5)). In particular this result implies that R_ε is a continuous function at $\varepsilon = 0$.

Note that (12) means that Yoccoz's estimate (9) can be improved for the standard family by observing that $\widetilde{R}_\varepsilon = \frac{\sigma(\varepsilon)}{\varepsilon}K(\varepsilon)$, with $K(\varepsilon) = R_0 + \mathcal{O}(\varepsilon \log \varepsilon)$, and hence, $K(\varepsilon) \to R_0$ as $\varepsilon \to 0$.

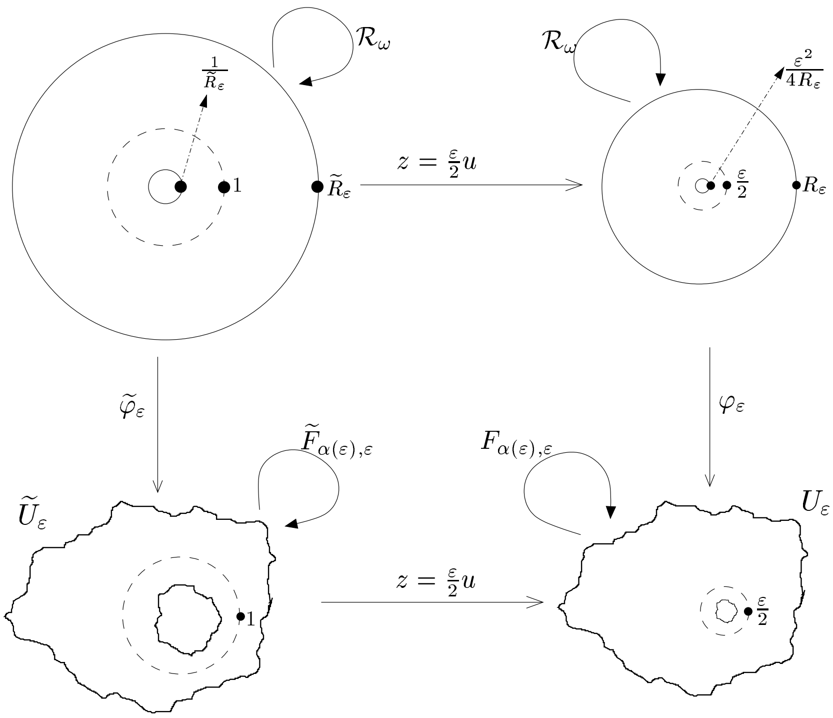

Figure 3: Scaling the dynamical plane and the linearizing plane.

4 Proof of Theorem A

The proof of Theorem A is based in an explicit (quantitative) version of the (qualitative) surgery construction [G] that relates a member of the (non scaled) complex standard family $\widetilde{F}_{\alpha(\varepsilon),\varepsilon}$, with the semistandard map G. First, in Section 4.1 we explain Geyer's construction, slightly modified and adapted to the scaled map $F_{\alpha(\varepsilon),\varepsilon}$. In Section 4.2 we re-formulate Theorem A in terms of the previous surgery construction. Section 4.3 gives an explicit version of Geyer's construction, which allows us to obtain the quantitative results. In Section 4.4 we obtain Theorem A as an easy consequence of Proposition B and the results of Section 4.3. Finally, Section 5 contains the proof of Proposition B.

4.1 Surgery construction

The idea of Geyer's construction to relate $F_{\alpha(\varepsilon),\varepsilon}$ to G is basically to "fill up the hole" of the Herman ring U_ε in order to transform the Herman ring into a "Siegel disk". For our purposes, "the hole" is the disk of radius $\varepsilon/2$ (denoted by $\mathbb{D}_{\frac{\varepsilon}{2}}$), given the fact that its boundary is simpler than the boundary of U_ε, and that it is invariant under the map. This can be accomplished by defining a new map H_ε which consists of the old one $F_{\alpha(\varepsilon),\varepsilon}$ everywhere outside $\mathbb{D}_{\frac{\varepsilon}{2}}$, and a suitable quasiconformal map conjugate to a rotation of angle ω inside this disk.

Due to the fact that the behaviour of the scaled standard map and the semistandard map at ∞ are the same, the map H_ε thus obtained is then "morally" equivalent (in the dynamical sense) to the semistandard map. However, the map H_ε constructed in this way will be quasiregular, but not holomorphic. We shall make it holomorphic by means of the Ahlfors-Bers' theorem (see Theorem 2.12), as we explained in Remark 2.13. So, we will construct a Beltrami form

μ_ε, invariant by H_ε, and by integration of μ_ε we will obtain a quasiconformal map h_ε such that $h_\varepsilon \circ H_\varepsilon \circ h_\varepsilon^{-1}$ is holomorphic, and has the same dynamics than H_ε (*holomorphic smoothing*). By choosing H_ε appropiatelly, we will prove that this new map is the semistandard map G.

We now proceed to make this construction precise. To define a rotation inside the small disk $\mathbb{D}_{\frac{\varepsilon}{2}}$, we first choose a "gluing" map ψ_ε. Let $\psi_\varepsilon : \overline{\mathbb{D}_{\frac{\varepsilon}{2}}} \to \overline{\mathbb{D}_{\frac{\varepsilon}{2}}}$ be any quasiconformal map that agrees with φ_ε on the boundary (i.e., $\psi_{\varepsilon|\mathbf{C}_{\frac{\varepsilon}{2}}} = \varphi_\varepsilon$) and sends 0 to 0. Since φ_ε is the projection of a real analytic map, the existence of ψ_ε is guaranteed (see [Pom]). Then we define the new map H_ε as:

$$H_\varepsilon = \begin{cases} F_{\alpha(\varepsilon),\varepsilon} & \text{on } \mathbb{C} \setminus \mathbb{D}_{\frac{\varepsilon}{2}} \\ \psi_\varepsilon \circ \mathcal{R}_\omega \circ \psi_\varepsilon^{-1} & \text{on } \overline{\mathbb{D}_{\frac{\varepsilon}{2}}}. \end{cases}$$

See Figure 4.

By the choice of ψ_ε, the map H_ε is continuous and quasiregular. By construction, it has a fixed point at $z = 0$, and it is conjugate to a rotation of angle ω on the set (topological disk) $\mathbb{D}_{\frac{\varepsilon}{2}} \cup U_\varepsilon$ by means of the conjugacy maps φ_ε and ψ_ε, which match up continuously. Last, we notice that H_ε has only one critical point: the former critical point $c_-(\varepsilon)$ of $F_{\alpha(\varepsilon),\varepsilon}$ (given in (10)) which was outside $\mathbb{D}_{\frac{\varepsilon}{2}}$, since the symmetric one $c_+(\varepsilon)$ has been annihilated.

To start the second part of the surgery construction (holomorphic smoothing) we define the Beltrami form μ_ε on $\mathbb{C}$ as follows. First we define it on the surgery region by pulling back $\mu_0 = 0$ to $\mathbb{D}_{\frac{\varepsilon}{2}}$ by means of ψ_ε^{-1}. We extend this almost complex structure to every preimage of $\mathbb{D}_{\frac{\varepsilon}{2}}$ using H_ε (or equivalently $F_{\alpha(\varepsilon),\varepsilon}$, as both maps coincide outside of $\mathbb{D}_{\frac{\varepsilon}{2}}$); and finally we set $\mu_\varepsilon = 0$ at the remaining points. That is,

$$\mu_\varepsilon = \begin{cases} \left(\psi_\varepsilon^{-1}\right)^* (0) & \text{on } \mathbb{D}_{\frac{\varepsilon}{2}} \\ \left(H_\varepsilon^n\right)^* (\mu_\varepsilon) & \text{on } H_\varepsilon^{-n}(\mathbb{D}_{\frac{\varepsilon}{2}}), \text{ if } n \geq 1 \\ 0 & \text{on } \mathbb{C} \setminus \bigcup_{n\geq 0} H_\varepsilon^{-n}(\mathbb{D}_{\frac{\varepsilon}{2}}), \end{cases}$$

where $H_\varepsilon^{-n}(\mathbb{D}_{\frac{\varepsilon}{2}})$ should be understood as the set of points whose n$^{\text{th}}$ iterate falls (for the first time) in $\mathbb{D}_{\frac{\varepsilon}{2}}$. Notice that with this definition, and using that $H_\varepsilon(U_\varepsilon \setminus \mathbb{D}_{\frac{\varepsilon}{2}}) = U_\varepsilon \setminus \mathbb{D}_{\frac{\varepsilon}{2}}$, we have that the points in $U_\varepsilon \setminus \mathbb{D}_{\frac{\varepsilon}{2}}$ (and all their preimages) satisfy $\mu_\varepsilon(z) = 0$. By construction, we have that μ_ε is measurable and invariant under the pull-back by H_ε, for it is spread out by the dynamics.

Remark 4.1. Since ψ_ε is k_ε-quasiconformal in $\mathbb{D}_{\frac{\varepsilon}{2}}$, for some $0 < k_\varepsilon < 1$, then μ_ε has maximal dilatation $\|\mu_\varepsilon\|_{\mathbb{D}_{\frac{\varepsilon}{2}}} = \|\mu_{\psi_\varepsilon^{-1}}\|_{\mathbb{D}_{\frac{\varepsilon}{2}}} = \|\mu_{\psi_\varepsilon}\|_{\mathbb{D}_{\frac{\varepsilon}{2}}} \leq k_\varepsilon < 1$, and also in the remainder of the plane since it is pulled-back by a holomorphic map (see Remark 2.9).

Therefore, we may apply Ahlfors-Bers' theorem (see Theorem 2.12) to $\widetilde{\mu}_\varepsilon(z) := \mu_\varepsilon(c_-(\varepsilon)z)$, obtaining a (unique) quasiconformal mapping $\widetilde{h}_\varepsilon : \mathbb{C} \to \mathbb{C}$ which integrates $\widetilde{\mu}_\varepsilon$, that is, $\mu_{\widetilde{h}_\varepsilon} = \widetilde{\mu}_\varepsilon$, and such that $\widetilde{h}_\varepsilon(0) = 0$ and $\widetilde{h}_\varepsilon(1) = 1$. Now, we define $h_\varepsilon(z) = -\widetilde{h}_\varepsilon(z/c_-(\varepsilon))$. We notice that h_ε integrates μ_ε and verifies $h_\varepsilon(0) = 0$ and $h_\varepsilon(c_-(\varepsilon)) = -1$.

Remark 4.2. As $\mu_\varepsilon = 0$ in $U_\varepsilon \setminus \overline{\mathbb{D}_{\frac{\varepsilon}{2}}}$, we have from the Beltrami equation (7) that h_ε is holomorphic in this region.

As explained in Remark 2.13, the composition map

$$G_\varepsilon := h_\varepsilon \circ H_\varepsilon \circ h_\varepsilon^{-1}$$

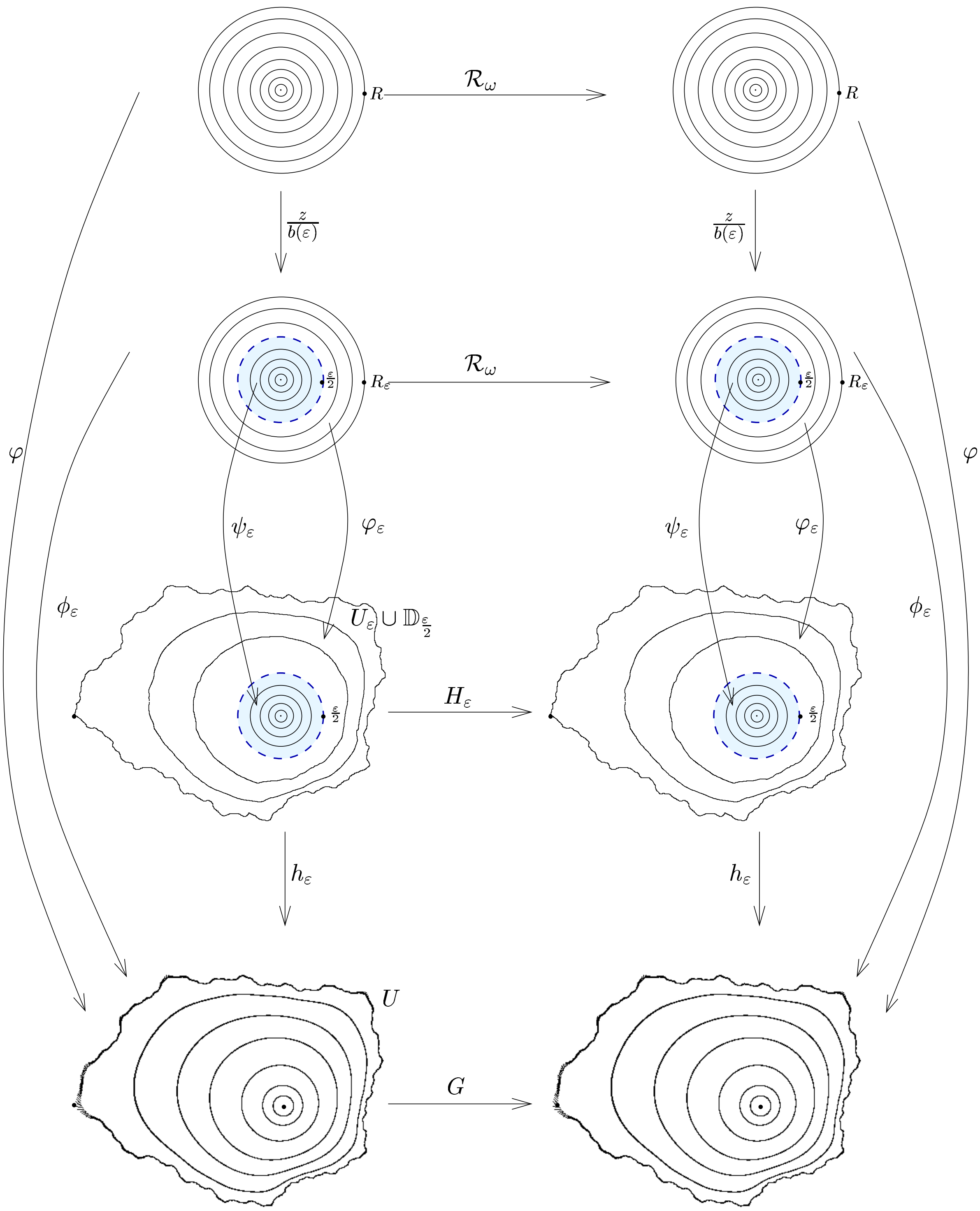

Figure 4: Commutative diagram showing all maps involved in the surgery construction.

is not only quasiconformally conjugate to H_ε but also holomorphic in $\mathbb{C}$. Moreover, if $\|\mu_\varepsilon\| < 1/3$ then we have that this map does not depend on the parameter ε as shown in the following proposition.

Proposition 4.3. *With the previous notations, if $\|\mu_\varepsilon\| < 1/3$, then for all $z \in \mathbb{C}$*

$$G_\varepsilon(z) = G(z) = z\mathrm{e}^{\mathrm{i}\omega}\mathrm{e}^{z}.$$

The proof of this proposition is similar to an analogous result for Geyer's construction in [G]. However, for the sake of completeness, we include here a proof adapted to our context. The key tool in the proof is an estimate on the growth of a quasiconformal map at ∞. This estimate is a consequence of the following basic property of quasiconformal maps.

Theorem 4.4 ([Ah], p. 71). *A k-quasiconformal mapping in a domain $\mathcal{U} \subset \mathbb{C}$ is uniformly Hölder continuous with exponent $\frac{1-k}{1+k}$ in every compact subset of $\mathcal{U}$.*

This result implies the desired bound for a quasiconformal map at ∞.

Lemma 4.5. *Let ϕ be a k-quasiconformal mapping of $\mathbb{C}$, fixing 0 and ∞. Then there exists $C > 0$ such that for any $|z| >> 1$ we have $|\phi(z)| \le C|z|^{\frac{1+k}{1-k}}$.*

Proof : We set $h = \phi^{-1}$ and $g(z) = \frac{1}{h(1/z)}$. It is easy to check that g is k-quasiconformal, with $g(0) = 0$ and $g(\infty) = \infty$. Applying Theorem 4.4 we have that there exists a constant $m > 0$ such that

$$|g(z_1) - g(z_2)| \le m|z_1 - z_2|^{\frac{1-k}{1+k}}, \qquad \text{if } |z_1|, |z_2| \le 1.$$

We take $z_1 = z$ and $z_2 = 0$, and we replace $z = \frac{1}{\phi(w)}$ to obtain the desired bound:

$$|\phi(w)| \le C|w|^{\frac{1+k}{1-k}},$$

with $C = m^{\frac{1+k}{1-k}}$. However, this estimate holds provided that $|\phi(w)| \ge 1$. As $\{w \in \mathbb{C} : |\phi(w)| \le 1\}$ is a compact set, we can assure that $|\phi(w)| \ge 1$ if $|w| >> 1$.

□

Proof of Proposition 4.3: By construction we know the following properties of G_ε: a) G_ε is entire; b) $G_\varepsilon(z) = 0$ if and only if $z = 0$; c) G_ε has a Siegel disk $h_\varepsilon(\mathbb{D}_{\frac{\varepsilon}{2}} \cup U_\varepsilon)$ around $z = 0$, with rotation number θ, and hence $G'_\varepsilon(0) = \mathrm{e}^{\mathrm{i}\omega}$; and d) $G'_\varepsilon(-1) = 0$ because G_ε is not univalent (it has degree two) around -1.

Combining the first two properties of G_ε, we have that

$$G_\varepsilon(z) = z g_\varepsilon(z),$$

with g_ε entire and without zeros. Now, we can estimate the growth order of g_ε. To this end, we use that if $z \in \mathbb{C} \setminus \mathbb{D}_{\frac{\varepsilon}{2}}$ then:

$$H_\varepsilon(z) = F_{\alpha(\varepsilon),\varepsilon}(z) = z\mathrm{e}^{\mathrm{i}\alpha(\varepsilon)}\mathrm{e}^{z-\frac{\varepsilon^2}{4z}}.$$

So, we have that if $|z| >> 1$ then:

$$G_\varepsilon(z) = h_\varepsilon\left(h_\varepsilon^{-1}(z)\mathrm{e}^{\mathrm{i}\alpha(\varepsilon)}\mathrm{e}^{h_\varepsilon^{-1}(z)-\frac{\varepsilon^2}{4h_\varepsilon^{-1}(z)}}\right).$$

From Lemma 4.5 we have that if $|z| >> 1$ there exists some constant $C > 0$ such that $|h_\varepsilon(z)| \leq C|z|^{K_\varepsilon}$, where $K_\varepsilon = \frac{1+\|\mu_\varepsilon\|}{1-\|\mu_\varepsilon\|}$. For the other values of z the map h_ε is bounded. This is also true for h_ε^{-1}, and both facts can be summarized by saying that there exists $M > 0$, which depends only on ε, such that

$$|h_\varepsilon(z)| \leq M \max\{|z|^{K_\varepsilon}, 1\}, \qquad |h_\varepsilon^{-1}(z)| \leq M \max\{|z|^{K_\varepsilon}, 1\}.$$

Moreover, we may also ask $|h_\varepsilon^{-1}(z)| \geq 1$ if $|z| >> 1$, obtaining

$$|G_\varepsilon(z)| \leq m_1 |z|^{K_\varepsilon^2} \mathrm{e}^{m_2 |z|^{K_\varepsilon}},$$

where m_1 and m_2 may depend on ε (of course, the condition $|z| >> 1$ is not necessarily uniform on ε). As we are assuming $\|\mu_\varepsilon\| < 1/3$, we have that $1 \leq K_\varepsilon < 2$, and so, we deduce that g_ε has growth order controlled by

$$|g_\varepsilon(z)| \leq \mathrm{e}^{|z|^p}, \qquad \text{if } |z| >> 1,$$

with $1 \leq p < 2$. The known properties of $g_\varepsilon(z)$ (entire function without zeros and with exponential growth of order $1 \leq p < 2$) imply that it is of the form:

$$g_\varepsilon(z) = \mathrm{e}^{P_\varepsilon(z)},$$

(see [D]) with $P_\varepsilon(z)$ a polynomial of degree not greater than 1. Now, the proposition follows from the remaining properties.

□

Remark 4.6. Assuming that $\|\mu_\varepsilon\| < 1/3$, we just proved that $G = h_\varepsilon \circ H_\varepsilon \circ h_\varepsilon^{-1}$, and so H_ε and G are conjugated by h_ε. Then, as the invariant curves are preserved by conjugation, we have that the rotation domain $\mathbb{D}_{\frac{\varepsilon}{2}} \cup U_\varepsilon$ of H_ε is mapped by h_ε to the Siegel Disk U of G (see (5)). That is:

$$U = h_\varepsilon(\mathbb{D}_{\frac{\varepsilon}{2}} \cup U_\varepsilon).$$

This concludes the surgery construction relating the (scaled) standard map $F_{\alpha(\varepsilon),\varepsilon}$ and the semistandard map G. In the following section we shall see which quantities we need to estimate in order to obtain quantitative information from the surgery we just performed.

4.2 Restatement of the problem

Using the same notation of Section 4.1 and assuming $\|\mu_\varepsilon\| < 1/3$, we observe (see Figure 4) that the map defined as

$$\phi_\varepsilon(z) = \begin{cases} (h_\varepsilon \circ \varphi_\varepsilon)(z) & \text{if } z \in \mathbb{D}_{R_\varepsilon} \setminus \mathbb{D}_{\frac{\varepsilon}{2}} \\ (h_\varepsilon \circ \psi_\varepsilon)(z) & \text{if } z \in \mathbb{D}_{\frac{\varepsilon}{2}} \end{cases}$$

is holomorphic in $\mathbb{D}_{R_\varepsilon}$. From Remark 4.2, this assertion is obviously true in $\mathbb{D}_{R_\varepsilon} \setminus \mathbb{D}_{\frac{\varepsilon}{2}}$. To prove the analyticity of ϕ_ε in $\mathbb{D}_{\frac{\varepsilon}{2}}$, we can check that $\phi_\varepsilon^* \mu_0 = \mu_0$ (see Remark 2.10), which follows from the fact that $\mu_\varepsilon = (\psi^{-1})^* \mu_0$ in $\mathbb{D}_{\frac{\varepsilon}{2}}$.

By construction, ϕ_ε conjugates the semistandard map G on the Siegel disk $U = h_\varepsilon(\mathbb{D}_{\frac{\varepsilon}{2}} \cup U_\varepsilon)$ to the rotation $\mathcal{R}_\omega$ on $\mathbb{D}_{R_\varepsilon}$.

But we can not be sure that the map ϕ_ε is the normalized linearizing map φ of G (see (5)), since, among other reasons, we expect the radius R_ε to move with ε while G, U and R_0 do not.

This is equivalent to say that $\phi'_\varepsilon(0) \neq 1$. Then, to recuperate the (normalized) linearizing map φ let us define

$$b(\varepsilon) = \phi'_\varepsilon(0),$$

and so by the preceding argument,

$$\varphi(z) = \phi_\varepsilon\left(\frac{z}{b(\varepsilon)}\right),$$

given that $\phi_\varepsilon\left(\frac{z}{b(\varepsilon)}\right)$ satisfies both normalization conditions ($\varphi(0) = 0$, $\varphi'(0) = 1$). See Figure 4.

From here, it is clear that $R_\varepsilon = \frac{R_0}{|b(\varepsilon)|}$ so that, to relate R_0 and R_ε we need to have control over $b(\varepsilon)$, i.e. over $(h_\varepsilon \circ \psi_\varepsilon)'(0)$.

Remark 4.7. Let us observe that $(h_\varepsilon \circ \psi_\varepsilon)'(0)$ does not depend on the particular quasiconformal map ψ_ε used in the surgery construction (which is of course not unique). This allows us to compute this derivative by constructing explicitly a convenient ψ_ε.

From the previous observations, Theorem A follows immediately from the next proposition.

Proposition 4.8. *With the previous notation, we have*

$$b(\varepsilon) = (h_\varepsilon \circ \psi_\varepsilon)'(0) = 1 + \mathcal{O}(\varepsilon \log \varepsilon).$$

To prove Proposition 4.8, we shall study the quantity

$$\frac{d}{dz}\left(h_\varepsilon \circ \psi_\varepsilon(z) - z\right)_{|z=0}$$

by means of the Cauchy integral formula. This will be done in Section 4.4. The estimates we use come from studying the quantities $|\psi_\varepsilon(z) - z|$ and $|h_\varepsilon(z) - z|$ or, equivalently, how far the maps ψ_ε and h_ε are from the identity map in a neighborhood of zero.

To obtain such an estimate for ψ_ε we construct ψ_ε explicitly in Section 4.3. The estimate for h_ε is a direct application of Proposition B.

4.3 Explicit surgery construction

The main purpose of this section is to construct explicitly the quasiconformal extension ψ_ε used in the surgery construction of Section 4.1, and to give the explicit estimates that measure how far ψ_ε is from the identity map.

Let us recall that we have a circle of radius $\varepsilon/2$ on which the real analytic (scaled) conjugacy φ_ε is defined. Our goal is to find a quasi-conformal map $\psi_\varepsilon : \mathbb{D}_{\frac{\varepsilon}{2}} \to \mathbb{D}_{\frac{\varepsilon}{2}}$ that extends φ_ε.

We define the "gluing map" ψ_ε to be the most natural extension: the radial one. More explicitly, given $z \in \mathbb{D}_{\frac{\varepsilon}{2}}$ we define

$$\psi_\varepsilon(z) = \frac{2}{\varepsilon}|z| \ \ \varphi_\varepsilon\left(\frac{\varepsilon}{2}\frac{z}{|z|}\right). \tag{13}$$

This map is clearly continuous, it agrees with φ_ε on the boundary of $\mathbb{D}_{\frac{\varepsilon}{2}}$ and sends 0 to 0. We observe also that it leaves all circles in $\mathbb{D}_{\frac{\varepsilon}{2}}$ invariant. In Proposition 4.9 we will prove that ψ_ε is a quasiconformal mapping if ε is small enough (even more, it is $\mathcal{C}^\infty$ at all points except at $z = 0$). Moreover, this result will show that for small values of ε we have $\|\mu_\varepsilon\| < 1/3$ (this condition is needed to show that $G_\varepsilon = G$, as stated in Proposition 4.3), and so all the results derived from the quasiconformal construction (see Sections 4.1 and 4.2) hold.

Our goal in the remainder of this section is to prove the following result.

Proposition 4.9. *There exists a constant $C_1 > 0$, independent of ε, such that for any ε small enough ψ_ε is a $(C_1\varepsilon)$-quasiconformal mapping in $\mathbb{D}_{\frac{\varepsilon}{2}}$, and it verifies*

$$|\psi_\varepsilon(z) - z| \leq C_1\varepsilon|z|. \tag{14}$$

Remark 4.10. One could also use an alternative, more dynamically meaningful, quasi-conformal extension of φ_ε given by $\psi_\varepsilon(z) = \varphi_{2|z|}(z)$. In this case, the map H_ε would be explicitly given by $H_\varepsilon(z) = F_{\alpha(2|z|),2|z|}(z) = z\mathrm{e}^{\mathrm{i}\alpha(2|z|)}\mathrm{e}^{z-\bar{z}}$ on $\mathbb{D}_{\frac{\varepsilon}{2}}$. This extension also preserves circles in $\mathbb{D}_{\frac{\varepsilon}{2}}$ and, on each of these circles, it is the linearizing map of a scaled standard map. In this case, one can show that $|\psi_\varepsilon(z) - z| \leq C_1|z|^2$.

The proof of this proposition will be an easy consequence of the following lemma.

Lemma 4.11. *The linearization φ_ε (see (11)) of the scaled standard map $F_{\alpha(\varepsilon),\varepsilon}(z) = z\mathrm{e}^{\mathrm{i}\alpha(\varepsilon)}\mathrm{e}^{z-\frac{\varepsilon^2}{4z}}$, verifies the following bounds if ε is small enough and $|z| = \frac{\varepsilon}{2}$:*

$$|\varphi_\varepsilon(z) - z| \leq C_2\varepsilon^2, \qquad \left|\varphi'_\varepsilon(z) - 1\right| \leq C_2\varepsilon, \tag{15}$$

where C_2 is a constant independent of ε.

The proof of Lemma 4.11 is deferred to the end of the section.

Proof of Proposition 4.9: First of all, we stress that the estimates given by Lemma 4.11 are only valid if we evaluate $\varphi_\varepsilon(z)$ for $|z| = \varepsilon/2$. From the definition of $\psi_\varepsilon(z)$ (see (13)) this is precisely the case which we are interested in.

Let us see that ψ_ε is a quasiconformal mapping, and obtain a bound for its distortion. In $\mathbb{D}_{\frac{\varepsilon}{2}}$ we have:

$$\frac{\partial\psi_\varepsilon}{\partial z}(z) = \frac{1}{2}\varphi'_\varepsilon\left(\frac{\varepsilon}{2}\frac{z}{|z|}\right) + \frac{1}{\varepsilon}\frac{\bar{z}}{|z|}\varphi_\varepsilon\left(\frac{\varepsilon}{2}\frac{z}{|z|}\right),$$

and

$$\frac{\partial\psi_\varepsilon}{\partial\bar{z}}(z) = -\frac{1}{2}\frac{z^2}{|z|^2}\varphi'_\varepsilon\left(\frac{\varepsilon}{2}\frac{z}{|z|}\right) + \frac{1}{\varepsilon}\frac{z}{|z|}\varphi_\varepsilon\left(\frac{\varepsilon}{2}\frac{z}{|z|}\right).$$

Then, applying Lemma 4.11, we can bound:

$$\begin{aligned}\left|\frac{\partial\psi_\varepsilon}{\partial z}(z) - 1\right| &= \left|\frac{1}{2}\left(\varphi'_\varepsilon\left(\frac{\varepsilon}{2}\frac{z}{|z|}\right) - 1\right) + \frac{1}{\varepsilon}\frac{\bar{z}}{|z|}\left(\varphi_\varepsilon\left(\frac{\varepsilon}{2}\frac{z}{|z|}\right) - \frac{\varepsilon}{2}\frac{z}{|z|}\right)\right| \\ &\leq \frac{1}{2}C_2\varepsilon + \frac{1}{\varepsilon}C_2\varepsilon^2 \leq \frac{3}{2}C_2\varepsilon,\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\partial\psi_\varepsilon}{\partial\bar{z}}(z)\right| &= \left|-\frac{1}{2}\frac{z^2}{|z|^2}\left(\varphi'_\varepsilon\left(\frac{\varepsilon}{2}\frac{z}{|z|}\right) - 1\right) + \frac{1}{\varepsilon}\frac{z}{|z|}\left(\varphi_\varepsilon\left(\frac{\varepsilon}{2}\frac{z}{|z|}\right) - \frac{\varepsilon}{2}\frac{z}{|z|}\right)\right| \\ &\leq \frac{1}{2}C_2\varepsilon + \frac{1}{\varepsilon}C_2\varepsilon^2 \leq \frac{3}{2}C_2\varepsilon.\end{aligned}$$

So, if we assume ε to be small enough in order to have that $\frac{3}{2}C_2\varepsilon \leq \frac{1}{2}$, we can bound the distortion of ψ_ε by:

$$\left|\frac{\frac{\partial\psi_\varepsilon}{\partial\bar{z}}(z)}{\frac{\partial\psi_\varepsilon}{\partial z}(z)}\right| \leq \frac{\frac{3}{2}C_2\varepsilon}{1 - \frac{3}{2}C_2\varepsilon} \leq 3C_2\varepsilon.$$

Hence, ψ_ε is a $(C_1\varepsilon)$-quasiconformal mapping with $C_1 = 3C_2$.

In order to estimate how far ψ_ε is from the identity map, we apply the first inequality of (15) (see Lemma 4.11), obtaining

$$|\psi_\varepsilon(z) - z| = \left|\frac{2}{\varepsilon}|z|\left(\varphi_\varepsilon\left(\frac{\varepsilon}{2}\frac{z}{|z|}\right) - \frac{\varepsilon}{2}\frac{z}{|z|}\right)\right| \le \frac{2}{\varepsilon}|z|\,C_2\varepsilon^2 \le C_1\varepsilon\,|z|$$

which concludes the proof.

□

The rest of the section is dedicated to prove Lemma 4.11. For this purpose, we will need a Diophantine-like bound for Brjuno numbers, that is weaker than the Brjuno condition. More precisely, if θ is a Brjuno number, there exist constants c_1, c_2, depending only on θ, such that for any $k \in \mathbb{Z}\setminus\{0\}$, the following inequality is satisfied (see [Br1, p. 140])

$$|\mathrm{e}^{2\pi \mathrm{i} k\theta} - 1|^{-1} \le c_1 \mathrm{e}^{2\pi c_2 |k|}. \tag{16}$$

This inequality leads to the following Lemma:

Lemma 4.12. *Let $m(x)$ be a 1-periodic function with zero average, and $\theta \in \mathbb{R}$ be a Brjuno number, hence verifying (16). We assume that m is analytic in the complex strip $\mathcal{A}_{c_3} = \{x \in \mathbb{C}/\mathbb{Z} : |\mathrm{Im}(x)| < c_3\}$, being $c_3 > c_2$, and that $B = \sup_{x\in\mathcal{A}_{c_3}} |m(x)| < +\infty$.*

Consider the 1-periodic solution $\xi(x)$ of the difference equation:

$$\xi(x+\theta) - \xi(x) = m(x), \qquad \xi(0) = 0. \tag{17}$$

Then, ξ is analytic in $\mathcal{A}_{c_3-c_2}$, and verifies:

$$|\xi(x)| \le 4Bc_1 \frac{\mathrm{e}^{-2\pi(c_3-c_2-|\mathrm{Im}(x)|)}}{1-\mathrm{e}^{-2\pi(c_3-c_2-|\mathrm{Im}(x)|)}}, \qquad x \in \mathcal{A}_{c_3-c_2}.$$

Proof : First, we expand m in Fourier series:

$$m(x) = \sum_{k\in\mathbb{Z}} m_k \mathrm{e}^{2\pi \mathrm{i} kx}.$$

Then, using the bound of $|m|$ in the strip $\mathcal{A}_{c_3}$, and that it has zero average, one has that its Fourier coefficients verify $m_0 = 0$ and $|m_k| \le B\mathrm{e}^{-2\pi c_3|k|}$ if $k \ne 0$. On the other hand, writing ξ also in Fourier series, we can solve equation (17) for the coefficients of ξ, obtaining

$$\xi_k = \frac{m_k}{\mathrm{e}^{2\pi \mathrm{i} k\theta} - 1}, \quad \text{if } k \ne 0.$$

Moreover, condition $\xi(0) = 0$ gives $\xi_0 = -\sum_{k\in\mathbb{Z}\setminus\{0\}} \xi_k$.

Using (16) we obtain that

$$|\xi_k| \le c_1\,|m_k|\,\mathrm{e}^{2\pi c_2|k|} \le Bc_1\mathrm{e}^{-2\pi(c_3-c_2)|k|}, \quad \text{if } k \ne 0,$$

and, for ξ_0,

$$|\xi_0| \le \sum_{k\in\mathbb{Z}\setminus\{0\}} |\xi_k| \le 2Bc_1 \frac{\mathrm{e}^{-2\pi(c_3-c_2)}}{1-\mathrm{e}^{-2\pi(c_3-c_2)}}.$$

Joining these bounds, the estimate for $|\xi(x)|$ in $\mathcal{A}_{c_3-c_2}$ follows immediately.

□

Now, we have all the ingredients to prove Lemma 4.11.

Proof of Lemma 4.11: To obtain the estimates of this Lemma, it will be more convenient for us to work with the lift on the circle of the scaled standard map, that is, the Arnold standard family (1). To this end, we make the transformations we have done backwards in order to go from $\widetilde{f}_{\alpha(\varepsilon),\varepsilon}(x) = x + \frac{\alpha(\varepsilon)}{2\pi} + \frac{\varepsilon}{2\pi}\sin(2\pi x)$ to $F_{\alpha(\varepsilon),\varepsilon}(z)$. Then, if we make the change $z = \frac{\varepsilon}{2}u$, we have $\widetilde{\varphi}_\varepsilon(u) = \frac{2}{\varepsilon}\varphi_\varepsilon(\frac{\varepsilon}{2}u)$, where $\widetilde{\varphi}_\varepsilon$ is the linearization of the standard map $\widetilde{F}_{\alpha(\varepsilon),\varepsilon}$ given in (3). On the other hand, writing $u = \mathrm{e}^{\mathrm{i}x}$ we have $\widetilde{\varphi}_\varepsilon(\mathrm{e}^{2\pi\mathrm{i}x}) = \mathrm{e}^{2\pi\mathrm{i}\widetilde{\eta}_\varepsilon(x)}$, being $\widetilde{\eta}_\varepsilon$ a conjugation of $\widetilde{f}_{\alpha(\varepsilon),\varepsilon}$ to the rotation $\mathcal{T}_\theta$ in $\mathbb{T}^1$ (see (2)).

Then, we obtain that:

$$\varphi_\varepsilon(z) - z = \frac{\varepsilon}{2}(\widetilde{\varphi}_\varepsilon(u) - u) = \frac{\varepsilon}{2}(\mathrm{e}^{2\pi\mathrm{i}\widetilde{\eta}_\varepsilon(x)} - \mathrm{e}^{2\pi\mathrm{i}x}) = z(\mathrm{e}^{2\pi\mathrm{i}(\widetilde{\eta}_\varepsilon(x)-x)} - 1). \tag{18}$$

Our purpose is to bound $\widetilde{\eta}_\varepsilon(x) - x$ using the Local Conjugacy Theorem 2.1, and to derive from these bounds the ones for $\varphi_\varepsilon(z) - z$ and its derivative.

First of all, we observe that $\widetilde{f}_{\alpha(\varepsilon),\varepsilon}(x)$ is univalent in the strip $\mathcal{A}_{\frac{1}{2\pi}\log(\sigma(\varepsilon)/\varepsilon)}$, where $\sigma(\varepsilon) = 1 + \sqrt{1-\varepsilon^2}$ is defined in (9).

In order to apply Theorem 2.1, let us observe that, if ε is small enough, we have that $\frac{1}{2\pi}\log(\sigma(\varepsilon)/\varepsilon) > \frac{1}{2\pi}\Phi(\theta) + C_0$. Thus, Theorem 2.1 ensures that $\widetilde{\eta}_\varepsilon(x)$ is analytic if

$$|\mathrm{Im}(x)| < \frac{1}{2\pi}\log(\sigma(\varepsilon)/\varepsilon) - \frac{1}{2\pi}\Phi(\theta) - C_0.$$

In particular, if we take any constant $0 < c_4 \leq \frac{\sigma(\varepsilon)}{\mathrm{e}^{\Phi(\theta)+2\pi C_0}}$, the linearization $\widetilde{\eta}$ is defined in $\mathcal{A}_{\frac{1}{2\pi}\log(c_4/\varepsilon)}$ and its range is contained in $\mathcal{A}_{\frac{1}{2\pi}\log(\sigma(\varepsilon)/\varepsilon)}$. Moreover, in this domain it verifies $\widetilde{f}_{\alpha(\varepsilon),\varepsilon} \circ \widetilde{\eta}_\varepsilon = \widetilde{\eta}_\varepsilon \circ \mathcal{T}_\theta$.

Now, calling $\xi(x) = \widetilde{\eta}_\varepsilon(x) - x$, it is straightforward to check that

$$\xi(x+\theta) - \xi(x) = m(x),$$

where $m(x) = \widetilde{f}_{\alpha(\varepsilon),\varepsilon}(\widetilde{\eta}_\varepsilon(x)) - \widetilde{\eta}_\varepsilon(x) - \theta$. Then, $\xi(x)$ verifies the hypotheses of Lemma 4.12 if we define $c_3 \equiv c_3(\varepsilon) = \frac{1}{2\pi}\log(c_4/\varepsilon)$ and we take B a bound of $|\widetilde{f}_{\alpha(\varepsilon),\varepsilon}(x) - x|$ in the domain $\mathcal{A}_{\frac{1}{2\pi}\log(\sigma(\varepsilon)/\varepsilon)}$, which can be taken independent of ε. We remark that being $c_3 = \frac{1}{2\pi}\log(c_4/\varepsilon)$, the hypothesis $c_3 > c_2$ of Lemma 4.12 always hold if we assume $\varepsilon < c_4\mathrm{e}^{-2\pi c_2}$.

Applying the conclusions of Lemma 4.12, we obtain the following bound for the function $\xi(x)$ when $|\mathrm{Im}(x)| \leq c_3 - c_2 = \frac{1}{2\pi}\log(c_4/\varepsilon) - c_2$,

$$|\xi(x)| \leq 4Bc_1\frac{\mathrm{e}^{-(\log(c_4/\varepsilon)-2\pi(c_2+|\mathrm{Im}(x)|))}}{1-\mathrm{e}^{-(\log(c_4/\varepsilon)-2\pi(c_2+|\mathrm{Im}(x)|))}} = \frac{4Bc_1\varepsilon}{c_4\mathrm{e}^{-2\pi(c_2+|\mathrm{Im}(x)|)} - \varepsilon}.$$

Let us now take c_5 any constant, independent of ε, verifying $0 < c_5 < \frac{1}{2\pi}\log\left(\frac{c_4}{2\varepsilon}\right) - c_2$. We observe that c_5 can be taken arbitrarily large, provided that ε is sufficiently small, and that, with this definition of c_5, we always have the inequality $\varepsilon \leq \frac{c_4}{2}\mathrm{e}^{-2\pi(c_2+c_5)}$.

Then, the previous considerations imply that the function $\widetilde{\eta}_\varepsilon(x) - x = \xi(x)$ is defined in $\mathcal{A}_{c_5}$ and verifies, in this domain, the bound

$$|\widetilde{\eta}_\varepsilon(x) - x| \leq \frac{4Bc_1\varepsilon}{c_4\mathrm{e}^{-2\pi(c_2+c_5)} - \varepsilon} \leq \frac{8Bc_1\mathrm{e}^{2\pi(c_2+c_5)}}{c_4}\varepsilon. \tag{19}$$

Using the changes $u = \mathrm{e}^{2\pi \mathrm{i} x}$ and $z = \frac{\varepsilon}{2} u$, we have that if $x \in \mathcal{A}_{c_5}$ then $u \in A(\mathrm{e}^{-2\pi c_5}, \mathrm{e}^{2\pi c_5})$ and $z \in A\left(\frac{\varepsilon}{2}\mathrm{e}^{-2\pi c_5}, \frac{\varepsilon}{2}\mathrm{e}^{2\pi c_5}\right)$ (see (4)). Now, we assume ε small enough such that $\frac{8Bc_1 \mathrm{e}^{2\pi(c_2+c_5)}}{c_4}\varepsilon \leq 1$. So, equation (19) implies $|\widetilde{\eta}_\varepsilon(x) - x| \leq 1$. Then, for these values of z, inequality (18) leads to

$$|\varphi_\varepsilon(z) - z| \leq |z| \mathrm{e}^{2\pi |\widetilde{\eta}_\varepsilon(x) - x|} |\widetilde{\eta}_\varepsilon(x) - x| \leq \frac{16\pi B c_1 \mathrm{e}^{2\pi(c_2+c_5+1)}}{c_4} \varepsilon |z|, \tag{20}$$

So, the first bound in (15) follows from (20) as a particular case when $|z| = \frac{\varepsilon}{2}$, taking $C_2 \geq \frac{8\pi B c_1 \mathrm{e}^{2\pi(c_2+c_5+1)}}{c_4}$.

Using the bound of the function $\varphi_\varepsilon(z) - z$ in the annulus $A\left(\frac{\varepsilon}{2}\mathrm{e}^{-2\pi c_5}, \frac{\varepsilon}{2}\mathrm{e}^{2\pi c_5}\right)$, we proceed to bound its derivative for $|z| = \frac{\varepsilon}{2}$. To this end, we pick a particular value of z, and we consider the disk of center z and radius $r = \frac{\varepsilon}{2}(1 - \mathrm{e}^{-2\pi c_5})$, which we denote by $\mathbf{C}_r(z)$. One can easily check that if we take a point $t \in \mathbf{C}_r(z)$, then t belongs to $A\left(\frac{\varepsilon}{2}\mathrm{e}^{-2\pi c_5}, \frac{\varepsilon}{2}\mathrm{e}^{2\pi c_5}\right)$, and then $|\varphi_\varepsilon(t) - t|$ verifies (20) with $z \equiv t$.

Then, we can bound the derivative of $\varphi_\varepsilon(z) - z$ using the Cauchy integral formula. We have:

$$\varphi_\varepsilon'(z) - 1 = \frac{d}{dt}(\varphi_\varepsilon(t) - t)_{|t=z} = \frac{1}{2\pi \mathrm{i}} \int_{\mathbf{C}_r(z)} \frac{\varphi_\varepsilon(t) - t}{(t-z)^2} dt.$$

In order to bound the integral above, we may apply inequality (20) to any $t \in \mathbf{C}_r(z)$, obtaining:

$$\begin{aligned} \left|\varphi_\varepsilon'(z) - 1\right| &\leq \frac{1}{r} \max_{t \in \mathbf{C}_r(z)} |\varphi_\varepsilon(t) - t| \leq \frac{1}{r} \max_{t \in \mathbf{C}_r(z)} \left\{ \frac{16\pi B c_1 \mathrm{e}^{2\pi(c_2+c_5+1)}}{c_4} \varepsilon \, |t| \right\} \\ &\leq \frac{2 - \mathrm{e}^{-2\pi c_5}}{1 - \mathrm{e}^{-2\pi c_5}} \frac{16\pi B c_1 \mathrm{e}^{2\pi(c_2+c_5+1)}}{c_4} \varepsilon, \end{aligned}$$

which gives the second part of (15) taking $C_2 \geq \frac{2 - \mathrm{e}^{-2\pi c_5}}{1 - \mathrm{e}^{-2\pi c_5}} \frac{16\pi B c_1 \mathrm{e}^{2\pi(c_2+c_5+1)}}{c_4}$.

□

4.4 Proof of Theorem A

From the restatement of the problem in Section 4.1, we only need to prove Proposition 4.8 in order to proof Theorem A.

Proof of Proposition 4.8: As we know that $h_\varepsilon \circ \psi_\varepsilon$ is holomorphic, we can express $b(\varepsilon) - 1$ in terms of the Cauchy integral formula:

$$\begin{aligned} b(\varepsilon) - 1 &= \frac{d}{dz}\left(h_\varepsilon \circ \psi_\varepsilon(z) - z\right)_{|z=0} \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\mathbf{C}_r} \frac{h_\varepsilon \circ \psi_\varepsilon(z) - z}{z^2} dz \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\mathbf{C}_r} \frac{h_\varepsilon \circ \psi_\varepsilon(z) - \psi_\varepsilon(z)}{z^2} dz + \frac{1}{2\pi \mathrm{i}} \int_{\mathbf{C}_r} \frac{\psi_\varepsilon(z) - z}{z^2} dz, \end{aligned}$$

where $\mathbf{C}_r$ is any circle contained in $\overline{\mathbb{D}_{\frac{\varepsilon}{2}}}$. From now on we take $r = \frac{\varepsilon}{2}$.

We can bound the second integral by using inequality (14) of Proposition 4.9, obtaining:

$$\left| \frac{1}{2\pi \mathrm{i}} \int_{\mathbf{C}_{\frac{\varepsilon}{2}}} \frac{\psi_\varepsilon(z) - z}{z^2} dz \right| \leq \frac{2}{\varepsilon} \sup_{z \in \mathbf{C}_{\frac{\varepsilon}{2}}} \left\{ |\psi_\varepsilon(z) - z| \right\} \leq C_1 \varepsilon.$$

For the first integral, we use that $|\psi_\varepsilon(z)| = |z|$ if $|z| \leq \frac{\varepsilon}{2}$ (see (13)), and so,

$$\left| \frac{1}{2\pi \mathrm{i}} \int_{\mathbf{C}_{\frac{\varepsilon}{2}}} \frac{h_\varepsilon \circ \psi_\varepsilon(z) - \psi_\varepsilon(z)}{z^2} dz \right| \leq \frac{2}{\varepsilon} \sup_{z \in \mathbf{C}_{\frac{\varepsilon}{2}}} \{|h_\varepsilon(z) - z|\}.$$

In order to bound $h_\varepsilon(z) - z$ we first recall that $\psi_\varepsilon(z)$ is $(C_1\varepsilon)$-quasiconformal (see Proposition 4.9). Hence, the maximal dilatation of μ_ε is bounded by the same constant, that is, $\|\mu_\varepsilon\| \leq C_1\varepsilon$. Now, we want to apply Proposition B. However, we can not do it directly because h_ε has been constructed in such a way that $h_\varepsilon(0) = 0$ and $h_\varepsilon(c_-(\varepsilon)) = -1$, where $c_-(\varepsilon) = (-1 - \sqrt{1-\varepsilon^2})/2$ is the critical point of $F_{\alpha(\varepsilon),\varepsilon}$ given in (10). We can arrange h_ε to apply Proposition B simply by defining $\widetilde{h}_\varepsilon(z) = -h_\varepsilon(c_-(\varepsilon)z)$. We observe that $\widetilde{h}_\varepsilon$ is a quasiconformal map of $\mathbb{C}$, solving the Beltrami equation (7) with the Beltrami form $\widetilde{\mu}_\varepsilon(z) = \mu_{\widetilde{h}_\varepsilon}(z) = \mu_\varepsilon(c_-(\varepsilon)z)$ and verifies $\widetilde{h}_\varepsilon(0) = 0$ and $\widetilde{h}_\varepsilon(1) = 1$. Then, if we assume that ε is small enough such that $\|\widetilde{\mu}_\varepsilon\| = \|\mu_\varepsilon\| \leq C_1\varepsilon \leq \rho$, we can apply Proposition B to $\widetilde{\mu}_\varepsilon$, obtaining that if $|z| \leq \rho$ and $\|\widetilde{\mu}_\varepsilon\| |\log|z|| \leq C_1\varepsilon|\log|z|| \leq \rho$, then

$$|\widetilde{h}_\varepsilon(z) - z| \leq C\|\widetilde{\mu}_\varepsilon\||z||\log|z|| \leq CC_1\varepsilon|z||\log|z||. \tag{21}$$

Taking into account that $\left|\frac{1}{c_-(\varepsilon)} + 1\right| \leq \varepsilon^2$, we have that if $|z| = \frac{\varepsilon}{2}$ with ε small, then

$$\left|\frac{z}{c_-(\varepsilon)}\right| \leq \frac{\varepsilon}{2}(1+\varepsilon^2) \leq \rho, \qquad \|\mu_\varepsilon\| \left|\log\left|\frac{z}{c_-(\varepsilon)}\right|\right| \leq C_1\varepsilon \left|\log\left(\frac{\varepsilon}{2}(1+\varepsilon^2)\right)\right| \leq \rho.$$

So, we can apply formula (21) to $\frac{z}{c_-(\varepsilon)}$, obtaining:

$$\begin{aligned} |h_\varepsilon(z) - z| &= \left|\widetilde{h}_\varepsilon\left(\frac{z}{c_-(\varepsilon)}\right) + z\right| \leq \left|\widetilde{h}_\varepsilon\left(\frac{z}{c_-(\varepsilon)}\right) - \frac{z}{c_-(\varepsilon)}\right| + |z|\left|1 + \frac{1}{c_-(\varepsilon)}\right| \\ &\leq CC_1\varepsilon\left|\frac{z}{c_-(\varepsilon)}\right| \log\left(\left|\frac{z}{c_-(\varepsilon)}\right|\right) + |z|\varepsilon^2 \leq C'\varepsilon^2|\log\varepsilon|, \end{aligned}$$

with C' depending on C and C_1. As a consequence of this, we can bound:

$$\left| \frac{1}{2\pi \mathrm{i}} \int_{\mathbf{C}_{\frac{\varepsilon}{2}}} \frac{h_\varepsilon \circ \psi_\varepsilon(z) - \psi_\varepsilon(z)}{z^2} dz \right| \leq 2C'\varepsilon|\log\varepsilon|,$$

obtaining $b(\varepsilon) = 1 + \mathcal{O}(\varepsilon|\log\varepsilon|)$, and therefore ending the proof of Proposition 4.8.

$\square$

This concludes the proof of Theorem A, up to proving Proposition B.

5 Proof of Proposition B

Our goal in this section is to prove Proposition B.

5.1 Proof of (a)

Let us consider the following one-parameter family of Beltrami forms:

$$\mu_t = t\frac{\mu}{||\mu||},$$

where $t \in \mathbb{D}$. As $||\mu_t|| = |t| < 1$ for all $t \in \mathbb{D}$, we can apply Theorem 2.12 and obtain a one-parameter family of integrating maps $h_t : \mathbb{C} \to \mathbb{C}$ fixing 0 and 1, and such that $\frac{\partial h_t}{\partial \bar{z}} = \mu_t \frac{\partial h_t}{\partial z}$. Moreover, $h_t(z)$ depends analytically on t. Observe that, as $\mu_t = 0$ if $t = 0$ and $\mu_t = \mu$ if $t = \|\mu\|$, we have that $h_0 = \mathrm{Id}$ and $h_{||\mu||} = h$.

Now, let $z \in \mathbb{C} \setminus \{0, 1\}$ fixed, and consider the holomorphic map

$$\begin{array}{rccc} f_z : & \mathbb{D} & \longrightarrow & \mathbb{C} \setminus \{0,1\} \\ & t & \longmapsto & h_t(z). \end{array}$$

Since both $\mathbb{D}$ and $\mathbb{C} \setminus \{0, 1\}$ are hyperbolic sets, we conclude from Big Schwartz-Pick lemma (see Theorem 2.23) that f_z is a contraction in the Poincaré metrics, that is

$$\delta_{\mathbb{C}\setminus\{0,1\}}(h_{t_1}(z), h_{t_2}(z)) \le \delta_{\mathbb{D}}(t_1, t_2)$$

for all $t_1, t_2 \in \mathbb{D}$. If we take $t_1 = 0$ and $t_2 = ||\mu||$, the statement follows.

5.2 Proof of (b)

The second part of Proposition B will be deduced from the first one by comparing the Euclidean distance, between points close to 0, with the hyperbolic distance in $\mathbb{C} \setminus \{0, 1\}$, and using the explicit formula for $\delta_{\mathbb{D}}(0, \|\mu\|)$ given in Proposition 2.18. We notice that it is not easy to work directly with $\delta_{\mathbb{C}\setminus\{0,1\}}$, since there is no explicit formula for this hyperbolic distance. However, as stated in the following proposition, $\delta_{\mathbb{C}\setminus\{0,1\}}$ is comparable –close to the origin– to $\delta_{\mathbb{D}^*}$, for which we have an explicit expression (see Proposition 2.22).

Proposition 5.1. *There exist constants $0 < c < 1/2$, $M > 0$ and $\sigma > 0$ such that:*

(a) *For all $z_1, z_2 \in \mathbb{D}_c^* = \mathbb{D}_c \setminus \{0\}$*

$$1 \le \frac{\delta_{\mathbb{D}^*}(z_1, z_2)}{\delta_{\mathbb{C}\setminus\{0,1\}}(z_1, z_2)} \le 1 + M.$$

(b) *If $z_1, z_2 \in \mathbb{D}^*$, with $|z_1| \le c/2$ and $\delta_{\mathbb{C}\setminus\{0,1\}}(z_1, z_2) \le \sigma$, then $|z_2| \le c$.*

The comparison between the Euclidean distance and $\delta_{\mathbb{D}^*}$ is given by the following lemma.

Lemma 5.2. *Let $z_1, z_2 \in \mathbb{D}^*$ satisfying*

$$|\log |z_1||\delta_{\mathbb{D}^*}(z_1, z_2) \le 1/\sqrt{2}, \qquad \delta_{\mathbb{D}^*}(z_1, z_2) \le 2\log(1 + \sqrt{2}),$$

then:

$$|z_1 - z_2| \le \frac{\sqrt{2}}{\sqrt{2} - 1}|z_1|| \log |z_1||\delta_{\mathbb{D}^*}(z_1, z_2).$$

Remark 5.3. We point out that part **(a)** of Proposition B *only* provides information about $\delta_{\mathbb{C}\setminus\{0,1\}}(z,h(z))$, and thus on $\delta_{\mathbb{D}^*}(z,h(z))$ (after using Proposition 5.1). This is the reason for which the hypotheses of Lemma 5.2 are formulated in terms of the hyperbolic distance. If, for instance, *a-priori* estimates on $|z_1-z_2|/|z_1|$ were known, then the statement (and the proof) of the lemma could be simplified.

The proof of Proposition 5.1 and Lemma 5.2 are postponed to the following section. Now, we prove the second part of Proposition B.

The key point to prove (**b**) is the estimate provided by (**a**):

$$\delta_{\mathbb{C}\setminus\{0,1\}}(z,h(z)) \leq \delta_{\mathbb{D}}(0,\|\mu\|). \tag{22}$$

Using this inequality, we will check that the desired result follows by taking $\rho = \min\left\{\frac{\sigma}{4},\frac{c}{2},\frac{\sqrt{2}}{8(1+M)}\right\}$, where $0<c<1/2$, $M>0$ and $\sigma>0$ are the constants provided by Proposition 5.1.

First of all, by Proposition 2.18 we have

$$\delta_{\mathbb{D}}(0,\|\mu\|) = \log\left(\frac{1+\|\mu\|}{1-\|\mu\|}\right).$$

Now, as we are assuming $\|\mu\|\leq\rho$, then we have $\|\mu\|\leq c/2<1/2$. So, applying the mean value theorem to the logarithm, we deduce that $\delta_{\mathbb{D}}(0,\|\mu\|)\leq 4\|\mu\|$.

As we also suppose $\|\mu\|\leq\sigma/4$, the previous inequality on $\delta_{\mathbb{D}}(0,\|\mu\|)$ jointly with (22) implies that $\delta_{\mathbb{C}\setminus\{0,1\}}(z,h(z))\leq\sigma$. Then, from part **(b)** of Proposition 5.1 we deduce that if $|z|\leq c/2$, then $|h(z)|\leq c$. This a-priori estimate on the size of $h(z)$ allows us to apply part **(a)** of Proposition 5.1, obtaining

$$\delta_{\mathbb{D}^*}(z,h(z)) \leq (1+M)\delta_{\mathbb{C}\setminus\{0,1\}}(z,h(z)) \leq 4(1+M)\|\mu\|.$$

Finally, if we combine this last inequality with the hypotheses $\|\mu\|\leq\rho$ and $\|\mu\||\log|z||\leq\rho$, we obtain:

$$\delta_{\mathbb{D}^*}(z,h(z))|\log|z|| \leq 4(1+M)\|\mu\||\log|z|| \leq 4(1+M)\rho \leq 1/\sqrt{2},$$

and

$$\delta_{\mathbb{D}^*}(z,h(z)) \leq 4(1+M)\|\mu\| \leq 4(1+M)\rho \leq \frac{\sqrt{2}}{2} \leq 2\log(1+\sqrt{2}).$$

Thus, the hypotheses of Lemma 5.2 are verified for $z_1=z$ and $z_2=h(z)$, giving:

$$|h(z)-z| \leq \frac{\sqrt{2}}{\sqrt{2}-1}|z||\log|z||\delta_{\mathbb{D}^*}(z,h(z)) \leq \frac{4\sqrt{2}(1+M)}{\sqrt{2}-1}\|\mu\||z||\log|z|| \equiv C\|\mu\||z||\log|z||.$$

5.3 Proof of Proposition 5.1 and Lemma 5.2

We recall that Ahlfors' Lemma (see Proposition 2.24) provides a way to compare the *line elements* corresponding to the hyperbolic distances when $\mathcal{U}=\mathbb{D}^*$ and $\mathcal{V}=\mathbb{C}\setminus\{0,1\}$. To prove Proposition 5.1 we have to obtain a similar comparison result for the hyperbolic distances between two points. In addition to Ahlfors' Lemma we need the following result, whose prove is postponed to the end of this section.

Lemma 5.4. *There exists a constant* $0<\kappa<1$ *and a topological closed disk* D, $0\in D\subset\mathbb{D}_{1-\kappa}$, *such that if* $z_1,z_2\in D^*=D\setminus\{0\}$, *then the geodesic (with respect to* $\lambda_{\mathbb{C}\setminus\{0,1\}}$*) that joins* z_1 *and* z_2 *is entirely contained in* D.

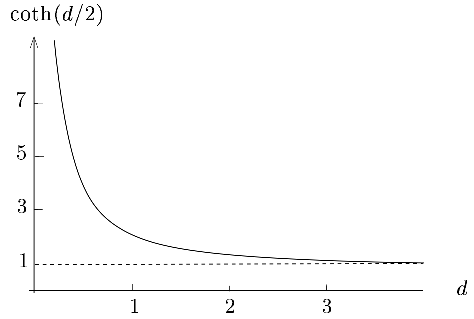

Figure 5: The graph of $\coth(d/2)$ or equivalently $\frac{e^d+1}{e^d-1}$.

Proof of Proposition 5.1: During the proof, let us set $\mathcal{U} = \mathbb{D}^*$ and $\mathcal{V} = \mathbb{C} \setminus \{0, 1\}$.
(a) The left hand inequality follows from its analogue in Proposition 2.24. Indeed, let $\gamma \subset \mathcal{U}$ be a path joining z_1 and z_2. Since $\mathcal{U} \subset \mathcal{V}$, we have

$$l_{\mathcal{U}}(\gamma) = \int_a^b \lambda_{\mathcal{U}}(\gamma(t))|\gamma'(t)|dt \geq \int_a^b \lambda_{\mathcal{V}}(\gamma(t))|\gamma'(t)|dt = l_{\mathcal{V}}(\gamma) \geq \delta_{\mathcal{V}}(z_1, z_2).$$

As this holds for any γ, we obtain $\delta_{\mathcal{U}}(z_1, z_2) \geq \delta_{\mathcal{V}}(z_1, z_2)$.

To prove the right hand inequality we need to work a little harder. From Proposition 2.24 we only obtain for any $\gamma \subset \mathcal{U}$ connecting z_1 and z_2:

$$\begin{aligned} l_{\mathcal{U}}(\gamma) &= \int_a^b \lambda_{\mathcal{U}}(\gamma(t))|\gamma'(t)|dt \leq \int_a^b \coth\left(\frac{d(t)}{2}\right) \lambda_{\mathcal{V}}(\gamma(t))|\gamma'(t)|dt \\ &\leq \coth\left(\frac{d_\gamma}{2}\right) l_{\mathcal{V}}(\gamma) = K_\gamma \, l_{\mathcal{V}}(\gamma), \end{aligned} \tag{23}$$

where we define $d(t) = \delta_{\mathcal{V}}(\gamma(t), \partial\mathcal{U})$, $d_\gamma = \min_{t\in[a,b]} d(t)$ and $K_\gamma = \coth\left(\frac{d_\gamma}{2}\right)$.

Notice that if γ is a curve that comes very close to $\partial\mathcal{U} \setminus \partial\mathcal{V} = \mathbf{C}_1 \setminus \{1\}$, then d_γ is a constant very close to 0, and consequently, K_γ is very close to infinity (see Figure 5). Thus, to assert that K_γ is finite we need to have γ bounded away from the set $\mathbf{C}_1 \setminus \{1\}$.

Let $0 < c < 1/2$ be a constant such that the disc $\mathbb{D}_c$ is contained in the topological disk D, $D \subset \mathbb{D}_{1-\kappa}$, of Lemma 5.4. For any two points $z_1, z_2 \in \mathbb{D}_c^*$ it is not difficult to check, by combining Proposition 2.19 with Proposition 2.22, that the $\mathbb{D}^*$-geodesic path joining z_1 and z_2 is entirely contained in a disk whose radius is the maximum of the moduli of z_1 and z_2, which is at most c. Consequently,

$$\delta_{\mathcal{U}}(z_1, z_2) = \inf_{\gamma \subset \mathcal{U}} l_{\mathcal{U}}(\gamma) = \inf_{\gamma \subset \mathbb{D}_c^*} l_{\mathcal{U}}(\gamma) = \inf_{\gamma \subset \mathbb{D}_{1-\kappa}^*} l_{\mathcal{U}}(\gamma),$$

where the infimums are always taken on paths γ joining z_1 and z_2. Since for any $\gamma \subset \mathbb{D}_{1-\kappa}^*$, the constant d_γ is bounded away from zero and therefore $K_\gamma \leq 1 + M$ for a certain constant $M = M(\kappa) > 0$, equation (23) reads as:

$$\inf_{\gamma \subset \mathbb{D}_{1-\kappa}^*} l_{\mathcal{U}}(\gamma) \leq \inf_{\gamma \subset \mathbb{D}_{1-\kappa}^*} K_\gamma \, l_{\mathcal{V}}(\gamma) \leq (1 + M) \inf_{\gamma \subset \mathbb{D}_{1-\kappa}^*} l_{\mathcal{V}}(\gamma).$$

To conclude the proof we observe that for any $z_1, z_2 \in \mathbb{D}_c^*$, Lemma 5.4 guarantees that the $\mathcal{V}$-geodesic joining z_1 and z_2 is entirely contained in $D \subset \mathbb{D}_{1-\kappa}$. Then, this implies:

$$\begin{aligned}\delta_{\mathcal{U}}(z_1,z_2) &= \inf_{\gamma\subset\mathbb{D}^*_{1-\kappa}} l_{\mathcal{U}}(\gamma) \le (1+M) \inf_{\gamma\subset\mathbb{D}^*_{1-\kappa}} l_{\mathcal{V}}(\gamma) \\ &= (1+M) \inf_{\gamma\subset\mathcal{V}} l_{\mathcal{V}}(\gamma) = (1+M)\delta_{\mathcal{V}}(z_1,z_2).\end{aligned}$$

(b) The geometrical definition of the Poincaré metric makes this result straightforward. We just have to define

$$\sigma = \inf_{\substack{|z_1|=c \\ |z_2|=c/2}} \delta_{\mathcal{V}}(z_1,z_2).$$

We stress that this proof is independent on the particular value of $0 < c < 1/2$ provided by **(a)**.

□

To conclude the first part of this section, we now prove Lemma 5.4.

Proof of Lemma 5.4: The key to proving this lemma lies on the understanding of how geodesics in the thrice punctured sphere $\mathcal{V} = \mathbb{C} \setminus \{0,1\}$ look like.

We first observe that the vertical line l going through the point $\frac{1}{2}$ is a geodesic since it is a line of symmetry in $\mathcal{V}$. Two more geodesics can be obtained by considering the Möbius transformations of $\overline{\mathbb{C}}$, $g(z) = \frac{az+b}{cz+d}$, such that map $\mathcal{V}$ onto itself (i.e., those that permute 1,0 and ∞). These maps are isometries and hence they send geodesics to geodesics. In particular, some of them map l to the unit circle and the others to the circle centered at 1 with radius 1 (check this using, for instance, the transformations $\frac{z-1}{z}$ and $\frac{1}{1-z}$). See Figure 7.

This already says that if two points are in $\mathbb{D}^*$, the geodesic path that joins them must lie entirely in $\mathbb{D}^*$. But this is not enough for our purposes since we need to have a domain D strictly contained in $\mathbb{D}^*$ with the same property. Hence we need to understand more about the other geodesics. This would be easy if we knew an explicit expression for a universal covering of $\mathcal{V}$, which could be used to transfer the geodesics of the covering space, $\mathbb{D}$ or equivalently $\mathbb{H}$, to the geodesics in $\mathcal{V}$. But such an expression does not exist, although a universal covering $\varphi : \mathbb{D} \to \mathcal{V}$ can be obtained from the so-called *modular map*, $\mathcal{M} : \mathbb{H} \to \mathcal{V}$. This map has been extensely studied, and we refer the reader to [Be1] or [C] for a detailed investigation of $\mathcal{M}$. Here we shall only recall the main facts that lead us to specifically prove Lemma 5.4.

Let Γ denote the *modular group*, i.e., the group of Möbius transformations such that $a, d \in 2\mathbb{Z}+1$, $b, c \in 2\mathbb{Z}$ and $ad - bc = 1$. Then Γ is generated by the maps

$$u(z) = z+2, \qquad v(z) = \frac{z}{2z+1}.$$

If we consider the region Σ in the upper half plane depicted in Figure 6, it can be shown that u and v pair the sides of Σ and that the images of Σ by Γ tessellate $\mathbb{H}$.

The modular map $\mathcal{M} : \mathbb{H} \to \mathcal{V}$ is constructed as follows. Consider the open right-half piece of Σ, or more precisely $\Sigma_0 = \Sigma \cap \{\mathrm{Re}(z) > 0\}$, and choose a conformal map $\mathcal{M} : \Sigma_0 \to \mathbb{H}$ (whose existence follows from the *Riemann Mapping Theorem*). Then, $\mathcal{M}$ extends to a homeomorphism between the boundaries of these domains and, by combining it with an appropriate conformal automorphism of $\mathbb{H}$, we may assume that $\mathcal{M}$ fixes 0, 1 and ∞. Hence, the positive imaginary axis is mapped to the interval $(-\infty, 0)$. By the *Schwartz reflection principle*, we can extend $\mathcal{M}$

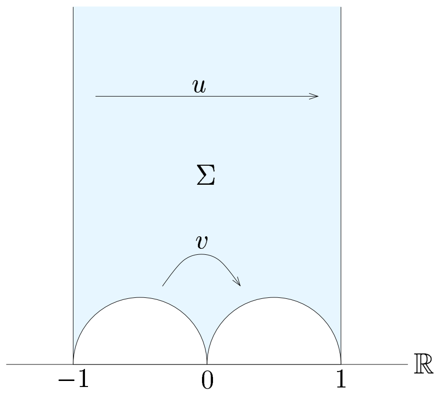

Figure 6: The region Σ.

across this axis so that it maps the left part of Σ to the lower half plane. So, we have that $\mathcal{M}$ (abusing notation again) conformally maps Σ onto $\mathbb{C} \setminus [0, +\infty)$. Using the transformations u and v we can analytically continue $\mathcal{M}$ to the whole upper half plane, obtaining a holomorphic covering $\mathcal{M} : \mathbb{H} \to \mathcal{V}$. One can show that $\mathcal{M}(z) = \mathcal{M}(w)$ if and only if $w = g(z)$ for some $g \in \Gamma$.

Now, let us see which geodesics in $\mathbb{H}$ correspond to those we know in $\mathcal{V}$. Recall that geodesics in $\mathbb{H}$ are either vertical lines or half circles perpendicular to the real axis (see Proposition 2.19). It is not hard to check that the half circle going through -1, i and 1 is mapped by $\mathcal{M}$ to the unit circle, while the vertical segments $\{\pm\frac{1}{2} + \mathrm{i}t : \frac{1}{2} \le t \le +\infty\}$ are mapped to l. See Figure 7.

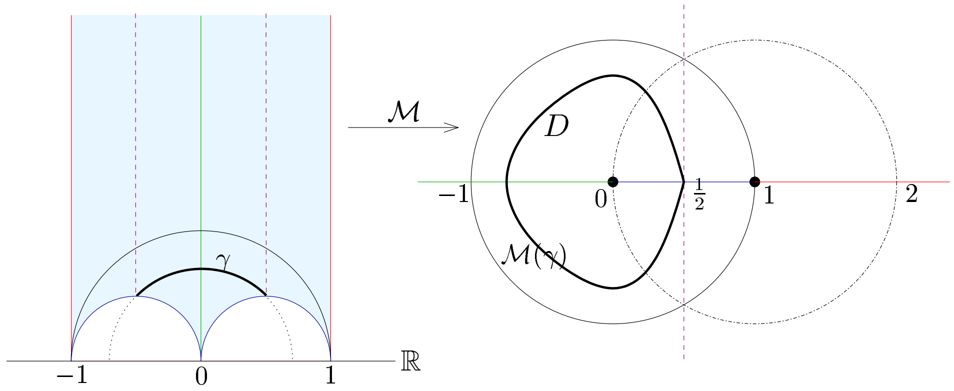

Figure 7: The map $\mathcal{M} : \mathbb{H} \to \mathcal{V}$, some geodesics in $\mathcal{V}$ and the topological disk D.

Notice that in this setting, if we take γ a piece of $\mathbb{H}$-geodesic joining two symmetric points on the lower boundary of Σ, it would be mapped by $\mathcal{M}$ to a simple closed curve inside $\mathbb{D}$ surrounding 0, $\mathcal{M}(\gamma)$, which will be a $\mathcal{V}$-geodesic. We define D as the topological disk bounded by the curve $\mathcal{M}(\gamma)$. Thus, the $\mathcal{V}-$geodesic path joining two points in D lies entirely in D. Now, if we pick γ, for instance, the $\mathbb{H}-$geodesic connecting $\frac{1}{2} + \frac{\mathrm{i}}{2}$ and $-\frac{1}{2} + \frac{\mathrm{i}}{2}$ (see Figure 7), then the lemma is proved.

□

The second part of this section is devoted to prove Lemma 5.2. The proof of this lemma will need the following auxiliary results.

Lemma 5.5. *Given $z_1, z_2 \in \mathbb{D}^*$, then we have:*

(*i*) $|\log|z_1| - \log|z_2||^2 \leq \log|z_1| \log|z_2| \sinh^2\left[\frac{\delta_{\mathbb{D}^*}(z_1,z_2)}{2}\right]$.

(*ii*) $(\arg(z_1) - \arg(z_2))^2 \leq \log|z_1| \log|z_2| \sinh^2\left[\frac{\delta_{\mathbb{D}^*}(z_1,z_2)}{2}\right]$.

(*iii*) $|z_1 - z_2|^2 \leq 2(\max\{|z_1|, |z_2|\})^2 \log|z_1| \log|z_2| \sinh^2\left[\frac{\delta_{\mathbb{D}^*}(z_1,z_2)}{2}\right]$.

Proof : Items (*i*) and (*ii*) are immediate by applying the formula for $\delta_{\mathbb{D}^*}(z_1, z_2)$ given in Proposition 2.22, and using that

$$|\log z_1 - \log z_2|^2 = |\log|z_1| - \log|z_2||^2 + (\arg(z_1) - \arg(z_2))^2.$$

On the other hand, from the cosine rule, we can write

$$\begin{aligned} |z_1 - z_2|^2 &= |z_1|^2 + |z_2|^2 - 2\cos(\arg(z_1) - \arg(z_2))|z_1||z_2| \\ &= (|z_1| - |z_2|)^2 + 2(1 - \cos(\arg(z_1) - \arg(z_2)))|z_1||z_2|. \end{aligned}$$

Then, item (*iii*) follows by combining this expression with (*i*) and (*ii*), the main value theorem,

$$|\log|z_1| - \log|z_2|| = \frac{1}{\xi}||z_1| - |z_2||, \qquad \xi \in \langle |z_1|, |z_2| \rangle,$$

and the bound

$$1 - \cos x \leq \frac{x^2}{2}, \qquad x \in [-\pi, \pi].$$

□

The third part of Lemma 5.5 is very close to what it is stated in Lemma 5.2, but it is not exactly what we need: we want a bound for $|z_1 - z_2|$ depending just on z_1 and $\delta_{\mathbb{D}^*}(z_1, z_2)$. This requires the following *a-priori* estimate.

Corollary 5.6. *Given $z_1, z_2 \in \mathbb{D}^*$ such that $\delta_{\mathbb{D}^*}(z_1, z_2) \leq 2\log(1 + \sqrt{2})$, we have:*

$$|\log|z_2|| \leq \frac{3 + \sqrt{5}}{2} |\log|z_1||.$$

Proof : If we set $x = -\log|z_2| > 0$ and $d = \delta_{\mathbb{D}^*}(z_1, z_2)$, then from part (*i*) of Lemma 5.5 we have that

$$x^2 + \log|z_1|(2 + \sinh^2(d/2))x + \log^2|z_1| \leq 0. \tag{24}$$

Thus, $x_- \leq -\log|z_2| \leq x_+$, where $x_\pm$ are, respectively, the two (positive) zeros for x of the left-hand side of equation (24). This provides lower and upper bounds for $-\log|z_2|$, but we are only interested in the upper one:

$$x_+ = \frac{|\log|z_1||}{2}\left(2 + \sinh^2(d/2) + \sinh(d/2)\sqrt{\sinh^2(d/2) + 4}\right) \leq \frac{3 + \sqrt{5}}{2} |\log|z_1||,$$

where we have used that $d \leq 2\log(1 + \sqrt{2})$ implies $\sinh(d/2) \leq 1$.

□

Proof of Lemma 5.2: In the proof we set $d = \delta_{\mathbb{D}^*}(z_1, z_2)$. To prove this result we have to deal with two different cases:
(i) If $|z_2| \geq |z_1|$, then from part (iii) of Lemma 5.5 we have:

$$|z_1 - z_2| \leq \sqrt{2}|z_2|| \log |z_1|| \sinh(d/2).$$

From here, it follows the expression

$$|z_1 - z_2| \left(1 - \sqrt{2}| \log |z_1|| \sinh(d/2)\right) \leq \sqrt{2}|z_1|| \log |z_1|| \sinh(d/2).$$

We point out that from the hypotheses on the statement we have that $\cosh(d/2) \leq \sqrt{2}$. Thus, using the main value theorem we deduce:

$$\sqrt{2}| \log |z_1|| \sinh(d/2) \leq \sqrt{2}| \log |z_1|| \cosh(d/2)d/2 \leq | \log |z_1||d \leq 1/\sqrt{2}.$$

From here, we obtain

$$|z_1 - z_2| \leq \frac{2}{\sqrt{2}-1}|z_1|| \log |z_1|| \sinh(d/2).$$

(i) If $|z_1| \geq |z_2|$, applying again part (iii) of Lemma 5.5 and Corollary 5.6, we obtain

$$|z_1 - z_2| \leq \sqrt{2}|z_1|| \log |z_2|| \sinh(d/2) \leq \frac{3+\sqrt{5}}{\sqrt{2}}|z_1|| \log |z_1|| \sinh(d/2).$$

The proof ends by applying again the main value theorem.

□

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