SCATTERING MATRIX FOR MAGNETIC POTENTIALS WITH COULOMB DECAY AT INFINITY

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Abstract

We consider the Schrödinger operator H in the space $L_2(\mathbb{R}^d)$ with a magnetic potential $A(x)$ decaying as $|x|^{-1}$ at infinity and satisfying the transversal gauge condition $< A(x), x >= 0$. Such potentials correspond, for example, to magnetic fields $B(x)$ with compact support and hence are quite general. Our goal is to study properties of the scattering matrix $S(\lambda)$ associated to the operator H. In particular, we find the essential spectrum σ_{ess} of $S(\lambda)$ in terms of the behaviour of $A(x)$ at infinity. It turns out that $\sigma_{ess}(S(\lambda))$ is normally a rich subset of the unit circle $\mathbb{T}$ or even coincides with $\mathbb{T}$. We find also the diagonal singularity of the scattering amplitude (of the kernel of $S(\lambda)$ regarded as an integral operator). In general, $S(\lambda)$ is a sum of a multiplication operator and of a singular integral operator. However, if the magnetic field decreases faster than $|x|^{-2}$ for $d \geq 3$ (and the total magnetic flux is an integer times 2π for $d = 2$), then this singular integral operator disappears. In this case the scattering amplitude has only a weak singularity (the diagonal Dirac function is neglected) in the forward direction and hence scattering is essentially of short-range nature. An important point of our approach is that we consider $S(\lambda)$ as a pseudodifferential operator on the unit sphere and find an explicit expression of its principal symbol in terms of A.

1. INTRODUCTION

Let the Schrödinger operator H be defined by differential expression

$$H = (i\nabla + A(x))^2 + V(x),$$

where the scalar function $V(x)$ and the vector-valued function $A(x) = (A_1(x), \dots, A_d(x))$, $d \geq 2$, are real valued and called the electrostatic and magnetic potentials, respectively. We assume that the potentials are C^∞-functions satisfying the estimates

$$|\partial^\alpha A(x)| + |\partial^\alpha V(x)| \leq C_\alpha (1 + |x|)^{-\rho-|\alpha|}, \quad \rho > 1/2, \tag{1.1}$$

for all multi-indices α. Then H is a self-adjoint operator in the space $\mathcal{H} = L_2(\mathbb{R}^d)$ on domain $\mathcal{D}(H) = \mathcal{D}(H_0)$ of the "free" operator $H_0 = -\Delta$. Under assumption (1.1) stationary scattering theory for the pair H_0, H was constructed in the paper [11] where it was supposed that $A = 0$. For the general case where V and A satisfy (1.1) for arbitrary $\rho > 0$, see [8].

Our goal is to study the scattering matrix (SM) $S(\lambda) = S(H, H_0; \lambda)$, $\lambda > 0$, for the operators H_0, H. Actually, we discuss two different but intimately connected problems. The first is a description of the diagonal singularity of the kernel $s(\omega, \omega'; \lambda)$ of the SM (known as the scattering amplitude) regarded as an integral operator on the unit sphere $\mathbb{S}^{d-1}$. The second problem is a localization of the essential spectrum of the SM.

Let us recall first of all the well-known results. Under assumption (1.1) the scattering amplitude is a smooth function off the diagonal $\omega \neq \omega'$ but might be very singular as $\omega - \omega' \to 0$. In the short-range case ($\rho > 1$ in (1.1)), $S(\lambda)$ differs from the identity operator I by a compact term (see, e.g., [12]). To be more precise, the kernel of this term is $O(|\omega - \omega'|^{\rho - d})$ as $\omega - \omega' \to 0$. This implies that in the short-range case the spectrum of $S(\lambda)$ consists of eigenvalues accumulating at the point 1 only.

In the opposite, long-range case, when $V(x)$ and $A(x)$ are, for example, asymptotically homogeneous potentials of degree $-\rho$, $\rho < 1$, the diagonal singularity of the scattering amplitude $s(\omega, \omega'; \lambda)$ is very wild (see, [11, 13]). It turns out that for long-range potentials, it is more convenient to study the SM as a pseudodifferential operator (PDO) on the unit sphere. Under assumption (1.1) its principal symbol (defined on the cotangent bundle to the unit sphere) is determined [11, 8] by the expression

$$p(\omega, z; \lambda) = \exp\Big(i\Theta(-k^{-1}z, k\omega)\Big), \quad \omega \in \mathbb{S}^{d-1},\ z \in \mathbb{R}^d,\ <\omega, z>= 0, \quad \lambda = k^2,\ k > 0, \tag{1.2}$$

where

$$\Theta(x, \xi) = 2^{-1} \int_{-\infty}^{\infty} \Big(V(t\xi) - V(x + t\xi) + 2 < A(x + t\xi) - A(t\xi), \xi >\Big) dt, \quad \xi \neq 0. \tag{1.3}$$

We emphasize that the function $\Theta(x, \xi)$ does not depend on the projection of x on the direction of ξ. Below this function is always considered off some conical neighbourhood of the set $x = \gamma\xi$, $\gamma \in \mathbb{R}$, where $\Theta(\gamma\xi, \xi) = 0$. Moreover, we can replace in (1.2) the function $\Theta(-k^{-1}z, k\omega)$ by its asymptotics $\Theta_\infty(-k^{-1}z, k\omega)$ as $|z| \to \infty$ which is determined by the asymptotics of $V(x)$ and $A(x)$ as $|x| \to \infty$. For homogeneous potentials of degree $-\rho$, $\rho \in (1/2, 1)$, the function Θ_∞ is homogeneous of degree $1 - \rho$, so that for long-range potentials $\Theta_\infty(-k^{-1}z, k\omega) \to \infty$ as $|z| \to \infty$. This implies [11] that the spectrum of the SM covers the whole unit circle $\mathbb{T}$, that is

$$\sigma(S(\lambda)) = \mathbb{T}. \tag{1.4}$$

Note that for short-range potentials the principal symbol of PDO $S(\lambda)$ equals 1 which corresponds to the Dirac-function in the scattering amplitude. The intermediary case $\rho = 1$ is also essentially long-range since for such potentials the function Θ_∞ has, generically, a logarithmic growth as $|z| \to \infty$, and hence again the spectrum of the SM covers the whole unit circle $\mathbb{T}$.

The main goal of this paper is to study the SM in the critical, but physically very important situation, where a magnetic potential $A(x)$ is asymptotically homogeneous of degree -1 and satisfies, additionally, transversal gauge condition

$$< A(x), x >= 0. \tag{1.5}$$

Such potentials correspond, for example, to magnetic fields

$$B(x) = \operatorname{curl} A(x) \tag{1.6}$$

(we use always three-dimensional notation although, strictly speaking, for $d > 3$ one has to consider A as a 1-form and $B = dA$ as a 2-form) with compact support and hence are quite general. Everywhere, except the last section, we assume $V(x) = 0$. Then the usual wave operators

$$W_\pm(H, H_0) = s - \lim_{t\to\pm\infty} e^{iHt} e^{-iH_0 t} \tag{1.7}$$

exist [6], so that the SM can be defined in their terms.

Alternatively, our paper can be considered as devoted to a study of the Aharonov-Bohm effect [1, 9] in a sufficiently general framework and in all dimensons d. From mathematical point of view this effect consists exactly in unusual properties of the SM for magnetic potentials $A(x)$ such that

$$A(x) = a(\hat{x})(-x_2, x_1)/|x|^2, \quad \hat{x} = x|x|^{-1}, \quad x \in \mathbb{R}^2, \tag{1.8}$$

for sufficiently large $|x|$. Here a is C^∞-function on the unit circle. Actually, for $d = 2$ formula (1.8) gives all magnetic potential which are asymptotically homogeneous of degree -1 and satisfy (1.5). For potentials (1.8) the essential spectrum σ_{ess} of $S(\lambda)$ consists of two complex conjugated (and perhaps overlapping) closed intervals of the unit circle. These intervals were explicitly calculated (but complete proofs were omitted) in [7] in terms of the function a. In particular, if $a(\hat{x}) = \alpha =$ const, then $\sigma_{ess}(S(\lambda))$ consists of the two points $e^{\pm i\pi\alpha}$. As far as the singilarity of the scattering amplitude is concerned, it is a sum of a multiplication operator and of a singular integral operator (but the latter disappears if the total magnetic flux is an integer times 2π). This is of course qualitatively different both from short-range and generic long-range cases.

It seems to be a common belief that exotic properties of the SM for potentials (1.8) are related to the equality $\operatorname{curl} A(x) = 0$ satisfied by these potentials. We argue that similar phenomena hold in any space dimension and for all magnetic potentials which are asymptotically homogeneous of degree -1 and satisfy, additionally, transversal gauge condition (1.5).

Our approach to the study of the SM relies on a consideration of $S(\lambda)$ as of a PDO on the unit sphere. Actually, we proceed from general results of [13] where all necessary information about the SM is collected. Under our assumptions on $A(x)$, function (1.3) is asymptotically homogeneous of degree zero in both variables. It turns out (Theorem 4.4) that in this case the essential spectrum σ_{ess} of the SM $S(\lambda)$ does not depend on λ and coincides with the image of the function $\exp\Big(i\Theta_\infty(z, \omega)\Big)$ for all $\omega, z \in \mathbb{S}^{d-1}$ such that $< z, \omega >= 0$. Thus, in general, relation (1.4) is violated. Next, making the Fourier transform of function (1.2) of the variable z in the hyperplane orthogonal to ω, we find (Theorem 4.5) the leading diagonal singularity of the scattering amplitude $s(\omega, \omega'; \lambda)$.

As for potentials (1.8), in the general case the singular part of the SM is a sum of a multiplication operator and of a singular integral operator.

However, if

$$B(x) = o(|x|^{-2}), \quad x \in \mathbb{R}^d, \quad d \geq 3, \quad |x| \to \infty, \tag{1.9}$$

(of course, the left-hand side here can be replaced by $\partial A_i/\partial x_j - \partial A_j/\partial x_i$ for all i, j) then this singular integral operator disappears. This result is particularly transparent if $B(x) = 0$ for sufficiently large $|x|$ and hence $A(x) = \operatorname{grad} U(x)$. So for decay (1.9) of a magnetic field, properties of the SM are quite close to those for short-range potentials. Note that if a magnetic field $B(x)$ is created by a sufficiently well localized electric current $J(x)$, that is $\operatorname{curl} B(x) = J(x)$, then it always satisfies (for $d = 3$) the condition $B(x) = O(|x|^{-3})$ as $|x| \to \infty$ (see, e.g., [5]). The cases when (1.9) is fulfilled or violated are called, respectively, regular or singular in the paper. In the case $d = 2$ the singular part of the SM reduces to a multiplication operator if the total flux is an integer times 2π. Thus, a long-range behaviour (as $b(\hat{x})|x|^{-2}$) of a magnetic field for $d \geq 3$ plays the same role as a topological obstruction (non-integer magnetic flux) for $d = 2$.

Our results show that the essential spectrum of the SM, as well as the leading diagonal singularity of its kernel, are determined by the asymptotics of $A(x)$ at infinity only; in particular, values of $A(x)$ for bounded x are irrelevant. We emphasize that the SM is not determined by a magnetic field only; its behaviour with respect to gauge transformations is discussed in subs. 3.4.

Similar results are obtained (see the last section) for odd electric potentials $V(x)$ with the asymptotics at infinity $v(\hat{x})|x|^{-1}$.

2. PSEUDODIFFERENTIAL OPERATORS

2.1. We start with an elementary result on the Fourier transform (in the sense of distributions) of homogeneous functions of degree zero.

Lemma 2.1 *Let $f \in C^\infty(\mathbb{R}^d \setminus \{0\})$ and $f(tx) = f(x)$ for $t > 0$. Then the Fourier transform*

$$\hat{f}(\xi) = (\mathcal{F} f)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i<x,\xi>} f(x) dx$$

is

$$\hat{f}(\xi) = (2\pi)^{d/2} f_0 \delta(\xi) + \hat{f}_1(\xi),$$

where $\delta(\cdot)$ is the Dirac function,

$$f_0 = |\mathbb{S}^{d-1}|^{-1} \int_{\mathbb{S}^{d-1}} f(\psi) d\psi,$$

and

$$\hat{f}_1(\xi) = (2\pi)^{-d/2} (-i)^d (d-1)! \int_{\mathbb{S}^{d-1}} (f(\psi) - f_0)(< \psi, \xi > -i0)^{-d} d\psi. \tag{2.1}$$

In particular,

$$\int_{\mathbb{S}^{d-1}} \hat{f}_1(\varphi) d\varphi = 0. \tag{2.2}$$

Proof. – Let $f_1(x) = f(x) - f_0$. The Fourier transform of f_0 equals of course $(2\pi)^{d/2} f_0 \delta(\xi)$. Thus, it remains to find

$$\hat{f}_1(\xi) = (2\pi)^{-d/2} \int_{\mathbb{S}^{d-1}} f_1(\psi) \Big(\int_0^\infty e^{-ir<\psi,\xi>} r^{d-1} dr \Big) d\psi, \tag{2.3}$$

where we have used that $f_1(r\psi) = f_1(\psi)$. Integrating here first in the variable r and setting $\alpha =< \psi, \xi >$, we see that

$$\begin{aligned} \int_0^\infty e^{-ir<\psi,\xi>} r^{d-1} dr &= \int_0^\infty e^{-i\rho\alpha} \rho^{d-1} d\rho = i^{d-1} \frac{\partial^{d-1}}{\partial \alpha^{d-1}} \int_0^\infty e^{-i\rho\alpha} d\rho \\ &= -i^d \frac{\partial^{d-1}}{\partial \alpha^{d-1}} (\alpha - i0)^{-1} = (-i)^d (d-1)! (\alpha - i0)^{-d}. \end{aligned}$$

Plugging this expression into (2.3), we obtain formula (2.1) for $\hat{f}_1(\xi)$. According to (2.1), up to a constant coefficient, integral (2.2) equals

$$\int_{\mathbb{S}^{d-1}} d\varphi f_1(\varphi) \int_{\mathbb{S}^{d-1}} d\psi (< \psi, \varphi > -i0)^{-d}.$$

Since the integral over ψ does not depend on φ, equality (2.2) is true because the integral of f_1 over $\mathbb{S}^{d-1}$ is zero. $\square$

Clearly, the function $\hat{f}(\xi)$ is homogeneous of degree $-d$. It is correctly defined as a distribution. Actually, for a test function $u(\xi)$, the integral $(\hat{f}_1, u)$ is understood in the sense of the principal value which is possible due to condition (2.2).

We need also the following technical assertion. Below C are different positive constants, whose precise values are of no importance.

Lemma 2.2 *Let $f \in C^\infty(\mathbb{R}^d)$ and*

$$|\partial^\alpha f(x)| \leq C(1 + |x|)^{-\rho - |\alpha|}, \quad \rho \in (0, 1], \quad |\alpha| \leq d.$$

Then

$$|\hat{f}(\xi)| \leq C|\xi|^{-d+\rho'}, \quad \forall \rho' < \rho, \quad |\xi| \leq 1. \tag{2.4}$$

Proof. – Let us choose a coordinate system such that, for example, the first axis is directed along ξ so that $\xi_1 = \xi$. Then integrating d times by parts in the variable x_1, we find that

$$\int_{-\infty}^\infty e^{-i<x,\xi>} f(x) dx_1 = \int_{-\infty}^\infty e^{-ix_1\xi_1} f(x) dx_1 = (i\xi_1)^{-d} \int_{-\infty}^\infty e^{-ix_1\xi_1} (\partial_1^d f)(x) dx_1. \tag{2.5}$$

Since

$$\int_{-\infty}^\infty (\partial_1^d f)(x) dx_1 = 0,$$

the right-hand side of (2.5) can be estimated by

$$|\xi|^{-d} \int_{-\infty}^\infty |e^{-i<x,\xi>} - 1| (1 + |x|)^{-\rho - d} dx_1 \leq |\xi|^{-d+\rho'} \int_{-\infty}^\infty |x_1|^{\rho'} (1 + |x|)^{-\rho - d} dx_1. \tag{2.6}$$

Integrating (2.5) and (2.6) over $x_2, \dots, x_d$, we obtain (2.4). $\square$

2.2. We need some elementary facts (see, e.g., [2] or [10]) about PDO. PDO on a domain $\Sigma \subset \mathbb{R}^d$ is defined by the equality

$$(Pf)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i<\xi,x>} p(x,\xi) \hat{f}(\xi) d\xi,$$

where a function f is from the class $C_0^\infty(\Sigma)$. We denote by $\mathcal{S}^m_{\rho,\delta}$ the class of symbols $p \in C^\infty(\Sigma \times \mathbb{R}^d)$ satisfying, for all multi-indices α and β, the estimates

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leq C_{\alpha,\beta}(1+|x|)^{m-\rho|\alpha|+\delta|\beta|}.$$

Moreover, we assume that $p(x,\xi) = 0$ for sufficiently large $|\xi|$. We always suppose that $1 \geq \rho > 1/2 > \delta \geq 0$ and set $\mathcal{S}^m = \mathcal{S}^m_{1,0}$.

In view of our applications, we also consider PDO P acting on the unit sphere $\mathbb{S}^{d-1}$. For any $\omega_0 \in \mathbb{S}^{d-1}$, let Π_{ω_0} be the hyperplane orthogonal to ω_0, and let $\Omega = \Omega(\omega_0, \gamma) \subset \mathbb{S}^{d-1}$ be determined by the condition $\langle \omega, \omega_0 \rangle > \gamma > 0$. We denote by $\zeta = \varkappa(\omega)$ the orthogonal projection of ω on Π_{ω_0}; in particular, we assume that $\varkappa(\omega_0) = 0$. We denote by Σ the orthogonal projection of Ω on the hyperplane Π_{ω_0} and identify points $\omega \in \Omega$ and $\zeta = \varkappa(\omega)$. The hyperplane Π_{ω_0} can be identified with $\mathbb{R}^{d-1}$. Let us also consider the unitary mapping $Z = Z_\varkappa : L_2(\Omega) \to L_2(\Sigma)$ defined by

$$(Zu)(\zeta) = (1-|\zeta|^2)^{-1/4} u(\omega), \quad \zeta = \varkappa(\omega).$$

We suppose that for every diffeomorphism $\varkappa$ the operator $P_\varkappa = Z_\varkappa P Z_\varkappa^*$ is a PDO on $\Sigma \subset \mathbb{R}^{d-1}$, that is

$$(P_\varkappa u)(\zeta) = (2\pi)^{-(d-1)/2} \int_{\mathbb{R}^{d-1}} p_\varkappa(\zeta, y) e^{i<y,\zeta>} \hat{u}(y) dy, \tag{2.7}$$

with symbol $p_\varkappa(\zeta, y)$ from the class $\mathcal{S}^0_{\rho,\delta} = \mathcal{S}^0_{\rho,\delta}(\Sigma \times \mathbb{R}^{d-1})$ for some $1 \geq \rho > 1/2 > \delta \geq 0$. It is invariant with respect to diffeomorphisms of Σ up to terms from the class $\mathcal{S}^{-\rho+\delta}_{\rho,\delta}$. This invariant part, considered modulo functions from $\mathcal{S}^{-\rho+\delta}_{\rho,\delta}$, is called the principal symbol of the PDO $P_\varkappa$ and will be denoted $p_\varkappa^{(pr)}$. The principal symbol of the PDO P is correctly defined on the cotangent bundle $T^*\mathbb{S}^{d-1}$ of $\mathbb{S}^{d-1}$ by the equality

$$p(\omega, z) = p_\varkappa^{(pr)}(\zeta, y), \quad |\omega| = 1, \quad <\omega, z> = 0, \tag{2.8}$$

where $\zeta = \varkappa(\omega)$ and $z = {}^t\varkappa'(\omega) y$ is the orthogonal projection of y on the hyperplane Π_ω. We emphasize that ζ plays the role of the space variable and y is the dual one. Note also that kernels $g(\omega, \omega')$ and $g_\varkappa(\zeta, \zeta')$ of the operators P and $P_\varkappa$ regarded as integral operators in $L_2(\Omega)$ and $L_2(\Sigma)$, respectively, are related by the equation

$$g(\omega, \omega') = g_\varkappa(\zeta, \zeta')(1-|\zeta|^2)^{1/4}(1-|\zeta'|^2)^{1/4}, \quad \omega, \omega' \in \Omega. \tag{2.9}$$

It is required that $g(\omega, \omega')$ be a C^∞-function off the diagonal $\omega = \omega'$.

We need information on the essential spectrum of a PDO with a homogeneous symbol of degree zero. Below, by definition, a function $f \in C^\infty$ is called homogeneous of degree k if $f(tz) = t^k f(z)$ for $t \geq 1$ and $|z| \geq 1/2$. Of course, $1/2$ is chosen here for convenience of notation. Actually, only the behaviour of $f(z)$ for large $|z|$ is essential. Let us denote $T_1^*\mathbb{S}^{d-1} \subset T^*\mathbb{S}^{d-1}$ the set of pairs (ω, z) such that $\omega, z \in \mathbb{S}^{d-1}$ and $<\omega, z> = 0$.

Proposition 2.3 *Let P be a PDO on $\mathbb{S}^{d-1}$ from the class $\mathcal{S}^0_{\rho,\delta}$ with the principal symbol such that*

$$p(\omega, z) = p(\omega, \hat{z}), \quad \hat{z} = z|z|^{-1}, \quad |z| \geq 1/2.$$

Then the essential spectrum $\sigma_{ess}(P)$ of the operator P in the space $L_2(\mathbb{S}^{d-1})$ coincides with the image Γ of the function $p(\omega, z)$ considered on the set $T^*_1\mathbb{S}^{d-1}$.

Proof. – Let $\mu_0 = p(\omega_0, z_0)$ for some $\omega_0, z_0 \in \mathbb{S}^{d-1}$, $<\omega_0, z_0>= 0$. Let $\varkappa$ be the orthogonal projection of some neighbourhood Ω of ω_0 on a part Σ of Π_{ω_0}. Up to a compact term, the operator $P_\varkappa = Z_\varkappa P Z^*_\varkappa$ is defined by equality (2.7) where $p_\varkappa(\zeta, y) = p^{(pr)}_\varkappa(\zeta, y)$ is related to $p(\omega, z)$ by formula (2.8). It follows from (2.8) and the definition of μ_0 that $\mu_0 = p_\varkappa(0, y_0)$ where $y_0 = {}^t\varkappa'(\omega)^{-1}z_0$. Let us set

$$u_{\lambda,\varepsilon}(\zeta) = \varepsilon^{-(d-1)/2} f(\zeta/\varepsilon) e^{i\lambda<y_0,\zeta>},$$

where $f \in C^\infty_0(\mathbb{R}^{d-1})$, $||f|| = 1$. Then

$$\hat{u}_{\lambda,\varepsilon}(y) = \varepsilon^{(d-1)/2} \hat{f}(\varepsilon(y - \lambda y_0)).$$

We suppose that $\varepsilon \to 0$ and $\lambda = \varepsilon^{-\tau}$ for some $\tau > (d + 1)/2$. Plugging the function $\hat{u}_{\lambda,\varepsilon}(y)$ in (2.7), we find that

$$(P_\varkappa u_{\lambda,\varepsilon})(\zeta) = p_\varkappa(\zeta, \lambda y_0) u_{\lambda,\varepsilon}(\zeta) + v_{\lambda,\varepsilon}(\zeta), \tag{2.10}$$

where

$$v_{\lambda,\varepsilon}(\zeta) = (2\pi)^{-(d-1)/2} e^{i\lambda<y_0,\zeta>} \int_{\mathbb{R}^{d-1}} \Big(p_\varkappa(\zeta, y + \lambda y_0) - p_\varkappa(\zeta, \lambda y_0)\Big) e^{i<y,\zeta>} \varepsilon^{(d-1)/2} \hat{f}(\varepsilon y) dy. \tag{2.11}$$

Let us first estimate the function $v_{\lambda,\varepsilon}(\zeta)$. Since $\hat{f} \in \mathcal{S}$, the integral (2.11) over the set $|y| \geq \varepsilon^{-1-\sigma}$ for any $\sigma > 0$ tends to zero as $\varepsilon \to 0$ faster than any power of ε. In the integral over the ball $|y| \leq \varepsilon^{-1-\sigma}$, we use that $p_\varkappa(\zeta, y)$ is a homogeneous function of degree zero of the variable y and hence

$$|p_\varkappa(\zeta, y + \lambda y_0) - p_\varkappa(\zeta, \lambda y_0)| \leq C|y|\lambda^{-1} \leq C\varepsilon^{-1-\sigma+\tau}.$$

Therefore, for any N,

$$||v_{\lambda,\varepsilon}|| \leq C_N\Big(\varepsilon^{-(d+1)/2-\sigma+\tau} + \varepsilon^N\Big),$$

and the right-hand side tends to zero as $\varepsilon \to 0$ provided $\sigma < \tau - (d + 1)/2$. The first term in the right-hand side of (2.10) equals for sufficiently large λ

$$\mu_0 u_{\lambda,\varepsilon}(\zeta) + (p_\varkappa(\zeta, y_0) - p_\varkappa(0, y_0)) u_{\lambda,\varepsilon}(\zeta).$$

The L_2-norm of the second term is estimated by

$$\max_{|\zeta|\leq\varepsilon} |p_\varkappa(\zeta, y_0) - p_\varkappa(0, y_0)|$$

which tends to zero as $\varepsilon \to 0$. Combining the results obtained, we see that

$$\lim_{\varepsilon\to 0} ||P_\varkappa u_{\lambda,\varepsilon} - \mu_0 u_{\lambda,\varepsilon}|| = 0.$$

Since $||u_{\lambda,\varepsilon}|| = 1$ and $u_{\lambda,\varepsilon} \to 0$ weakly as $\varepsilon \to 0$, we have that $u_{\lambda,\varepsilon}$ is a Weyl sequence for the operator $P_\varkappa$ and the point μ_0. It follows that $Z_\varkappa^* u_{\lambda,\varepsilon}$ is a Weyl sequence for the operator P and the same point μ_0. Thus, $\mu_0 \in \sigma_{ess}(P)$.

Conversely, let us prove that $\sigma_{ess}(P) \subset \Gamma$. Suppose that $\mu_0 \notin \Gamma$. Let $\mathcal{R}(\mu_0)$ be a PDO on the unit sphere with the principal symbol

$$r(\omega, z) = (p(\omega, z) - \mu_0)^{-1}, \quad |\omega| = 1, \quad <\omega, z>= 0.$$

Since $r \in \mathcal{S}^0_{\rho,\delta}$, the operator $\mathcal{R}(\mu_0)$ is bounded in the space $L_2(\mathbb{S}^{d-1})$. The product $\mathcal{R}(\mu_0)(P - \mu_0 I)$ is a PDO with the principal symbol which equals 1. It follows that

$$\mathcal{R}(\mu_0)(P - \mu_0 I) = I + K, \tag{2.12}$$

where K is a compact operator on $L_2(\mathbb{S}^{d-1})$. Now suppose that $\mu_0 \in \sigma_{ess}(P)$. Then there exists a sequence u_n such that

$$\|u_n\| = 1, \quad w - \lim_{n\to\infty} u_n = 0, \quad \lim_{n\to\infty} \|Pu_n - \mu_0 u_n\| = 0. \tag{2.13}$$

It follows from (2.12) that

$$\|u_n\| \le \|\mathcal{R}(\mu_0)\| \|Pu_n - \mu_0 u_n\| + \|Ku_n\|,$$

which contradicts (2.13). $\square$

As is well known, the kernel $g(\omega, \omega')$ of a PDO P regarded as an integral operator can be very singular on the diagonal $\omega = \omega'$. Let us find this singularity under the assumptions of Proposition 2.3. By virtue of equation (2.9) it suffices to consider kernel $g_\varkappa$ of the operator $P_\varkappa$ satisfying according to (2.7) the equality

$$g_\varkappa(\zeta, \zeta') = (2\pi)^{-d+1} \int_{\mathbb{R}^{d-1}} p_\varkappa(\zeta, y) e^{i<y,\zeta-\zeta'>} dy, \quad \zeta, \zeta' \in \Sigma. \tag{2.14}$$

The next result follows from Lemma 2.1 applied to the function $p_\varkappa(\zeta, y)$ in the variable y.

Proposition 2.4 *Suppose that the symbol $p_\varkappa(\zeta, y)$ is homogeneous of degree 0 in the variable y. Set*

$$p_\varkappa^{(0)}(\zeta) = |\mathbb{S}^{d-2}|^{-1} \int_{\mathbb{S}^{d-2}} p_\varkappa(\zeta, \psi) d\psi,$$

$$q_\varkappa(\zeta, \xi) = (2\pi)^{-d+1}(-i)^{d-1}(d-2)! \int_{\mathbb{S}^{d-2}} (p_\varkappa(\zeta, \psi) - p_\varkappa^{(0)}(\zeta))(<\psi, \xi> -i0)^{-d+1} d\psi,$$

so that, in particular,

$$\int_{\mathbb{S}^{d-2}} q_\varkappa(\zeta, \varphi) d\varphi = 0$$

for any $\zeta \in \Sigma$. Then kernel (2.14) satisfies the representation

$$g_\varkappa(\zeta, \zeta') = p_\varkappa^{(0)}(\zeta)\delta(\zeta' - \zeta) + \mathrm{P.V.} q_\varkappa(\zeta, \zeta' - \zeta), \tag{2.15}$$

up to a C^∞-term. The integral operator corresponding to the second term in (2.15) is understood in the sense of the principal value (in the variable $\zeta' - \zeta$).

This result extends easily to PDO defined on $\mathbb{S}^{d-1}$.

Proposition 2.5 *Under the assumptions of Proposition* 2.3 *where* $\rho = 1$, $\delta = 0$, *the kernel* $g(\omega, \omega')$ *of a PDO* P *with principal symbol* $p(\omega, z)$ *admits the representation*

$$g(\omega, \omega') = p^{(0)}(\omega)\delta(\omega, \omega') + \text{P.V.} q(\omega, \omega' - \omega), \tag{2.16}$$

up to terms of order $O(|\omega - \omega'|^{-d+1+\nu})$ *for any* $\nu < 1$. *Here* $\delta(\omega, \omega')$ *is the Dirac function on the unit sphere,*

$$p^{(0)}(\omega) = |\mathbb{S}^{d-2}|^{-1} \int_{\mathbb{S}_\omega^{d-2}} p(\omega, \psi) d\psi, \quad \mathbb{S}_\omega^{d-2} = \mathbb{S}^{d-1} \cap \Pi_\omega,$$

$$q(\omega, \tau) = (2\pi)^{-d+1}(-i)^{d-1}(d-2)! \int_{\mathbb{S}_\omega^{d-2}} (p(\omega, \psi) - p^{(0)}(\omega))(< \psi, \tau > -i0)^{-d+1} d\psi,$$

so that, in particular,

$$\int_{\mathbb{S}_\omega^{d-2}} q(\omega, \varphi) d\varphi = 0 \tag{2.17}$$

for any $\omega \in \mathbb{S}^{d-1}$.

Proof. – Let us consider as usual the PDO $P_\varkappa = Z_\varkappa P Z_\varkappa^*$. We have that $P_\varkappa = P_\varkappa^{(0)} + P_\varkappa^{(1)}$ where the symbol $p_\varkappa^{(pr)}(\zeta, y)$ of $P_\varkappa^{(0)}$ is related to $p(\omega, z)$ by formula (2.8) and $P_\varkappa^{(1)} \in \mathcal{S}^{-1}$. Then we apply Proposition 2.4 to the operator $P_\varkappa^{(0)}$ and Lemma 2.2 to the operator $P_\varkappa^{(1)}$. According to these assertions the kernel of $P_\varkappa^{(0)}$ is given by formula (2.15) and the kernel of $P_\varkappa^{(1)}$ is $O(|\zeta - \zeta'|^{-d+1+\nu})$ for any $\nu < 1$. Next we use that kernels of the operators P and $P_\varkappa$ are related by equation (2.9). Thus, it remains to express the functions in the right-hand side of (2.15) in terms of $p(\omega, \psi)$, $\omega, \psi \in \mathbb{S}^{d-1}$, $< \omega, \psi >= 0$. Without loss of generality, we may assume that $\omega_0 = \omega$ so that $\zeta(\omega) = 0$. Then $p(\omega, \psi) = p_\varkappa(\zeta, \psi)$ for $\psi \in \Pi_\omega$ and hence $p^{(0)}(\omega) = p_\varkappa^{(0)}(\zeta)$. Moreover,

$$\omega' - \omega = \omega < \omega, \omega' - \omega > + \zeta' - \zeta,$$

and therefore

$$< \psi, \omega' - \omega >=< \psi, \zeta' - \zeta >, \quad \psi \in \Pi_\omega.$$

It follows that the expressions for $q_\varkappa(\zeta, \zeta' - \zeta)$ in Proposition 2.4 and for $q(\omega, \omega' - \omega)$ coincide. □

We emphasize that the function $q(\omega, \omega' - \omega)$ in (2.16) is homogeneous of degree $-d+1$ in $\omega' - \omega$, so that due to condition (2.17) the integral operator with this kernel is correctly defined (as a bounded operator in $L_2(\mathbb{S}^{d-1})$) in terms of the principal value. Thus, under the assumptions of Proposition 2.5, P is essentially the sum P_0 of the operator of multiplication by $p^{(0)}(\omega)$ and of the singular integral operator. To be explicit,

$$(P_0 f)(\omega) = p^{(0)}(\omega) f(\omega) + \lim_{\varepsilon \to 0} \int_{|\omega' - \omega| > \varepsilon} q(\omega, \omega' - \omega) f(\omega') d\omega'. \tag{2.18}$$

Remark 2.6 If $d = 2$, then, for each $\omega \in \mathbb{S}$, the principal symbol $p(\omega, z)$ of a PDO on $\mathbb{S}$ takes only two values $p_+(\omega) = p(\omega, \omega^{(+)})$ and $p_-(\omega) = p(\omega, \omega^{(-)})$ where $\omega^{(+)}$ and $\omega^{(-)} = -\omega^{(+)}$ are obtained from ω by rotation at the angle $\pi/2$ and $-\pi/2$ in the positive (counterclockwise) direction. In this case integration over $\mathbb{S}_\omega^{d-2}$ reduces to a sum over two points $\omega^{(+)}$ and $\omega^{(-)}$. According to Proposition 2.5, the singular part of the kernel of the operator P can be written for $d=2$ in the form

$$g(\omega,\omega') = p^{(0)}(\omega)\delta(\omega,\omega') + p^{(1)}(\omega)\mathrm{P.V.}|\omega'-\omega|^{-1}\mathrm{sgn}\{\omega,\omega'\},$$

where

$$p^{(0)}(\omega) = 2^{-1}(p_+(\omega)+p_-(\omega)), \quad p^{(1)}(\omega) = (2\pi i)^{-1}(p_+(\omega)-p_-(\omega))$$

and $\{\omega,\omega'\}$ is the oriented angle between an initial vector ω and a final vector ω'.

3. SCATTERING MATRIX

3.1. Let us return to the operators H_0, $H=(i\nabla + A)^2+V$ where A and V satisfy condition (1.1). Although in this paper we are interested in properties of the SM for $\rho=1$ only, it is convenient to describe the analytical background for all $\rho>1/2$. To obtain a stationary representation for $S(\lambda)$, we have to introduce non-trivial identifications $J_\pm$ (depending on the sign of t) and consider first modified wave operators

$$W_\pm = W_\pm(H,H_0;J_\pm) = s-\lim_{t\to\pm\infty} e^{iHt}J_\pm e^{-iH_0t}. \tag{3.1}$$

The operators $J_\pm$ emerge naturally as PDO defined by the formula

$$(J_\pm f)(x) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{i<x,\xi>+i\Phi_\pm(x,\xi)}\zeta_\pm(x,\xi)\hat{f}(\xi)d\xi. \tag{3.2}$$

Set

$$\varphi_\pm(x,\xi) =<\xi,x> +\Phi_\pm(x,\xi). \tag{3.3}$$

Essentially, the symbol of the PDO $J_\pm$ is constructed in terms of approximate eigenfunctions $\Psi_\pm(x,\xi)=e^{i\varphi_\pm(x,\xi)}$ of the operator H. Substituting $\Psi=e^{i\varphi}$ in the Schrödinger equation $H\Psi=|\xi|^2\Psi$ and neglecting imaginary terms and the short-range term $|A(x)|^2$, we obtain the eikonal equation

$$|\nabla\varphi|^2 - 2<A,\nabla\varphi> +V(x)-|\xi|^2=0, \quad \nabla=\nabla_x, \tag{3.4}$$

for the phase function $\varphi=\varphi_\pm$. We choose approximate solutions of this equation in the form (3.3) where

$$\Phi_\pm(x,\xi) = \pm 2^{-1}\int_0^\infty \Big(V(x\pm t\xi)-V(\pm t\xi) - 2<A(x\pm t\xi)-A(\pm t\xi),\xi>\Big)dt. \tag{3.5}$$

Note that for all multi-indices α,β and any $\rho\in(1/2,1)$ we have in the region $\pm<\hat{x},\hat{\xi}>\geq\kappa$ (for any $\kappa>-1$) the estimates

$$|\partial_x^\alpha\partial_\xi^\beta\Phi_\pm(x,\xi)| \leq C_{\alpha,\beta}(\kappa)\,(1+|x|)^{1-\rho-|\alpha|}. \tag{3.6}$$

It follows from (3.4) and (3.6) that

$$((i\nabla + A(x))^2 + V(x) - |\xi|^2)\Psi_\pm(x,\xi) = \Psi_\pm(x,\xi)r_\pm(x,\xi),$$

where

$$|\partial_x^\alpha \partial_\xi^\beta r_\pm(x,\xi)| \le C_{\alpha,\beta}(\kappa)\,(1+|x|)^{-2\rho-|\alpha|}$$

in the same region of (x,ξ).

Following [3, 4], to get rid of the "bad" direction $\hat{x} = \mp\hat{\xi}$, we have introduced in the symbol of PDO (3.2) the cut-off function

$$\zeta_\pm(x,\xi) = \sigma_\pm(<\hat{\xi},\hat{x}>)\eta(x)\psi(|\xi|^2), \quad \hat{\xi} = \xi/|\xi|,\ \hat{x} = x/|x|. \tag{3.7}$$

Here $\sigma_\pm \in C^\infty$ is such that $\sigma_\pm(\tau) = 1$ near ± 1 and $\sigma_\pm(\tau) = 0$ near ∓ 1, so that it "kills" a conical neighbourhood of the bad direction. The function $\eta \in C^\infty(\mathbb{R}^d)$ such that $\eta(x) = 0$ near zero and $\eta(x) = 1$ for large $|x|$ is introduced only to get rid of the singularity of the function $\hat{x}$ at the point $x = 0$. Finally, $\psi \in C_0^\infty(\mathbb{R}_+)$ and $\psi(\lambda) = 1$ for $\lambda \in \Lambda \subset (0,\infty)$ where Λ is some compact interval. Thus, our considerations are localized on a bounded disjoint from zero energy interval. Since the function $\Phi_\pm$ satisfies estimates (3.6) on the support of $\zeta_\pm$, the operator $J_\pm$ belongs to the class $S^0_{\rho,1-\rho}$. In particular, $J_\pm$ is a bounded operator on $\mathcal{H}$.

The following assertion is well known (see, e.g., [3, 8, 11]).

Theorem 3.1 *Suppose that A and V satisfy estimates* (1.1)*. Then the wave operators $W_\pm(H,H_0;J_\pm)$ exist, are isometric on the subspace $E_0(\Lambda)\mathcal{H}$ and the asymptotic completeness holds*:

$$\mathrm{Ran}\,(W_\pm(H,H_0;J_\pm)E_0(\Lambda)) = E(\Lambda)\mathcal{H}.$$

Moreover, the operators $W_\pm(H,H_0;J_\pm)$ do not depend on the choice of functions $\sigma_\pm$ and η in definitions (3.2), (3.7)*.*

3.2. It follows from Theorem 3.1 that the scattering operator

$$\mathbf{S}(J_+,J_-) = W_+^*(H,H_0;J_+)W_-(H,H_0;J_-)$$

commutes with H_0 and is unitary on the subspace $E_0(\Lambda)\mathcal{H}$. The diagonal representation of H_0 can be constructed in the following way. Let $\mathfrak{N} = L_2(\mathbb{S}^{d-1})$, $\mathfrak{H} = L_2(\mathbb{R}_+;\mathfrak{N})$ and $\mathcal{U} : \mathcal{H} \to \mathfrak{H}$ be the unitary operator defined by the formula

$$(\mathcal{U}f)(\lambda;\omega) = 2^{-1/2}k^{(d-2)/2}\hat{f}(k\omega), \quad k = \lambda^{1/2}.$$

Since $\mathbf{S}(J_+,J_-)$ commutes with H_0, the operator $\mathcal{U}\mathbf{S}(J_+,J_-)\mathcal{U}^*$ acts in the space $\mathfrak{H}$ as multiplication by the operator-function $S(\lambda;J_+,J_-) : \mathfrak{N} \to \mathfrak{N}$, known as the scattering matrix (SM). The SM is defined for almost all $\lambda \in (0,\infty)$ and is unitary for almost all $\lambda \in \Lambda$. We suppose that $\lambda \in \Lambda$.

As was already mentioned, we actually treat the SM as a PDO on the unit sphere. A detailed study of the SM requires that an approximate solution of the corresponding transport equation be also introduced in the symbol of the operator $J_\pm$. The result formulated below about the structure of the symbol of the SM is a very particular case of the general result of [13] where a complete description of the amplitude of the PDO $S(\lambda;J_+,J_-)$ was obtained for an arbitrary $\rho > 0$. Moreover, this result can also be deduced from [11] or [8]. Let us start with a preliminary assertion.

Proposition 3.2 *Let condition* (1.1) *with* $\rho > 1/2$ *hold, and let us regard* $S(\lambda; J_+, J_-)$ *as an integral operator. Then its kernel* $s(\omega, \omega'; \lambda)$ *is a* C^∞*-function off the diagonal* $\omega = \omega'$.

Theorem 3.3 *Let condition* (1.1) *with* $\rho \in (1/2, 1)$ *hold, and let* $\omega_0 \in \mathbb{S}^{d-1}$ *be arbitrary. Define for* $\omega, \omega' \in \Omega(\omega_0)$ *the kernel*

$$s_0(\omega, \omega'; \lambda) = (2\pi)^{-d+1} k^{d-1} < \omega_0, \omega > \int_\Pi e^{ik<y,\omega'-\omega>} e^{i\Theta(y,k\omega)} dy, \tag{3.8}$$

where $y \in \Pi = \Pi_{\omega_0}$ *and* Θ *is function* (1.3). *Let* $S_0(\lambda)$ *be the operator with this kernel. Then*

$$S(\lambda; J_+, J_-) = S_0(\lambda) + S_1(\lambda) + S_2(\lambda), \tag{3.9}$$

where $S_1(\lambda)$ *is a PDO from the class* $\mathcal{S}^{1-2\rho}_{\rho,1-\rho}$ *and the kernel of the operator* $S_2(\lambda)$ *is continuous in* $\omega, \omega' \in \mathbb{S}^{d-1}$.

Remark 3.4 *If* $\rho = 1$ *and* $\Theta(x, \xi)$ *is a homogeneous function of degree zero of the variable* x, *then Theorem* 3.3 *remains true.*

3.3. To motivate our assumptions on a magnetic potential $A(x)$, suppose for a moment that a magnetic field $B(x)$ is given. Then $A(x)$ can be constructed by standard formulas. Let $B(x)$ satisfy the condition

$$|\partial^\alpha B(x)| \leq C_\alpha (1 + |x|)^{-\rho_0 - |\alpha|}, \quad \rho_0 > 2, \quad \forall \alpha. \tag{3.10}$$

In the case $d = 2$ we set

$$A_1(x) = x_2 \int_0^1 B(sx) s ds, \quad A_2(x) = -x_1 \int_0^1 B(sx) s ds. \tag{3.11}$$

Then $\partial A_1(x)/\partial x_2 - \partial A_2(x)/\partial x_1 = B(x)$ and transversal gauge condition (1.5) is satisfied. It follows from (3.10), (3.11) that $A(x)$ admits representation

$$A(x) = A_\infty(x) + A_r(x), \tag{3.12}$$

where $A_\infty(x)$ is given for $|x| \geq 1$ by formula (1.8) with

$$a(\hat{x}) = -\int_0^\infty B(s\hat{x}) s ds \tag{3.13}$$

and $A_r(x)$ obeys estimates (1.1) with $\tilde{\rho} = \rho_0 - 1 > 1$. Clearly, $\operatorname{curl} A_\infty(x) = 0$.

Similar formulas are true in a high-dimensional case $d \geq 3$. For simplicity of notation, we usually suppose that $d = 3$. If a magnetic field is given by a vector-function $B(x) = (B_1(x), B_2(x), B_3(x))$ such that $\operatorname{div} B(x) = 0$, then a magnetic potential $A(x) = (A_1(x), A_2(x), A_3(x))$ can be reconstructed by the formula

$$A_1(x) = \int_0^1 \Big(B_2(sx) x_3 - x_2 B_3(sx) \Big) s ds. \tag{3.14}$$

Expressions for components $A_2(x)$ and $A_3(x)$ are obtained by the cyclic permutation of indices in (3.14). Then $A(x)$ is one of magnetic potentials satisfying both conditions (1.5) and (1.6). Moreover, $A(x)$ admits representation (3.12) where

$$A_\infty(x) = |x|^{-2}(x_2a_3(\hat{x}) - x_3a_2(\hat{x}), x_3a_1(\hat{x}) - x_1a_3(\hat{x}), x_1a_2(\hat{x}) - x_2a_1(\hat{x})), \quad |x| \geq 1,$$

$$a_j(\hat{x}) = -\int_0^\infty B_j(s\hat{x})sds, \quad j = 1, 2, 3,$$

and $A_r(x)$ obeys estimates (1.1) with $\tilde{\rho} = \rho_0 - 1 > 1$.

Thus, we are led to the following

Assumption 3.5 *Let a magnetic potential $A(x)$ admit representation* (3.12) *where $A_\infty(x)$ is a homogeneous function of degree -1 satisfying transversal condition*

$$< A_\infty(x), x >= 0 \tag{3.15}$$

and the short-range potential $A_r(x)$ obeys estimates (1.1) *with some $\tilde{\rho} > 1$. Additionally, we assume that the electric potential $V = 0$.*

Of course, it suffices to impose condition (3.15) for sufficiently large $|x|$ only. Indeed, let $\eta \in C^\infty$, $\eta(x) = 0$ in a neighbourhood of zero and $\eta(x) = 1$ off a larger neighbourhood. If $A_\infty(x) = |x|^{-1}a(\hat{x})$ where $< a(\hat{x}), \hat{x} >= 0$ for large $|x|$, then $\tilde{A}_\infty(x) = |x|^{-1}a(\hat{x})\eta(x)$ is a homogeneous function of degree -1 (off a neighbourhood of zero) and satisfies (3.15) for all x. The remainder $\tilde{A}_r = A - \tilde{A}_\infty$ obeys estimates (1.1) with $\tilde{\rho} > 1$.

Let Assumption 3.5 hold. Then the usual wave operators $W_\pm(H, H_0)$ defined by (1.7) exist and are simply related to wave operators (3.1). Indeed, let us set

$$(Jf)(x) = e^{i\Xi(\hat{x})}f(x), \quad \text{where} \quad \Xi(\hat{x}) = \int_0^\infty < A_r(t\hat{x}), \hat{x} > dt. \tag{3.16}$$

As shown in [8],

$$W_\pm(H, H_0; J_\pm) = W_\pm(H, H_0; J)\psi(H_0). \tag{3.17}$$

To pass to the wave operators $W_\pm(H, H_0)$, we use the following

Lemma 3.6 *Let $\phi(x)$ be an asymptotically homogeneous function of degree zero, that is, more precisely, $\phi \in C^1$, $\phi(x) = \phi_0(x) + \tilde{\phi}(x)$ where $\phi_0(tx) = \phi_0(x)$ for all $t > 0$ and $\tilde{\phi}(x) = o(1)$, $\operatorname{grad}\tilde{\phi}(x) = o(|x|^{-1})$ as $|x| \to \infty$. Then*

$$e^{-i\phi}\exp(-iH_0t)f = \exp(-iH_0t)f^{(\pm)} + o(1), \tag{3.18}$$

where

$$\hat{f}^{(\pm)}(\xi) = e^{-i\phi_0(\pm\xi)}\hat{f}(\xi) \tag{3.19}$$

and the remainder $o(1)$ tends to 0 in $L_2(\mathbb{R}^d)$ as $t \to \pm\infty$.

Proof. – Since

$$(\exp(-iH_0t)f)(x) = e^{i|x|^2/(4t)}(2it)^{-d/2}\hat{f}(x/(2t)) + o(1),$$

we have that

$$
\begin{aligned}
(e^{-i\phi}\exp(-iH_0t)f)(x) &= e^{i|x|^2/(4t)}(2it)^{-d/2}e^{-i\phi_0(\pm x/(2t))}\hat{f}(x/(2t))+o(1)\\
&= e^{i|x|^2/(4t)}(2it)^{-d/2}\hat{f}^{(\pm)}(x/(2t))+o(1)
\end{aligned}
$$

with $\hat{f}^{(\pm)}$ defined by (3.19). This is equivalent to (3.18). $\square$

Applying Lemma 3.6 to the operator $J\exp(-iH_0t)$, we find that $W_\pm(H,H_0)$ exist and

$$
W_\pm(H,H_0;J)=W_\pm(H,H_0)\mathcal{F}^*e^{i\Xi(\pm\hat{\xi})}\mathcal{F}.
$$

Comparing this equation with (3.17), we see that

$$
W_\pm(H,H_0)\psi(H_0)=W_\pm(H,H_0;J_\pm)\mathcal{F}^*e^{-i\Xi(\pm\hat{\xi})}\mathcal{F}. \tag{3.20}
$$

Below

$$
\mathbf{S}=W_+^*(H,H_0)W_-(H,H_0)
$$

and $S(\lambda)$ corresponds to $\mathbf{S}$. Equality (3.20) implies the following relations between corresponding scattering operators and scattering matrices:

$$
\mathcal{F}\mathbf{S}\mathcal{F}^*\psi(|\xi|^2)=e^{i\Xi(\hat{\xi})}\mathcal{F}\mathbf{S}(J_+,J_-)\mathcal{F}^*e^{-i\Xi(-\hat{\xi})},
$$

$$
S(\lambda)=e^{i\Xi(\omega)}S(\lambda;J_+,J_-)e^{-i\Xi(-\omega)},\quad \lambda\in\Lambda. \tag{3.21}
$$

3.4. Here we discuss the behaviour of the SM with respect to gauge trasformations defined by the formula

$$
\tilde{A}(x)=A(x)+\operatorname{grad}\phi(x),\quad \phi\in C^1(\mathbb{R}^d). \tag{3.22}
$$

Then $\operatorname{curl}\tilde{A}(x)=\operatorname{curl}A(x)$ and the Schrödinger operator $\tilde{H}=(i\nabla+\tilde{A}(x))^2$ with magnetic potential $\tilde{A}$ is related to the operator $H=(i\nabla+A(x))^2$ by the formula

$$
\tilde{H}=e^{i\phi}He^{-i\phi}. \tag{3.23}
$$

If $A(x)$ satisfies Assumption 3.5 and $\phi(x)$ is an asymptotically homogeneous function of degree zero, then

$$
\tilde{A}_\infty(x)=A_\infty(x)+\operatorname{grad}\phi_0(x)
$$

also satisfies transversal gauge condition (3.15). Note that in the case $d=2$ one may admit that only a function $e^{i\phi_0(\hat{x})}$ is defined on the unit circle (that is, $\phi_0(\hat{x})$ is defined up to a constant from $2\pi\mathbb{Z}$) and of course $\phi_0(\hat{x})$ is smooth. However, we do not need such multi-valued functions $\phi_0(\hat{x})$.

It is easy to find a relation between the wave operators $W(H,H_0)$ and $W(\tilde{H},H_0)$.

Proposition 3.7 *Let the wave operators $W_\pm(H,H_0)$ exist, and let $\phi(x)$ be an asymptotically homogeneous function of degree zero with homogeneous part $\phi_0(x)$. Then the wave operators $W_\pm(\tilde{H},H_0)$ also exist and*

$$
W_\pm(\tilde{H},H_0)=e^{i\phi(x)}W_\pm(H,H_0)\mathcal{F}^*e^{-i\phi_0(\pm\xi)}\mathcal{F}. \tag{3.24}
$$

Proof. – It follows from Lemma 3.6 and formula (3.23) that

$$W_\pm(\tilde{H}, H_0)f = \lim_{t\to\pm\infty} e^{i\tilde{H}t}e^{-iH_0t}f = \lim_{t\to\pm\infty} e^{i\phi}e^{iHt}e^{-i\phi}e^{-iH_0t}f$$
$$= \lim_{t\to\pm\infty} e^{i\phi}e^{iHt}e^{-iH_0t}f^{(\pm)} = e^{i\phi}W_\pm(H, H_0)f^{(\pm)},$$

where $f^{(\pm)}$ is defined by (3.19). This proves (3.24). □

As an immediate consequence of Proposition 3.7, we obtain a relation between the corresponding scattering operators and matrices.

Proposition 3.8 *Under the assumptions of Proposition* 3.7*, the scattering operators for the pairs H_0, H and $H_0, \tilde{H}$ are related by the equation*

$$\mathcal{F}\mathbf{S}(\tilde{H}, H_0)\mathcal{F}^* = e^{i\phi_0(\xi)}\mathcal{F}\mathbf{S}(H, H_0)\mathcal{F}^*e^{-i\phi_0(-\xi)}. \tag{3.25}$$

The SM $S(\lambda) = S(H, H_0; \lambda)$ and $\tilde{S}(\lambda) = S(\tilde{H}, H_0; \lambda)$ satisfy for all $\lambda > 0$ the relations

$$\tilde{S}(\lambda) = e^{i\phi_0(\omega)}S(\lambda)e^{-i\phi_0(-\omega)} \tag{3.26}$$

or, in terms of the scattering amplitudes,

$$\tilde{s}(\omega, \omega'; \lambda) = e^{i\phi_0(\omega)-i\phi_0(-\omega')}s(\omega, \omega'; \lambda). \tag{3.27}$$

In particular, $S(\lambda)$ and $\tilde{S}(\lambda)$ are unitarily equivalent if $\phi_0(\omega)$ is an even function.

We emphasize that relations (3.25)-(3.27) depend only on the asymptotics ϕ_0 of the phase function ϕ.

Formulas (3.26) or (3.27) show that the SM is not determined by the magnetic field $B(x) = \operatorname{curl} A(x)$ only. This seems to contradict the following mental experiment. Suppose that a quantum particle interacts with a magnetic field. Note that it is exactly a field but not a potential which can be created by our hands. After interaction we calculate the SM which depends on a potential. So it appears that a particle itself chooses a gauge convenient for it. There could be (at least) two possible explanations of this similar contradiction. The first is that the scattering amplitude $s(\omega, \omega'; \lambda)$ cannot be measured experimentally although it is widely believed to be possible. From this point of view only the scattering cross-section $|s(\omega, \omega'; \lambda)|^2$ can be practically found which is compatible with (3.27). Another point of view is that experimental devices used for observation of a quantum particle are not harmless and fix some specific gauge.

As a by-side remark, we mention also the following consequence of Propositions 3.7 and 3.8.

Proposition 3.9 *Let $\phi(x)$ be an asymptotically homogeneous function of degree zero with homogeneous part $\phi_0(x)$ and $A(x) = \operatorname{grad}\phi(x)$. Then the wave operators $W_\pm(H, H_0)$ exist and*

$$W_\pm(H, H_0) = e^{i\phi(x)}\mathcal{F}^*e^{-i\phi_0(\pm\xi)}\mathcal{F}$$

and the SM $S(H, H_0, \lambda)$ acts as multiplication by $e^{i\phi_0(\omega)-i\phi_0(-\omega)}$ for all $\lambda > 0$.

Note that the SM (but not the wave operators) depends only on the asymptotics ϕ_0 of ϕ.

4. MAIN RESULT

4.1. Let us first discuss the structure of the SM $S(\lambda)$ for the pair H_0, H under Assumption 3.5. We proceed from Theorem 3.3. Then we replace in (3.8) the function $\Theta(x,\xi)$ by

$$\Theta_\infty(x,\xi) = \int_{-\infty}^{\infty} < A_\infty(x+t\xi), \xi > dt, \quad \xi \neq 0, \tag{4.1}$$

which is the circulation of the magnetic potential $A_\infty(x)$ over the straight line $x + t\xi$ where $t \in \mathbb{R}$. Note that

$$\Theta_\infty(x,-\xi) = -\Theta_\infty(x,\xi). \tag{4.2}$$

Lemma 4.1 *Integral* (4.1) *converges and the function* $\Theta_\infty(x,\xi)$ *is homogeneous of degree* 0 *in both variables, that is*

$$\Theta_\infty(x,\xi) = \Theta_\infty(\hat{x},\hat{\xi}), \quad |x| \geq 1/2, \quad \xi \neq 0. \tag{4.3}$$

Proof. – It follows from condition (3.15) that

$$< A_\infty(x+t\xi), \xi >= -t^{-1} < A_\infty(x+t\xi), x >,$$

which is $O(|t|^{-2})$ as $|t| \to \infty$, and hence integral (4.1) converges. Making in (4.1) the change of variables $t = |x||\xi|^{-1}s$, we arrive at (4.3). □

Under Assumption 3.5, Theorem 3.3 can be formulated in a more concrete way. Clearly, formula (1.3) implies that

$$\Theta(x,\xi) = \Theta_\infty(x,\xi) + \int_{-\infty}^{\infty} < A_r(x+t\xi), \xi > dt - \int_{-\infty}^{\infty} < A_r(t\xi), \xi > dt. \tag{4.4}$$

The integral of $< A_r(x+t\xi), \xi >$ can be omitted here since the arising error belongs to the class $\mathcal{S}^{-\tilde{\rho}+1}$ and can be included in the operator $S_1(\lambda)$. By definition (3.16),

$$\int_{-\infty}^{\infty} < A_r(t\xi), \xi > dt = \Xi(\hat{\xi}) - \Xi(-\hat{\xi}). \tag{4.5}$$

Since Θ_∞ is a homogeneous function, Theorem 3.3 for $S(\lambda; J_+, J_-)$ remains valid (see Remark 3.4) for $\rho = 1$. To formulate it in terms of the SM $S(\lambda)$, we use relations (3.21), (4.4) and (4.5).

Theorem 4.2 *Let Assumption* 3.5 *hold. Define the function* $\Theta_\infty(x,\xi)$ *by equation* (4.1), *and let* S_{00} *be the PDO on the unit sphere with principal symbol*

$$p(\omega, z; \lambda) = \exp\Big(i\Theta_\infty(-z,\omega)\Big), \quad \omega \in \mathbb{S}^{d-1}, \ z \in \mathbb{R}^d, \ < \omega, z >= 0.$$

Then

$$S(\lambda) = e^{i\Xi(\omega)} S_{00} e^{-i\Xi(\omega)} + S_1(\lambda) + S_2(\lambda),$$

where $S_1(\lambda)$ *is a PDO from the class* $\mathcal{S}^{-\nu_0}$, $\nu_0 = \min\{\tilde{\rho} - 1, 1\}$, *and the kernel of the operator* $S_2(\lambda)$ *is continuous in* $\omega, \omega' \in \mathbb{S}^{d-1}$.

Corollary 4.3 *The operator $S(\lambda) - e^{i\Xi(\omega)} S_{00} e^{-i\Xi(\omega)}$ is compact in the space $\mathfrak{N}$.*

Let us apply Proposition 2.3 to the PDO S_{00}. We remark that the sets $T_1^*\mathbb{S}^{d-1}$ are connected for $d \geq 3$ and $T_1^*\mathbb{S}$ has two disjoint connected components $T_1^*\mathbb{S}_+$ and $T_1^*\mathbb{S}_-$. Indeed, $T_1^*\mathbb{S}_\pm$ consists of points (ω, z) where $\omega \in \mathbb{S}$ is arbitrary and $z = \omega^{(\pm)}$. Therefore for $d \geq 3$ the function Θ_∞ on $T_1^*\mathbb{S}^{d-1}$ takes all the values between its maximum γ and minimum γ'. Moreover, according to (4.2) $\gamma' = -\gamma$. If $d = 2$, then, similarly, Θ_∞ on $T_1^*\mathbb{S}_+$ takes all the values between its maximum γ_+ and minimum γ_-. According to (4.2) the maximum of Θ_∞ on $T_1^*\mathbb{S}_-$ equals $-\gamma_-$ and its minimum equals $-\gamma_+$. Below for any $\mu_1, \mu_2 \in \mathbb{S}$ we denote by $[\mu_1, \mu_2]$ the arc of the unit circle obtained as μ_1 moves to μ_2 in the positive direction.

Theorem 4.4 *Let Assumption 3.5 hold and let $\lambda > 0$ be arbitrary. If $d \geq 3$, we set*

$$\gamma = \max_{(\omega,z)\in T_1^*\mathbb{S}^{d-1}} \Theta_\infty(z, \omega). \tag{4.6}$$

Then $\sigma_{ess}(S(\lambda)) = [\exp(-i\gamma), \exp(i\gamma)]$ if $\gamma < \pi$ and relation (1.4) holds if $\gamma \geq \pi$. If $d = 2$, we set

$$\gamma_+ = \max_{\omega\in\mathbb{S}} \Theta_\infty(\omega^{(+)}, \omega), \quad \gamma_- = \min_{\omega\in\mathbb{S}} \Theta_\infty(\omega^{(+)}, \omega). \tag{4.7}$$

If $\gamma_+ - \gamma_- < 2\pi$, then

$$\sigma_{ess}(S(\lambda)) = [\exp(i\gamma_-), \exp(i\gamma_+)] \cup [\exp(-i\gamma_+), \exp(-i\gamma_-)], \tag{4.8}$$

that is $\sigma_{ess}(S(\lambda))$ consists of the two complex conjugated and perhaps overlapping intervals. If $\gamma_+ - \gamma_- \geq 2\pi$, then relation (1.4) holds.

Using Proposition 2.5, we can describe the diagonal singularity of the scattering amplitude $s(\omega, \omega'; \lambda)$.

Theorem 4.5 *Under the assumptions of Theorem 4.2, let us set*

$$p^{(0)}(\omega) = |\mathbb{S}^{d-2}|^{-1} \int_{\mathbb{S}^{d-2}_\omega} \exp(i\Theta_\infty(\psi, \omega)) d\psi, \quad \mathbb{S}^{d-2}_\omega = \mathbb{S}^{d-1} \cap \Pi_\omega, \tag{4.9}$$

$$q(\omega, \tau) = (2\pi i)^{-d+1}(d-2)! \int_{\mathbb{S}^{d-2}_\omega} \Big(\exp(i\Theta_\infty(\psi, \omega) - p^{(0)}(\omega)\Big)(< \psi, \tau > -i0)^{-d+1} d\psi \tag{4.10}$$

and

$$s_0(\omega, \omega') = p^{(0)}(\omega)\delta(\omega, \omega') + \text{P.V.} q(\omega, \omega' - \omega). \tag{4.11}$$

Then

$$|s(\omega, \omega'; \lambda) - s_0(\omega, \omega')| \leq C|\omega - \omega'|^{-d+1+\nu} \tag{4.12}$$

for any $\nu < \nu_0 = \min\{\tilde{\rho} - 1, 1\}$.

We emphasize that the function $q(\omega, \tau)$ is homogeneous in τ of degree $-d + 1$ and satisfies condition (2.17). Therefore the integral operator in the sense of principal value (see (2.18)) is well defined. Theorem 4.5 implies

Corollary 4.6 *Let $\omega \neq \omega'$, $\omega - \omega' \to 0$ and $\nu < \nu_0$. Then*

$$s(\omega, \omega'; \lambda) = q(\omega, \omega' - \omega) + O(|\omega - \omega'|^{-d+1+\nu}).$$

Now we can give an explicit asymptotics of the scattering cross section

$$\Sigma_{diff}(\omega;\omega_0,\lambda)=(2\pi)^{d-1}\lambda^{-(d-1)/2}|s(\omega,\omega_0;\lambda)|^2,\quad \omega\neq\omega_0,$$

for incident direction ω_0 of a beam of particles and direction of observation ω.

Corollary 4.7 *Let $\omega\to\omega_0$ and $\nu<\nu_0$. Then*

$$\Sigma_{diff}(\omega;\omega_0,\lambda)=(2\pi)^{d-1}\lambda^{-(d-1)/2}|q(\omega_0,\omega_0-\omega)|^2+O(|\omega-\omega_0|^{-2d+2+\nu}). \tag{4.13}$$

Note that the order of singularity $|\omega-\omega_0|^{-2d+2}$ in (4.13) is the same as for electric Coulomb potentials.

If magnetic potentials A and $\tilde A$ are related by equality (3.22), then corresponding functions (4.1) differ by the term

$$\begin{aligned}\int_{-\infty}^{\infty} <\operatorname{grad}\phi_0(x+t\xi),\xi> dt &= \lim_{T\to\infty}\int_{-T}^{T}\frac{d}{dt}\phi_0(x+t\xi)dt\\ &=\lim_{T\to\infty}(\phi_0(x+T\xi)-\phi_0(x-T\xi))=\phi_0(\xi)-\phi_0(-\xi)\end{aligned} \tag{4.14}$$

and hence

$$\tilde\Theta_\infty(x,\xi)=\Theta_\infty(x,\xi)+\phi_0(\xi)-\phi_0(-\xi).$$

According to (4.9), (4.10), this implies that

$$\tilde p^{(0)}(\omega)=p^{(0)}(\omega)e^{i\phi_0(\omega)-i\phi_0(-\omega)},\quad \tilde q(\omega,\tau)=q(\omega,\tau)e^{i\phi_0(\omega)-i\phi_0(-\omega)},$$

and hence, according to (4.11), the singular parts of the SM $S(\lambda)$ and $\tilde S(\lambda)$ are related by the equality

$$\tilde s_0(\omega,\omega')=e^{i\phi_0(\omega)-i\phi_0(-\omega)}s_0(\omega,\omega'),$$

which is of course conformal with exact equality (3.27).

4.2. In the case $d=2$ the results of the previous subsection can be made more explicit. Note first that for $d=2$ condition (1.5) implies that $A_\infty(x)$ is defined by formula (1.8) with a function $a\in C^\infty(\mathbb{S})$. Recall that $\omega^{(\pm)}$ is obtained from $\omega\in\mathbb{S}$ by rotation at the angle $\pm\pi/2$ in the positive direction. Set

$$f(\omega)=\int_{\mathbb{S}(\omega^{(-)},\omega^{(+)})}a(\psi)d\psi,\quad \omega\in\mathbb{S}, \tag{4.15}$$

where the integral is taken in the positive (counter-clockwise) direction over the half-circle between the points $\omega^{(-)}$ and $\omega^{(+)}$. Then for any $\omega\in\mathbb{S}$

$$f(\omega)+f(-\omega)=\int_{\mathbb{S}}a(\psi)d\psi=\lim_{R\to\infty}\int_{|x|=R}<A(x),dx>=\Phi \tag{4.16}$$

is the total magnetic flux.

Lemma 4.8 *Let Θ_∞ be defined by formula* (4.1), *where $A_\infty(x)$ is potential* (1.8), *and let f be function* (4.15). *Then*

$$\Theta_\infty(\omega,\omega^{(\pm)})=\pm f(\omega). \tag{4.17}$$

Proof. – By virtue of (4.2), it suffices to consider the case of the upper sign. $\tan\theta_t = t$. Since

$$< (-\omega_2 - t\omega_2^{(+)}, \omega_1 + t\omega_1^{(+)}), (\omega_1^{(+)}, \omega_2^{(+)}) >= \omega_1\omega_2^{(+)} - \omega_2\omega_1^{(+)} = 1,$$

we have that for potentials (1.8)

$$\Theta_\infty(\omega, \omega^{(\pm)}) = \int_{-\infty}^{\infty} a(\frac{\omega + t\omega^{(+)}}{\sqrt{t^2+1}})\frac{dt}{t^2+1}.$$

Making the change of variables $t = \tan\psi$, we get formula (4.17). $\square$

If a magnetic potential is given by equality (3.11), then comparing formulas (3.13) and (4.15), we can express the function $f(\omega)$ in terms of the magnetic field

$$f(\omega) = -\int_{<x,\omega>\geq 0} B(x)dx.$$

The following two assertions are immediate consequences of Theorems 4.4 and 4.5 (see also Remark 2.6).

Theorem 4.9 *Let $A(x)$ admit representation* (3.12) *where $A_\infty(x)$ is function* (1.8) (*for sufficiently large $|x|$*) *and $A_r(x)$ obeys estimates* (1.1) *with $\tilde\rho > 1$. Let the function $f(\omega)$ be defined by formula* (4.15)*, and let $\gamma_+ = -\min f(\omega)$ and $\gamma_- = -\max f(\omega)$. Then for all $\lambda > 0$ relation* (4.8) *holds if $\gamma_+ - \gamma_- < 2\pi$, and relation* (1.4) *holds if $\gamma_+ - \gamma_- \geq 2\pi$.*

Theorem 4.10 *Let assumptions of Theorem* 4.9 *hold. Let S_0 be the integral operator on $L_2(\mathbb{S})$ with kernel*

$$\mathrm{s}_0(\omega,\omega') = e^{i(f(\omega^{(-)})-f(\omega^{(+)}))/2}\Big(\cos(\Phi/2)\delta(\omega,\omega') + \pi^{-1}\sin(\Phi/2)\mathrm{P.V.}|\omega-\omega'|^{-1}\mathrm{sgn}\{\omega,\omega'\}\Big). \tag{4.18}$$

Then estimate (4.12) *is satisfied.*

It follows from formula (4.18) that, up to the phase factor, the singular part of the SM is determined by the magnetic flux Φ only. In particular, if $\Phi \in 2\pi\mathbb{Z}$, then, according to (4.16), the SM acts as multiplication by the function $e^{if(\omega^{(-)})}$, where

$$f(\omega^{(-)}) = \int_{\mathbb{S}(-\omega,\omega)} a(\psi)d\psi.$$

As we shall see in the next section, this situation is typical for dimensions $d \geq 3$. Note also that if $a(\hat{x})$ is even, then, according to (4.16), $f(\omega) = \Phi/2$ and hence the first factor in the right-hand side of (4.18) equals 1.

As a concrete example, let us consider the function

$$a(\hat{x}) = \alpha + < p, \hat{x} >, \quad \alpha \in \mathbb{R}, \quad p \in \mathbb{R}^2.$$

Then $\Phi = 2\pi\alpha$ and function (4.15) equals

$$f(\omega) = \pi\alpha + 2 < p, \omega >,$$

so that the conclusion of Theorem 4.9 is true with $\gamma_+ = 2|p| - \pi\alpha$ and $\gamma_2 = -2|p| - \pi\alpha$. In particular, relation (1.4) holds if, for example, $2|p| \geq \pi$ or, to the contrary, it consists of the two points $\exp(-\pi i\alpha)$ and $\exp(\pi i\alpha)$ if $p = 0$. The phase factor in (4.18) equals $\exp(2i < p, \omega^{(-)} >)$.

In the case $d = 2$ equality (3.22) implies that

$$\tilde{a}(\hat{x}) = a(\hat{x}) + \phi_0'(\hat{x}). \tag{4.19}$$

In particular, the magnetic fluxes for potentials $\tilde{\Phi}$ and Φ are the same, i.e., $\tilde{\Phi} = \Phi$. Conversely, if $\tilde{\Phi} = \Phi$, then defining $\phi_0(\omega)$ by the formula

$$\phi_0(\omega) = \int_{\mathbb{S}(\omega_0,\omega)} (\tilde{a}(\psi) - a(\psi))d\psi \tag{4.20}$$

(the point $\omega_0 \in \mathbb{S}$ is arbitrary but fixed) we obtain that, due to the condition $\tilde{\Phi} = \Phi$, function (4.20) is single-valued on the unit circle. Obviously, equality (4.19) is also satisfied. Relations (3.27) and (4.20) show that, for a given Φ, it suffices to prove Theorem 4.10 only for one function a satisfying (4.16). In particular, we can choose $a(\psi) = (2\pi)^{-1}\Phi$, which reduces the proof of Theorem 4.10 to the case of a constant function a.

5. DIMENSIONS LARGER THAN TWO

The three-dimensional case $\mathcal{H} = L_2(\mathbb{R}^3)$ (or of a higher dimension $d \geq 3$) is qualitatively different from the case $d = 2$. On the one hand, the topological obstruction created by a non-integer (modulo 2π) magnetic flux disappears since the exterior of any ball $|x| \leq R$ is simply connected. On the other hand, for dimensions $d \geq 3$, all magnetic potentials satisfying Assumption 3.5 cannot be described by a simple formula of type (1.8). Actually, we distinguish two qualitatively different cases: the regular one when condition (1.9) is fulfilled and the singular one when $B(x)$ behaves as $b(\hat{x})|x|^{-2}$ at infinity. As always, in this section Assumption 3.5 is supposed to be fulfilled.

5.1. Let us start with the regular case.

Lemma 5.1 *Let condition* (1.9) *for* $B(x) = \operatorname{curl} A(x)$ *be satisfied. Then the function* $\Theta_\infty(x, \xi) =: \Theta_\infty(\xi)$ *does not depend on* x *provided* $< x, \xi >= 0$, $|x| \geq 1/2$.

Proof. – By virtue of Lemma 4.1, it suffices to check that, for any $\hat{x}_1, \hat{x}_2 \in \Pi_\xi$,

$$\int_{-\infty}^{\infty} (< A_\infty(R\hat{x}_1 + t\xi), \xi > - < A_\infty(R\hat{x}_2 + t\xi), \xi >)dt \tag{5.1}$$

tends to zero as $R \to \infty$. Let us apply the Stokes theorem to a part of the lateral surface of the cylinder $R\theta + t\xi$, where $\theta \in (\hat{x}_1, \hat{x}_2) \subset \mathbb{S}$, $t \in \mathbb{R}$. Then the difference (5.1) can be rewritten as

$$-R \int_{-\infty}^{\infty} dt \int_{\theta \in (\hat{x}_1,\hat{x}_2)} d\theta < \operatorname{curl} A_\infty(R\theta + t\hat{\xi}), \theta > \tag{5.2}$$

where we have used the cylindrical coordinates in the three-dimensional subspace spanned by the vectors $\hat{x}_1, \hat{x}_2$ and ξ. By condition (1.9), the integrand here is bounded by $o((t^2+R^2)^{-1})$, so that expression (5.2) tends to zero as $R \to \infty$. $\square$

Combining this result with Theorem 4.5, we obtain

Theorem 5.2 *Let $d \geq 3$ be arbitrary. Under assumption* (1.9) *the function $\Theta_\infty(\psi, \omega)$ does not depend on ψ, and the SM is, up to a compact term (whose kernel is bounded by $C|\omega - \omega'|^{-d+1+\nu}$ for any $\nu < \nu_0 = \min\{\tilde{\rho} - 1, 1\}$), the operator of multiplication by the function $\exp(i\Theta_\infty(\omega))$. In this case the essential spectrum of the SM coincides with values of the function $\exp(i\Theta_\infty(\omega))$ for all $\omega \in \mathbb{S}^{d-1}$. Moreover, the differential cross section satisfies estimate*

$$\Sigma_{diff}(\omega; \omega_0, \lambda) = O(|\omega - \omega_0|^{-2d+2+\nu}).$$

We emphasize that according to (4.2) $\Theta_\infty(-\omega) = -\Theta_\infty(\omega)$.

Theorem 5.2 shows that under assumption (1.9) scattering has short-range nature although the magnetic potential decays as $|x|^{-1}$ at infinity.

We can get a slightly different expression for the function $\Theta_\infty(\omega)$ if $B \in C_0^\infty(\mathbb{R}^d)$. Then $\operatorname{curl} A_\infty(x) = 0$ for sufficiently large $|x|$, say $|x| \geq R$. Let us standardly define (for $|x| \geq R$) the function $U(x)$ as a curvilinear integral

$$U(x) = \int_{\Gamma_x} < A_\infty(y), dy > \tag{5.3}$$

taken between some fixed point x_0 and a variable point x. It is required that Γ_x lies outside the ball $|x| \leq R$, so that by the Stokes theorem $U(x)$ does not depend on a choice of Γ_x (here the condition $d \geq 3$ is used). Clearly,

$$A_\infty(x) = \operatorname{grad} U(x), \quad |x| \geq R. \tag{5.4}$$

Moreover, $U(x)$ is homogeneous of degree 0. Indeed, if $x_2 = \gamma x_1$, $\gamma > 1$, then, by transversal gauge condition (3.15), $A_\infty(x)$ for $x \in (x_1, x_2)$ is orthogonal to the line (x_1, x_2) and hence $U(x_1) = U(x_2)$. We extend $U(x)$ as a homogeneous function to all $|x| \geq 1/2$.

It follows from equalities (4.14) where the role of ϕ_0 is played by U that

$$\Theta_\infty(x, \xi) = U(\xi) - U(-\xi).$$

In particular, we see again that the function $\Theta_\infty(x, \xi)$ does not depend on x. Thus, we obtain (cf. Proposition 3.9) the following

Theorem 5.3 *If $B \in C_0^\infty(\mathbb{R}^d)$, $d \geq 3$, and U is defined by* (5.3)*, then the conclusion of Theorem* 5.2 *remains true with the function $\Theta_\infty(\omega) = U(\omega) - U(-\omega)$.*

Remark 5.4 If $d = 2$, then of course these arguments do not work since, by the Stokes theorem, the integral (5.3) over the circle $|x| = R$ tends as $R \to \infty$ to the total flux Φ of the magnetic field. This difficulty is inessential if $\Phi \in 2\pi\mathbb{Z}$. Indeed, if $d = 2$, then

Assumption 3.5 implies that $B(x) = O(|x|^{-\tilde{\rho}-1})$ where $\tilde{\rho} > 1$. Therefore, again by the Stokes theorem,

$$\Theta_\infty(x, \xi) - \Theta_\infty(-x, \xi) = \lim_{R\to\infty} \Big(\Theta_\infty(R\hat{x}, \xi) - \Theta_\infty(-R\hat{x}, \xi)\Big) \in 2\pi\mathbb{Z}.$$

It follows that the function $\exp(i\Theta_\infty(\psi, \omega)) = \exp(i\Theta_\infty(\omega))$ does not depend on ψ such that $< \psi, \omega >= 0$. In this case the SM acts, up to compact terms, as multiplication by $\exp(i\Theta_\infty(\omega))$, so that we recover a result of subs. 4.2 (see formula (4.18)) for the case $\Phi \in 2\pi\mathbb{Z}$.

5.2. Let us consider a concrete example of a toroidal solenoid $\mathbf{T}$ in the space $\mathbb{R}^3$ symmetric with respect to rotations around the axis x_3. We do not assume that the section of $\mathbf{T}$ by a half-plane passing through the x_3-axis is a disc. Suppose (which looks quite realistic) that

$$B(x_1, x_2, x_3) = \alpha(-x_2, x_1, 0), \quad \alpha = \text{const}, \tag{5.5}$$

inside of $\mathbf{T}$ and is zero outside. Then $\operatorname{div} B(x) = 0$. Of course such a field is not smooth, but we neglect this circumstance. According to formula (3.14) in this case we have for large $|x|$

$$\left.\begin{array}{rcl} A_1(x) & = & -x_1 x_3 |x|^{-3} g(z), \\ A_2(x) & = & -x_2 x_3 |x|^{-3} g(z), \\ A_3(x) & = & (x_1^2 + x_2^2)|x|^{-3} g(z), \end{array}\right\} \tag{5.6}$$

where the function $g \in C_0^\infty(\mathbb{R})$, depends on parameters of the solenoid and $z = z(x) = x_3(x_1^2 + x_2^2)^{-1/2}$. Indeed, plugging (5.5) into (3.14) and taking into account the rotational symmetry, we find that for sufficiently large $|x|$

$$A_1(x) = \alpha x_1 x_3 |x|^{-3} \int_{c_-(z)}^{c_+(z)} s^2 ds, \quad c_+(z) > c_-(z),$$

where $c_\pm(z)$ are the points where the half-line $s(1, 0, z)$, $s \in \mathbb{R}_+$, intersects the torus. Components $A_2(x)$ and $A_3(x)$ can be found quite similarly. Thus, we can set

$$g(z) = -3^{-1}\alpha(c_+^3(z) - c_-^3(z))$$

which determines the function g in (5.6). Clearly, $\pm g \geq 0$ if $\mp\alpha \geq 0$. Let $\operatorname{supp} g = [-z_0, z_0]$. Of course, for any function g in (5.6), we have that $< A(x), x >= 0$ and $\operatorname{curl} A(x) = 0$ for sufficiently large $|x|$. Note also that $A(x) = 0$ if $|z(x)| \geq |z_0|$. In case (5.6) a function $U(x)$ satisfying (5.4) can be constructed by the explicit formula

$$U(x) = G(x_3(x_1^2 + x_2^2)^{-1/2}),$$

where

$$G'(z) = g(z)(z^2 + 1)^{-3/2}.$$

Clearly, $G(z)$ increases (decreases) if $\alpha < 0$ ($\alpha > 0$) and it is a constant if $|z| \geq |z_0|$. We set $G(z) = 0$ for $z \leq -z_0$. Then $G(z) = G_0$ for $z \geq z_0$ where $-G_0$ is the magnetic flux over a section of our toroidal solenoid. Indeed, by the Stokes theorem it equals the circulation of the magnetic potential $A(x)$, for example, over the closed contour formed by the x_3-axis and the line $x_1 + \xi t$, $t \in \mathbb{R}$, where ξ is directed along the x_3-axis. We remark that x_1 is sufficiently large here and the integrals over pieces of lines connecting these infinite parallel lines tend to zero because $A(x) = O(|x|^{-1})$. As was already noted, the integral over the x_3-axis is zero. By (4.14), the integral over the line $x_1 + \xi t$ equals $U(\xi) - U(-\xi) = G_0$. Let us formulate the results obtained.

Proposition 5.5 *Let a magnetic potential be given (for large $|x|$) by equations (5.6). Then, up to a compact term (whose kernel is bounded by $C|\omega - \omega'|^{-d+\mu}$ for any $\mu < 2$), the SM is the operator of multiplication by the function*

$$\exp\Big(iG(\omega_3(1-\omega_3^2)^{-1/2}) - iG(-\omega_3(1-\omega_3^2)^{-1/2})\Big).$$

If the magnetic flux $-G_0$ over a section of our toroidal solenoid satisfies $|G_0| < \pi$, then the essential spectrum of the SM coincides with the arc $[\exp(-i|G_0|), \exp(i|G_0|)]$. If $|G_0| \geq \pi$, then relation (1.4) holds. The points $\exp(\pm iG_0)$ are always eigenvalues of infinite multiplicity.

5.3. Let us pass to the singular case. Here we consider two examples of magnetic potentials $A(x)$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, homogeneous (as always, for large $|x|$) of degree -1 and satisfying transversal gauge condition (1.5). However, the corresponding magnetic fields $B(x)$ will decay only as $|x|^{-2}$ at infinity, which will change completely the conclusion of Theorem 5.2.

We define the first of these potentials by the equation

$$A(x) = |x|^{-3}(\alpha_1 x_2 x_3, \alpha_2 x_3 x_1, \alpha_3 x_1 x_2), \quad |x| \geq 1, \tag{5.7}$$

where α_j are constants and

$$\alpha_1 + \alpha_2 + \alpha_3 = 0. \tag{5.8}$$

Let us calculate function (4.1). According to (5.7)

$$< A(x + t\xi), \xi >= (x^2 + t^2\xi^2)^{-3/2}$$
$$\times\Big(\alpha_1\xi_1(x_2 + t\xi_2)(x_3 + t\xi_3) + \alpha_2\xi_2(x_3 + t\xi_3)(x_1 + t\xi_1) + \alpha_3\xi_3(x_1 + t\xi_1)(x_2 + t\xi_2)\Big).$$

This function is a polynomial of the second degree in t. The sum of terms containing t^2 is zero due to condition (5.8). The integral over $\mathbb{R}$ of terms containing t is zero since the corresponding function is odd. So integrating $(x^2 + t^2\xi^2)^{-3/2}$, we find that function (4.1) equals

$$\Theta_\infty(x, \xi) = 2|\xi|^{-1}|x|^{-2}(\alpha_1\xi_1 x_2 x_3 + \alpha_2\xi_2 x_3 x_1 + \alpha_3\xi_3 x_1 x_2). \tag{5.9}$$

Let us find the essential spectrum of the corresponding SM. According to Theorem 4.4 we have to find the maximum of function (5.9) restricted to the set $T_1^*\mathbb{S}^2$ where $|x|^2 = |\xi|^2 = 1$ and condition

$$x_1\xi_1 + x_2\xi_2 + x_3\xi_3 = 0 \tag{5.10}$$

is satisfied. We fix x and consider first (5.9) as a function of ξ only. The method of Lagrange multipliers gives us the equations

$$\left.\begin{aligned}\alpha_1 x_2 x_3 &= \mu x_1 + \nu\xi_1,\\ \alpha_2 x_3 x_1 &= \mu x_2 + \nu\xi_2,\\ \alpha_3 x_1 x_2 &= \mu x_3 + \nu\xi_3.\end{aligned}\right\} \tag{5.11}$$

Let us multiply these equations by x_1, x_2, x_3, respectively, and take their sum. It follows from conditions (5.8) and (5.10) that $\mu = 0$ and hence

$$\nu^2(x) = (\alpha_1 x_2 x_3)^2 + (\alpha_2 x_3 x_1)^2 + (\alpha_3 x_1 x_2)^2. \tag{5.12}$$

Multiplying (5.11) by ξ_1, ξ_2, ξ_3, respectively, we find that

$$\Theta_\infty(x,\xi) = 2\nu(x), \tag{5.13}$$

which is the maximum (if $\nu(x) > 0$) of the function $\Theta_\infty(x,\xi)$ for fixed x. Next we have to find the maximum of function (5.12) on the sphere $|x| = 1$. It is an easy exercise to check that the function $\nu^2(x)$ attains the maximum $\alpha_1^2/4$ at the points $x_1 = 0, x_2^2 = x_3^2 = 2^{-1}$ and, similarly, for other permutations of indices. It follows from (5.13) that

$$\max \Theta_\infty(x,\xi) = \max\{|\alpha_1|, |\alpha_2|, |\alpha_3|\} =: \alpha_0. \tag{5.14}$$

Using now Theorem 4.4, we find the spectrum of the SM.

Proposition 5.6 *Let a magnetic potential be given (for large $|x|$) by equation (5.7) where the constants α_j satisfy condition (5.8), and let α_0 be defined by (5.14). Then the essential spectrum of the SM coincides with the arc $[\exp(-i\alpha_0), \exp(i\alpha_0)]$ if $\alpha_0 < \pi$ and relation (1.4) holds if $\alpha_0 \geq \pi$.*

The function (5.9) depends on variables x_1, x_2, x_3, so that it can be expected that the second term in representation (4.11) of the corresponding SM is non-trivial. Set $\omega = (\omega_1, \omega_2, \omega_3)$ and

$$n = (\omega_1^2+\omega_2^2)^{-1/2}(-\omega_2, \omega_1, 0), \quad m = \omega \times n = (\omega_1^2+\omega_2^2)^{-1/2}(-\omega_1\omega_3, -\omega_2\omega_3, \omega_1^2+\omega_2^2).$$

An arbitrary point $x \in \mathbb{S}_\omega = \mathbb{S}^2 \cap \Pi_\omega$ can be written as $x = x(\theta) = n\cos\theta + m\sin\theta$, $\theta \in [0, 2\pi)$, so that

$$\begin{gathered} x_1 = -(\omega_1^2+\omega_2^2)^{-1/2}(\omega_2\cos\theta + \omega_1\omega_3\sin\theta),\\ x_2 = (\omega_1^2+\omega_2^2)^{-1/2}(\omega_1\cos\theta - \omega_2\omega_3\sin\theta), \quad x_3 = (\omega_1^2+\omega_2^2)^{1/2}\sin\theta. \end{gathered} \tag{5.15}$$

Plugging these expressions in (5.9), we find that

$$\begin{gathered} \Theta_\infty(x,\omega) = (\omega_1^2+\omega_2^2)^{-1}\Big(-2\alpha_3\omega_1\omega_2\omega_3\cos(2\theta)\\ +(\alpha_1(\omega_1^2-\omega_2^2\omega_3^2) - \alpha_2(\omega_2^2-\omega_1^2\omega_3^2))\sin(2\theta)\Big) = \mathcal{A}(\omega)\sin(2\theta+\theta_0(\omega)) \end{gathered} \tag{5.16}$$

where

$$\mathcal{A}(\omega) = (\omega_1^2+\omega_2^2)^{-1}\Big(4\alpha_3^2\omega_1^2\omega_2^2\omega_3^2 + (\alpha_1(\omega_1^2-\omega_2^2\omega_3^2) - \alpha_2(\omega_2^2-\omega_1^2\omega_3^2))^2\Big)^{1/2}$$

and

$$\tan\theta_0(\omega) = -2\alpha_3\omega_1\omega_2\omega_3(\alpha_1(\omega_1^2-\omega_2^2\omega_3^2)-\alpha_2(\omega_2^2-\omega_1^2\omega_3^2))^{-1}.$$

Now we can calculate function (4.9). Using expression (5.16), we see that

$$p^{(0)}(\omega) = (2\pi)^{-1}\int_0^{2\pi}\Big(\cos(\mathcal{A}(\omega)\sin(2\theta+\theta_0(\omega)) + i\sin(\mathcal{A}(\omega)\sin(2\theta+\theta_0(\omega))\Big)d\theta.$$

The integral of the second term is zero, and the first one can be simplified by the change of variables $2\theta+\theta_0(\omega)\mapsto\theta$. Therefore Theorem 4.5 implies

Proposition 5.7 *Under the assumptions of Proposition* 5.6 *the singular part of the scattering amplitude is given by expression* (4.11) *where*

$$p^{(0)}(\omega) = (2\pi)^{-1}\int_0^{2\pi}\cos(\mathcal{A}(\omega)\sin\theta)d\theta,$$

$$q(\omega,\tau) = -(2\pi)^{-2}\int_0^{2\pi}\Big(\exp(i\mathcal{A}(\omega)\sin(2\theta+\theta_0(\omega))) - p^{(0)}(\omega)\Big)(<x(\theta),\tau> -i0)^{-2}d\theta \tag{5.17}$$

and the vector $x(\theta)$ is defined by formulas (5.15). *Estimate* (4.12) *holds for any $\nu<1$.*

As another example of a magnetic potential $A(x)$ homogeneous (as always, for large $|x|$) of degree -1 and satisfying transversal gauge condition (1.5), we choose a modification of the Aharonov-Bohm potential

$$A(x) = \alpha|x|^{-2}(-x_2,x_1,0),\quad |x|\geq 1,\quad x=(x_1,x_2,x_3)\in\mathbb{R}^3. \tag{5.18}$$

Once again the corresponding magnetic fields $B(x)$ decays only as $|x|^{-2}$ at infinity, so that the conclusion of Theorem 5.2 is violated.

Calculating function (4.1), we find that

$$\Theta_\infty(x,\xi) = \pi\alpha|\xi|^{-1}|x|^{-1}(x_1\xi_2-\xi_1x_2). \tag{5.19}$$

Let us first find the essential spectrum of the corresponding SM. According to Theorem 4.4, we have to find the image of function (5.19) restricted to the set $T_1^*\mathbb{S}^2$ where $|x|^2=|\xi|^2=1$ and condition (5.10) is satisfied. Easy calculations yield

Proposition 5.8 *Let a magnetic potential be given by equation* (5.18). *Then the essential spectrum of the SM coincides with the arc* $[\exp(-i\pi|\alpha|),\exp(i\pi|\alpha|)]$ *if* $|\alpha|<1$ *and relation* (1.4) *holds if* $|\alpha|\geq 1$.

To find the singularity of the scattering amplitude, we repeat, with significant simplifications, the construction of Proposition 5.7. In particular, it follows from (5.15) and (5.19) that

$$\Theta_\infty(x,\omega) = -\pi\alpha(1-\omega_3^2)^{1/2}\cos\theta.$$

Therefore Theorem 4.5 implies

Proposition 5.9 *Under the assumptions of Proposition* 5.8 *the singular part of the scattering amplitude is given by expression* (4.11) *where*

$$p^{(0)}(\omega) = (2\pi)^{-1} \int_0^{2\pi} \cos(\pi(1-\omega_3^2)^{1/2}\cos\theta)d\theta,$$

$$q(\omega,\tau) = -(4\pi)^{-1} \int_0^{2\pi} \Big(\exp(-\pi i\alpha(1-\omega_3^2)^{1/2}\cos\theta) - p^{(0)}(\omega)\Big)(< x(\theta),\tau > -i0)^{-2}d\theta \tag{5.20}$$

and the vector $x(\theta)$ *is defined by formulas* (5.15). *Estimate* (4.12) *holds for any* $\nu < 1$.

We emphasize that both terms (5.17) and (5.20) are not zero, so that the singular part of the SM in the examples of this subsection does not reduce to an operator of multiplication.

6. ELECTRIC POTENTIALS

The SM for electric potentials $V(x)$ which are odd functions and behave at infinity as homogeneous functions of order -1 possess the same properties as for magnetic potentials considered above. Suppose that a potential $V(x)$ satisfies condition (1.1) with $\rho = 1$ and is such that

$$V(x) = V_\infty(x) + V_r(x), \quad |x| \geq 1,$$

where

$$V_\infty(x) = v(\hat{x})|x|^{-1}, \quad v(-\hat{x}) = -v(\hat{x}),$$

and $V_r(x)$ obeys estimates (1.1) with some $\tilde{\rho} > 1$. We assume additionally that a magnetic potential $A(x) = 0$. Now the SM is defined in terms of wave operators (3.1) where $J_\pm$ are PDO (3.2) and $\Phi_\pm$ is integral (3.5) with $A = 0$.

Using again Theorem 3.3, we see that the singular part of the SM is determined by the operator $S_0(\lambda)$. Since $V(x) = -V(-x)$, function (1.3) now equals

$$\Theta(x,\xi) = -2^{-1} \int_0^\infty \Big(V(x+t\xi) + V(x-t\xi)\Big)dt.$$

As always, we assume $< x,\xi >= 0$ and replace here V by V_∞, since the arising error is $O(|x|^{-\epsilon})$, $\epsilon = \tilde{\rho} - 1 > 0$. This gives the expression

$$\Theta(x,\xi) = |\xi|^{-1}\Theta_\infty(x,\xi) + O(|x|^{-\epsilon}),$$

where

$$\Theta_\infty(x,\xi) = -2^{-1} \int_0^\infty \Big(v\Big(\frac{\hat{x}+t\hat{\xi}}{(t^2+1)^{1/2}}\Big) + v\Big(\frac{\hat{x}-t\hat{\xi}}{(t^2+1)^{1/2}}\Big)\Big)(t^2+1)^{-1/2}dt.$$

Making here the change of variables $t = \tan\theta$, we can rewrite the last expression as

$$\Theta_\infty(x,\xi) = -2^{-1} \int_0^{\pi/2} \Big(v(\cos\theta\hat{x} + \sin\theta\hat{\xi}) + v(\cos\theta\hat{x} - \sin\theta\hat{\xi})\Big)(\cos\theta)^{-1}d\theta. \tag{6.1}$$

The last integral converges at the upper limit $\theta = \pi/2$ by virtue of the condition $v(-\hat{x}) = -v(\hat{x})$. Note also that

$$\Theta_\infty(-x, \xi) = -\Theta_\infty(x, \xi).$$

Using Propositions 2.3 and 2.5, we can now formulate analogues of Theorems 4.4 and 4.5.

Theorem 6.1 *Under the assumptions above, let the function Θ_∞ and the numbers γ, γ_+, γ_- be defined by equations* (6.1) *and* (4.6), (4.7), *respectively. Then the assertion of Theorem* 4.4 *remains true if the numbers γ, γ_+, γ_- are replaced by the numbers $\lambda^{-1/2}\gamma$, $\lambda^{-1/2}\gamma_+$, $\lambda^{-1/2}\gamma_-$, respectively. The assertion of Theorem* 4.5 *also remains true if the function $\Theta_\infty(\psi, \omega)$ in definitions* (4.9) *and* (4.10) *is replaced by function $\lambda^{-1/2}\Theta_\infty(\psi, \omega)$ defined by* (6.1).

As a concrete example, let us consider the function

$$v(\hat{x}) = 2 < p, \hat{x} >, \quad p \in \mathbb{R}^d, \quad d \geq 2.$$

Calculating integral (6.1), we find that

$$\Theta_\infty(x, \xi) = -|\xi|^{-1}\pi < p, \hat{x} >, \quad |x| \geq 1/2.$$

Thus, the essential spectrum of the SM $S(\lambda)$ coincides with the arc $[\exp(-\pi i|p|\lambda^{-1/2}), \exp(\pi i|p|\lambda^{-1/2})]$ if $|p| < \lambda^{1/2}$, and it covers the whole unit circle if $|p| \geq \lambda^{1/2}$.

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