Algebraic methods for determining Hamiltonian Hopf bifurcations in three–degree–of–freedom systems

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Abstract

When considering bifurcations, the type of bifurcation is usually classified by comparing to standard situations or normal forms. It is shown how Hamiltonian Hopf bifurcations can be determined in three–degree–of–freedom systems, as is done in this paper for the $3D$ Hénon–Heiles family. After a careful formulation of the local once reduced system in terms of properly chosen invariants the system can be compared to the standard form to determine the presence of non–degenerate Hamiltonian Hopf bifurcations.

1 Introduction

The Hamiltonian Hopf bifurcation is present in many mechanical systems. Examples are the Lagrange top [11, 24, 9], the Kirchhoff top [2], the spinning orthogonal double pendulum [3] or the double sperical pendulum [12, 20, 21]. Examples related to fluid mechanics can be found in [4, 17, 13]. The latter two references actually deal with reversible systems, however it is shown in [5, 26] that this problem is equivalent to the Hamiltonian Hopf bifurcation. Examples also appear in buckling problems [8, 7]. Examples from celestial mechanics are the restricted problem of three bodies [23], where this problem finds its origin, and, more recently, the $3D$ Henon–Heiles family [14, 16] related to the movement of galaxies.

The mechanism of the Hamiltonian Hopf bifurcation is best described on $\mathbb{R}^4$ with standard symplectic form. Let a family of Hamiltonian systems be given, depending on a parameter λ. Suppose that the origin is a stationairy point and that for some value λ_0 the linearized system consists of two harmonic oscillators in 1:-1 resonance. Then for λ_0 the system linearized at the origin has a pair of purely imaginary eigenvalues $\pm \mathrm{i}\nu_0$ and generically the eigenvalues will behave as follows when the parameter λ passes through λ_0 : four purely imaginary eigenvalues, moving in pairs $\pm \mathrm{i}\nu_1$ and $\pm \mathrm{i}\nu_2$, will meet for $\lambda = \lambda_0$ at $\pm \mathrm{i}\nu_0$ and move off from the imaginary axis into the complex plane to form a complex quartet $\pm\alpha \pm \mathrm{i}\beta$. This bifurcation in two–degree–of freedom systems is known as the Hamiltonian Hopf bifurcation [23], which name was chosen because it is the Hamiltonian equivalent of the non–conservative (linear) Hopf bifurcation where a pair of eigenvalues crosses the imaginary axis. The eigenvalue movement in the Hamiltonian case is sometimes also called a Krein collision, cf. [18, 20]. Like in the non–conservative equivalent the equilibrium at the origin changes from stable to unstable. Whether this bifurcation is non–degenerate, and hence of co–dimension one, has to be decided by considering the higher order terms of the Hamiltonian. To this end one normalizes with respect to the linearized part, which allows to transform away the third order terms of the Hamiltonian. The non–degeneracy condition then requires that a certain coefficient of the (normalized) fourth order terms be nonzero. See [16] for a geometric criterion for this non–degeneracy condition and [25] for further transversality conditions ensuring that the co–dimension is 2 in the degenerate case where that coefficient does vanish.

The Lagrange and Kirchhoff tops, the double spherical pendulum and the $3D$ Henon–Heiles family are all systems with three or more degrees of freedom. In the case of the spherical pendulum only the behaviour of the eigenvalues has been considered, while for the Lagrange top rigorous proofs of the presence of the Hamiltonian Hopf bifurcation could be given in [11, 24, 9]. Here it is a periodic orbit that undergoes the bifurcation and the rôle of the unfolding parameter is played by the value of the third component of the angular momentum. In the $3D$ Hénon–Heiles family the unfolding parameter appears as an external parameter λ describing the relative strength of the two cubic terms in the potential. Based on phase space geometry the presence of three non–degenerate Hamiltonian Hopf bifurcations was conjectured in [14]. This led to the geometric criterion in [16] which allowed to prove that conjecture. A fourth (linear) Hamiltonian Hopf bifurcation at $\lambda = +1$ is degenerate and of infinite co–dimension.

To decide whether one has a non–degenerate (periodic) Hamiltonian Hopf bifurcation in a three–degree–of–freedom system is in general a hard problem because one has to find the higher order terms in the normal form of the Hamiltonian. In general this will be a normalized and once reduced system, *i.e.* a normal form in terms of invariants with respect to some non–standard symplectic form evaluated at some point of the first reduced phase space. Due to the special structure of the problem this could be done in the case of the Lagrange top. In this way, the proof in [11] and [24] is based on singularity theory, transforming the S^1–equivariant singularity of the energy–momentum mapping to its standard form. On the other hand, the proof in [9] consists in flattening the second order terms of the non–standard symplectic form and then computing the normalized fourth order terms of the reduced Hamiltonian.

The three–degree–of–freedom problems mentioned above all have the same structure. The problems are either integrable to start with or integrable after an initial normalization and truncation. The two extra integrals besides the Hamiltonian then generate two S^1–actions. Where the 2–torus action defined by these S^1–actions is not free this gives rise to singular reduced phase spaces. Our interest is in reduced phase spaces with conical singular points. For the $3D$ Hénon–Heiles family it is shown that degenerate and non–degenerate Hamiltonian Hopf bifurcations are present. The method consists in finding a representation of $\mathfrak{sl}(2,\mathbb{R})$ generated by the S^1–symmetry within the local Poisson structure on the first reduced phase space. A careful application of singularity theoretic results then brings the problem into standard form.

This paper is organized as follows. The next section contains the standard form of the Hamiltonian Hopf bifurcation and in Section 3 the reduction of the $3D$ Hénon–Heiles family from three to two degrees of freedom is recalled. In Section 4 co–ordinates are chosen adapted to the local Poisson structure at the bifurcating (relative) equilibrium, which is used in Section 5 to prove non–degeneracy of two Hamiltonian Hopf bifurcations. For the third non–degenerate Hamiltonian Hopf bifurcation in the $3D$ Hénon–Heiles family a higher order analysis is necessary and given in Section 6.

2 The standard form of the Hamiltonian Hopf bifurcation

The Hamiltonian Hopf bifurcation occurs in Hamiltonian systems where stationary points change stability type. Generically the whole bifurcation scenario is unfolded under variation of a single parameter. In its simplest form it is described by a parameter dependent two–degree–of–freedom system with Hamiltonian function $H(q_1, q_2, p_1, p_2; \nu)$ for which the linearized system at $\nu = 0$ has a semi–simple part in 1:-1 resonance and a non–vanishing nilpotent part. The eigenvalues of the linearized system change from four purely imaginary eigenvalues to four eigenvalues in the complex plane with nonzero real part. At the bifurcation value of the parameter one has two coinciding pairs of purely imaginary eigenvalues. As a consequence the normal form of the quadratic Hamiltonian at the stationary point has the form

$$H_2(q, p; \nu) \;=\; (q_1 p_2 - q_2 p_1) \;+\; \tfrac{1}{2}\,(q_1^2 + q_2^2) \;+\; \frac{\nu}{2}(p_1^2 + p_2^2) \;\;. \qquad (1)$$

We abbreviate $S = q_1p_2 - q_2p_1$, $N = \frac{1}{2}(q_1^2 + q_2^2)$, $M = \frac{1}{2}(p_1^2 + p_2^2)$ and introduce $P = \{N, M\} = q_1p_1 + q_2p_2$. Here $\{\ ,\ \}$ is the Poison bracket induced by the standard symplectic form $\mathrm{d}q_1 \wedge \mathrm{d}p_1 + \mathrm{d}q_2 \wedge \mathrm{d}p_2$. With this Poisson bracket one has

$$\{S, N\} = \{S, M\} = \{S, P\} = 0 \tag{2}$$

and the quantities N, M and P span a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, *i.e.* they satisfy the bracket relations of Table 1.

As shown in [22, 23] the normalized higher order terms of the Hamiltonian at the stationary point are polynomials in S and M. Thus the normal form is

$$\bar{H}(q, p; \nu) = S + N + \nu M + aM^2 + bSM + cS^2 + h.o.t. \tag{3}$$

This normal form is obtained through normalization with respect to both the semi–simple part S and the nilpotent part N of the linearization (*i.e.* the quadratic part H_2 at $\nu = 0$). The former introduces an S^1–symmetry as the normalized system becomes invariant under the periodic flow generated by

$$X_S : \begin{cases} \dot{q}_1 = -q_2 \\ \dot{q}_2 = q_1 \\ \dot{p}_1 = -p_2 \\ \dot{p}_2 = p_1 \end{cases} . \tag{4}$$

Recall from [9] that S, N, M and P form a *Hilbert basis* of the ring of invariant functions, *i.e.* every function $f(q, p)$ that is invariant under the flow of (4) can be expressed as a function $g(S, N, M, P)$:

$$f(q_1, q_2, p_1, p_2) \equiv g\left(q_1p_2 - q_2p_1, \tfrac{1}{2}(q_1^2 + q_2^2), \tfrac{1}{2}(p_1^2 + p_2^2), q_1p_1 + q_2p_2\right) .$$

In this way the S^1–symmetry is reduced and S, N, M, P can be used as co–ordinates on the reduced phase space, cf. [23, 1]. As expressed by (2) the generator S of the symmetry becomes a *Casimir* in the reduced phase space. The dimension of the quotient $\mathbb{R}^4/S^1$ is three and correspondingly the four generators are constrained by the syzygy

$$4NM = S^2 + P^2 . \tag{5}$$

This allows to identify the reduced phase space with $\mathfrak{sl}(2, \mathbb{R})$ — up to the sign of the Casimir S. The reduction from two to one degree of freedom amounts to fixing the value of S. While this value is of importance in the syzygy (5), it only constitutes an irrelevant constant in a reduced Hamiltonian function. We therefore indicate the reduced Hamiltonian obtained from *e.g.* $\hat{H}(S, N, M, P; \nu)$ by merely neglecting S and writing $\hat{H}(N, M, P; \nu)$.

If one is only interested in the bifurcation of periodic solutions, the Hamiltonian (3) can be further normalized. The first step is a reduction based on Liapunov–Schmidt reduction

Table 1: The bracket relations defining the structure matrix for $\mathfrak{sl}(2,\mathbb{R})$.

$\{\ ,\ \}$	N	M	P
N	0	P	$2N$
M	$-P$	0	$-2M$
P	$-2N$	$2M$	0

which relates the normalized non–symmetric Hamiltonian $\bar{H}$ to a fully symmetric function $\tilde{H}$ which equals the normal form up to arbitrary degree of normalization (see [23, 28]). The periodic solutions of the Hamiltonian system corresponding to $\tilde{H}$ are described by the singularity of the energy–momentum mapping $\mathcal{EM} = (\tilde{H}, S)$, which is equivariant with respect to the S^1–action given by the flow of (4). The second step is therefore to obtain a singularity theoretic normal form for this energy–momentum mapping. Such a singularity theoretic normal form is a mapping equivalent to the energy–momentum mapping under an appropriate element of the group of equivariant origin preserving diffeomorphisms, whence the singularities of both mappings are diffeomorphic. In this particular case one considers the group action $(\varphi,\psi)\cdot(\tilde{H},S) = \psi\circ(\tilde{H},S)\circ\varphi$, with φ an S^1–equivariant origin preserving diffeomorphism on $\mathbb{R}^4$ and ψ an origin preserving diffeomorphism on $\mathbb{R}^2$. The actual computations (see [23]) are performed on the tangent space showing in which basic directions one may transform the mapping and at the same time determining which are the missing directions, *i.e.* the co–dimension of the bifurcation. One only proves equivalence of mappings. The actual transformations are not explicitly computed, but see [19, 6]. The following theorem will be needed later on.

Theorem 2.1 *[Van der Meer [23]] Let ϱ denote the S^1–action given by the flow of* (4). *Consider a ϱ–invariant mapping*

$$\begin{array}{rccc} (\tilde{H},S): & \mathbb{R}^4 & \longrightarrow & \mathbb{R}^2 \\ & (S,N,M,P) & \mapsto & (\tilde{H}(S,N,M,P),S) \end{array}$$

with

$$\tilde{H}(S,N,M,P) \;=\; S + N + aM^2 + bSM + cS^2 + h.o.t.$$

where $a \neq 0$. Then there exists a ϱ–equivariant origin preserving diffeomorphism φ on $\mathbb{R}^4$ and a origin preserving diffeomorphism ψ on $\mathbb{R}^2$ such that

$$\psi\circ(\tilde{H},S)\circ\varphi \;=\; (\hat{G},S)$$

with

$$\hat{G}(S,N,M,P) \;=\; N + aM^2 \ . \tag{6}$$

The universal unfolding of $(\hat{G},S)$ is $(\hat{H},S)$ with

$$\hat{H}(S,N,M,P;\nu) \;=\; N + \nu M + aM^2 \ . \tag{7}$$

Thus imposing the genericity condition $a \neq 0$ on (3) produces a bifurcation which is equivalent to the standard form $\hat{H}$ for the non–degenerate Hamiltonian Hopf bifurcation. Certain degenerate cases (a=0) of low co–dimension are considered in [25]. Note that the semi–simple part has disappeared. From the singularity theory it is clear that any part in the ϱ–invariant function $\tilde{H}$ which only depends on S can be transformed away by a diffeomorphism on the target space of the energy–momentum mapping. Thus the reduction to (7) is also possible if the semi–simple part of the linearized bifurcating system vanishes — under the condition that the system is already S^1–symmetric to start with. Here two pairs of purely imaginary eigenvalues meet at the origin and, due to the symmetry (and the nilpotent linear part), are forced to split off from the imaginary axis.

This is precisely what happens in the $3D$ Hénon–Heiles family. The necessary S^1–symmetry is inherited from the axial symmetry (9) of the force field perturbing the 1:1:1 resonant oscillator. In the following we reduce the normalized $3D$ Hénon–Heiles family to two degrees of freedom. To study the occurring Hamiltonian Hopf bifurcations we transform the reduced phase space locally to the above standard form. This should produce a local representation of $\mathfrak{sl}(2,\mathbb{R})$ and thus allow us to put the Hamiltonian into the form (7).

3 The $3D$ Hénon–Heiles family

Following [14, 16] we consider the model of a perturbed isotropic harmonic oscillator with an additional axial symmetry. This is a family of Hamiltonian systems defined by the Hamiltonian function

$$H^\lambda(x,y) \;=\; \frac{1}{2}(y \mid y) + \frac{1}{2}(x \mid x) + \varepsilon\left(\frac{x_3^3}{3} + \lambda(x_1^2+x_2^2)x_3\right) \quad , \tag{8}$$

where $(..|..)$ denotes the standard inner product and λ is an external parameter the perturbing force field depends upon. The small parameter ε is introduced by a scaling $x \mapsto \varepsilon x\,,\ y \mapsto \varepsilon y\,,\ H \mapsto \varepsilon^2 H$, whence (8) describes the dynamics close to the origin. Similarly, the frequency of the resonant oscillator has been put to 1 through rescaling time.

As the $2D$ Hénon–Heiles family is embedded in (8) by taking *e.g.* $x_2 = y_2 = 0$, we will refer to (8) as the $3D$ Hénon–Heiles family. Note that the potential is chosen to be the most general cubic expression such that the system is axially symmetric. More precisely, the system is invariant under the axial S^1–action

$$\begin{array}{rccc} \rho: & S^1 \times T^*\mathbb{R}^3 & \longrightarrow & T^*\mathbb{R}^3 \\ & (\nu,(x,y)) & \mapsto & (\exp_\nu x, \exp_\nu y) \end{array} \tag{9}$$

where

$$\exp_\nu \;=\; \exp\begin{pmatrix} 0 & -\nu & 0 \\ \nu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;=\; \begin{pmatrix} \cos\nu & -\sin\nu & 0 \\ \sin\nu & \cos\nu & 0 \\ 0 & 0 & 1 \end{pmatrix} \ .$$

The action corresponds to spatial rotation about the third axis and the corresponding integral is the third component

$$S \;=\; x_1 y_2 \;-\; x_2 y_1$$

of the angular momentum. The family (8) has been normalized twice in [14], yielding a fourth order polynomial in the x_i and y_i. This normal form Hamiltonian commutes with S as well as with the quadratic part

$$H_0 \;=\; \tfrac{1}{2}(x_1^2 + y_1^2) \;+\; \tfrac{1}{2}(x_2^2 + y_2^2) \;+\; \tfrac{1}{2}(x_3^2 + y_3^2)$$

of H. Therefore, it can be written in terms of the invariants corresponding to the action of H_0. These invariants can be easily derived using complex conjugate co–ordinates, see [29, 15]. The H_0–invariants are

$$\begin{array}{lll} \pi_1 = x_1^2 + y_1^2 \;, & \pi_2 = x_2^2 + y_2^2 \;, & \pi_3 = x_3^2 + y_3^2 \;, \\ \pi_4 = x_1 x_2 + y_1 y_2 \;, & \pi_5 = x_1 x_3 + y_1 y_3 \;, & \pi_6 = x_2 x_3 + y_2 y_3 \;, \\ \pi_7 = x_1 y_2 - x_2 y_1 \;, & \pi_8 = x_1 y_3 - x_3 y_1 \;, & \pi_9 = x_2 y_3 - x_3 y_2 \;\; . \end{array} \tag{10}$$

The normal form of H with respect to H_0 can now be expressed in these invariants. Note that the normalization can be performed in such a way that the resulting normal form is still invariant under the S^1–action generated by $S = \pi_7$. The first nontrivial normal form is

$$\begin{aligned} \bar{H}^\lambda \;=\; & \frac{1}{2}(\pi_1 + \pi_2 + \pi_3) \;+\; \frac{\varepsilon^2}{48}\Big(-5\pi_3^2 \;+\; 2\lambda(\pi_5^2 + \pi_6^2 - (\pi_8^2 + \pi_9^2) - 6(\pi_1 + \pi_2)\pi_3) \\ & -\; \lambda^2(12(\pi_5^2 + \pi_6^2 - (\pi_8^2 + \pi_9^2)) + 5(\pi_1 + \pi_2)^2 + 8(\pi_1 + \pi_2)\pi_3 + 4\pi_7^2)\Big) \;\; , \end{aligned} \tag{11}$$

see [14]. By writing the normal form as a function of $\pi_1, \ldots, \pi_9$ we have effectively divided out the S^1–action generated by H_0. To complete the reduction to two degrees of freedom we put $H_0 = \eta$. This is a regular reduction, whence the reduced phase space is a (smooth) symplectic manifold. As shown in [27] this space is isomorphic to $\mathbb{CP}^2$. The invariants (10) provide an embedding of the reduced phase space as a submanifold of $\mathbb{R}^9$, with co–ordinates $\pi_1, \ldots, \pi_9$, determined by the relations

$$\begin{array}{lll} \pi_3\pi_1 = \pi_5^2 + \pi_8^2 \;, & \pi_3\pi_7 = \pi_6\pi_8 - \pi_5\pi_9 \;, & \pi_2\pi_8 = \pi_4\pi_9 + \pi_6\pi_7 \;, \\ \pi_3\pi_2 = \pi_6^2 + \pi_9^2 \;, & \pi_2\pi_1 = \pi_4^2 + \pi_7^2 \;, & \pi_1\pi_6 = \pi_4\pi_5 + \pi_7\pi_8 \;, \\ \pi_3\pi_4 = \pi_5\pi_6 + \pi_8\pi_9 \;, & \pi_2\pi_5 = \pi_4\pi_6 - \pi_7\pi_9 \;, & \pi_1\pi_9 = \pi_4\pi_8 - \pi_5\pi_7 \;, \\ \pi_1 + \pi_2 + \pi_3 = 2\eta \;\; . & & \end{array} \tag{12}$$

At every point of $\mathbb{CP}^2 \subseteq \mathbb{R}^9$ only five of the ten relations (12) are linearly independent and can be used to compute the tangent space, see Section 4 below.

In order to obtain the Poisson structure or symplectic structure on the reduced phase space the Poisson brackets for the invariants π_i have to be computed. This is done in Table 2

Table 2: The bracket relations $\{\pi_i, \pi_j\}$ defining the structure matrix Π.

$\{\ ,\ \}$	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9
π_1	0	0	0	$2\pi_7$	$2\pi_8$	0	$-2\pi_4$	$-2\pi_5$	0
π_2	0	0	0	$-2\pi_7$	0	$2\pi_9$	$2\pi_4$	0	$-2\pi_6$
π_3	0	0	0	0	$-2\pi_8$	$-2\pi_9$	0	$2\pi_5$	$2\pi_6$
π_4	$-2\pi_7$	$2\pi_7$	0	0	π_9	π_8	$\pi_1 - \pi_2$	$-\pi_6$	$-\pi_5$
π_5	$-2\pi_8$	0	$2\pi_8$	$-\pi_9$	0	π_7	$-\pi_6$	$\pi_1 - \pi_3$	π_4
π_6	0	$-2\pi_9$	$2\pi_9$	$-\pi_8$	$-\pi_7$	0	π_5	π_4	$\pi_2 - \pi_3$
π_7	$2\pi_4$	$-2\pi_4$	0	$\pi_2 - \pi_1$	π_6	$-\pi_5$	0	π_9	$-\pi_8$
π_8	$2\pi_5$	0	$-2\pi_5$	π_6	$\pi_3 - \pi_1$	$-\pi_4$	$-\pi_9$	0	π_7
π_9	0	$2\pi_6$	$-2\pi_6$	π_5	$-\pi_4$	$\pi_3 - \pi_2$	π_8	$-\pi_7$	0

which gives us the structure matrix Π for the Poisson structure on $\mathbb{R}^9$. This Poisson structure has a natural restriction to the reduced phase space $\mathbb{CP}^2$. On this submanifold the structure matrix is non–degenerate and thus defines a symplectic structure.

A further reduction of the S^1–action generated by $S = \pi_7$ from $\mathbb{CP}^2$ to the twice reduced phase space $V_{\sigma,\eta}$ is performed by fixing the value σ of S and expressing everything in the invariants

$$\begin{aligned} \mathrm{Re}\,\tau_1 &= (\pi_5^2 + \pi_6^2) \;-\; (\pi_8^2 + \pi_9^2) \\ \mathrm{Im}\,\tau_1 &= -2(\pi_8\pi_5 + \pi_9\pi_6) \\ \tau_2 &= \pi_1 + \pi_2 \ . \end{aligned} \tag{13}$$

Following [10, 14] the twice reduced phase space becomes in this way

$$V_{\sigma,\eta} \;=\; \{\, \tau = (\tau_1, \tau_2) \in \mathbb{C} \times \mathbb{R} \mid R_{\sigma,\eta}(\tau) = 0\,,\ 2|\sigma| \le \tau_2 \le 2\eta \,\} \ ,$$

with

$$R_{\sigma,\eta}(\tau) \;=\; (\mathrm{Re}\,\tau_1)^2 + (\mathrm{Im}\,\tau_1)^2 \;-\; (2\eta - \tau_2)^2(\tau_2^2 - 4\sigma^2)$$

and has the Poisson bracket

$$\{\, f\,,\, g \,\} \;=\; (\, \nabla f \times \nabla g \mid \nabla R_{\sigma,\eta}\,) \ . \tag{14}$$

Furthermore the (twice) reduced Hamiltonian is obtained from (11). We drop the constant terms $\eta - \frac{5\varepsilon^2\eta^2}{12}$ and divide by $\frac{\varepsilon^2}{48}$, but retain the term $-\frac{\varepsilon^2\lambda^2\sigma^2}{12}$ for later use and get

$$\mathcal{H}^\lambda(\tau) \;=\; 2\lambda(1-6\lambda)\mathrm{Re}\,\tau_1 \;+\; 4\eta(5-6\lambda-4\lambda^2)\tau_2 \;-\; (5-12\lambda-3\lambda^2)\tau_2^2 \;-\; 4\lambda^2\sigma^2 \ . \tag{15}$$

In [10] it was shown that these reduced phase spaces have conical singularities at $\tau_2 = 2\eta$ and, for $\sigma = 0$, at $\tau_2 = 0$. Thus, by perturbing σ away from zero the latter 'cone' becomes a smooth surface. It was proved in [16] that the transversality conditions for a non–degenerate Hamiltonian Hopf bifurcation in (15) are fulfilled at $\lambda = \frac{1}{2}, \frac{5}{2}$. By using geometric criteria the problem of bringing the problem locally in the standard form for the Hamiltonian Hopf bifurcation was in effect circumvented.

4 Local Poisson structure of the first reduced phase space

From the analysis in [14] it is clear that the Hamiltonian Hopf bifurcations take place at $\tau_2 = \pi_1 + \pi_2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 = 0$. Thus we are considering a bifurcation from the normal mode determined by $x_1 = x_2 = y_1 = y_2 = 0$ and $x_3^2 + y_3^2 = 2\eta$. At the (first) reduced phase space $\mathbb{CP}^2$ this mode corresponds to the point $\pi^\eta = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7, \pi_8, \pi_9) = (0, 0, 2\eta, 0, 0, 0, 0, 0, 0)$. Consequently, in order to study the bifurcation a chart is to be constructed for the reduced phase space at this point. Note that there is a strong analogy with the approach followed in the analysis of the Lagrange top in [11].

The fact that $\Pi(\pi^\eta)$ is non–degenerate if one restricts to the variables $\pi_5, \pi_6, \pi_8, \pi_9$ indicates that these are the appropriate variables for describing our local problem near $\tau_2 = \pi_1 + \pi_2 = 0$ on the reduced space with $\pi_1 + \pi_2 + \pi_3 = 2\eta$. The relations (12) give

$$\pi_1 = \frac{\pi_5^2 + \pi_8^2}{\pi_3}, \ \pi_2 = \frac{\pi_6^2 + \pi_9^2}{\pi_3}, \ \pi_4 = \frac{\pi_5\pi_6 + \pi_8\pi_9}{\pi_3}, \ \pi_7 = \frac{\pi_6\pi_8 - \pi_5\pi_9}{\pi_3} . \tag{16}$$

This allows to express the restriction Π^η of $\Pi(\pi^\eta)$ to the tangent space $T_{\pi^\eta}\mathbb{CP}^2$ in the variables $\pi_5, \pi_6, \pi_8, \pi_9$ and π_3, see Table 3.

Table 3: The structure matrix Π^η on the tangent space $T_{\pi^\eta}\mathbb{CP}^2$.

{ , }	π_5	π_6	π_8	π_9
π_5	0	$\frac{\pi_6\pi_8-\pi_5\pi_9}{\pi_3}$	$\frac{\pi_5^2+\pi_8^2-\pi_3^2}{\pi_3}$	$\frac{\pi_5\pi_6+\pi_8\pi_9}{\pi_3}$
π_6	$-\frac{\pi_6\pi_8-\pi_5\pi_9}{\pi_3}$	0	$\frac{\pi_5\pi_6+\pi_8\pi_9}{\pi_3}$	$\frac{\pi_6^2+\pi_9^2-\pi_3^2}{\pi_3}$
π_8	$-\frac{\pi_5^2+\pi_8^2-\pi_3^2}{\pi_3}$	$-\frac{\pi_5\pi_6+\pi_8\pi_9}{\pi_3}$	0	$\frac{\pi_6\pi_8-\pi_5\pi_9}{\pi_3}$
π_9	$-\frac{\pi_5\pi_6+\pi_8\pi_9}{\pi_3}$	$-\frac{\pi_6^2+\pi_9^2-\pi_3^2}{\pi_3}$	$-\frac{\pi_6\pi_8-\pi_5\pi_9}{\pi_3}$	0

Adding the first two equations of (16) and using that $\pi_1 + \pi_2 = 2\eta - \pi_3$ gives

$$\pi_3^2 - 2\eta\pi_3 + \pi_5^2 + \pi_6^2 + \pi_8^2 + \pi_9^2 = 0 .$$

As $\pi_1 + \pi_2 = 0$ implies $\pi_3 = 2\eta$ this quadratic equation for π_3 has the solution

$$\pi_3 = \eta + \sqrt{\eta^2 - (\pi_5^2 + \pi_6^2 + \pi_8^2 + \pi_9^2)} . \tag{17}$$

Write $\chi = \pi_5^2 + \pi_6^2 + \pi_8^2 + \pi_9^2$ then power series evaluation at $\chi = 0$ gives

$$\begin{aligned} \pi_3 &= 2\eta + \sum_{k=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-k+1)}{k!}(-1)^k\eta^{1-2k}\chi^k \\ &= 2\eta - \frac{\chi}{2\eta} - \frac{\chi^2}{8\eta^3} - \frac{\chi^3}{16\eta^5} + \ldots \end{aligned} \tag{18}$$

and hence

$$\pi_3^{-1} \;=\; \frac{1}{2\eta} \;+\; \frac{\chi}{8\eta^3} \;+\; \frac{\chi^2}{16\eta^5} \;+\; \frac{5\chi^3}{128\eta^7} \;+\; \dots\;. \tag{19}$$

Consequently, the restriction Π^η of Π to $(\pi_5,\pi_6,\pi_8,\pi_9)$–space can be developed into

$$\Pi^\eta \;=\; \Pi_0 \;+\; \Pi_1 \;+\; \Pi_2 \;+\; \dots$$

with

$$\Pi_0 \;=\; \begin{pmatrix} 0 & 0 & -2\eta & 0 \\ 0 & 0 & 0 & -2\eta \\ 2\eta & 0 & 0 & 0 \\ 0 & 2\eta & 0 & 0 \end{pmatrix}$$

$$\Pi_1 \;=\; \frac{1}{2\eta}\begin{pmatrix} 0 & (\pi_6\pi_8-\pi_5\pi_9) & -(\pi_6^2+\pi_9^2) & (\pi_5\pi_6+\pi_8\pi_9) \\ -(\pi_6\pi_8-\pi_5\pi_9) & 0 & (\pi_5\pi_6+\pi_8\pi_9) & -(\pi_5^2+\pi_8^2) \\ (\pi_6^2+\pi_9^2) & -(\pi_5\pi_6+\pi_8\pi_9) & 0 & (\pi_6\pi_8-\pi_5\pi_9) \\ -(\pi_5\pi_6+\pi_8\pi_9) & (\pi_5^2+\pi_8^2) & -(\pi_6\pi_8-\pi_5\pi_9) & 0 \end{pmatrix}$$

$$\Pi_2 \;=\; \frac{1}{4\eta^2}\left(\pi_5^2+\pi_6^2 \;+\; \pi_8^2+\pi_9^2\right)\Pi_1 \;=\; \frac{\chi}{4\eta^2}\Pi_1\;.$$

On $(\pi_5,\pi_6,\pi_8,\pi_9)$–space the S^1–action of $S=\pi_7$ is determined by the vector field

$$X_S\;:\;\begin{pmatrix}\dot\pi_5\\ \dot\pi_6\\ \dot\pi_8\\ \dot\pi_9\end{pmatrix} \;=\; \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix}\pi_5\\ \pi_6\\ \pi_8\\ \pi_9\end{pmatrix}\;.$$

The S^1–action of this vector field is given by its flow which is

$$\varrho_\nu\begin{pmatrix}\pi_5\\ \pi_6\\ \pi_8\\ \pi_9\end{pmatrix} \;=\; \begin{pmatrix} \cos\nu & -\sin\nu & 0 & 0 \\ \sin\nu & \cos\nu & 0 & 0 \\ 0 & 0 & \cos\nu & -\sin\nu \\ 0 & 0 & \sin\nu & \cos\nu \end{pmatrix}\begin{pmatrix}\pi_5\\ \pi_6\\ \pi_8\\ \pi_9\end{pmatrix}\;. \tag{20}$$

The invariants for this action are

$$\begin{aligned} X &= \tfrac{1}{2}(\pi_5^2+\pi_6^2) \\ Y &= \tfrac{1}{2}(\pi_8^2+\pi_9^2) \\ Z &= \pi_5\pi_8+\pi_6\pi_9 \\ \tilde S &= \pi_6\pi_8-\pi_5\pi_9 \end{aligned} \tag{21}$$

and related through

$$\begin{aligned} 2(X+Y) &= \chi \;=\; \pi_3\tau_2 \\ 2(X-Y) &= \operatorname{Re}\tau_1 \\ -2Z &= \operatorname{Im}\tau_1 \\ \tilde S &= \pi_3\pi_7 \;=\; \pi_3 S \end{aligned} \tag{22}$$

to the invariants defined in (10) and (13). The syzygy between the invariants (21) is

$$4XY \;=\; Z^2 \;+\; \tilde{S}^2 \tag{23}$$

which, using the identities (22), is equivalent to $R_{\sigma,\eta}(\tau) = 0$. For the bracket relations among the invariants we have

$$\begin{aligned}
\{\, X \,,\, Z \,\} \;&=\; -2\pi_3 X \;+\; \frac{1}{\pi_3}(4X^2 + Z^2 - \tilde{S}^2) \;, \\
\{\, Y \,,\, Z \,\} \;&=\; 2\pi_3 Y \;-\; \frac{1}{\pi_3}(4Y^2 + Z^2 - \tilde{S}^2) \;, \\
\{\, X \,,\, Y \,\} \;&=\; -\pi_3 Z \;+\; \frac{2}{\pi_3}(X+Y)Z \;, \\
\{\, \tilde{S} \,,\, X \,\} \;&=\; -\frac{2}{\pi_3}\tilde{S}Z \;, \\
\{\, \tilde{S} \,,\, Y \,\} \;&=\; \frac{2}{\pi_3}\tilde{S}Z \;, \\
\{\, \tilde{S} \,,\, Y \,\} \;&=\; \frac{8}{\pi_3}\tilde{S}(X-Y) \;\; .
\end{aligned} \tag{24}$$

By substituting the power series (18) for π_3 these brackets are completely expressed in terms of the invariants $X, Y, Z, \tilde{S}$; this shows that they indeed generate the Lie algebra of S^1–invariant functions for the S^1–action generated by π_7. Actually the above computed brackets then give the results for the nonlinear Poisson structure defined by Π. If we let $[\;,\;]_0$ denote the Poisson bracket with respect to Π_0 then the brackets (24) reduce to

$$\begin{aligned}
&[\, X \,,\, Z \,]_0 \;=\; -4\eta X \;, \qquad [\, Y \,,\, Z \,]_0 \;=\; 4\eta Y \;, \qquad [\, X \,,\, Y \,]_0 \;=\; -2\eta Z \;, \\
&[\, \tilde{S} \,,\, X \,]_0 \;=\; 0 \;, \qquad\quad\; [\, \tilde{S} \,,\, Y \,]_0 \;=\; 0 \;, \qquad\quad\; [\, \tilde{S} \,,\, Z \,]_0 \;=\; 0 \;\; .
\end{aligned}$$

Thus, by considering the linear parts in the invariants only, one obtains a standard representation of $\mathfrak{sl}(2,\mathbb{R})$. Expressing π_7 as a power series in the invariants $X, Y, Z, \tilde{S}$ we get

$$\pi_7 \;=\; \tilde{S}\,(\frac{1}{2\eta} \;+\; \frac{1}{4\eta^3}(X+Y) \;+\; \frac{1}{4\eta^5}(X+Y)^2 \;+\; \ldots) \;\; . \tag{25}$$

From (15) we obtain the normalized Hamiltonian

$$\begin{aligned}
\mathcal{H}^\lambda(X,Y,Z,\tilde{S}) \;&=\; 4\lambda(1-6\lambda)(X-Y) \;+\; \frac{8\eta}{\pi_3}(5-6\lambda-4\lambda^2)(X+Y) \\
&\quad -\; \frac{4}{\pi_3^2}(5-12\lambda-3\lambda^2)(X+Y)^2 \;-\; \frac{4}{\pi_3^2}\lambda^2\tilde{S}^2 \;\; ,
\end{aligned} \tag{26}$$

where π_3 still has to be replaced by its series expansion (18) in $\chi = 2(X+Y)$, which leads to

$$\begin{aligned}
\widetilde{\mathcal{H}}^\lambda(X,Y,Z,\tilde{S}) \;&=\; 20(1-\lambda-2\lambda^2)X \;+\; 4(5-7\lambda+2\lambda^2)Y \\
&\quad +\; \frac{5}{\eta^2}(1-\lambda^2)(X+Y)^2 \;-\; \frac{\lambda^2}{\eta^2}\tilde{S}^2 \;+\; h.o.t.
\end{aligned} \tag{27}$$

The *higher order terms* in (27) are at least cubic in the invariants $X, Y, X, \tilde{S}$ or, equivalently, contain η^{-4} as a common factor. Recall that we had obtained $\mathcal{H}^\lambda$ from the first nontrivial normal form (11) where the truncated remainder terms have the common factor ε^4 and are, after further normalization, at least cubic in the invariants π_i. Truncating the *h.o.t.* in (27) we obtain the standard form of the Hamiltonian Hopf bifurcation for those values of λ where either $1-\lambda-2\lambda^2=0$ or $5-7\lambda+2\lambda^2=0$, that is, if $\lambda=-1, \frac{1}{2}, 1, \frac{5}{2}$, which is in agreement with [16].

There are three procedures to make this precise. The first is by locally linearizing or flattening the Poisson structure Π on the first reduced phase space $\mathbb{CP}^2$ as in [9], ch. V.8. This is actually a constructive approach of the Darboux theorem which says that a non–degenerate Poisson structure can be transformed to standard form, *i.e.* to the one corresponding to the standard symplectic form. This transformation might influence the lower order terms of the Hamiltonian and therefore has to be computed explicitly.

The second approach locally analyses the Poisson structure (14) on the twice reduced phase spaces $V_{\sigma,\eta}$ in terms of the geometry of the conical singularity. This leads to the geometric criterion in [16] which relates the relative position of the (energy) level sets of (15) with respect to $V_{\sigma,\eta}$ to the coefficient a of M in (7). For this method it is not necessary to change co–ordinates.

The third method uses singularity theory as in [23], ch. 3. The bifurcation is described by the singularity of the energy–momentum map $\mathcal{EM} : \mathbb{R}^6 \longrightarrow \mathbb{R}^3 : (x, y) \mapsto (\mathcal{H}^\lambda, \pi_7, H_0)$. One has to transform the S^1–equivariant singularity to its standard form. In this method the symplectic form or Poisson structure plays no role anymore, but one needs to have the right S^1–action with its $\mathfrak{sl}(2,\mathbb{R})$ of invariants.

5 Singularities of energy–momentum maps

Consider the energy–momentum map locally within the chart chosen on the first reduced phase space. This gives

$$\begin{array}{cccc} \widetilde{\mathcal{EM}} : & \mathbb{R}^4 & \longrightarrow & \mathbb{R}^2 \\ & (\pi_5, \pi_6, \pi_8, \pi_9) & \mapsto & (\widetilde{\mathcal{H}}^\lambda, \pi_7) \end{array}$$

with $\widetilde{\mathcal{H}}^\lambda$ and π_7 given by (27) and (25), respectively, while $\pi_5, \pi_6, \pi_8, \pi_9$ enter through (21). Let ϱ denote the S^1–action given by (20). Then this map is ϱ–invariant. Furthermore this action can be considered as the flow of the Hamiltonian vector field corresponding to $\tilde{S}$ with respect to the symplectic form $-(\mathrm{d}\pi_5 \wedge \mathrm{d}\pi_8 + \mathrm{d}\pi_6 \wedge \mathrm{d}\pi_9)$ on $\mathbb{R}^4$ (which is, up to the factor 2η, the linear part Π_0 of the local Poison structure Π^η on the tangent space). Therefore we are exactly in the framework of the standard Hamiltonian Hopf bifurcation and we may apply Theorem 2.1.

As a consequence the type of bifurcation is determined by the coefficient of the quadratic term in the Hamiltonian (27). For the different cases we obtain the following values for the coefficients.

Table 4: Values of the coefficients of G.

	coeff. of X	coeff. of Y	coeff. of X^2 and Y^2
λ	$20(1-\lambda-2\lambda^2)$	$4(5-7\lambda+2\lambda^2)$	$\frac{5}{\eta^2}(1-\lambda^2)$
-1	0	56	0
$\frac{1}{2}$	0	8	$\frac{15}{4\eta^2}$
1	-40	0	0
$\frac{5}{2}$	-280	0	$-\frac{105}{4\eta^2}$

Consequently we have by Theorem 2.1 the following

Theorem 5.1 *In the Hamiltonian system corresponding to the normal form* (11) *two non–degenerate supercritical Hamiltonian Hopf bifurcations take place at* $\lambda = \frac{1}{2}, \frac{5}{2}$.

Furthermore for λ *close to each of the values* $\frac{1}{2}, \frac{5}{2}$ *there exist a* ϱ*–equivariant origin preserving diffeomorphism* φ *on* $\mathbb{R}^4$ *and a origin preserving diffeomorphism* ψ *on* $\mathbb{R}^2$ *such that*

$$\psi \circ (\widetilde{\mathcal{H}}^\lambda, \pi_7) \circ \varphi \;=\; (G_\lambda, \tilde{S})$$

with

$$G_\lambda \;=\; 4(5-7\lambda+2\lambda^2)Y \;+\; 20(1-\lambda-2\lambda^2)X \;+\; \frac{5}{\eta^2}(1-\lambda^2)X^2$$

for λ *close to* $\frac{1}{2}$, *and*

$$G_\lambda \;=\; 20(1-\lambda-2\lambda^2)X \;+\; 4(5-7\lambda+2\lambda^2)Y \;+\; \frac{5}{\eta^2}(1-\lambda^2)Y^2$$

for λ *close to* $\frac{5}{2}$.

Recall that a Hamiltonian Hopf bifurcation is supercritical if the nilpotent part and the quadratic term have the same sign. This is the case where a family of normally elliptic invariant 2–tori detaches from the bifurcating periodic orbit when the bifurcation parameter passes through its critical value. In addition we have the subcritical Hamiltonian Hopf bifurcation when the nilpotent part and the quadratic term have opposite sign. In this case there is a family of invariant 2–tori emanating and returning to the bifurcating periodic orbit, which for a part consists of hyperbolic invariant 2–tori.

It is instructive to compare the three approaches mentioned at the end of Section 4. The sign of the coefficient a of M^2 in (7) determines whether the parabola $N = -aM^2$

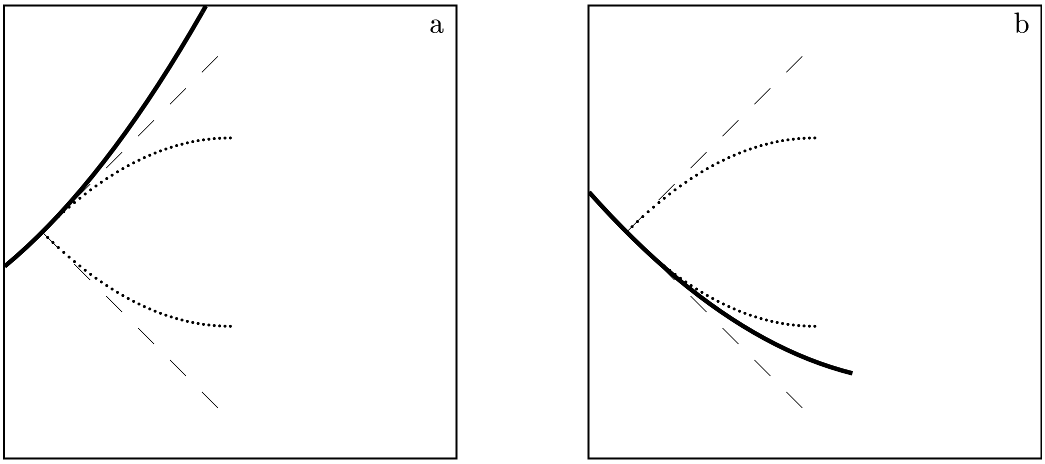

Figure 5.1: Relative position of the twice reduced phase space $V_{0,\eta}$ (dotted), the linearized cone (dashed) and the (energy) level set of (15) through the summit for a). $\lambda = \frac{1}{2}$ and b). $\lambda = \frac{5}{2}$.

touches the M–axis from below or above, *i.e.* touches the cone of $\mathfrak{sl}(2, \mathbb{R})$ from outside (supercritical case of the Hamiltonian Hopf bifurcation) or from inside (subcritical case). In particular, the level set $\hat{H}(N, M, P; 0) = 0$ has quadratic contact with the cone if and only if the non–degeneracy condition $a \neq 0$ is satisfied.

Let us match this standard setting with the situations of Figure 5.1. The linearized Poisson structure Π_0 corresponds to the dashed cone, while the full nonlinear Poisson structure Π^η corresponds to the dotted 'cone', the slightly curved cone–like part of $V_{0,\eta}$. In both cases the energy level lies outside the dotted 'cone', whence the geometric criterion of [16] allows to conclude that the Hamiltonian Hopf bifurcation is of supercritical type. Note that one should not compare the position of the energy level to the dashed (linearized) cone ; in case b). this would lead to an erroneous conclusion.

The first step of linearizing or flattening the Poisson structure Π^η would consist in a co–ordinate change that pushes the dotted 'cone' outside to better align with the dashed cone. Since it is exactly the second order terms that are transformed away in the first step the transformed energy level would then in both cases a). and b). lie outside the dashed cone (and the fourth order terms that distinguish the dashed cone from the transformed dotted 'cone' are not important).

While the dashed cone in Figure 5.1 has axes along $\tau_2 \pm \operatorname{Re} \tau_1$, the introduction of $X = \frac{1}{4}(\pi_3 \tau_2 + \operatorname{Re} \tau_1)$ and $Y = \frac{1}{4}(\pi_3 \tau_2 - \operatorname{Re} \tau_1)$ effectively yields co–ordinate axes along the dotted 'cone' — recall that for $\operatorname{Im} \tau_1 = 0$ (*i.e.* $Z = 0$) the syzygy $R_{0,\eta} = 0$ is equivalent to $X \cdot Y = 0$. Where the contact to the energy level is of second order the expressions (18) and (19) may be cut off after the second (and occasionally after the first) term. This leads to the Hamiltonian (27), the higher order terms of which are transformed away in

Theorem 5.1.

Table 4 yields a nonlinearly degenerate Hamiltonian Hopf bifurcation in the cases $\lambda = 1$ and $\lambda = -1$. As shown in [14] the degeneracy at $\lambda = 1$ cannot be removed ; this is related to the $3D$ Hénon–Heiles family being integrable when $\lambda = 1$. Geometrically this means that for all higher order normal forms the Hamiltonian will coincide with the reduced phase space along a curve. In the local normal form this is equivalent to the fact that no powers of Y will appear in the normal form at any order. One can prove, but this will not be done here, that in this case the bifurcation has infinite co–dimension.

For $\lambda = -1$ the normal form approximation (11) of (8) is not sufficient, we therefore now study a higher order normal form.

6 Higher order analysis

In [16] it was shown that the degeneracy at $\lambda = -1$ is removed by normalizing the $3D$ Hénon–Heiles family to higher order. Therefore we consider the next nontrivial normal form for (8), drop the constant terms $\eta - \frac{5}{12}\varepsilon^2\eta^2 - \frac{235}{432}\varepsilon^4\eta^3$ and divide by $\frac{\varepsilon^2}{48}$. This yields

$$\mathcal{K}^\lambda_\varepsilon(X, Y, Z, \tilde{S}) \;=\; \mathcal{H}^\lambda(X, Y, Z, \tilde{S}) + \frac{\varepsilon^2\eta}{36} K^\lambda(X, Y, Z, \tilde{S}) \tag{28}$$

where η is still the (constant) value of $H_0 = \frac{1}{2}(\pi_1 + \pi_2 + \pi_3)$, $\mathcal{H}^\lambda$ is given by (26) and

$$\begin{aligned} K^\lambda(X, Y, Z, \tilde{S}) \;=\; & 4\lambda(41 - 42\lambda - 656\lambda^2 - 48\lambda^3)(X - Y) \\ & + \frac{4\eta}{\pi_3}(705 - 720\lambda - 251\lambda^2 - 972\lambda^3 - 172\lambda^4)(X + Y) \\ & - \frac{4\lambda}{\eta\pi_3}(1 - \lambda)(41 - 61\lambda - 606\lambda^2)(X^2 - Y^2) \\ & - \frac{4}{\pi_3^2}(705 - 1440\lambda + 189\lambda^2 - 1764\lambda^3 + 900\lambda^4)(X + Y)^2 \\ & + \frac{20}{\eta\pi_3^3}(1 - \lambda)(47 - 97\lambda - 9\lambda^2 - 201\lambda^3)(X + Y)^3 \\ & + \frac{4\lambda^2}{\pi_3^2}(43 - 348\lambda + 220\lambda^2)\tilde{S}^2 \\ & - \frac{4\lambda^2}{\eta\pi_3^3}(43 - 372\lambda + 369\lambda^2)(X + Y)\tilde{S}^2 \;. \end{aligned} \tag{29}$$

We remark that we have corrected in (29) an inconsequential error in formula (13) of [16].

Again we replace π_3 by its series expansion (18) in $\chi = 2(X + Y)$ and obtain

$$\widetilde{\mathcal{K}}^\lambda_\varepsilon(X, Y, Z, \tilde{S}) \;=\; \widetilde{\mathcal{H}}^\lambda(X, Y, Z, \tilde{S}) + \frac{\varepsilon^2\eta}{36}\widetilde{K}^\lambda(X, Y, Z, \tilde{S}) \tag{30}$$

with $\widetilde{\mathcal{H}}^\lambda$ as in (27) and, up to order two in the invariants (21),

$$
\begin{aligned}
\widetilde{K}^\lambda(X,Y,Z,\tilde{S}) \;=\;& 4\lambda(41-42\lambda-656\lambda^2-48\lambda^3)(X-Y) \\
&+\; 2(705-720\lambda-251\lambda^2-972\lambda^3-172\lambda^4)(X+Y) \\
&-\; \frac{2\lambda}{\eta^2}(1-\lambda)(41-61\lambda-606\lambda^2)(X^2-Y^2) \\
&+\; \frac{8\lambda}{\eta^2}(90-55\lambda+99\lambda^2-134\lambda^3)(X+Y)^2 \\
&+\; \frac{\lambda^2}{\eta^2}(43-348\lambda+220\lambda^2)\tilde{S}^2 \;+\; h.o.t.
\end{aligned}
$$

We see that adding higher order terms with respect to ε in the normalized Hamiltonian changes coefficients at all orders (with respect to the invariants) of the reduced Hamiltonian on the first reduced phase space. We find as coefficient of X

$$
a_1(\lambda) \;=\; 20(1-\lambda-2\lambda^2) \;+\; \frac{\varepsilon^2\eta}{18}(705-638\lambda-335\lambda^2-2284\lambda^3-268\lambda^4) \tag{31}
$$

and as coefficient of Y

$$
b_1(\lambda) \;=\; 4(5-7\lambda+2\lambda^2) \;+\; \frac{\varepsilon^2\eta}{18}(705-802\lambda-167\lambda^2+340\lambda^3-76\lambda^4)\;. \tag{32}
$$

The coefficient of X^2 is given by

$$
a_2(\lambda) \;=\; \frac{5}{\eta^2}(1-\lambda^2) \;+\; \frac{\varepsilon^2\lambda}{18\eta}(319-118\lambda+941\lambda^2-1142\lambda^3) \tag{33}
$$

and the coefficient of Y^2 is

$$
b_2(\lambda) \;=\; \frac{5}{\eta^2}(1-\lambda^2) \;+\; \frac{\varepsilon^2\lambda}{18\eta}(401-322\lambda-149\lambda^2+70\lambda^3)\;. \tag{34}
$$

We may now formulate a theorem similar to Theorem 5.1.

Theorem 6.1 *In the Hamiltonian system corresponding to the normal form (28) two non–degenerate supercritical Hamiltonian Hopf bifurcations take place at $\lambda = \frac{1}{2}, \frac{5}{2}$ and a subcritical Hamiltonian Hopf bifurcation takes place at $\lambda^*_{-1}(\varepsilon)$ close to -1.*

Furthermore for suficiently small ε and λ close to each of the values $\frac{1}{2}, \frac{5}{2}, \lambda^*_{-1}(\varepsilon)$ there exist a ϱ–equivariant origin preserving diffeomorphism φ on $\mathbb{R}^4$ and a origin preserving diffeomorphism ψ on $\mathbb{R}^2$ such that

$$
\psi \circ (\widetilde{\mathcal{K}}^\lambda_\varepsilon, \pi_7) \circ \varphi \;=\; (G_\lambda, \tilde{S})
$$

with

$$
G_\lambda \;=\; b_1(\lambda)Y \;+\; a_1(\lambda)X \;+\; a_2(\lambda)X^2
$$

for λ close to $\frac{1}{2}$ or $\lambda^*_{-1}(\varepsilon)$, and

$$G_\lambda \;=\; a_1(\lambda)X \;+\; b_1(\lambda)Y \;+\; b_2(\lambda)Y^2$$

for λ close to $\frac{5}{2}$. Here $a_1(\lambda)$, $b_1(\lambda)$, $a_2(\lambda)$ and $b_2(\lambda)$ are given by (31), (32), (33), (34).

Proof. We have

$$\begin{aligned} G_{\frac{1}{2}} &= (8+\frac{50}{3}\varepsilon^2\eta)Y \;+\; \frac{15}{4\eta^2}(1+\frac{47}{18}\varepsilon^2\eta)X^2 \\ G_{\frac{5}{2}} &= -(280+2730\varepsilon^2\eta)X \;-\; \frac{35}{4\eta^2}(3+\frac{23}{6}\varepsilon^2\eta)Y^2 \;. \end{aligned}$$

Consequently, for these cases, the statement is a direct consequence of Theorem 2.1. Next consider the case $\lambda = -1$. Note that for $\varepsilon = 0$ the coefficient $a_1(-1) = 0$. Thus, as a consequence of the implicit function theorem, for ε close to zero there exists a value $\lambda^*_{-1}(\varepsilon)$ such that $a_1(\lambda^*_{-1}(\varepsilon)) = 0$. The linear system is again nilpotent and we may apply Theorem 2.1.

To prove non–degeneracy we have to study the coefficient of X^2 at $\lambda = \lambda^*_{-1}(\varepsilon)$, which is a rather complicated (though algorithmically computable) expression. To first order in $\delta = \varepsilon^2\eta$ this coefficient is determined by the sum of the two terms

$$a_2(-1) \;=\; -140\frac{\varepsilon^2}{\eta}$$

and

$$\frac{5}{\eta^2}\left(1-(\lambda^*_{-1}(\varepsilon))^2\right) \;\approx\; -28\frac{\varepsilon^2}{\eta}$$

Consequently the Hamiltonian Hopf bifurcation at $\lambda^*_{-1}(\varepsilon)$ is of subcritical type. □

Rescaling to the choices of [16] we can immediately relate the coefficient $b_2(\frac{5}{2})$ to the nominator of $l''_{\frac{5}{2},\delta}(0)$ given in [16], p. 691. The relation between $a_2(-1)$ and the expression $u''_{-1,\delta}(0) = -24\delta + \mathcal{O}(\delta^2)$ given in [16], p. 692 is less obvious since one has to account for $u'_{-1,\delta}(0) \neq 0$, whence one is led to $u''_{-1,\delta}(0) = -32\delta + \mathcal{O}(\delta^2)$ correcting an error in [16].

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