ON THE SURFACE PRESSURE FOR THE EDWARDS-ANDERSON MODEL

Pierluigi Contucci, Sandro Graffi

Dipartimento di Matematica

Università di Bologna, 40127 Bologna, Italy

e-mail: contucci@dm.unibo.it, graffi@dm.unibo.it

June 3th 2003

Abstract

For the Edwards-Anderson model we introduce an integral representation for the *surface pressure* (per unit surface) $\tau_{\partial\Lambda}$ in terms of a quenched moment of the bond-overlap on the surface. We consider free Φ, periodic Π and antiperiodic Π^* boundary conditions (by symmetry $\tau^{(\Pi)}_{\partial\Lambda} = \tau^{(\Pi^*)}_{\partial\Lambda}$), and prove the bounds

$$-\frac{1}{4} \;\leq\; \tau^{(\Phi)}_{\partial\Lambda} \;\leq 0 \;,$$

$$\tau^{(\Phi)}_{\partial\Lambda} \;\leq\; \tau^{(\Pi)}_{\partial\Lambda} \;\leq \frac{1}{2} \;,$$

We show moreover that at high temperatures $\tau^{(\Phi)}_{\partial\Lambda}$ is close to $-\beta^2/4$ and $\tau^{(\Pi)}_{\partial\Lambda}$ is close to $\beta^2/4$ uniformly in the volume Λ.

1 Introduction

In statistical mechanics once the existence of the thermodynamic limit has been proved for the free energy per unit volume a natural subsequent question is to establish at which rate with respect to the volume such limit is reached. In particular it is interesting to determine the next term in the expansion

$$\ln Z_\Lambda \;=\; p|\Lambda| + o(|\Lambda|) \; .$$

The problem has been analyzed since the pioneering work by Fisher and Lebowitz [FL] on classical particle systems and followed by a series of results in both Euclidean quantum field theories [G, GRS] and in ferromagnetic spin models [FC]. In those cases the basic properties of monotonicity and convexity of the thermodynamic quantities with respect to the strength of the interaction, namely the first and second Griffiths inequalities, made possible a rigorous proof of what thermodynamics suggests (see [Si]): for sufficiently regular potentials and (say) free boundary conditions the pressure varies with the volume as

$$\ln Z_\Lambda \;=\; p|\Lambda| + \tau|\partial\Lambda| + o(|\partial\Lambda|) \; , \tag{1.1}$$

where p is the thermodynamic limit of the pressure per unit volume

$$p_\Lambda \;=\; \frac{\ln Z_\Lambda}{|\Lambda|} \; , \tag{1.2}$$

and τ is the thermodynamic limit of the *surface pressure* per unit area

$$\tau_{\partial\Lambda} \;=\; \frac{\ln Z_\Lambda - p|\Lambda|}{|\partial\Lambda|} \; . \tag{1.3}$$

The quantity τ, unlike p, depends in general not only on the interaction but also on the boundary conditions and represents the contribution to the pressure due to the interaction of the system with its boundary.

In this paper we analyze the surface pressure problem for the Edwards-Anderson model with Gaussian couplings in the quenched ensemble. Basing on the property of existence, self averaging and independence on the boundary conditions of the thermodynamic limit for the random pressure per particle (see for instance [EH] and [CG]) we study the correction to the leading term for different boundary conditions (free, periodic and antiperiodic). Our main idea relies on an inequality which translates to random systems the contents of the *first* Griffiths inequality: in a ferromagnet the free energy decreases with the strength of each interaction, in a spin-glass the free energy decreases with the variance of each random coupling. Our technical tool is an interpolation method (similar to those in [GT] and [CG]) which plays, in spin glass statistical mechanics, the same role of the Griffiths interpolation method [Si, Gr] in classical ferromagnetic systems. Our main result is an integral representation theorem for the surface pressure in the quenched ensemble for different boundary conditions and rectangular boxes. As an immediate consequence we find that its value is bounded from above by 0 and from below by $-1/4$ and that for high temperatures it is non-zero. We prove moreover that the surface pressure for periodic or antiperiodic boundary conditions is larger than the free one and we provide an integral representation for their quenched difference which we control at high temperature.

2 Definitions and Results

Consider the Edwards-Anderson d-dimensional spin-glass model defined by configurations of Ising spins σ_n, $n \in \Lambda \subset \mathbb{Z}^d$ for some d-parallelepiped Λ. To be definite we locate it in the positive quadrant of $\mathbb{Z}^d$ with a vertex in the origin. We denote $L_1, L_2, ..., L_d$ the sides, $|\Lambda|$ the volume and $|\partial\Lambda|$ the surface. The interaction is described by the potential

$$U_\Lambda(J,\sigma) \ = \sum_{(n,n')\in B(\Lambda)} J_{n,n'}\sigma_n\sigma_{n'} \ , \tag{2.4}$$

where the $J_{n,n'}$ are independent normal Gaussian variables and the sum runs over all pairs of nearest neighbors sites $|n-n'| = 1$. We use here the standard identification of the space of nearest neighbors with the d-dimensional *bond*-lattice $b \in \mathbb{B}^d$ with $b = (n, n')$ and denote $B(\Lambda)$ the d-bond-parallelepiped associated to Λ. Given two spin configurations σ and τ introduce the notation $\sigma_b = \sigma_n\sigma_{n'}$ and $\tau_b = \tau_n\tau_{n'}$; the local bond-overlap between σ and τ is

$$q_b(\sigma,\tau) := \sigma_b\tau_b \ ; \tag{2.5}$$

for every $B \subset B(\Lambda)$ we define

$$q_B(\sigma,\tau) := \frac{1}{|B|}\sum_{b\in B} q_b(\sigma,\tau) \ . \tag{2.6}$$

The reason to introduce the bond overlap is related to the mathematical structure of the Hamiltonian (2.4): as a sum of Gaussian variables it is, for each σ-configuration, a Gaussian variable itself and thus by the Wick theorem completely identified by its covariance matrix which is proportional to the bond-overlap $q_{B(\Lambda)}(\sigma,\tau)$. Denoting Av the Gaussian average we have in fact:

$$\begin{aligned} Av(U_\Lambda(J,\sigma)U_\Lambda(J,\tau)) &= \sum_{b,b'} Av(J_b J_{b'})\sigma_b\tau_{b'} \\ &= \sum_{b,b'} \delta_{b,b'}\sigma_b\tau_{b'} = |B(\Lambda)| q_{B(\Lambda)}(\sigma,\tau) \ . \end{aligned}$$

Definitions.

For assigned boundary conditions Ξ we consider

1. the random partition function,

$$Z_\Lambda^{(\Xi)}(J) := \sum_\sigma e^{U_\Lambda^{(\Xi)}(\sigma,J)} \ , \tag{2.7}$$

2. the random pressure

$$P_\Lambda^{(\Xi)}(J) := \ln Z_\Lambda^{(\Xi)}(J) \ , \tag{2.8}$$

3. the quenched pressure

$$P_\Lambda^{(\Xi)} := \mathrm{Av}\left(\ln Z_\Lambda^{(\Xi)}(J)\right) \ , \tag{2.9}$$

4. the product (over the same disorder) random Gibbs-Boltzmann state

$$\omega_\Lambda^{(\Xi)}(-) := \frac{\sum_{\sigma,\tau} - e^{U_\Lambda^{(\Xi)}(\sigma)+U_\Lambda^{(\Xi)}(\tau)}}{[Z_\Lambda^{(\Xi)}]^2} \ , \tag{2.10}$$

5. the quenched equilibrium state

$$<->_\Lambda^{(\Xi)} := \mathrm{Av}\left(\omega(-)_\Lambda^{(\Xi)}\right) \ , \tag{2.11}$$

6. the random surface pressure

$$T_\Lambda^{(\Xi)}(J) \;:=\; P_\Lambda^{(\Xi)}(J) - p|\Lambda| \;, \tag{2.12}$$

7. and the quenched surface pressure

$$T_\Lambda^{(\Xi)} \;:=\; \mathrm{Av}\left(T_\Lambda^{(\Xi)}(J)\right) \;. \tag{2.13}$$

We will consider three type of boundary conditions. The free ones Φ in which the partition sum runs over all the spins inside the parallelepiped Λ:

$$Z_\Lambda^{(\Phi)}(J) \;:=\; \sum_\sigma e^{U_\Lambda(\sigma,J)} \;. \tag{2.14}$$

The periodic boundary conditions Π in which the partition sum runs over all the spin values in the torus $\Pi_\Lambda = \mathbb{Z}^d/\Lambda$:

$$Z_\Lambda^{(\Pi)}(J) \;:=\; \sum_\sigma e^{U_{\Pi_\Lambda}(\sigma,J)} \;. \tag{2.15}$$

To define the anti-periodic ones Π^* we introduce the set $\partial B(\Lambda)$ i.e. the standard orthogonal cut of the torus which unfolds Π to Λ. $\partial B(\Lambda)$ is defined as the collection of $b = (n, n')$ with $n < n'$ and $n = (n_1, n_2, ..., n_k)$ in which $n_i = 1 \; \forall i \neq k$ and $n_k = 0$. We observe that $|\partial\Lambda| = |\partial B(\Lambda)|$. Given

$$\alpha_b \;=\; \begin{cases} -1, & \text{if } b \in \partial B(\Lambda), \\ 1, & \text{otherwise}\,, \end{cases} \tag{2.16}$$

and the potential

$$U_{\Pi_\Lambda^*}(\sigma, J) \;=\; \sum_{b \in B(\Pi_\Lambda)} \alpha_b J_b \sigma_b \;, \tag{2.17}$$

the partition sum runs over all the spins in the torus $\Pi_\Lambda = \mathbb{Z}^d/\Lambda$

$$Z_\Lambda^{(\Pi^*)}(J) \;:=\; \sum_\sigma e^{U_{\Pi_\Lambda^*}(\sigma,J)} \;. \tag{2.18}$$

To state our results we first establish some further notation.

Consider the boundary bond-overlap

$$q_{\partial B(\Lambda)} \;=\; \frac{1}{|\partial B(\Lambda)|} \sum_{b \in \partial B(\Lambda)} q_b \;. \tag{2.19}$$

Let $k\Lambda$ be the k-magnified Λ defined, for each positive integer k, as the d-parallelepiped of sides $kL_1, kL_2, ..., kL_d$. The magnificated torus $\Pi_{k\Lambda}$ is partitioned into k^d non-empty disjoint cubes Λ_s all congruent to Λ:

$$k\Lambda = \bigcup_{s=1}^{k^d} \Lambda_s \ , \tag{2.20}$$

$$\Lambda_s \cap \Lambda_{s'} = \emptyset \qquad for \qquad s \neq s' \ . \tag{2.21}$$

$$\Pi_{k\Lambda} = \bigcup_{s=1}^{k^d} \Lambda_s \ , \tag{2.22}$$

Define the set

$$\mathcal{C}_{\Pi_{k\Lambda}} \ := \ B(\Pi_{k\Lambda}) \setminus \bigcup_{s=1}^{k^d} B(\Lambda_s) \ . \tag{2.23}$$

and associate with $\Pi_{k\Lambda}$ the potential

$$U_{\Pi_{k\Lambda}}(t) \ = \sum_{b \in B(\Pi_{k\Lambda})} \sqrt{t_b} J_b \sigma_b \ , \tag{2.24}$$

with

$$t_b \ = \ \begin{cases} t, & \text{if } b \in \mathcal{C}_{\Pi_{k\Lambda}}, \\ 1, & \text{otherwise,} \end{cases} \tag{2.25}$$

Finally let $< - >_t^{(\Pi_{k\Lambda})}$ be the corresponding quenched state.

Theorem 1 (Integral representation for $T_\Lambda^{(\Phi)}$) *The surface pressure per unit surface admits the representation*

$$T_\Lambda^{(\Phi)} \ = \ -\frac{|\partial\Lambda|}{4} \lim_{k\to\infty} \int_0^1 \left(1 - < q_{\partial B(\Lambda)} >_t^{(\Pi_{k\Lambda})}\right) dt \ ; \tag{2.26}$$

in particular the quantity

$$\tau_{\partial\Lambda}^{(\Phi)} \ = \ \frac{T_\Lambda^{(\Phi)}}{|\partial\Lambda|} \tag{2.27}$$

admits the bounds

$$-\frac{1}{4} \ \leq \ \tau_{\partial\Lambda}^{(\Phi)} \ \leq \ 0 \ . \tag{2.28}$$

Theorem 2 (Integral representation for $T_\Lambda^{(\Pi)}$ and $T_\Lambda^{(\Pi^*)}$) *For every Λ the symmetry of the Gaussian distribution implies*

$$T_\Lambda^{(\Pi)} \;=\; T_\Lambda^{(\Pi^*)}\;. \tag{2.29}$$

Consider in the torus Π_Λ the interpolating potential

$$U^{(\Pi_\Lambda)}(t) \;=\; \sum_{b\in B(\Pi_\Lambda)} \sqrt{t_b} J_b \sigma_b\,, \tag{2.30}$$

with

$$t_b \;=\; \begin{cases} t, & \textit{if } b \in \partial B(\Lambda), \\ 1, & \textit{otherwise}\,, \end{cases} \tag{2.31}$$

and let $< - >_t^{(\Pi_\Lambda)}$ be its quenched state . Then

$$T_\Lambda^{(\Pi)} \;=\; T_\Lambda^{(\Phi)} \;+\; \frac{|\partial\Lambda|}{2}\int_0^1 (1 - < q_{\partial B(\Lambda)} >_t^{(\Pi_\Lambda)})dt\;. \tag{2.32}$$

In particular the quantity

$$\tau_{\partial\Lambda}^{(\Pi)} \;=\; \frac{T_\Lambda^{(\Pi)}}{|\partial\Lambda|} \tag{2.33}$$

admits the bounds

$$\tau_{\partial\Lambda}^{(\Phi)} \;\le\; \tau_{\partial\Lambda}^{(\Pi)} \;\le\; \frac{1}{2}\;. \tag{2.34}$$

Theorem 3 (High temperatures) *Consider the potential*

$$U_\Lambda(J,\sigma) = \beta \sum_{(n,n')\in B(\Lambda)} J_{n,n'}\sigma_n\sigma_{n'} \tag{2.35}$$

Then:

(1) *There exist $\overline{\beta}$ and $C > 0$ depending only on d such that for all $\beta \le \overline{\beta}$*

$$\frac{\tau_{\partial\Lambda}^{(\Phi)}}{\beta^2} \;\le\; -C \;<\; 0 \tag{2.36}$$

(2) *For any $\varepsilon > 0$ there exists $\beta^{(\varepsilon)} > 0$ such that for all $\beta \le \beta^{(\varepsilon)}$*

$$\frac{\tau_{\partial\Lambda}^{(\Phi)}}{\beta^2} \;\le\; -\frac{1}{4}(1-\varepsilon)\;, \tag{2.37}$$

and equivalently

$$\frac{\tau_{\partial\Lambda}^{(\Pi)}}{\beta^2} \;\ge\; \frac{1}{4}(1-\varepsilon)\;, \tag{2.38}$$

uniformly in Λ.

3 Proof of the results

We start by stating and proving the basic result.

Lemma 3.1 (Monotonicity in the variance) *Let $t_b \geq 0$ $\forall b \in B(\Lambda)$ and J_b be a normal Gaussian variable. The Gaussian variable $\sqrt{t_b} J_b$ has variance t_b. Consider the potential $U_\Lambda = \sum_{b \in B(\Lambda)} \sqrt{t_b} J_b \sigma_b$ with its associated quenched thermodynamics. The quenched pressure P_Λ is monotone increasing with respect to all the variances t_b:*

$$\frac{d}{dt_b} P_\Lambda \;=\; \frac{1}{2\sqrt{t_b}} \mathrm{Av}\left(J_b \omega(\sigma_b)\right) \;=\; \frac{1}{2}(1 - < q_b >) \;\geq\; 0\,. \tag{3.39}$$

Proof of Lemma 3.1.

The first equality is the chain rule on the logarithm of an exponential of a square root:

$$\frac{d}{dt_b} P_\Lambda \;=\; \frac{1}{2\sqrt{t_b}} \mathrm{Av}\left(J_b \frac{\sum_\sigma \sigma_b e^{U(\sigma)}}{\sum_\sigma e^{U(\sigma)}} \right) \tag{3.40}$$

Next we recall the integration by parts formula for normal Gaussian variables

$$\mathrm{Av}\left(J f(J)\right) \;=\; \mathrm{Av}\left(\frac{df(J)}{dJ}\right)\,, \tag{3.41}$$

the correlation derivative formula

$$\frac{d\omega(\sigma_b)}{dJ_b} \;=\; \sqrt{t_b}\left(1 - \omega(\sigma_b)^2\right) \;\geq 0\,, \tag{3.42}$$

and the identity

$$\omega(\sigma_b)^2 \;=\; \left(\frac{\sum_\sigma \sigma_b e^{U(\sigma)}}{\sum_\sigma e^{U(\sigma)}}\right)^2 \;=\; \frac{\sum_{\sigma,\tau} \sigma_b \tau_b e^{U(\sigma)+U(\tau)}}{\sum_{\sigma,\tau} e^{U(\sigma)+U(\tau)}} \;=\; \omega(q_b)\,. \tag{3.43}$$

By applying successively (3.41), (3.42) and (3.43) we obtain lemma 3.1.

Proof of Theorem 1.

Given the d-parallelepiped Λ consider its magnification $k\Lambda$ defined, for each positive integer k, as the d-parallelepiped of sides $kL_1, kL_2, ..., kL_d$. Clearly $k\Lambda$ and $\Pi_{k\Lambda}$ are partitioned into k^d non-empty disjoint cubes Λ_s all congruent to Λ as explained in the definitions before Theorem 1. In finite volume and with free boundary conditions we have by definition

$$P_\Lambda^{(\Phi)} \;=\; \mathrm{Av}\left(\ln Z_\Lambda\right) \;=\; k^{-d} \mathrm{Av}\left(\ln Z_\Lambda^{k^d}\right)\,. \tag{3.44}$$

The limiting pressure per particle is independent on the boundary conditions. Hence:

$$p|\Lambda| \;=\; \lim_{k\to\infty} k^{-d}\mathrm{Av}\left(\ln Z_\Lambda^{(\Pi)}\right) \tag{3.45}$$

By (3.44) and (3.45) we obtain

$$\begin{aligned} T_\Lambda^{(\Phi)} \;&=\; \left(P_\Lambda^{(\Phi)} - p|\Lambda|\right) \\ &=\; \lim_{k\to\infty} k^{-d}\mathrm{Av}\left(\ln Z_\Lambda^{k^d} - \ln Z_\Lambda^{(\Pi)}\right)\ . \end{aligned} \tag{3.46}$$

For each $0 \le t \le 1$ we define the interpolating potential

$$U^{(\Pi_{k\Lambda})}(t) \;=\; \sum_{b\in B(\Pi_{k\Lambda})} \sqrt{t_b} J_b \sigma_b\,, \tag{3.47}$$

with

$$t_b \;=\; \begin{cases} t, & \text{if } b \in \mathcal{C}_{\Pi_{k\Lambda}}, \\ 1, & \text{otherwise,} \end{cases} \tag{3.48}$$

the interpolating partition function

$$Z^{(\Pi_{k\Lambda})}(t) \;=\; \sum_\sigma e^{U^{(\Pi_{k\Lambda})}(t)}\ , \tag{3.49}$$

the interpolating pressure

$$P^{(\Pi_{k\Lambda})}(t) := \;\mathrm{Av}\left(\ln Z^{(\Pi_{k\Lambda})}(t)\right)\ , \tag{3.50}$$

and the corresponding states $\omega_t^{(\Pi_{k\Lambda})}(-)$ and $< - >_t^{(\Pi_{k\Lambda})}$. We observe that

$$Z^{(\Pi_{k\Lambda})}(0) \;=\; \prod_{s=1}^{k^d} Z_{\Lambda_s}\ , \qquad Z^{(\Pi_{k\Lambda})}(1) = Z_{\Pi_{k\Lambda}}\ , \tag{3.51}$$

or equivalently

$$P^{(\Pi_{k\Lambda})}(0) \;=\; k^d P_\Lambda\ , \qquad P^{(\Pi_{k\Lambda})}(1) \;=\; P_{\Pi_{k\Lambda}}\ , \tag{3.52}$$

and by (3.46)

$$T_\Lambda^{(\Phi)} \;=\; -\lim_{k\to\infty} k^{-d}\left[P^{(\Pi_{k\Lambda})}(1) - P^{(\Pi_{k\Lambda})}(0)\right] \;=\; -\lim_{k\to\infty} k^{-d}\int_0^1 \frac{d}{dt}P^{(\Pi_{k\Lambda})}(t)dt\ . \tag{3.53}$$

We remark now that

$$\frac{d}{dt}P^{(\Pi_{k\Lambda})}(t) \;=\; \sum_{b\in\mathcal{C}_{\Pi_{k\Lambda}}} \frac{1}{2\sqrt{t}} < J_b\sigma_b >_t^{(\Pi_{k\Lambda})}\ , \tag{3.54}$$

and by Lemma 3.1

$$\frac{d}{dt}P^{(\Pi_{k\Lambda})}(t) \;=\; \frac{1}{2}\sum_{b\in\mathcal{C}_{\Pi_{k\Lambda}}}(1-<q_b>_t^{(\Pi_{k\Lambda})})\;. \tag{3.55}$$

The translation symmetry over the torus and the equality

$$2|\mathcal{C}_{\Pi_{k\Lambda}}| = k^d|B(\partial\Lambda)|$$

imply by (3.53)

$$\tau^{(\Phi)}_{\partial\Lambda} = -\frac{1}{4}\lim_{k\to\infty}\int_0^1\left(1-<q_{\partial B(\Lambda)}>_t^{(\Pi_{k\Lambda})}\right)dt\;. \tag{3.56}$$

Proof of Theorem 2. We first notice that the potential

$$U^{(\Pi_\Lambda)}(\alpha,\sigma,J) \;=\; \sum_{b\in B(\Pi_\Lambda)}\alpha_b J_b\sigma_b\,, \tag{3.57}$$

has a quenched pressure independent of α for each choice of $\alpha_b = \pm 1$. That is a simple consequence of the symmetry $J_b \to -J_b$ of the Gaussian distribution. The previous observations entail in particular (2.29). Consider in the torus Π_Λ the interpolating potential

$$U^{(\Pi_\Lambda)}(t) \;=\; \sum_{b\in B(\Pi_\Lambda)}\sqrt{t_b}J_b\sigma_b\,, \tag{3.58}$$

with

$$t_b \;=\; \begin{cases} t, & \text{if } b\in\partial B(\Lambda),\\ 1, & \text{otherwise}\,,\end{cases} \tag{3.59}$$

with the relative pressure $P^{(\Pi_\Lambda)}(t)$ and quenched state $<->_t^{(\Pi_\Lambda)}$. Since

$$P^{(\Pi_\Lambda)}(0) \;=\; P^{(\Phi)}_\Lambda\,,\qquad P^{(\Pi_\Lambda)}(1) \;=\; P^{(\Pi)}_\Lambda\,, \tag{3.60}$$

and

$$P'(t) \;=\; \frac{1}{2}\sum_{b\in\partial B(\Lambda)}(1-<q_b>_t^{(\Pi_\Lambda)})\,, \tag{3.61}$$

we have

$$P^{(\Pi)}_\Lambda - P^{(\Phi)}_\Lambda \;=\; \frac{1}{2}\sum_{b\in\partial B(\Lambda)}\int_0^1(1-<q_b>_t^{(\Pi_\Lambda)})dt\,, \tag{3.62}$$

which immediately entails theorem 2.

Proof of Theorem 3. The cluster expansion of [Be] and [FI] overcomes the well known difficulty due the infinite range of the Gaussian variable. We apply it to the present case to show that, regardless of the boundary conditions, each $< q_b >$ is small for small β and definitely away from 1. Applying Proposition 1 of [Be] to our problem we may write

$$< q_b >_t^{(\Pi_{k\Lambda})} \; = \; A_{k\Lambda}(b, \beta^2, t)\beta^2 + C_{k\Lambda}(b, \beta^2, t) \; , \tag{3.63}$$

where:

(1) for every ε we may choose

$$|C_{k\Lambda}(b, \beta^2, t)| \; \leq \; \frac{\varepsilon}{2} \; , \tag{3.64}$$

uniformly in all the variables and

(2) $A_{k\Lambda}(b, \beta^2, t)\beta^2$ is analytic in β for $\beta < \beta_0$, where β_0 depends only on the dimension d and not on Λ and t. Remark once again that the parity of the Gaussian variables yields the parity in β of each thermodynamic function so that the odd powers of the cluster expansion vanish.

After integrating in t we take the $k \to \infty$ limit of the previous relation (which exists by Theorem 1 of [Be] $\beta < \beta_0$), sum over all bonds in $\partial B(\Lambda)$ and obtain:

$$\tau_{\partial\Lambda}^{(\Phi)} = -\frac{\beta^2}{4}\left[1 - (A_\Lambda(\beta^2)\beta^2 + C_\Lambda(\beta^2))\right] \; , \tag{3.65}$$

with

$$C_\Lambda(\beta^2) \; = \; \lim_{k\to\infty} \int_0^1 dt \frac{1}{|\partial B(\Lambda)|} \sum_{b \in \partial B(\Lambda)} C_{k\Lambda}(b, \beta^2, t) \; , \tag{3.66}$$

and

$$A_\Lambda(\beta^2) \; = \; \lim_{k\to\infty} \int_0^1 dt \frac{1}{|\partial B(\Lambda)|} \sum_{b \in \partial B(\Lambda)} A_{k\Lambda}(b, \beta^2, t) \; , \tag{3.67}$$

We remind that the multiplicative β^2 factor in (3.65) comes from the fact that the potential (2.35) has interactions coefficients βJ whose variance is β^2. From (3.64) we derive the bound $|C_\Lambda| \leq \varepsilon/2$. On the other hand since the correlation is bounded by one,

$| < q_b >_t^{(\Pi_{k\Lambda})} | \leq 1$, and $A_{k\Lambda}(b, \beta^2, t)$ is bounded uniformly in (Λ, t), there is $K > 0$ independent of Λ such that

$$|A_\Lambda(\beta^2)| < K\beta^2 \ . \tag{3.68}$$

Hence there is a $\overline{\beta} > 0$ such that the quantity $|\beta^2 A_{k\Lambda}(b, \beta^2, t)| < C_1 < 1 - \varepsilon/2$ if $\beta < \overline{\beta}$, uniformly in Λ. Hence, by (3.65) we get the existence of $C > 0$ independent of Λ such

$$\frac{\tau^{(\Phi)}_{\partial\Lambda}}{\beta^2} \ < -C \ < 0 \tag{3.69}$$

This proves Assertion (1).

To prove assertion (2), remark that by (3.68) we can always choose $\beta^{(\varepsilon)}$ in such a way that

$$|A_\Lambda(\beta^2)\beta^2| \ \leq \ \varepsilon/2 \tag{3.70}$$

uniformly with respect to Λ. Hence by (3.65) we can conclude

$$\frac{\tau^{(\Phi)}_{\partial\Lambda}}{\beta^2} \ \leq \ -\frac{1}{4}(1-\varepsilon) \tag{3.71}$$

if $\beta < \beta^{(\varepsilon)}$. The proof of (2.38) is completely analogous.

Outlook

Our results show that the surface pressure has the expected *surface size* in dimension d. A change in the size dependence at low temperatures is very unlikely. In fact our integral representation would force the quenched overlap moments $< q >$ to be identically equal to one, a situation which is not generally expected in the mean field picture [MPV] nor in the droplet one [FH]. A further step along the present line would be the understanding of the variance of the difference of the pressure computed with two boundary conditions, for example periodic and antiperiodic. This would yield a *surface tension* like contribution. Bounds on the size dependence of such a quantity already exist (see ref [74] in [NS]) and it would be interesting to investigate if the interpolating method can be used to obtain the correct size; we hope to return elsewhere on that point and also on the existence of the thermodynamic limit for the quenched surface pressure especially in view to obtain an analogous of the second Griffiths inequality.

Acknowledgments. We thank M.Aizenman, A.Berretti, A.Bovier, A.C.D.van Enter, C. Giardina, F.Guerra, J.Imbrie, C.Newman, E.Olivieri and E.Presutti for interesting discussions.

References

[Be] A.Berretti, Jou. Stat. Phys., Vol. 38, Nos.3/4, 483-496 (1985).

[CG] P.Contucci, S.Graffi, http://arxiv.org/abs/math-ph/0302013 To appear in Jou. Stat. Phys.

[CDGG] P.Contucci, M.Degli Esposti, C.Giardinà and S.Graffi, Commun. in Math. Phys. 236, 55-63, (2003)

[EH] A.C.D.van Enter, and J.L.van Hemmen, Jou.Stat.Phys. 32, 141-152 (1983)

[FC] M.E.Fisher and G.Caginalp, Commun. Math. Phys., 56, no. 1, 11-56 (1977)

[FH] D.S.Fisher and D.H.Huse, Phys. Rev. Lett., 56, 1601 (1986)

[FI] J.Fröhlich, J.Z.Imbrie, Commun. Math. Phys. 96, 145-180 (1984).

[FL] M.Fisher, J.Lebowitz, Commun. Math. Phys. 19, 251-272 (1970).

[G] F.Guerra, Phys. Rev. Lett. 28, no.18, 1213-1215, (1972)

[Gr] R.B.Griffiths, Phys. Rev. Lett. 23, 17 (1969)

[GRS] F.Guerra, L.Rosen, B.Simon, Ann. Inst. H. Poincare A25, no.3, 231-334 (1976).

[GT] F.Guerra and F.Toninelli, Commun.Math.Phys. 230, 71-79 (2002)

[MPV] M.Mezard, G.Parisi, M.A.Virasoro, Spin Glass theory and beyond, World Scien. (1987)

[NS] C.M.Newmnan D.L.Stein, http://arxiv.org/abs/cond-mat/0301403

[Si] B.Simon, *The statistical mechanics of lattice gases.* Princeton Univ. Press. (1992)