Generalized eigenfunctions of relativistic Schrödinger operators I

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Abstract

Generalized eigenfunctions of the 3-dimensional relativistic Schrödinger operator $\sqrt{\Delta} + V(x)$ with $|V(x)| \leq C\langle x\rangle^{-\sigma}$, $\sigma > 1$, are considered. We construct the generalized eigenfunctions by exploiting results on the limiting absorption principle. We compute explicitly the integral kernel of $(\sqrt{-\Delta} - z)^{-1}$, $z \in \mathbb{C} \setminus [0, +\infty)$, which has nothing in common with the integral kernel of $(-\Delta - z)^{-1}$, but the leading term of the integral kernels of the boundary values $(\sqrt{-\Delta} - \lambda \mp i0)^{-1}$, $\lambda > 0$, turn out to be the same, up to a constant, as the integral kernels of the boundary values $(-\Delta - \lambda \mp i0)^{-1}$. This fact enables us to show that the asymptotic behavior, as $|x| \to +\infty$, of the generalized eigenfunction of $\sqrt{\Delta} + V(x)$ is equal to the sum of a plane wave and a spherical wave when $\sigma > 3$.

Contents

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1 Introduction

This is the first part of a paper, consisting of two parts, on the operator

$$\sqrt{-\Delta}+V(x), \quad x \in \mathbb{R}^3, \tag{1.1}$$

with a short range potential $V(x)$, the operator which we shall call the relativistic Schrödinger operator. The first part, the present paper, is concerned with asymptotic behaviors, as $|x| \to +\infty$, of the generalized eigenfunctions of $\sqrt{-\Delta}+V(x)$, whereas the second part [28] deals with the completeness of the generalized eigenfunctions, i.e., the eigenfunction expansion for the absolutely continuous spectrum. Part of the present and coming papers was announced in [27].

We remark here that a prototype of generalized eigenfunction expansions is provided by the Fourier inversion formula

$$u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot k}\, \hat{u}(k)\, dk,$$

where $e^{ix\cdot k}$ should be regarded as a generalized eigenfunction of the Laplace operator $-\Delta_x$ in the sense that $e^{ix\cdot k}$ satisfies $-\Delta_x e^{ix\cdot k} = |k|^2 e^{ix\cdot k}$, but does not belong to $L^2(\mathbb{R}^n_x)$. It has to be noted that the absolutely continuous spectrum of $-\Delta$ is given by the interval $[0, +\infty)$.

Although relativistic Schrödinger operators have received a substantial amount of attention in recent years, there have been only a few works on the decay of eigenfunctions associated to the discrete spectra of these operators; see Nardini[16], [17] Carmona-Masters-Simon[4] and Helffer-Parisse[8]). And it is a surprise that up to now there seems to have been no results on asymptotic behaviors of the generalized eigenfunctions of these operators and on the completeness of the generalized eigenfunctions.

For the purpose of making a comparison, let us briefly recall some results of Ikebe[7] on the asymptotic behaviors of the generalized eigenfunctions of the Schrödinger operator

$$-\Delta + V(x), \quad x \in \mathbb{R}^3$$

in connection with the eigenfunction expansion for the absolutely continuous spectrum. In [7], the generalized eigenfunction of $-\Delta+V(x)$ was constructed as a solution to the Lippmann-Schwinger equation

$$\varphi(x,\, k) = e^{ix\cdot k} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}\, V(y)\, \varphi(y,\, k)\, dy, \tag{1.2}$$

the solution being unique if $\varphi(x,\, k) - e^{ix\cdot k}$ belongs to $C_\infty(\mathbb{R}^3_x)$, the space of all continuous functions vanishing at infinity. Then the generalized Fourier transform, of which kernel is the generalized eigenfunctions obtained, was introduced and the generalized Fourier inversion formula, i.e., the eigenfunction expansion for the absolutely continuous spectrum of the operator $-\Delta + V(x)$ was established.

Ikebe's discussions on asymptotic behaviors of the generalized eigenfunctions were based upon the Lippmann-Schwinger equation (1.2). Roughly speaking, we see that his assumption on the potential function is that $V(x)$ is locally Hölder continuous and $V(x) = O(|x|^{-\sigma}),\ \ \sigma > 2$, at infinity (see Ikebe[7, §1] for the precise description of his assumption).

It is apparent that the term

$$\frac{1}{4\pi}\cdot\frac{e^{i|k||x-y|}}{|x-y|}$$

in (1.2) comes from the integral kernel of the resolvent of $-\Delta$:

$$(-\Delta - z)^{-1}u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i\sqrt{z}\,|x-y|}}{|x-y|}\, u(y)\, dy, \quad \operatorname{Im}\sqrt{z} > 0$$

for $z \in \mathbb{C} \setminus [0,\, +\infty)$. In other words, the limiting absorption principle for $-\Delta$ shows that the boundary value of the resolvent $(-\Delta - z)^{-1}$, as $z = \lambda + i\mu$ $(\lambda,\, \mu > 0)$ tends to $\lambda + i0$, is expressed as the integral operator

$$(-\Delta - \lambda - i0)^{-1}u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i\sqrt{\lambda}\,|x-y|}}{|x-y|}\, u(y)\, dy.$$

It was also shown in [7], by appealing to the Lippmann-Schwinger equation (1.2), that if $\sigma > 3$ then the generalized eigenfunction has the asymptotics

$$\varphi(x,\, k) = e^{ix\cdot k} + f(|k|,\, \omega_x,\, \omega_k)\, \frac{e^{i|k||x|}}{|x|} + o(\frac{1}{|x|}) \tag{1.3}$$

as $|x| \to +\infty$, where $\omega_x = x/|x|$, and $\omega_k = k/|k|$. From the view point of physics, (1.3) is interpreted to mean that $\varphi(x,\, k)$ is asymptotically equal to a superposition of the incoming plane wave $e^{ix\cdot k}$ and the outgoing spherical wave $e^{i|k||x|}/|x|$ (cf. Yafaev[29, §1.3]).

What we have recalled above indicates that computing the integral kernel of $(\sqrt{-\Delta} - z)^{-1}$ is naturally a starting point to investigate asymptotic behaviors of the generalized eigenfunctions of $\sqrt{-\Delta} + V(x)$. Our computations show that the integral kernel of the resolvent of $\sqrt{-\Delta}$ is given by

$$(\sqrt{-\Delta} - z)^{-1} u(x) = \int_{\mathbb{R}^3} g_z(x-y)\, u(y)\, dy$$

for $z \in \mathbb{C} \setminus [0,\, +\infty)$, where

$$\begin{aligned} g_z(x) = & \frac{1}{2\pi^2|x|^2} + \\ & \frac{z}{2\,\pi^2\,|x|}\Big[\, \sin(z|x|)\,\mathrm{ci}(-z|x|) - \cos(z|x|)\,\mathrm{si}(-z|x|)\,\Big] \end{aligned}$$

(see Section 2). For the definitions of the cosine and sine integral functions $\mathrm{ci}(z)$ and $\mathrm{si}(z)$, see Subsection A.1 in Appendix.

The integral kernel $g_z(x-y)$ has nothing in common with the integral kernel of $(-\Delta - z)^{-1}$, but if we take the limit of $g_z(x-y)$ as z approaches the positive half of the real axis ($z = \lambda + i\mu \to \lambda + i0$), then the term

$$\frac{\lambda}{2\pi} \cdot \frac{e^{i\lambda|x-y|}}{|x-y|}$$

emerges as the leading term of $g_{\lambda+i0}(x-y)$, which is actually the integral kernel of the boundary value $(\sqrt{-\Delta} - \lambda - i0)^{-1}$:

$$(\sqrt{-\Delta} - \lambda - i0)^{-1} u(x) = \int_{\mathbb{R}^3} g_{\lambda+i0}(x-y)\, u(y)\, dy, \quad \lambda > 0,$$

where

$$\begin{aligned} g_{\lambda+i0}(x) &= \frac{\lambda}{2\pi} \cdot \frac{e^{i\lambda|x|}}{|x|} + \frac{1}{2\pi^2|x|^2} + m_\lambda(x), \qquad (1.4) \\ m_\lambda(x) &= O(|x|^{-2}) \quad \text{as } |x| \to +\infty. \end{aligned}$$

This fact enables us to investigate asymptotic behaviors of the generalized eigenfunctions of $\sqrt{-\Delta} + V(x)$ by utilizing the integral equation which we shall call the modified Lippmann-Schwinger equation.

Unfortunately, the term $1/(2\pi^2|x|^2)$ in (1.4) is quite troublesome. The reason for this is that our generalized eigenfunctions must be bounded functions of x since they are expected to be distorted plane waves in physics terminology. However, the integral operator

$$\frac{1}{2\pi^2}\int_{\mathbb{R}^3}\frac{1}{|x-y|^2}\,u(y)\,dy,$$

which is known as the Riesz potential, cannot be a bounded operator from $L^p(\mathbb{R}^3)$ to $L^\infty(\mathbb{R}^3)$ for any $p \geq 1$ (see Stein[23, p.119]). To overcome this difficulty, we shall introduce a few inequalities for the Riesz potentials in Section 5.

We should like to remark here that one might ignore the formula $-\Delta_x\, e^{ix\cdot k} = |k|^2 e^{ix\cdot k}$ plays a significant role in discussing the generalized eigenfunction expansion for the Schrödinger operator $-\Delta + V(x)$, because the formula is so trivial. On the contrary, it is far from trivial to show that

$$\sqrt{-\Delta_x}\, e^{ix\cdot k} = |k| e^{ix\cdot k} \quad \text{in the distribution sense.} \tag{1.5}$$

Indeed, the left hand side of (1.5) is formally defined by

$$\int e^{ix\cdot\xi}\,|\xi|\,\delta(\xi-k)\,d\xi, \quad (\,\delta(\cdot) \text{ is the delta function}\,),$$

while the symbol $|\xi|$ of $\sqrt{-\Delta}$ is singular at the origin $\xi = 0$. Therefore, making sense of the expression $\sqrt{-\Delta_x}\, e^{ix\cdot k}$ is one of the main tasks in the present paper, and it will be accomplished in Section 8 with the aid of a theorem in Section 6.

Assumption Throughout the paper we shall assume that $V(x)$ is a real-valued measurable function on $\mathbb{R}^3$ satisfying

$$|V(x)| \leq C\langle x\rangle^{-\sigma}, \qquad \sigma > 1, \tag{1.6}$$

though σ will be required to satisfy the assumption $\sigma > 2$ when we investigate asymptotic behaviors of the generalized eigenfunctions in precise manners. We emphasize that we do not require any smoothness assumption on the potential V. Although we could allow some local singularities of V in the sense that $V(x)$ behaves like $|x-x_0|^{-\beta}$ with $0 < \beta < 1$ near some isolated points x_0's, we shall not do so for the sake of simplicity.

The plan of the paper is as follows. In Section 2, we compute the integral kernel of the resolvent $(\sqrt{-\Delta} - z)^{-1}$ for $z \in \mathbb{C} \setminus [0, +\infty)$. In Section 3, we derive expressions of the boundary values $(\sqrt{-\Delta} - \lambda \mp i0)^{-1}$ on the half positive axis in terms of the boundary values $(-\Delta - \lambda \mp i0)^{-1}$. The expressions will be used in Section 6. In Section 4, we compute the integral kernels of $(\sqrt{-\Delta} - \lambda \mp i0)^{-1}$. In order to show that our generalized eigenfunctions are bounded functions, we shall prove some inequalities, in Section 5, for the Riesz potential and the integral operator appearing as a part of $(\sqrt{-\Delta} - \lambda \mp i0)^{-1}$. In Section 6, we establish the radiation conditions for $\sqrt{-\Delta}$, which implies that the second term of the generalized eigenfunction of $\sqrt{-\Delta} + V(x)$ is a spherical wave in a certain sense. In Section 7, we establish the radiation conditions for $\sqrt{-\Delta} + V(x)$, which is of some interest on its own. We construct the generalized eigenfunctions of $\sqrt{-\Delta} + V(x)$, and characterize them as unique solutions to the modified Lippmann-Schwinger equations in Section 8. In Section 9, we show that the generalized eigenfunctions are bounded functions of x, and continuous functions of the both variables x and k. Our discussions here are based on the modified Lippmann-Schwinger equations. In Section 10, we give estimates on the difference between the generalized eigenfunction and the plane wave when $\sigma > 2$. Also, we give estimates on the difference between the generalized eigenfunction and the sum of a plane wave and a spherical wave when $\sigma > 3$. In Appendix, we illustrate some properties of the cosine and sine integral functions, and prove inequalities for a convolution which are used several times in the present paper.

It is worthwhile to mention that all the results and the discussions in Sections 3, 6 and 7 remain valid for the n-dimensional case with $n \geq 2$ with trivial changes. However, we shall confine our attention, throughout the present paper, to the 3-dimensional case for the sake of clarity of description.

Notation We introduce the notation which will be used in the present paper. Although the discussions in the present paper will be made for the 3-dimensional case, the notation introduced here are given in the n-dimensional setting.

For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x and

$$\langle x \rangle = \sqrt{1 + |x|^2}.$$

The Fourier transform of a function u is denoted by $\mathcal{F}u$ or $\hat{u}$, and defined by

$$[\mathcal{F}u](\xi) = \hat{u}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\, u(x)\, dx.$$

For s and ℓ in $\mathbb{R}$, we define the weighted L^2-space and the weighted Sobolev space by

$$L^{2,s}(\mathbb{R}^n) = \{ f \mid \langle x \rangle^s f \in L^2(\mathbb{R}^n) \}$$

and

$$H^{\ell,s}(\mathbb{R}^n) = \{ f \mid \langle x \rangle^s \langle D \rangle^\ell f \in L^2(\mathbb{R}^n) \}$$

respectively, where D stands for $-i\partial/\partial x$ and $\langle D \rangle = \sqrt{1 + |D|^2} = \sqrt{1 - \Delta}$. When $s = 0$, we write $L^2(\mathbb{R}^n) = L^{2,0}(\mathbb{R}^n)$ and $H^\ell(\mathbb{R}^n) = H^{\ell,0}(\mathbb{R}^n)$. The inner products and the norms in $L^{2,s}(\mathbb{R}^n)$ and $H^{\ell,s}(\mathbb{R}^n)$ are given by

$$\begin{cases} (f,\, g)_{L^{2,s}} = \displaystyle\int_{\mathbb{R}^n} \langle x\rangle^{2s}\, f(x)\,\overline{g(x)}\,dx \\ \\ \|f\|_{L^{2,s}} = \{(f,\, f)_{L^{2,s}}\}^{1/2} \end{cases}$$

and

$$\begin{cases} (f,\, g)_{H^{\ell,s}} = \displaystyle\int_{\mathbb{R}^n} \langle x\rangle^{2s}\, \langle D\rangle^{\ell} f(x)\,\overline{\langle D\rangle^{\ell} g(x)}\,dx \\ \\ \|f\|_{H^{\ell,s}} = \{(f,\, f)_{H^{\ell,s}}\}^{1/2} \end{cases}$$

respectively.

By $C_0^\infty(\mathbb{R}^n)$ we mean the space of C^∞-functions of compact support. By $\mathcal{S}(\mathbb{R}^n)$ we mean the Schwartz space of rapidly decreasing functions, and by $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions. For a pair of $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$, we denote the duality bracket by $\langle f,\, \psi\rangle$. For a pair of $f \in L^{2,-s}(\mathbb{R}^n)$ and $g \in L^{2,s}(\mathbb{R}^n)$, we define the anti-duality bracket by

$$(f,\, g)_{-s,s} := \int_{\mathbb{R}^n} f(x)\,\overline{g(x)}\,dx.$$

For a pair of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, $\mathbf{B}(\mathcal{H},\, \mathcal{K})$ denotes the Banach space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. We set $\mathbf{B}(\mathcal{H}) = \mathbf{B}(\mathcal{H},\, \mathcal{H})$.

For a selfadjoint operator T in a Hilbert space, $\sigma(T)$ and $\rho(T)$ denote the spectrum of T and the resolvent set of T respectively. The point spectrum, i.e., the set of all eigenvalues of T, will be denoted by $\sigma_{\mathrm{p}}(T)$. The essential spectrum, the continuous spectrum and the absolutely continuous spectrum of T will be denoted by $\sigma_{\mathrm{ess}}(T)$, $\sigma_{\mathrm{c}}(T)$ and $\sigma_{\mathrm{ac}}(T)$ respectively.

2 Integral kernels of the resolvents of $\boldsymbol{H_0}$

This section is devoted to the computation of the resolvent kernel of $H_0 = \sqrt{-\Delta}$ on $\mathbb{R}^3$. We shall start with the definition of the operator H_0, and the description of its basic properties from the view point of spectral theory.

Let H_0 be the selfadjoint operator in $L^2(\mathbb{R}^3)$ given by

$$H_0 := \sqrt{-\Delta} \quad \text{with domain } H^1(\mathbb{R}^3).$$

Since H_0 is unitarily equivalent, through the Fourier transform $\mathcal{F}$, to the multiplication operator by $|\xi|\times$ in $L^2(\mathbb{R}^3_\xi)$, it follows from Kato[10, p. 520, Example 1.9] that H_0 is absolutely continuous, and that

$$\sigma(H_0) = \sigma_{\mathrm{ac}}(H_0) = [0,\, \infty).$$

Furthermore, we see that H_0 restricted on $C_0^\infty(\mathbb{R}^3)$ is essentially selfadjoint. Indeed, with a $C^\infty-$function χ satisfying

$$\chi(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| \le 1, \\ 0 & \text{if} \quad |\xi| \ge 2, \end{cases}$$

we can decompose $\sqrt{-\Delta}$ into a regular part and a singular part:

$$\sqrt{-\Delta} = (1 - \chi(D))\sqrt{-\Delta} + \chi(D)\sqrt{-\Delta},$$

which enables us to regard $\sqrt{-\Delta}$ as a sum of a essentially selfadjoint operator on $C_0^\infty(\mathbb{R}^3)$ (see Nagase and Umeda[15, Theorem 3.4]) and a bounded selfadjoint operator. The resolvent of H_0 will be denoted by

$$R_0(z) = (H_0 - z)^{-1} \qquad (z \in \rho(H_0) = \mathbb{C} \setminus [0,\, \infty)\ ).$$

By virtue of the fact that $R_0(z) = \mathcal{F}^{-1}(|\xi| - z)^{-1}\mathcal{F}$. it would be possible to obtain the resolvent kernel, i.e., the integral kernel of $R_0(z)$ by direct computation of $[\mathcal{F}^{-1}(|\xi| - z)^{-1}](x)$. We shall, however, avoid this computation. Instead, we take advantage of the fact that the strongly continuous semigroup generated by $-H_0$ is expressed as a convolution with the Poisson kernel (Stein[23, p.61], Strichartz[24, p.50]):

$$e^{-tH_0}u(x) = P_t * u(x) = \int_{\mathbb{R}^3} P_t(x-y)u(y)\,dy, \quad t > 0,\ u \in L^2(\mathbb{R}^3),$$

where

$$P_t(x) = \frac{t}{\pi^2\,(t^2 + |x|^2)^2}. \tag{2.1}$$

We then take the Laplace transform of e^{-tH_0} to get the resolvent:

$$R_0(z) = \int_0^{\infty} e^{tz} e^{-tH_0}\, dt \qquad \text{if } \operatorname{Re} z < 0.$$

Thus we need the following prerequisite.

Lemma 2.1 *If* $\operatorname{Re} z < 0$*, then*

$$\int_0^{+\infty} e^{tz} \frac{t}{\pi^2\,(t^2+a^2)^2}\, dt = \frac{1}{2\,\pi^2\, a^2} + \frac{z}{2\,\pi^2\, a}[\,\sin(za)\,\mathrm{ci}(-za) - \cos(za)\,\mathrm{si}(-za)\,],$$

where a is a positive constant.

Proof. Since

$$\frac{t}{(t^2+a^2)^2} = \frac{d}{dt}\Big\{ -\frac{1}{2(t^2+a^2)} \Big\},$$

we get, by integration by parts,

$$\int_0^{+\infty} e^{tz} \frac{t}{(t^2+a^2)^2}\, dt = \frac{1}{2a^2} + \frac{z}{2} \int_0^{\infty} e^{tz} \frac{1}{t^2+a^2}\, dt. \tag{2.2}$$

Applying the formula (A.4) in Appendix to the integral on the right-hand side of (2.2) and noting the remark after the formula (A.4), we obtain the lemma. □

In accordance with Lemma 2.1, we need to introduce two functions, which constitute the integral kernel of $R_0(z)$ as we shall see in Theorem 2.1 below.

Definition 2.1 For $z \in \mathbb{C} \setminus [0, +\infty)$, we define

$$\ell_z(x) := \frac{z}{2\,\pi^2\,|x|}\big[\,\sin(z|x|)\,\mathrm{ci}(-z|x|) - \cos(z|x|)\,\mathrm{si}(-z|x|)\,\big], \tag{2.3}$$

$$g_z(x) := \frac{1}{2\pi^2|x|^2} + \ell_z(x). \tag{2.4}$$

By G_0 we mean the operator defined by

$$G_0 u(x) := \frac{1}{2\,\pi^2} \int_{\mathbb{R}^3} \frac{1}{|x-y|^2} u(y)\, dy. \tag{2.5}$$

By G_z we mean the operator defined by

$$G_z u(x) := G_z * u(x) = \int_{\mathbb{R}^3} g_z(x-y)u(y)\,dy. \tag{2.6}$$

Note that G_0 is the Riesz potential. See Stein[23, p.117], in which I_1 is the same as the operator G_0 in the present paper. Note also that (2.3), (2.4) and Lemma 2.1 yield

$$\int_0^{+\infty} e^{tz}\,\frac{t}{\pi^2\,(t^2+|x|^2)^2}\,dt = g_z(x) \tag{2.7}$$

if $\operatorname{Re} z < 0$.

Theorem 2.1 *If $z \in \mathbb{C} \setminus [0, +\infty)$, then*

$$R_0(z)u = G_z u$$

for all $u \in C_0^\infty(\mathbb{R}^3)$.

Proof. It is sufficient to show that

$$(R_0(z)u,\, v)_{L^2} = (G_z u,\, v)_{L^2} \tag{2.8}$$

for all $z \in \mathbb{C} \setminus [0, +\infty)$ and all $u,\, v \in C_0^\infty(\mathbb{R}^3)$.

As mentioned before Lemma 2.1, we have

$$\begin{aligned}(R_0(z)u,\, v)_{L^2} &= \int_0^{+\infty} e^{tz}\,(e^{-tH_0}u,\, v)_{L^2}\,dt \\ &= \int_0^{+\infty} e^{tz}\,\Big\{\int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3} P_t(x-y)u(y)\,dy\Big)\,\overline{v(x)}\,dx\Big\}\,dt \qquad (2.9)\end{aligned}$$

if $\operatorname{Re} z < 0$. In order to make a change of order of integration in (2.9), we shall show that the function $e^{tz}P_t(x-y)u(y)\overline{v(x)}$ is absolutely integrable with respect the variables x, y and t if $\operatorname{Re} z < 0$ and $u,\, v \in C_0^\infty(\mathbb{R}^3)$. To this end, we see (by integration by parts as in (2.2)) that

$$\begin{aligned}\int_0^{+\infty} e^{t(\operatorname{Re} z)}\,\frac{t}{(t^2+a^2)^2}\,dt &= \frac{1}{2a^2} + \frac{\operatorname{Re} z}{2}\int_0^{+\infty} e^{t(\operatorname{Re} z)}\,\frac{1}{t^2+a^2}\,dt \\ &\le \frac{1}{2a^2} + \frac{|\operatorname{Re} z|}{2a^2}\int_0^{+\infty} e^{t(\operatorname{Re} z)}\,dt \\ &= \frac{1}{a^2}.\end{aligned}$$

This estimate, together with (2.1), implies that

$$
\begin{aligned}
&\iiint_{\mathbb{R}^6\times(0,\,\infty)} \left| e^{tz}P_t(x-y)u(y)\overline{v(x)} \right| dx\,dy\,dt \\
&\quad\le \frac{1}{\pi^2}\iint_{\mathbb{R}^6} \frac{|u(y)v(x)|}{|x-y|^2}\,dx\,dy \\
&\quad= \frac{1}{\pi^2}\int_{\mathbb{R}^3} |v(x)|\,dx \Big(\int_{|x-y|\le 1} + \int_{|x-y|\ge 1} \Big) \frac{|u(y)|}{|x-y|^2}\,dy \\
&\quad\le \frac{1}{\pi^2}\|v\|_{L^1}\Big(\|u\|_{L^\infty}\int_{|y|\le 1}\frac{1}{|y|^2}\,dy + \|u\|_{L^1} \Big) < +\infty.
\end{aligned}
$$

Therefore we can make a change of order of integration in (2.9), and we get

$$
\begin{aligned}
&(R_0(z)u,\, v)_{L^2} \\
&= \int_{\mathbb{R}^3}\Big\{ \int_{\mathbb{R}^3}\Big(\int_0^{+\infty} e^{tz}\frac{t}{\pi^2(t^2+|x-y|^2)^2}\,dt \Big) u(y)\,dy \Big\} \overline{v(x)}\,dx
\end{aligned}
\tag{2.10}
$$

when Re $z < 0$. If we apply Lemma 2.1 to the integral with respect to the t variable in (2.10) and appeal to (2.7), we obtain

$$
(R_0(z)u,\, v)_{L^2} = (G_z u,\, v)_{L^2} \quad \text{on } \{\, z \in \mathbb{C} \mid \text{Re } z < 0 \,\}. \tag{2.11}
$$

Differentiating

$$
\iint_{\mathbb{R}^6} g_z(x-y)u(y)\,\overline{v(x)}\,dxdy
$$

with respect to z under the sign of integration (recall (2.3), (2.4) and that u, $v \in C_0^\infty(\mathbb{R}^3)$), we can deduce that $(G_z u,\, v)_{L^2}$ is a holomorphic function of z in $\mathbb{C}\setminus[0,\,+\infty)$. In view of the fact that $(R_0(z)u,\, v)_{L^2}$ is also a holomorphic function of z in $\mathbb{C}\setminus[0,\,+\infty)$, (2.11) implies that (2.8) holds on $\mathbb{C}\setminus[0,\,+\infty)$ for all u, $v \in C_0^\infty(\mathbb{R}^3)$. □

Remark. Since $R_0(z)$ is a bounded operator in $L^2(\mathbb{R}^3)$ for any $z \in \mathbb{C}\setminus[0,\,+\infty)$, Theorem 2.1 implies that so is G_z. On the other hand, it is a well-known fact (Stein[23, Chapter V, §1.2]) that the Riesz potential G_0 cannot be a bounded operator in $L^2(\mathbb{R}^3)$. This makes it difficult to show directly from (2.3)–(2.6) that G_z is a bounded operator in $L^2(\mathbb{R}^3)$.

3 Properties of the resolvents of H_0

This section is devoted to investigating properties of the resolvents of $H_0 = \sqrt{-\Delta}$. We put emphasis on expressions of the extended resolvents $R_0^{\pm}(z)$ in the forms which will be useful for establishing the radiation conditions for $\sqrt{-\Delta}$ as well as $\sqrt{-\Delta} + V(x)$.

We shall begin with the limiting absorption principle for $\sqrt{-\Delta}$, which assures the existence of the extended resolvents $R_0^{\pm}(z)$, that is, the existence of the boundary values of $R_0(z)$ on the positive axis. The limiting absorption principle for $\sqrt{-\Delta + m^2}$ was first proved by Umeda[25] in the case where $m > 0$. The results in [25] were greatly generalized by Ben-Artzi and Nemirovski[3], where they were able to treat $\sqrt{-\Delta}$. Actually, Theorem 3.1 below is a corollary to results in Ben-Artzi and Nemirovski[3, Section 2], which is based on a general theory developed by Ben-Artzi and Devinatz[2].

Theorem 3.1 (Ben-Artzi and Nemirovski) *Let $s > 1/2$. Then*

(i) *For any $\lambda > 0$, there exist the limits*

$$R_0^{\pm}(\lambda) = \lim_{\mu \downarrow 0} R_0(\lambda \pm i\mu) \quad \textit{in } \mathbf{B}(L^{2,\,s},\ H^{1,\,-s}).$$

(ii) *The operator-valued functions $R_0^{\pm}(z)$ defined by*

$$R_0^{\pm}(z) = \begin{cases} R_0(z) & \textit{if } \ z \in \mathbb{C}^{\pm} \\ \\ R_0^{\pm}(\lambda) & \textit{if } \ z = \lambda > 0 \end{cases}$$

are $\mathbf{B}(L^{2,\,s},\ H^{1,\,-s})$-valued continuous functions, where $\mathbb{C}^{+}$ and $\mathbb{C}^{-}$ are the upper and the lower half-planes respectively:

$$\mathbb{C}^{\pm} = \{\, z \in \mathbb{C} \mid \ \pm \operatorname{Im} z > 0 \,\}.$$

Theorem 3.2 below gives representation formulae for the extended resolvents $R_0^{\pm}(z)$ of $\sqrt{-\Delta}$ in terms of the extended resolvents $\Gamma_0^{\pm}(z)$ of $-\Delta$ (see Agmon[1, Section 4] for the limiting absorption principle for $-\Delta$). The advantage of Theorem 3.2 is that its representation formulae are convenient tools to derive the radiation conditions for $\sqrt{-\Delta}$, which we shall need in later sections. It should be noted that Theorem 3.2 provides an alternative proof of Theorem 3.1.

Theorem 3.2 *Let $s > 1/2$. Suppose that $b > a > 0$, and define*

$$D_{ab} := \{\, z = \lambda + i\mu \in \mathbb{C} \mid a \le \lambda \le b,\ |\mu| \le \frac{a}{2} \,\}.$$

Then there exist operator-valued functions $A(z)$ and $B(z)$ such that

(i) *$A(z)$ is a $\mathbf{B}(L^{2,s})$-valued continuous function on $\mathbb{C}$,*

(ii) *$B(z)$ is a $\mathbf{B}(L^{2,s},\ H^{1,-s})$-valued continuous function on D_{ab},*

(iii) *$R_0^{\pm}(z) = \Gamma_0^{\pm}(z^2)\, A(z) + B(z)$ for all $z \in D_{ab}^{\pm}$, where*

$$D_{ab}^{\pm} := \{\ z \in D_{ab} \mid \pm\,\mathrm{Im}\ z \ge 0\ \}.$$

Following the idea in Umeda[25, Section 2], we shall give a proof of Theorem 3.2 by means of a series of lemmas. We first note that for $z \in \mathbb{C}^{\pm}$

$$\begin{aligned} R_0(z) &= \mathcal{F}^{-1}\Big[\frac{|\xi| + z}{|\xi|^2 - z^2}\Big]\mathcal{F} \\ &= \mathcal{F}^{-1}\Big[\frac{1}{|\xi|^2 - z^2}\Big]\mathcal{F}\cdot\mathcal{F}^{-1}\Big[z + \gamma(\xi)|\xi|\Big]\mathcal{F} \\ &\quad + \mathcal{F}^{-1}\Big[\frac{\big(1-\gamma(\xi)\big)|\xi|}{|\xi|^2 - z^2}\Big]\mathcal{F}, \end{aligned} \tag{3.1}$$

where γ is a C_0^∞-function, which will be specified soon. It is easy to see that

$$\frac{3}{4}a^2 \le \mathrm{Re}\ z^2 \le b^2 \quad \text{for } \forall z \in D_{ab}, \tag{3.2}$$

and that

$$\pm\,\mathrm{Im}\ z^2 > 0 \quad \text{for } \forall z \in D_{ab} \cap \mathbb{C}^{\pm} \tag{3.3}$$

In view of (3.2) and (3.3), we choose $\gamma \in C_0^\infty(\mathbb{R}^3)$ so that

$$\gamma(\xi) = \begin{cases} 1 & \text{if } \ \frac{1}{2}a^2 \le |\xi|^2 \le \frac{3}{2}b^2 \\ \\ 0 & \text{if } \ |\xi|^2 \le \frac{1}{4}a^2 \text{ or } 2b^2 \le |\xi|^2. \end{cases}$$

One can easily find that

$$\Big|\,|\xi|^2 - z^2\,\Big| \ge \frac{1}{4}a^2 \quad \text{for } \forall z \in D_{ab},\ \ \forall \xi \in \mathrm{supp}[1-\gamma], \tag{3.4}$$

and that

$$\Big|\,|\xi|^2 - z^2\,\Big| \ge \frac{1}{3}|\xi|^2 \qquad \text{if } \ z \in D_{ab},\ \ |\xi|^2 \ge \frac{3}{2}b^2. \tag{3.5}$$

In accordance with (3.1), we now define $A(z)$ and $B(z)$ by

$$A(z) := \mathcal{F}^{-1}\Big[z + \gamma(\xi)|\xi|\Big]\mathcal{F} = zI + \mathcal{F}^{-1}\Big[\gamma(\xi)|\xi|\Big]\mathcal{F} \tag{3.6}$$

and

$$B(z) := \mathcal{F}^{-1}\Big[\frac{(1-\gamma(\xi))|\xi|}{|\xi|^2 - z^2}\Big]\mathcal{F} \tag{3.7}$$

respectively. With

$$\varGamma_0(z) = (-\Delta - z)^{-1}, \quad z \in \mathbb{C} \setminus [0, +\infty), \tag{3.8}$$

we have

$$R_0(z) = \varGamma_0(z^2)\, A(z) + B(z) \quad \text{for } \forall z \in D_{ab} \text{ with } \mathrm{Im}\; z \neq 0 \tag{3.9}$$

by (3.1). In order to treat $A(z)$ and $B(z)$ in weighted L^2-spaces and weighted Sobolev spaces, we need terminology and a boundedness result on pseudodifferential operators in these spaces.

Definition. A C^∞-function $p(x,\xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be in the class $S^\mu_{0,0}$ ($\mu \in \mathbb{R}$) if for any pair α and β of multi-indices there exists a constant $C_{\alpha\beta} \geq 0$ such that

$$\Big|\Big(\frac{\partial}{\partial\xi}\Big)^\alpha\Big(\frac{\partial}{\partial x}\Big)^\beta p(x,\xi)\Big| \leq C_{\alpha\beta}\,\langle\xi\rangle^\mu.$$

The class $S^\mu_{0,0}$ is a Fréchet space equipped with the seminorms

$$|p|^{(\mu)}_\ell = \max_{|\alpha|,|\beta|\leq\ell}\sup_{x,\xi}\Big\{\Big|\Big(\frac{\partial}{\partial\xi}\Big)^\alpha\Big(\frac{\partial}{\partial x}\Big)^\beta p(x,\xi)\Big|\langle\xi\rangle^{-\mu}\Big\} \qquad (\ell = 0, 1, 2, \cdots).$$

For $p(x,\, \xi) \in S^\mu_{0,0}$, a pseudodifferential operator $p(x,\, D)$ is defined by

$$p(x,\, D)u(x) = (2\pi)^{-n/2}\int e^{ix\cdot\xi}p(x,\, \xi)\hat{u}(\xi)\, d\xi.$$

It is well-known (Kumano-go [12, Theorem 1.3, p.57]) that $p(x,\, D)$ maps $\mathcal{S}(\mathbb{R}^n)$ continuously into itself, and by duality, maps $\mathcal{S}'(\mathbb{R}^n)$ into itself.

Lemma 3.1 *Let $p(x,\, \xi)$ belong to $S^{-m}_{0,0}$ for some integer $m \geq 0$, and let $s \in \mathbb{R}$. Then there exist a nonnegative constant $C = C_{ms}$ and a positive integer $\ell = \ell_{ms}$ such that*

$$\|p(x,\, D)u\|_{H^{m,s}} \;\leq\; C\,|p|^{(-m)}_\ell\,\|u\|_{L^{2,s}}$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$.

Proof. We first prove the lemma in the case where $m = 0$. If $s \geq 0$, the lemma is a special case of [25, Lemma 2.2], where $\langle x\rangle^s p(x, D)\langle x\rangle^{-s}$ was shown to be a bounded operator in $L^2(\mathbb{R}^n)$, of which norm is estimated by a constant times $|p|_\ell^{(0)}$ with some integer ℓ.

If $s < 0$, we consider $\langle x\rangle^{-s} p^*(x, D)\langle x\rangle^s$, where $p^*(x, D)$ is a formal adjoint operator of $p(x, D)$ in the sense that

$$(p(x, D)u, v)_{L^2} = (u, p^*(x, D)v)_{L^2}, \qquad u, v \in \mathcal{S}(\mathbb{R}^n).$$

It is well-known that the symbol $p^*(x, \xi)$ of the operator $p^*(x, D)$ belongs to $S^0_{0,0}$ (see [12, Theorem 2.6, p.74]), and that each seminorm of $p^*(x, \xi)$ is estimated by a seminorm of $p(x, \xi)$ (see [12, Theorem 2.5, p.73]). Hence, for all u and v in $\mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned}
\left|(\langle x\rangle^s p(x, D)\langle x\rangle^{-s}u, v)_{L^2}\right| &= \left|(u, \langle x\rangle^{-s} p^*(x, D)\langle x\rangle^s v)_{L^2}\right| \\
&\leq \|u\|_{L^2} \, \|\langle x\rangle^{-s} p^*(x, D)\langle x\rangle^{-(-s)} v\|_{L^2} \\
&\leq \|u\|_{L^2} \, C|p^*|_\ell^{(0)} \, \|v\|_{L^2} \qquad (\because -s > 0) \\
&\leq \|u\|_{L^2} \, C'|p|_{\ell'}^{(0)} \, \|v\|_{L^2},
\end{aligned}$$

where in the second inequality the result in the preceding paragraph was used. We have thus shown that for $s < 0$, the operator $\langle x\rangle^s p(x, D)\langle x\rangle^{-s}$ is bounded in $L^2(\mathbb{R}^n)$, and its norm is estimated by a constant times $|p|_{\ell'}^{(0)}$ with some integer ℓ'.

All that remains is to prove the lemma in the case where m is a positive integer. This can be done in the same manner as in the proof of [25, Lemma 2.2]. We omit the details. □

We now turn to the proof of Theorem 3.2. Note that s in Lemma 3.2 below can be negative. This is due to Lemma 3.1

Lemma 3.2 *For any $s \in \mathbb{R}$, $A(z)$ is a $\mathbf{B}(L^{2,s})$-valued continuous function on $\mathbb{C}$.*

Proof. Since the support of the function γ is away from the origin, it is evident that $\gamma(\xi)|\xi| \in C_0^\infty(\mathbb{R}^3)$, which one can regard as a subset of $S^0_{0,0}$. Then it follows from Lemma 3.1 that $\gamma(D)|D|$ defines a bounded operator in $L^{2,s}(\mathbb{R}^3)$. This immediately implies the lemma, because of the fact that $A(z) = zI + \gamma(D)|D|$. □

Lemma 3.3 *For any $s \geq 0$, $B(z)$ is a $\mathbf{B}(L^{2,s}, H^{1,-s})$-valued continuous function on D_{ab}.*

Proof. In order to decompose the symbol of $B(z)$ into a regular part and a singular part, we shall use the same function $\chi \in C_0^\infty(\mathbb{R}^3)$ as in the beginning of Section 2. We thus define

$$\begin{aligned} B_1(z) &:= \mathcal{F}^{-1}\Big[\frac{(1-\gamma(\xi))\,(1-\chi(\xi))\,|\xi|}{|\xi|^2 - z^2}\Big]\mathcal{F}, \\ B_2(z) &:= \mathcal{F}^{-1}\Big[\frac{(1-\gamma(\xi))\,\chi(\xi)\,|\xi|}{|\xi|^2 - z^2}\Big]\mathcal{F}. \end{aligned}$$

It is obvious that

$$B(z) = B_1(z) + B_2(z). \tag{3.10}$$

Therefore, it is sufficient to show that both $B_1(z)$ and $B_2(z)$ are $\mathbf{B}(L^{2,s}, H^{1,-s})$-valued continuous functions on D_{ab}.

As for $B_1(z)$, we note that the symbol of $B_1(z)$ is a C^∞-function, and we shall apply Lemma 3.1. To this end, we exploit the inequalities (3.4) and (3.5), and obtain

$$\Big|\Big(\frac{\partial}{\partial\xi}\Big)^\alpha \Big\{\frac{\big(1-\gamma(\xi)\big)\,\big(1-\chi(\xi)\big)\,|\xi|}{|\xi|^2 - z^2}\Big\}\Big| \le C_\alpha\,\langle\xi\rangle^{-1-|\alpha|} \tag{3.11}$$

for all α, where C_α is a constant independent of $z \in D_{ab}$. It then follows from (3.11) and Lemma 3.1 with $m = 1$ that for every $s \in \mathbb{R}$

$$\|B_1(z)u\|_{H^{1,\,s}} \le C_s \|u\|_{L^{2,\,s}}, \qquad u \in \mathcal{S}(\mathbb{R}^3), \tag{3.12}$$

where C_s is a constant independent of $z \in D_{ab}$. Therefore, for each $z \in D_{ab}$, $B_1(z)$ can be extended to a bounded operator from $L^{2,\,s}(\mathbb{R}^3)$ to $H^{1,s}(\mathbb{R}^3)$. In a similar fashion, we can see that for $z,\, z' \in D_{ab}$

$$\begin{aligned} &\Big|\Big(\frac{\partial}{\partial\xi}\Big)^\alpha \Big\{\frac{(1-\gamma(\xi))\,(1-\chi(\xi))\,|\xi|}{|\xi|^2 - z^2} - \frac{(1-\gamma(\xi))\,(1-\chi(\xi))\,|\xi|}{|\xi|^2 - z'^2}\Big\}\Big| \\ &\qquad \le C_\alpha |z - z'|\,\langle\xi\rangle^{-3-|\alpha|} \end{aligned} \tag{3.13}$$

for all α, where the constant C_α is independent of $z,\, z' \in D_{ab}$. Lemma 3.1 with $m = 3$, together with (3.13), gives

$$\|\{B_1(z) - B_1(z')\}u\|_{H^{3,\,s}} \le C_s |z - z'|\,\|u\|_{L^{2,\,s}}, \qquad u \in \mathcal{S}(\mathbb{R}^3),$$

for every $s \in \mathbb{R}$, where C_s is a constant being uniform for $z,\, z' \in D_{ab}$. In particular, $B_1(z)$ is a $\mathbf{B}(L^{2,s},\, H^{1,s})$-valued continuous function on D_{ab} for every $s \in \mathbb{R}$. As a result, we can deduce that $B_1(z)$ is a $\mathbf{B}(L^{2,s},\, H^{1,-s})$-valued continuous function on D_{ab} for every $s \ge 0$.

As for $B_2(z)$, we exhibit it as a product of a pseudodifferential operator with a smooth symbol and a pseudodifferential operator with a singular symbol:

$$\begin{aligned} B_2(z) &= \mathcal{F}^{-1}\Big[\frac{(1-\gamma(\xi))}{|\xi|^2 - z^2}\Big]\mathcal{F}\cdot\mathcal{F}^{-1}\Big[\chi(\xi)\,|\xi|\Big]\mathcal{F} \\ &=: B_{2,1}(z)\cdot B_{2,2}. \end{aligned}$$

Note that $B_{2,1}(z)$ can be treated in a similar fashion to $B_1(z)$, and one can deduce that for every $s \in \mathbb{R}$

$$\|B_{2,1}(z)u\|_{H^{2,\,s}} \le C_s\|u\|_{L^{2,\,s}}, \qquad u \in \mathcal{S}(\mathbb{R}^3), \tag{3.14}$$

where C_s is a constant independent of $z \in D_{ab}$, and that

$$\|\{B_{2,1}(z) - B_{2,1}(z')\}u\|_{H^{4,\,s}} \le C'_s|z-z'|\,\|u\|_{L^{2,\,s}}, \qquad u \in \mathcal{S}(\mathbb{R}^3),$$

for every $s \in \mathbb{R}$, where C'_s is a constant independent of $z,\ z' \in D_{ab}$. In particular, $B_{2,1}(z)$ is a $\mathbf{B}(L^{2,s},\, H^{2,s})$-valued continuous function on D_{ab} for every $s \in \mathbb{R}$. Taking into account the fact that $\chi(\xi)|\xi|$ is a bounded function, we see that for $s \ge 0$

$$\begin{aligned} \|B_{2,2}\,u\|_{L^{2,\,-s}} &\le \|B_{2,2}\,u\|_{L^2} \\ &\le (\max_\xi \chi(\xi)|\xi|\,)\,\|u\|_{L^2} \\ &\le (\max_\xi \chi(\xi)|\xi|\,)\,\|u\|_{L^{2,\,s}}. \end{aligned}$$

Hence $B_{2,2} \in \mathbf{B}(L^{2,s},\, L^{2,-s})$ for every $s \ge 0$, which implies that $B_2(z) = B_{2,1}(z)\,B_{2,2}$ is a $\mathbf{B}(L^{2,s},\, H^{2,-s})$-valued continuous function on D_{ab} for every $s \ge 0$. Summing up the arguments, we have completed the proof of the lemma. □

It is clear that we have actually showed the following assertion in the proof of Lemma 3.3.

Corollary to Lemma 3.3 *There exist a $\mathbf{B}(L^{2,s},\, H^{1,s})$-valued continuous function $B_1(z)$ on D_{ab} for every $s \in \mathbb{R}$ and a $\mathbf{B}(L^{2,s},\, H^{2,-s})$-valued continuous function $B_2(z)$ on D_{ab} for every $s \ge 0$ such that $B(z)u = B_1(z)u + B_2(z)u$ for all $u \in L^{2,s}(\mathbb{R}^3)$ with $s \ge 0$.*

Proof of Theorem 3.2 Assertions (i) and (ii) are special cases of Lemmas 3.2 and 3.3 respectively, since we assume $s > 1/2$ in the theorem.

To prove assertion (iii), we recall a well-known result by Agmon[1, Theorem 4.1]: the extended resolvents $\Gamma_0^{\pm}(z)$ defined by

$$\Gamma_0^{\pm}(z) = \begin{cases} \Gamma_0(z) & \text{if } \ z \in \mathbb{C}^{\pm} \\ \\ \Gamma_0^{\pm}(\lambda) & \text{if } \ z = \lambda > 0 \end{cases} \tag{3.15}$$

are $\mathbf{B}(L^{2,s},\, H^{2,-s})$-valued continuous function on $\mathbb{C}^{\pm} \cup (0, +\infty)$ provided that $s > 1/2$. In view of assertions (i) and (ii), the theorem follows from this fact and (3.9), together with (3.3). $\square$

It is worthwhile to improve assertion (ii) of Theorem 3.2.

Theorem 3.3 *Under the same assumptions and with the same notation as in Theorem 3.2, the operator-valued function $B(z)$ has the following property: If $0 \le s < 5/2$ and $t < s - 3/2$, then $B(z)$ is a $\mathbf{B}(L^{2,s},\ H^{1,t})$-valued continuous function on D_{ab}.*

Proof. We utilize the decomposition (3.10) of $B(z)$ made in the proof of Lemma 3.3, where it was actually shown that $B_1(z)$ is a $\mathbf{B}(L^{2,s},\ H^{1,s})$-valued continuous function on D_{ab} for any $s \in \mathbb{R}$ (see Corollary to Lemma 3.3). It is therefore sufficient to prove that $B_2(z)$ has the property described in the theorem.

We use the same factorization as in the proof of Lemma 3.3: $B_2(z) = B_{2,1}(z)\, B_{2,2}$. Apparently, we have shown in the proof of Lemma 3.3 that $B_{2,1}(z)$ is a $\mathbf{B}(L^{2,s},\ H^{2,s})$-valued continuous function on D_{ab} for every $s \in \mathbb{R}$. Since $B_{2,2}$ is equal to a pseudodifferential operator $\chi(D)\sqrt{-\Delta}$, we can apply Umeda[26, Lemma 5.2]. Thus we see that $B_{2,2} \in \mathbf{B}(L^{2,s},\ L^{2,t})$ if $0 \le s < n/2+1$ and $t < s - n/2$. It then follows that $B_2(z)$ is a $\mathbf{B}(L^{2,s},\ H^{2,t})$-valued continuous function on D_{ab} under the assumption of the theorem. $\square$

4 Integral kernels of $R_0^{\pm}(\lambda)$

In this section, we shall derive the integral kernels of the boundary values $R_0^{\pm}(\lambda)$ of the resolvent $R_0(z)$ on the positive half axis (recall that the existence of $R_0^{\pm}(\lambda)$ was assured in the previous section). We have to start with examining the boundary values of the complex variable function $\mathrm{ci}(-z)$, $z \in \mathbb{C} \setminus [0, +\infty)$, since the integral kernel $g_z(x)$ of $R_0(z)$ contains the term $\mathrm{ci}(-z|x|)$ as was shown in Section 2. In connection with the integral kernel $g_z(x)$, it is worthwhile noting that all of $\sin(z)$, $\cos(z)$ and $\mathrm{si}(z)$ are entire functions, but $\mathrm{ci}(z)$ is a many-valued function with a logarithmic branch point at $z = 0$; we shall choose the principal branch (see Subsection A.1 in Appendix).

By (A.1) in Appendix and the definition of the function $h_e(z)$ introduced in Appendix, we have

$$\mathrm{ci}(-z) = -i\,\mathrm{Arg}\,(-z) - \gamma - \log|z| + h_e(z) \tag{4.1}$$

for $z \in \mathbb{C} \setminus [0, +\infty)$. It follows from (4.1) that if $\lambda > 0$, then

$$\mathrm{ci}(-(\lambda \pm i\mu)) \to \pm i\pi + \mathrm{ci}(\lambda) \quad \text{as } \mu \downarrow 0, \tag{4.2}$$

where we have used that fact that h_e is an even function, as is remarked in Appendix.

We now turn to the boundary values of $\ell_z(x)$ on the positive axis (see (2.3) for the definition of $\ell_z(x)$). Putting $z = \lambda \pm i\mu$ with λ, $\mu > 0$, we take the limit of $\ell_z(x)$ as $\mu \downarrow 0$. We then see that

$$\begin{aligned} \ell_z(x) \quad \to \quad & \frac{\lambda}{2\,\pi^2\,|x|}\Big[\,\sin(\lambda|x|)\,\{\pm i\pi + \mathrm{ci}(\lambda|x|)\} \\ & \qquad\qquad - \cos(\lambda|x|)\,\{-\pi - \mathrm{si}(\lambda|x|)\}\,\Big] \end{aligned} \tag{4.3}$$

for each $x \neq 0$ as $\mu \downarrow 0$, where we have used (4.2) and (A.3) in Appendix. By the fact that $e^{\pm i\lambda|x|} = \cos(\lambda|x|) \pm i\sin(\lambda|x|)$, we get

$$\ell_z(x) \to \frac{\lambda}{2\pi} \cdot \frac{e^{\pm i\lambda|x|}}{|x|} + m_\lambda(x), \tag{4.4}$$

for each $x \neq 0$ as $\mu \downarrow 0$, where

$$m_\lambda(x) := \frac{\lambda}{2\,\pi^2\,|x|}\Big[\,\sin(\lambda|x|)\,\mathrm{ci}(\lambda|x|) + \cos(\lambda|x|)\,\mathrm{si}(\lambda|x|)\,\Big]. \tag{4.5}$$

In accordance with (2.4) in Section 2, we define

$$g_\lambda^{\pm}(x) := \frac{1}{2\pi^2|x|^2} + \frac{\lambda}{2\pi} \cdot \frac{e^{\pm i\lambda|x|}}{|x|} + m_\lambda(x). \tag{4.6}$$

(Recall that $g_{\lambda+i0}(x)$ in Introduction, which is exactly the same as $g_\lambda^+(x)$ defined above.) It follows immediately from (2.4), (4.4) and (4.6) that for $\lambda > 0$

$$g_{\lambda\pm i\mu}(x) \to g_\lambda^\pm(x), \qquad x \neq 0 \tag{4.7}$$

as $\mu \downarrow 0$. From the view point of the time independent theory of scattering, it is very important that the leading term of (4.6) at infinity is the second term $\lambda e^{\pm i\lambda|x|}/(2\pi|x|)$, which is the same, up to a constant, as the integral kernels of the boundary values of the resolvent $\Gamma_0(z)$ of $-\Delta$ on $\mathbb{R}^3$.

We finally state a result on the integral representations of the boundary values of the resolvent $R_0(z)$.

Theorem 4.1 *Let $s > 1/2$. If $\lambda > 0$, then*

$$(R_0^\pm(\lambda)u,\, v)_{-s,s} = \int_{\mathbb{R}^3} \Big\{ \int_{\mathbb{R}^3} g_\lambda^\pm(x-y)\, u(y)\, dy \Big\} \overline{v(x)}\, dx$$

for all u and $v \in C_0^\infty(\mathbb{R}^3)$.

Proof. It follows from (2.6) and Theorem 2.1 that

$$(R_0(\lambda \pm i\mu)u,\, v)_{L^2} = \int_{\mathbb{R}^3} \Big\{ \int_{\mathbb{R}^3} g_{\lambda\pm i\mu}(x-y)\, u(y)\, dy \Big\} \overline{v(x)}\, dx, \tag{4.8}$$

where $\mu > 0$. Since $R_0^\pm(z)$ defined in Theorem 3.1 are $\mathbf{B}(L^{2,s},\, L^{2,-s})$-valued continuous functions on $\mathbb{C}^\pm \cup (0, +\infty)$ respectively, we see that

$$(R_0(\lambda \pm i\mu)u,\, v) \to (R_0^\pm(\lambda)u,\, v)_{-s,s} \tag{4.9}$$

as $\mu \downarrow 0$. As for the right hand side of (4.8), we shall apply the Lebesgue dominated convergence theorem. To this end, we first note that $g_z(x)$ is locally integrable. More precisely, in view of (2.3), (2.4), (4.1) and the fact that $h_e(z)$ and $\mathrm{si}(z)$ are entire functions, we find that for each pair of $\lambda > 0$ and $a > 1$, there corresponds a positive constant $C_{\lambda a}$, independent of μ with $0 < \mu < 1$, such that

$$|g_{\lambda\pm i\mu}(x)| \leq C_{\lambda a} \begin{cases} 1/|x|^2 & \text{if } \ |x| \leq 1 \\ \\ 1 & \text{if } \ 1 \leq |x| \leq a. \end{cases} \tag{4.10}$$

Since u and v lie in $C_0^\infty(\mathbb{R}^3)$, it follows from (4.10) that

$$|g_{\lambda\pm i\mu}(x-y)u(x)\overline{v(y)}| \leq C_{\lambda uv} \begin{cases} \dfrac{|u(x)v(y)|}{|x-y|^2} & \text{if } \ |x-y| \leq 1 \\ \\ |u(x)v(y)| & \text{otherwise} \end{cases} \tag{4.11}$$

where $C_{\lambda uv} > 0$ is a constant, being dependent on λ, u and v, but independent of μ with $0 < \mu < 1$. Note that the function on the right hand side of (4.11) is integrable on $\mathbb{R}^3 \times \mathbb{R}^3$. By virtue of (4.7) and (4.11), we can apply the Lebesgue dominated convergence theorem, and see that

$$\iint_{\mathbb{R}^6} g_{\lambda \pm i\mu}(x-y)u(x)\overline{v(y)}\,dxdy \to \iint_{\mathbb{R}^6} g_\lambda^\pm(x-y)u(x)\overline{v(y)}\,dxdy \quad (4.12)$$

as $\mu \downarrow 0$. Combining (4.8) with (4.9), (4.12) gives the theorem. $\square$

It follows from Theorem 4.1 that the integral operators defined by

$$G_\lambda^\pm u(x) := \int_{\mathbb{R}^3} g_\lambda^\pm(x-y)\,u(y)\,dy, \qquad u \in C_0^\infty(\mathbb{R}^3) \quad (4.13)$$

can be extended to bounded operators from $L^{2,\,s}(\mathbb{R}^3)$ to $H^{1,\,-s}(\mathbb{R}^3)$ for $s > 1/2$, since $R_0^\pm(\lambda) \in \mathbf{B}(L^{2,\,s},\, H^{1,\,-s})$ for $s > 1/2$, and

$$R_0^\pm(\lambda)u = G_\lambda^\pm u, \qquad u \in C_0^\infty(\mathbb{R}^3). \quad (4.14)$$

5 Estimates on the integral operators

In this section, we consider the Riesz potential G_0 (see (2.5)) and the integral operators $K_\lambda^\pm$, M_λ defined by

$$(K_\lambda^\pm u)(x) := \frac{\lambda}{2\pi}\int_{\mathbb{R}^3} \frac{e^{\pm i\lambda|x-y|}}{|x-y|}\, u(y)\,dy, \tag{5.1}$$

$$(M_\lambda u)(x) := \int_{\mathbb{R}^3} m_\lambda(x-y)\,u(y)\,dy. \tag{5.2}$$

(For the definition of $m_\lambda(x)$, see (4.5).) Our task here is to derive estimates of these operators in weighted L^2-spaces as well as pointwise estimates of $(G_0u)(x)$, $(K_\lambda^\pm u)(x)$ and $(M_\lambda u)(x)$ for u belonging to some weighted L^2-space or to a suitable class of functions. We shall apply these estimates in the later sections in order to examine asymptotic behaviors of the generalized eigenfunctions of $\sqrt{-\Delta}+V(x)$ on $\mathbb{R}^3$. In connection with this, it is important to notice that we have formal identities

$$R_0^\pm(\lambda) = G_\lambda^\pm = G_0 + K_\lambda^\pm + M_\lambda, \tag{5.3}$$

which hold at least on $C_0^\infty(\mathbb{R}^3)$; see (4.6), (4.13) and (5.17).

It is well-known (Stein[23, p.119]) that the inequality

$$\|G_0u\|_{L^\infty} \le C\|u\|_{L^p}$$

cannot hold for any $p \ge 1$. Furthermore, we make a remark that if one defines

$$u_0(x) := \begin{cases} 1/|x| & (|x| \le 1) \\ 0 & (\text{otherwise}), \end{cases}$$

then $u_0 \in L^{2,s}(\mathbb{R}^3)$ for all $s \in \mathbb{R}$, and $(G_0u_0)(0) = +\infty$. In spite of these facts, we need to find a class of functions u for which $(G_0u)(x)$ are bounded functions of x. Actually, we shall obtain two sufficient conditions (see Lemmas 5.2 and 5.3 below), either of which is suitable for showing the boundedness of generalized eigenfunctions of $\sqrt{-\Delta}+V(x)$ on $\mathbb{R}^3$. It is also well-known (Stein [23, p. 119]) that the inequality

$$\|G_0u\|_{L^q} \le C\|u\|_{L^p}$$

holds only if $q^{-1} = p^{-1} - 3^{-1}$ in the context of the present paper. When $p = 2$, we actually have

$$\|G_0u\|_{L^6} \le C\|u\|_{L^2}. \tag{5.4}$$

On the other hand, we are going to show a few boundedness results on G_0 in the framework of weighted L^2-spaces as well as in some other frameworks.

Lemma 5.1 *Let $s > 3/2$. Then*

(i) $G_0 \in \mathbf{B}(L^{2,\,s},\, L^2)$.

(ii) $G_0 \in \mathbf{B}(L^2,\, L^{2,\,-s})$.

Proof. Let $u \in L^{2,\,s}(\mathbb{R}^3)$. Since $s > 3/2$, the Schwarz inequality gives

$$\int_{\mathbb{R}^3} |u(x)|\,dx = \int_{\mathbb{R}^3} \langle x\rangle^{-s} \cdot \langle x\rangle^{s}|u(x)|\,dx \le C_s\|u\|_{L^{2,s}}, \tag{5.5}$$

hence $u \in L^1(\mathbb{R}^3)$. With $B = \{\, x \mid |x| \le 1 \,\}$ and $E = \{\, x \mid |x| \ge 1 \,\}$, we decompose the function $1/(2\pi^2|x|^2)$ into two parts:

$$\begin{aligned}\frac{1}{2\pi^2|x|^2} &= \frac{1_B(x)}{2\pi^2|x|^2} + \frac{1_E(x)}{2\pi^2|x|^2} \\ &=: h_B(x) + h_E(x),\end{aligned} \tag{5.6}$$

where $1_B(x)$ and $1_E(x)$ are the characteristic functions of the sets B and E respectively. It is clear that $h_B(x) \in L^1(\mathbb{R}^3)$ and $h_E(x) \in L^2(\mathbb{R}^3)$, and that

$$G_0 u = h_B * u + h_E * u. \tag{5.7}$$

If we regard u as a function belonging to $L^2(\mathbb{R}^3)$, we can apply the Young inequality (see Stein[23, p.271]) to $h_B * u$, and obtain

$$\|h_B * u\|_{L^2} \le \|h_B\|_{L^1}\|u\|_{L^2} \le \|h_B\|_{L^1}\|u\|_{L^{2,s}}. \tag{5.8}$$

If we regard u as a function belonging to $L^1(\mathbb{R}^3)$ (recall (5.5)), we can also apply the Young inequality to $h_E * u$, and obtain

$$\|h_E * u\|_{L^2} \le \|h_E\|_{L^2}\|u\|_{L^1} \le \|h_E\|_{L^2}\|u\|_{L^{2,s}}, \tag{5.9}$$

where we have used (5.5). Combining (5.7)–(5.9), we conclude that assertion (i) is true.

To prove assertion (ii), we note that G_0 is symmetric on $C_0^\infty(\mathbb{R}^3)$:

$$(G_0 u,\, v)_{L^2} = (u,\, G_0 v)_{L^2} \quad \text{for } u,\ v \in C_0^\infty(\mathbb{R}^3),$$

which, together with assertion (i), implies

$$|(u,\, G_0 v)_{L^2}| \le \|G_0 u\|_{L^2}\|v\|_{L^2} \le C\|u\|_{L^{2,s}}\|v\|_{L^2} \tag{5.10}$$

for all $u,\ v \in C_0^\infty(\mathbb{R}^3)$. We can regard the left hand side of (5.10) as the anti-duality bracket $(u,\, G_0 v)_{s,-s}$. Hence, by the density argument, it follows from (5.10) that

$$\|G_0 v\|_{L^{2,-s}} \le C\|v\|_{L^2}$$

for all $v \in C_0^\infty(\mathbb{R}^3)$. This yields assertion (ii). $\square$

Lemma 5.2 *If u satisfies*

$$|u(x)| \le C\langle x\rangle^{-\ell}, \quad \ell > 1,\ C > 0, \tag{5.11}$$

then

$$|G_0u(x)| \le C_\ell \,\|\langle\cdot\rangle^\ell u\|_{L^\infty} \times \begin{cases} \langle x\rangle^{-(\ell-1)} & \text{if } 1 < \ell < 3, \\ \langle x\rangle^{-2}\log(1+\langle x\rangle) & \text{if } \ell = 3, \\ \langle x\rangle^{-2} & \text{if } \ell > 3. \end{cases}$$

Proof. It is evident from the definition (2.5) that we have

$$|G_0u(x)| \le \frac{1}{2\pi^2}\|\langle\cdot\rangle^\ell u\|_{L^\infty}\int_{\mathbb{R}^3}\frac{1}{|x-y|^2\langle y\rangle^\ell}\,dy. \tag{5.12}$$

If we apply Lemma A.1 in Appendix, with $n = 3$, $\beta = 2$ and $\gamma = \ell$, to the function defined by the integral on the right hand side of (5.12), then the lemma follows. □

Lemma 5.3 *Suppose that*

$$u \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3), \quad q > 3. \tag{5.13}$$

Then there exists a constant C_q, independent of u, such that

$$\|G_0u\|_{L^\infty} \le C_q(\|u\|_{L^2} + \|u\|_{L^q}) \tag{5.14}$$

Proof. We exploit the same decomposition of G_0u as in (5.7). If we apply the Hölder inequality to $h_B * u$, we obtain

$$|h_B * u(x)| \le \frac{1}{2\pi^2}\left\{\int_{|x-y|\le 1}\frac{1}{|x-y|^{2p}}\,dy\right\}^{1/p}\|u\|_{L^q}, \tag{5.15}$$

where $p^{-1} = 1 - q^{-1}$. Since $q > 3$, it follows that $2p < 3$. Hence the inequality (5.15), together with the assumption (5.13), implies that $h_B * u(x)$ is a bounded function. Similarly, if we apply the Schwarz inequality to $h_E * u$, we can deduce that $h_E * u(x)$ is a bounded function. Summing up, we have shown the inequality (5.14). □

In order to derive estimates of the operator M_λ, we need the inequality

$$|\sin(\rho)\,\mathrm{ci}(\rho) + \cos(\rho)\,\mathrm{si}(\rho)| \le \text{const.}(1+\rho)^{-1}, \qquad 0 < \rho < +\infty, \tag{5.16}$$

which follows from the inequalities in the subsections A.1 and A.2 in Appendix. The inequality (5.16), together with (4.5), immediately implies that for each $\lambda > 0$, there is a positive constant C_λ such that

$$|m_\lambda(x)| \le C_\lambda \, |x|^{-1} \, \langle x \rangle^{-1}. \tag{5.17}$$

It is apparent that one can take the constant C_λ in (5.17) to be uniform for λ in each compact interval in $(0, +\infty)$.

Lemma 5.4 *There exists a positive constant C'_λ, being uniform for λ in each compact interval in $(0, +\infty)$, such that*

$$|M_\lambda \, u(x)| \le C'_\lambda \, \|u\|_{L^2} \tag{5.18}$$

for all $u \in L^2(\mathbb{R}^3)$.

Proof. It follows from (5.17) that $m_\lambda \in L^2(\mathbb{R}^3)$. Applying the Schwarz inequality to the right hand side of (5.2) gives the lemma. □

Lemma 5.5 *Let $s > 3/2$. Then there exists a constant $C_{s\lambda}$ such that*

$$|M_\lambda u(x)| \le C_{s\lambda}(\langle x \rangle^{-2} + \langle x \rangle^{-s}) \, \|u\|_{L^{2,s}}$$

for all $u \in L^{2,s}(\mathbb{R}^3)$, $C_{s\lambda}$ being uniform for λ in each compact interval in $(0, +\infty)$.

Proof. Let $u \in L^{2,s}(\mathbb{R}^3)$. We first note that $M_\lambda u(x)$ satisfies the inequality (5.18), since we can regard u as an element in $L^2(\mathbb{R}^3)$. Hence, we have

$$|M_\lambda u(x)| \le C'_\lambda \|u\|_{L^{2,s}}. \tag{5.19}$$

We shall next show the inequality

$$|M_\lambda u(x)| \le C_\lambda \widetilde{C}_s (|x|^{-2} + \langle x \rangle^{-s}) \|u\|_{L^{2,s}}, \tag{5.20}$$

where C_λ is the same constant as in (5.17) and $\widetilde{C}_s$ is a constant depending only on s. The inequality (5.20), together with the inequality (5.19), gives the lemma. In order to show (5.20), we decompose $M_\lambda u(x)$ into three terms:

$$M_\lambda u(x) = I(x) + II(x) + III(x), \tag{5.21}$$

where

$$I(x) := \int_{|y| \le |x|/2} m_\lambda(x - y) \, u(y) \, dy, \tag{5.22}$$

$$II(x) := \int_{\substack{|y|\geq|x|/2 \\ |x-y|\geq|x|/2}} m_\lambda(x-y)\, u(y)\, dy, \tag{5.23}$$

and

$$III(x) := \int_{\substack{|y|\geq|x|/2 \\ |x-y|\leq|x|/2}} m_\lambda(x-y)\, u(y)\, dy. \tag{5.24}$$

To deal with $I(x)$, we note that $|x-y| \geq |x| - |y| \geq |x|/2$ if $|y| \leq |x|/2$. This fact, together with (5.17), yields

$$\begin{aligned} |I(x)| &\leq C_\lambda \int_{|y|\leq|x|/2} |x-y|^{-2}\, |u(y)|\, dy \\ &\leq 4\, C_\lambda\, |x|^{-2} \int_{|y|\leq|x|/2} |u(y)|\, dy \\ &\leq 4\, C_\lambda\, C_s\, |x|^{-2}\, \|u\|_{L^{2,s}}, \end{aligned} \tag{5.25}$$

where we have used (5.5) in the last inequality and the constant C_s is the same one as in (5.5). It follows from (5.17) that

$$\begin{aligned} |II(x)| &\leq C_\lambda \int_{|x-y|\geq|x|/2} |x-y|^{-2}\, |u(y)|\, dy \\ &\leq 4\, C_\lambda\, C_s\, |x|^{-2}\, \|u\|_{L^{2,s}}. \end{aligned} \tag{5.26}$$

To get an estimate of $III(x)$, we should note that if $|x-y| \leq |x|/2$, then $|y| \geq |x| - |x-y| \geq |x|/2$, hence $\langle y\rangle \geq \langle x\rangle/2$. By using this fact and (5.17), we have

$$\begin{aligned} |III(x)| &\leq C_\lambda \int_{|x-y|\leq|x|/2} |x-y|^{-1} \langle x-y\rangle^{-1}\, |u(y)|\, dy \\ &\leq C_\lambda \Big\{ \int_{|x-y|\leq|x|/2} \frac{\langle y\rangle^{-2s}}{|x-y|^2 \langle x-y\rangle^2}\, dy \Big\}^{1/2} \|u\|_{L^{2,s}} \\ &\leq 2^s\, C_\lambda\, \langle x\rangle^{-s} \|u\|_{L^{2,s}}, \end{aligned} \tag{5.27}$$

where we have used the Schwarz inequality in the second inequality. Finally we deduce from (5.21) – (5.27) that (5.20) is verified. □

As an immediate corollary to Lemma 5.5, we obtain a boundedness result on the operator M_λ.

Lemma 5.6 *If $s > 3/2$, then $M_\lambda \in \mathbf{B}(L^{2,s}, L^2)$. Moreover, the operator norm of M_λ is bounded by a constant $C_{s\lambda}$, which is uniform for λ in each compact interval in* $(0, +\infty)$.

We shall close this section with estimates of the operator $K_\lambda^{\pm}$.

Lemma 5.7 *Let* $s > 1/2$. *Then there exists a positive constant* C_s *such that*

$$|K_\lambda^{\pm}u(x)| \le C_s\,\lambda\,\|u\|_{L^{2,\,s}} \begin{cases} \langle x\rangle^{-(s-1/2)} & \textit{if}\ \ 1/2 < s < 3/2, \\ \langle x\rangle^{-1}\{\log(1+\langle x\rangle)\}^{1/2} & \textit{if}\ \ s = 3/2, \\ \langle x\rangle^{-1} & \textit{if}\ \ s > 3/2 \end{cases}$$

for all $u \in L^{2,\,s}(\mathbb{R}^3)$.

Proof. Let $u \in L^{2,\,s}(\mathbb{R}^3)$. Then applying the Schwarz inequality to (5.1), we have

$$|K_\lambda^{\pm}u(x)| \le \frac{\lambda}{2\pi}\Big\{\int_{\mathbb{R}^3} \frac{1}{|x-y|^2\langle y\rangle^{2s}}\,dy\Big\}^{1/2}\|u\|_{L^{2,s}}. \tag{5.28}$$

We now apply Lemma A.1 in Appendix with $n = 3$, $\beta = 2$ and $\gamma = 2s > 1$, and obtain the lemma. □

As an immediate consequence of Lemma 5.7, we obtain a boundedness result on the operators $K_\lambda^{\pm}$.

Lemma 5.8 *If* $s > 1$, *then* $K_\lambda^{\pm} \in \mathbf{B}(L^{2,\,s},\, L^{2,\,-s})$. *Moreover, the operator norms of* $K_\lambda^{\pm}$ *are bounded by* $C_s\lambda$, *where* C_s *is a constant depending only on* s.

Summing up all the results of Lemma 5.1(ii) and Lemmas 5.4 and 5.7, we see that (5.3) hold on $L^{2,\,s}(\mathbb{R}^3)$, $s > 1/2$, i.e.,

$$R_0^{\pm}(\lambda)u = G_\lambda^{\pm}u = G_0u + K_\lambda^{\pm}u + M_\lambda u \tag{5.29}$$

for all $\lambda > 0$ and all $u \in L^{2,\,s}(\mathbb{R}^3)$ with $s > 1/2$.

6 Radiation conditions for $\sqrt{-\Delta}$

This section is devoted to discussing radiation conditions for $\sqrt{-\Delta}$ on $\mathbb{R}^3$. The main result in this section is Theorem 6.3.

It is well-known that the radiation condition

$$\Big(\frac{\partial}{\partial r} - i\lambda\Big)u = O(r^{-2}) \ \text{ as } r = |x| \to \infty$$

was first introduced in order to single out an outgoing solution of the Helmholtz equation $(-\Delta - \lambda^2)u = f$ in $\mathbb{R}^3$, where $\lambda > 0$. The outgoing solution is the one which behaves as $e^{i\lambda r}/r$ at infinity. In the present paper we shall exploit the Ikebe-Saitō's formulation of the radiation conditions for the Helmholtz equation, which we regard as a special case of the time-independent Schrödinger equations investigated in Ikebe-Saitō[9, Theorems 1.4, 1.5 and Remark 1.6]. See also Saitō[20], [21] and Pladdy-Saitō-Umeda[18] for the formulation of the radiation conditions.

Theorem 6.1 (Ikebe-Saitō) *Let* $1/2 < s < 1$.

(i) *Suppose that* u *belongs* $L^{2,-s}(\mathbb{R}^3) \cap H^2_{\text{loc}}(\mathbb{R}^3)$ *and satisfies the equation*

$$(-\Delta - \lambda^2)u = 0, \qquad \lambda > 0, \tag{6.1}$$

and, in addition, that u *satisfies either the outgoing radiation condition*

$$\Big(\frac{\partial}{\partial x_j} - i\lambda\omega_j\Big)u \in L^{2,\,s-1}(\mathbb{R}^3), \quad j = 1,\, 2,\, 3, \tag{6.2}$$

or the incoming radiation condition

$$\Big(\frac{\partial}{\partial x_j} + i\lambda\omega_j\Big)u \in L^{2,\,s-1}(\mathbb{R}^3), \quad j = 1,\, 2,\, 3, \tag{6.3}$$

where $\omega = x/|x|$. *Then* u *vanishes identically.*

(ii) *Suppose that* $f \in L^{2,\,s}(\mathbb{R}^3)$ *and* $\lambda > 0$. *Then* $v^+(\lambda,\, f) := \Gamma_0^+(\lambda^2)f$ *and* $v^-(\lambda,\, f) := \Gamma_0^-(\lambda^2)f$ *satisfy the equation*

$$(-\Delta - \lambda^2)u = f, \qquad \lambda > 0 \tag{6.4}$$

with the outgoing radiation condition (6.2) *and the incoming radiation condition* (6.3) *respectively.* (*For the definition of* $\Gamma_0^{\pm}(z)$, *see* (3.8) *and* (3.15)).

It is not difficult to find radiation conditions for $\sqrt{-\Delta}$ in a formal manner, because it is easy to see that $\sqrt{-\Delta}\,(\sqrt{-\Delta} u) = -\Delta u$ is formally valid. Actually a difficulty arises if one tries to make sense of $\sqrt{-\Delta}\,(\sqrt{-\Delta} u)$ for $u \in L^{2,-s}(\mathbb{R}^3)$ with $s < 0$. The difficulty comes from the fact that the symbol $|\xi|$ is singular at the origin $\xi = 0$ (see Lieb-Loss[14, §7.15]). In order to overcome the difficulty, we need to clarify the function spaces to which $\sqrt{-\Delta} u$ belongs when u belongs to $L^{2,-s}(\mathbb{R}^3)$ with $s < 0$. The clarification was made in Umeda[26]. We reproduce [26, Theorem 5.8] for the reader's convenience.

Theorem 6.2 (Umeda) *Let $\ell \in \mathbb{R}$. If s and t satisfy either*

$$s \geq 0, \quad t < \min\{1,\, s - 3/2\} \tag{6.5}$$

or

$$-5/2 < s < 0, \quad t < s - 3/2, \tag{6.6}$$

then $\sqrt{-\Delta}$ is a bounded operator from $H^{\ell,\,s}(\mathbb{R}^3)$ to $H^{\ell-1,\,t}(\mathbb{R}^3)$.

With the aid of Theorem 6.2, we shall first make sense of $\sqrt{-\Delta}\,(\sqrt{-\Delta} u)$ for $u \in \mathcal{S}(\mathbb{R}^3)$.

Lemma 6.1 *If $\varphi \in \mathcal{S}(\mathbb{R}^3)$, then $\sqrt{-\Delta}\,(\sqrt{-\Delta}\varphi) \in H^{-1,t}(\mathbb{R}^3)$ for all $t < 1$, and*

$$\sqrt{-\Delta}\,(\sqrt{-\Delta}\,\varphi) = -\Delta\varphi \quad in \;\; \mathcal{S}'(\mathbb{R}^3). \tag{6.7}$$

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^3)$. By virtue of [26, Theorem 4.4], we find that $\sqrt{-\Delta}\,\varphi \in L^{2,s}(\mathbb{R}^3)$ for any $s < 5/2$. It follows from Theorem 6.2 that $\sqrt{-\Delta}\,(\sqrt{-\Delta}\varphi)$ makes sense, and that $\sqrt{-\Delta}\,(\sqrt{-\Delta}\varphi)$ belongs to $H^{-1,t}(\mathbb{R}^3)$ for all $t < 1$. It follows, in particular, that $\sqrt{-\Delta}\,(\sqrt{-\Delta}\varphi) \in \mathcal{S}'(\mathbb{R}^3)$.

To prove (6.7), we take a test function $\psi \in \mathcal{S}(\mathbb{R}^3)$. By definition of the action of $\sqrt{-\Delta}$ on distributions we have

$$\langle \sqrt{-\Delta}\,(\sqrt{-\Delta}\varphi),\, \psi \rangle = (\sqrt{-\Delta}\varphi,\, \sqrt{-\Delta}\,\overline{\psi}\,)_{-s,s} \tag{6.8}$$

if $-5/2 < s < 5/2$. (It follows from [26, Theorem 4.4] that the mapping $\psi \mapsto (\sqrt{-\Delta}\varphi,\, \sqrt{-\Delta}\,\overline{\psi}\,)_{-s,s}$ is a continuous linear functional on $\mathcal{S}(\mathbb{R}^3)$, because one can regard $\sqrt{-\Delta}\varphi$ as a function belonging to $L^{2,-s}(\mathbb{R}^3)$ for any $s > -5/2$, and because one finds that

$$|(\sqrt{-\Delta}\varphi,\, \sqrt{-\Delta}\,\overline{\psi}\,)_{-s,s}| \leq \|\sqrt{-\Delta}\varphi\|_{L^{2,-s}} \|\sqrt{-\Delta}\,\overline{\psi}\|_{L^{2,s}}$$

for any s with $-5/2 < s < 5/2$.) It is clear that we can regard the right hand side of (6.8) as the inner product in $L^2(\mathbb{R}^3)$, and we have

$$\begin{aligned}
\langle \sqrt{-\Delta}\,(\sqrt{-\Delta}\varphi),\, \psi\rangle &= (\sqrt{-\Delta}\varphi,\, \sqrt{-\Delta}\,\overline{\psi})_{L^2} \\
&= (|\xi|\,\mathcal{F}[\varphi],\, |\xi|\,\mathcal{F}[\overline{\psi}])_{L^2_\xi} \\
&= (-\Delta\varphi,\, \overline{\psi})_{L^2} \\
&= \langle -\Delta\varphi,\, \psi\rangle,
\end{aligned}$$

where we have used the Plancherel formula twice. This proves (6.7). □

Lemma 6.2 *Suppose that $\ell \in \mathbb{R}$ and $0 < s < 1$. If $u \in H^{\ell,-s}(\mathbb{R}^3)$, then*

$$\sqrt{-\Delta}\,(\sqrt{-\Delta}\,u) = -\Delta u \quad \textit{in } \mathcal{S}'(\mathbb{R}^3). \tag{6.9}$$

Proof. Let u be in $H^{\ell,-s}(\mathbb{R}^3)$. Since $\mathcal{S}(\mathbb{R}^3)$ is dense in $H^{\ell,-s}(\mathbb{R}^3)$, we can choose a sequence $\{\varphi_j\} \subset \mathcal{S}(\mathbb{R}^3)$ so that $\varphi_j \to u$ in $H^{\ell,-s}(\mathbb{R}^3)$ as $j \to \infty$. It follows from Theorem 6.2 that

$$\sqrt{-\Delta}\varphi_j \to \sqrt{-\Delta}u \quad \text{in } H^{\ell-1,\,t}(\mathbb{R}^3) \tag{6.10}$$

for any $t < -s - 3/2$. In view of the hypothesis that $0 < s < 1$, we can find that (6.10) holds for any t satisfying $-5/2 < t < -s - 3/2$. Therefore, it follows from Theorem 6.2 again that

$$\sqrt{-\Delta}\,(\sqrt{-\Delta}\varphi_j) \to \sqrt{-\Delta}\,(\sqrt{-\Delta}u) \quad \text{in } H^{\ell-2,\,t}(\mathbb{R}^3) \tag{6.11}$$

for any $t < -s - 3$. In particular, we have

$$-\Delta\varphi_j \to \sqrt{-\Delta}\,(\sqrt{-\Delta}u) \quad \text{in } \mathcal{S}'(\mathbb{R}^3), \tag{6.12}$$

where we have used Lemma 6.1. On the other hand, by using the fact that $\varphi_j \to u$ in $H^{\ell,-s}(\mathbb{R}^3)$, we obtain

$$-\Delta\varphi_j \to -\Delta u \quad \text{in } \mathcal{S}'(\mathbb{R}^3). \tag{6.13}$$

Combining (6.12) with (6.13) gives (6.9). □

We shall now establish the radiation conditions for $\sqrt{-\Delta}$ in the same formulation as in Theorem 6.1.

Theorem 6.3 *Let $1/2 < s < 1$.*

(i) *Suppose that u belongs to $L^{2,-s}(\mathbb{R}^3) \cap H^1_{\text{loc}}(\mathbb{R}^3)$ and satisfies the equation*

$$(\sqrt{-\Delta} - \lambda)u = 0 \quad in \;\; \mathcal{S}'(\mathbb{R}^3), \qquad \lambda > 0, \tag{6.14}$$

and, in addition, that u satisfies either of the outgoing radiation condition (6.2) *or the incoming radiation condition* (6.3). *Then u vanishes identically.*

(ii) *Suppose that $f \in L^{2,s}(\mathbb{R}^3)$ and $\lambda > 0$. Then $u_0^+(\lambda, f) := R_0^+(\lambda)f$ and $u_0^-(\lambda, f) := R_0^-(\lambda)f$ satisfy the equation*

$$(\sqrt{-\Delta} - \lambda)u = f \quad in \;\; \mathcal{S}'(\mathbb{R}^3) \tag{6.15}$$

with the outgoing radiation condition (6.2) *and the incoming radiation condition* (6.3) *respectively.*

A very important consequence of Theorem 6.3 is the fact that the radiation conditions (6.2) and (6.3) characterize the boundary values $R_0^+(\lambda)$ and $R_0^-(\lambda)$ respectively.

In order to prove Theorem 6.3, we need to prepare two lemmas. One might regard the equality (6.16) below as straightforward. Unfortunately, this is not the case. Indeed, there exists a diffuculty to make sense of $\sqrt{-\Delta}R_0^\pm(\lambda)f$. The reason for this difficulty is the same as the ones mentioned before Theorem 6.2, namely, the fact that $R_0^\pm(\lambda)f$ merely belong to $L^{2,-s}(\mathbb{R}^3)$ with $s > 1/2$. Nevertheless we can prove, with the aid of theorems in Umeda[26], that (6.16) is true.

Lemma 6.3 *Suppose that $\lambda > 0$ and $f \in L^{2,s}(\mathbb{R}^3)$, $s > 1/2$. Then*

$$(\sqrt{-\Delta} - \lambda)R_0^\pm(\lambda)f = f \quad in \;\; \mathcal{S}'(\mathbb{R}^3). \tag{6.16}$$

Proof. We can assume, without loss of generality, that $1/2 < s < 5/2$. It then follows from Theorem 6.2 (cf. [26, Theorem 4.6]) that $\sqrt{-\Delta}R_0^\pm(\lambda)f$ make sense. In order to show (6.16), we take a test function $\psi \in \mathcal{S}(\mathbb{R}^3)$. We then have

$$\langle\, (\sqrt{-\Delta} - \lambda \mp i\mu)R_0(\lambda \pm i\mu)f,\, \psi\rangle = \langle f,\, \psi\rangle \tag{6.17}$$

for all $\mu > 0$, since $R_0(\lambda \pm i\mu)f$ belong to $H^1(\mathbb{R}^3)$, the domain of the selfadjoint operator H_0, and since

$$\begin{aligned}(\sqrt{-\Delta} - \lambda \mp i\mu)R_0(\lambda \pm i\mu)f &= (H_0 - (\lambda \pm i\mu))R_0(\lambda \pm i\mu)f \\ &= f.\end{aligned}$$

By definition of the action of $\sqrt{-\Delta}$ on $L^{2,\,-s}(\mathbb{R}^3)$, the left hand side of (6.17) becomes

$$(R_0(\lambda \pm i\mu)f,\, \sqrt{-\Delta}\,\overline{\psi}\,)_{-s,s} - (R_0(\lambda \pm i\mu)f,\, (\lambda \mp i\mu)\overline{\psi})_{-s,s}. \tag{6.18}$$

(Note that $\sqrt{-\Delta}\,\overline{\psi} \in L^{2,\,t}(\mathbb{R}^3)$ for any $t < 5/2$; see [26, Theorem 4.4].) It follows from Theorem 3.1 that

$$\lim_{\mu \downarrow 0} (R_0(\lambda \pm i\mu)f,\, \sqrt{-\Delta}\,\overline{\psi}\,)_{-s,s} = (R_0^{\pm}(\lambda)f,\, \sqrt{-\Delta}\,\overline{\psi}\,)_{-s,s} \tag{6.19}$$

Combining (6.18), (6.19) with (6.17), we conclude that

$$(R_0^{\pm}(\lambda)f,\, \sqrt{-\Delta}\,\overline{\psi}\,)_{-s,s} - (R_0^{\pm}(\lambda)f,\, \lambda\overline{\psi})_{-s,s} = \langle f,\, \psi\rangle$$

for any test function $\psi \in \mathcal{S}(\mathbb{R}^3)$. This completes the proof. □

Lemma 6.4 *Suppose that* $1/2 < s < 1$ *and* $\lambda > 0$. *If* u *belongs to* $\mathrm{Ran}\big(R_0^+(\lambda)\big)$, *then* u *satisfies the the outgoing radiation condition* (6.2). *Similarly, if* u *belongs to* $\mathrm{Ran}\big(R_0^-(\lambda)\big)$, *then* u *satisfies the incoming radiation condition* (6.3).

Proof. We only give the proof for $u \in \mathrm{Ran}\big(R_0^+(\lambda)\big)$. The proof for $u \in \mathrm{Ran}\big(R_0^-(\lambda)\big)$ is similar.

By assumption, one can find an $f \in L^{2,\,s}(\mathbb{R}^3)$ such that $u = R_0^+(\lambda)f$. It follows from Theorem 3.2, together with Corollary to Lemma 3.3, that there exist $A(\lambda) \in \mathbf{B}(L^{2,s})$, $B_1(\lambda) \in \mathbf{B}(L^{2,s},\ H^{1,s})$ and $B_2(\lambda) \in \mathbf{B}(L^2,\ H^2)$ such that

$$u = \Gamma_0^+(\lambda^2)\, A(\lambda)f + B_1(\lambda)f + B_2(\lambda)f. \tag{6.20}$$

By Theorem 6.1(ii), the first term on the right hand side of (6.20) satisfies the outgoing radiation condition (6.2). Since $B_1(\lambda)f \in H^{1,\,s}(\mathbb{R}^3)$, it is straightforward to see that

$$\Big(\frac{\partial}{\partial x_j} - i\lambda\omega_j\Big) B_1(\lambda)f \in L^{2,\,s}(\mathbb{R}^3) \subset L^{2,\,s-1}(\mathbb{R}^3), \quad j = 1,\, 2,\, 3,$$

that is, the second term on the right hand side of (6.20) satisfies (6.2). Finally, since $B_2(\lambda)f \in H^2(\mathbb{R}^3)$, it follows that

$$\Big(\frac{\partial}{\partial x_j} - i\lambda\omega_j\Big) B_2(\lambda)f \in H^1(\mathbb{R}^3) \subset L^{2,\,s-1}(\mathbb{R}^3), \quad j = 1,\, 2,\, 3,$$

where we have used the assumption that $s < 1$. Hence the last term on the right hand side of (6.20) satisfies (6.2). □

Proof of Theorem 6.3 It follows from (6.14) that $\sqrt{-\Delta}u = \lambda u$, hence $\sqrt{-\Delta}u$ belongs to $L^{2,-s}(\mathbb{R}^3)$ with $1/2 < s < 1$. By Lemma 6.2, it makes sense to consider $\sqrt{-\Delta}\,(\sqrt{-\Delta}u)$, and we see that u satisfies

$$(-\Delta - \lambda^2)u = 0 \quad in \;\; \mathcal{S}'(\mathbb{R}^3),$$

which implies that $-\Delta u = \lambda^2 u$ belongs to $L^2_{\mathrm{loc}}(\mathbb{R}^3)$. Therefore, we find that $u \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$. It is evident that we can apply Theorem 6.1(i) and obtain assertion (i) of the theorem.

Assertion (ii) of the theorem is an immediate consequence of Lemmas 6.3 and 6.4. □

7 Radiation conditions for $\sqrt{-\Delta}+V$

This section is devoted to discussing radiation conditions for $\sqrt{-\Delta}+V$ on $\mathbb{R}^3$. As mentioned in Introduction, we assume that $V(x)$ is a real-valued measurable function on $\mathbb{R}^3$ satisfying (1.6). Under this assumption, it is obvious that $V=V(x)\times$ is a bounded selfadjoint operator in $L^2(\mathbb{R}^3)$, and that $H:=H_0+V$ defines a selfadjoint operator in $L^2(\mathbb{R}^3)$, of which domain is $H^1(\mathbb{R}^3)$. For $z\in\rho(H)$, we write

$$R(z)=(H-z)^{-1}.$$

It is clear that H is essentially selfadjoint on $C_0^\infty(\mathbb{R}^3)$, since H is a bounded selfadjoint perturbation of H_0, which is essentially selfadjoint on $C_0^\infty(\mathbb{R}^3)$ (see Section 2). Since V is relatively compact with respect to H_0, it follows from Reed-Simon[19, p.113, Corollary 2] that

$$\sigma_{\mathrm{ess}}(H)=\sigma_{\mathrm{ess}}(H_0)=[0,\,+\infty).$$

Before establishing the radiation conditions for $\sqrt{-\Delta}+V(x)$, we need to remark that $\sigma_{\mathrm{p}}(H)\cap(0,\,+\infty)$ is a discrete set. This fact was first proved by Simon[22, Theorem 2.1] in a general setting, and later recovered by Ben-Artzi and Nemirovsky[3, Theorem 4A] also in a general setting. Moreover, Simon[22, Theorem 2.1] proved that each eigenvalue in the set $\sigma_{\mathrm{p}}(H)\cap(0,\,+\infty)$ has finite multiplicity.

To formulate the main theorem in this section, we exploit a result, which is a special case of Ben-Artzi and Nemirovsky[3, Theorem 4A].

Theorem 7.1 (Ben-Artzi and Nemirovski) *Let $\sigma>1$ and $s>1/2$. Then*

(i) *The continuous spectrum $\sigma_{\mathrm{c}}(H)=[0,\,+\infty)$ is absolutely continuous, except possibly for a discrete set of embedded eigenvalues $\sigma_{\mathrm{p}}(H)\cap(0,\,+\infty)$, which can accumulate only at 0 and $+\infty$.*

(ii) *For any $\lambda\in(0,\,+\infty)\setminus\sigma_{\mathrm{p}}(H)$, there exist the limits*

$$R^{\pm}(\lambda)=\lim_{\mu\downarrow 0}R(\lambda\pm i\mu)\quad\textit{in }\mathbf{B}(L^{2,s},\,H^{1,-s}).$$

(iii) *The operator-valued functions $R^{\pm}(z)$ defined by*

$$R^{\pm}(z)=\begin{cases}R(z) & \textit{if } z\in\mathbb{C}^{\pm}\\ \\ R^{\pm}(\lambda) & \textit{if } z=\lambda\in(0,\,+\infty)\setminus\sigma_{\mathrm{p}}(H)\end{cases}$$

are $\mathbf{B}(L^{2,s},\,H^{1,-s})$-valued continuous functions.

We now state the main result in this section, which establishes the radiation conditions for $\sqrt{-\Delta} + V(x)$.

Theorem 7.2 *Let $\sigma > 1$ and $1/2 < s < \min(\sigma/2,\, 1)$.*

(i) *Suppose that u belongs to $L^{2,-s}(\mathbb{R}^3) \cap H^1_{\mathrm{loc}}(\mathbb{R}^3)$ and satisfies the equation*

$$(\sqrt{-\Delta} + V(x) - \lambda)u = 0 \quad in\ \mathcal{S}'(\mathbb{R}^3), \qquad \lambda \in (0,\, +\infty) \setminus \sigma_{\mathrm{p}}(H) \tag{7.1}$$

and, in addition, that u satisfies either of the outgoing radiation condition (6.2) *or the incoming radiation condition* (6.3)*. Then u vanishes identically.*

(ii) *Suppose that $f \in L^{2,s}(\mathbb{R}^3)$ and $\lambda \in (0,\, +\infty) \setminus \sigma_{\mathrm{p}}(H)$. Then $u^+(\lambda,\, f) := R^+(\lambda)f$ and $u^-(\lambda,\, f) := R^-(\lambda)f$ satisfy the equation*

$$(\sqrt{-\Delta} + V(x) - \lambda)u = f \quad in\ \mathcal{S}'(\mathbb{R}^3) \tag{7.2}$$

with the outgoing radiation condition (6.2) *and the incoming radiation condition* (6.3) *respectively.*

The same remark after Theorem 6.3 applies to Theorem 7.2, namely, Theorem 7.2 gives the characterization of the boundary values $R^+(\lambda)$ and $R^-(\lambda)$ in terms of the radiation conditions (6.2) and (6.3) respectively.

We shall give a proof of Theorem 7.2 by means of a series of lemmas, but only for u satisfying the outgoing radiation condition (6.2). The proof for u satisfying the incoming radiation condition (6.3) is similar.

Lemma 7.1 *Let $\sigma > 1$, and suppose that $1/2 < s < \sigma/2$. Then*

$$R^{\pm}(z)\big(I + VR_0^{\pm}(z)\big) \;=\; R_0^{\pm}(z) \quad \text{on } L^{2,s}(\mathbb{R}^3), \tag{7.3}$$

$$R_0^{\pm}(z)\big(I - VR^{\pm}(z)\big) \;=\; R^{\pm}(z) \quad \text{on } L^{2,s}(\mathbb{R}^3) \tag{7.4}$$

for all $z \in \mathbb{C}^{\pm} \cup \{(0,\, +\infty) \setminus \sigma_p(H)\}$.

Proof. We shall give the proof only in the case where the superscripts are "+," the plus sign. If $z \in \mathbb{C}^+$, it is apparent that

$$\begin{aligned} (H - z)R_0(z) &= I + VR_0(z) \quad \text{on } L^2(\mathbb{R}^3), \\ (H_0 - z)R(z) &= I - VR(z) \quad \text{on } L^2(\mathbb{R}^3), \end{aligned}$$

from which it follows that

$$R(z)\big(I + VR_0(z)\big) \;=\; R_0(z) \quad \text{on } L^2(\mathbb{R}^3), \tag{7.5}$$

$$R_0(z)\big(I - VR(z)\big) \;=\; R(z) \quad \text{on } L^2(\mathbb{R}^3). \tag{7.6}$$

In order to proceed to the extended resolvents, we now regard that $R_0^+(z)$ and $R^+(z)$ are $\mathbf{B}(L^{2,\,s}, L^{2,\,-s})$-valued continuous functions on $\mathbb{C}^+ \cup (0,\,+\infty)$ and $\mathbb{C}^+ \cup \{(0,\,+\infty) \setminus \sigma_p(H)\}$ respectively. By (1.6), and by the assumption that $1/2 < s < \sigma/2$, we see that $V \in \mathbf{B}(L^{2,-s}, L^{2,s})$, and hence $VR_0^+(z)$ and $VR^+(z)$ are $\mathbf{B}(L^{2,\,s})$-valued continuous functions on $\mathbb{C}^+ \cup (0,\,+\infty)$ and $\mathbb{C}^+ \cup \{(0,\,+\infty) \setminus \sigma_p(H)\}$ respectively. Therefore, we conclude from (7.5) and (7.6) that the assertion of the lemma is valid. □

As a corollary to Lemma 7.1, we obtain the following result.

Lemma 7.2 *Let $\sigma > 1$, and suppose that $1/2 < s < \sigma/2$. Then*

$$\mathrm{Ran}\big(R^{\pm}(z)\big) = \mathrm{Ran}\big(R_0^{\pm}(z)\big)$$

for every $z \in \mathbb{C}^{\pm} \cup \{(0,\,+\infty) \setminus \sigma_p(H)\}$.

Lemma 7.3 *Let $\sigma > 1$, and suppose that $1/2 < s < \sigma/2$. Then*

$$\big(I - R^{\pm}(z)V\big)\big(I + R_0^{\pm}(z)V\big) = I \quad \text{on } L^{2,\,-s}(\mathbb{R}^3), \tag{7.7}$$

$$\big(I + R_0^{\pm}(z)V\big)\big(I - R^{\pm}(z)V\big) = I \quad \text{on } L^{2,\,-s}(\mathbb{R}^3) \tag{7.8}$$

for every $z \in \mathbb{C}^{\pm} \cup \{(0,\,+\infty) \setminus \sigma_p(H)\}$.

Proof. We shall only give the proof of (7.7) in the case where the superscripts are "+." The proof of (7.7) in the other case and the proof of (7.8) are similar.

We first show that for every $z \in \mathbb{C}^+$

$$\big(I - R(z)V\big)\,\big(I + R_0(z)V\big) = I \quad \text{on } L^2(\mathbb{R}^3). \tag{7.9}$$

In fact, if u belongs to $H^1(\mathbb{R}^3)$, we then have

$$\begin{aligned}(I - R(z)V)u &= R(z)(H - z)u - R(z)Vu \\ &= R(z)(H_0 - z)u\end{aligned}$$

and

$$\begin{aligned}(I + R_0(z)V)u &= R_0(z)(H_0 - z)u + R_0(z)Vu \\ &= R_0(z)(H - z)u\end{aligned}$$

(recall that $\mathrm{Dom}(H) = \mathrm{Dom}(H_0) = H^1(\mathbb{R}^3)$). Hence we get

$$\begin{aligned}\big(I - R(z)V\big)\big(I + R_0(z)V\big)u &= R(z)(H_0 - z)R_0(z)(H - z)u \\ &= u\end{aligned}$$

for all $u \in H^1(\mathbb{R}^3)$, where we have used the fact that $\big(I + R_0(z)V\big)u \in H^1(\mathbb{R}^3)$ when $u \in H^1(\mathbb{R}^3)$. Since $H^1(\mathbb{R}^3)$ is dense in $L^2(\mathbb{R}^3)$, we can deduce that (7.9) is true.

We next work in the weighted L^2-spaces. As mentioned in the proof of Lemma 7.1, we have $V \in \mathbf{B}(L^{2,-s}, L^{2,s})$. Also, as mentioned in the second half of the proof of Lemma 7.1, we can regard that $R_0^+(z)$ and $R^+(z)$ are $\mathbf{B}(L^{2,s}, L^{2,-s})$-valued continuous functions on $\mathbb{C}^+ \cup (0, +\infty)$ and $\mathbb{C}^+ \cup \{(0, +\infty) \setminus \sigma_p(H)\}$ respectively. Therefore $R_0^+(z)V$ and $R^+(z)V$ are $\mathbf{B}(L^{2,-s})$-valued continuous functions on $\mathbb{C}^+ \cup (0, +\infty)$ and $\mathbb{C}^+ \cup \{(0, +\infty) \setminus \sigma_p(H)\}$ respectively. Thus, we can conclude from (7.9) that (7.7) in the case where the superscripts are the plus sign is true. $\square$

Proof of Theorem 7.2 We first prove assertion (i) of the theorem. Let u belong to $L^{2,-s}(\mathbb{R}^3) \cap H^1_{\mathrm{loc}}(\mathbb{R}^3)$ and satisfy the equation (7.1) together with the outgoing radiation condition (6.2). By (7.1), we have

$$(\sqrt{-\Delta} - \lambda)u = -Vu \quad \text{in } \mathcal{S}'(\mathbb{R}^3). \tag{7.10}$$

Since Vu belongs to $L^{2,s}(\mathbb{R}^3)$ by the fact that $V \in \mathbf{B}(L^{2,-s}, L^{2,s})$, it follows from Lemma 6.3 that

$$(\sqrt{-\Delta} - \lambda)R_0^+(\lambda)Vu = Vu \quad \text{in } \mathcal{S}'(\mathbb{R}^3). \tag{7.11}$$

Combining (7.10) with (7.11) gives

$$(\sqrt{-\Delta} - \lambda)(u + R_0^+(\lambda)Vu) = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}^3). \tag{7.12}$$

By virtue of Lemma 6.4 and the fact that $R_0^+(\lambda)Vu \in H^{1,-s}(\mathbb{R}^3)$, it follows that $u + R_0^+(\lambda)Vu$ belongs to $L^{2,-s}(\mathbb{R}^3) \cap H^1_{\mathrm{loc}}(\mathbb{R}^3)$ and satisfies the outgoing radiation condition (6.2). Hence we can apply Theorem 6.3 and conclude that

$$u + R_0^+(\lambda)Vu = 0. \tag{7.13}$$

Since u belongs to $L^{2,-s}(\mathbb{R}^3)$, it follows from (7.13) and Lemma 7.3 that u vanishes identically.

We next prove assertion (ii). It follows from Lemmas 7.2 and 6.4 that $u^+(\lambda, f)$ satisfies the outgoing radiation condition (6.2). In order to show that $u^+(\lambda, f)$ is a solution to the equation (7.2), we follow the idea exploited in the proof of Lemma 6.3. Thus we start with

$$(\sqrt{-\Delta} + V - \lambda - i\mu)R(\lambda + i\mu)f = f, \quad \forall \mu > 0,$$

which implies that

$$\langle\, (\sqrt{-\Delta} + V - \lambda - i\mu) R(\lambda + i\mu) f,\, \psi \rangle = \langle f,\, \psi \rangle \tag{7.14}$$

for any test function $\psi \in \mathcal{S}(\mathbb{R}^3)$. By definition of the action of $\sqrt{-\Delta}$ on $L^{2,\,-s}(\mathbb{R}^3)$, the left hand side of (7.14) becomes

$$\begin{aligned} (R(\lambda + i\mu) f,\, \sqrt{-\Delta}\,\overline{\psi}\,)_{-s,s} \quad &+ \quad (V R(\lambda + i\mu) f,\, \overline{\psi}\,)_{-s,s} \\ &- \quad (R(\lambda + i\mu) f,\, (\lambda - i\mu)\overline{\psi}\,)_{-s,s}. \end{aligned} \tag{7.15}$$

(Note again that $\sqrt{-\Delta}\,\overline{\psi} \in L^{2,\,t}(\mathbb{R}^3)$ for any $t < 5/2$.) It follows from Theorem 7.1 that

$$\lim_{\mu \downarrow 0} (R(\lambda + i\mu) f,\, \sqrt{-\Delta}\,\overline{\psi}\,)_{-s,s} = (R^+(\lambda) f,\, \sqrt{-\Delta}\,\overline{\psi}\,)_{-s,s}. \tag{7.16}$$

Similarly, we have

$$\lim_{\mu \downarrow 0} \{ (V R(\lambda + i\mu) f,\, \overline{\psi}\,)_{-s,s} - (R(\lambda + i\mu) f,\, (\lambda - i\mu)\overline{\psi})_{-s,s} \} \tag{7.17}$$

$$= (V R^+(\lambda) f,\, \overline{\psi}\,)_{-s,s} - (R^+(\lambda) f,\, \lambda\overline{\psi}\,)_{-s,s}. \tag{7.18}$$

Combining (7.14) with (7.15) – (7.18) yields

$$\langle\, (\sqrt{-\Delta} + V - \lambda) R^+(\lambda) f,\, \psi \rangle = \langle f,\, \psi \rangle$$

for any test function $\psi \in \mathcal{S}(\mathbb{R}^3)$. Thus we have shown that $u^+(\lambda,\, f) = R^+(\lambda) f$ satisfies the equation (7.2). $\square$

8 Generalized eigenfunctions

Two tasks are set in this section. One of them is to construct generalized eigenfunctions of $\sqrt{-\Delta} + V(x)$ on $\mathbb{R}^3$, which are the superposition of plane waves and solutions of the equation (6.15), for some λ and f, satisfying the outgoing or the incoming radiation condition. To this end, we shall adopt the idea in Agmon[1] (cf. Kato and Kuroda[11]). The other task is to show that the generalized eigenfunctions to be constructed are characterized as the unique solutions to integral equations, which we shall call the modified Lippmann-Schwinger equations.

We shall write the plane wave $e^{ix\cdot k}$ as $\varphi_0(x, k)$:

$$\varphi_0(x, k) := e^{ix\cdot k}. \tag{8.1}$$

It should be noted that one can easily sees that

$$-\Delta_x \varphi_0(x, k) = |k|^2 \varphi_0(x, k),$$

which is a starting point when one discusses the generalized eigenfunction expansion for the Schrödinger operator $-\Delta + V(x)$. On the contrary, it is not trivial to justify

$$\sqrt{-\Delta_x}\varphi_0(x, k) = |k|\varphi_0(x, k) \quad \text{in } \mathcal{S}'(\mathbb{R}^3_x), \tag{8.2}$$

which is formally obvious though. The reason why (8.2) is nontrivial is that $\varphi_0(x, k)$ does not belong to the Sobolev space $H^\ell(\mathbb{R}^3_x)$ for any $\ell \in \mathbb{R}$. In fact, the Fourier transform of $\varphi_0(x, k)$ with respect to the variable x is a Delta-function $(2\pi)^{3/2}\delta(\xi - k)$, which is obviously not a function in $L^1_{\mathrm{loc}}(\mathbb{R}^3_\xi)$, whereas we have

$$H^\ell(\mathbb{R}^3) = \{\, f \mid \langle\xi\rangle^\ell \hat{f} \in L^2(\mathbb{R}^3_\xi) \,\}$$

by definition.

By virtue of some results in Umeda[26] we shall be able to make sense of $\sqrt{-\Delta_x}\varphi_0(x, k)$ and prove that (8.2) is valid.

Lemma 8.1 *For every $k \in \mathbb{R}^3$, $\varphi_0(x, k)$ satisfies the pseudodifferential equation* (8.2).

Proof. It is straightforward to see that $\varphi_0(x, k)$ belongs to $L^{2,s}(\mathbb{R}^3_x)$ for every $s < -3/2$. This fact, together with Theorem 6.2, implies that $\sqrt{-\Delta_x}\varphi_0(x, k)$ makes sense. Taking a test function $\psi \in \mathcal{S}(\mathbb{R}^3)$, we get

$$\langle \sqrt{-\Delta_x}\varphi_0(\cdot, k), \psi\rangle = (\varphi_0(\cdot, k), \sqrt{-\Delta_x}\,\overline{\psi})_{s,-s} \tag{8.3}$$

for all s with $-5/2 < s < -3/2$, where we have used the fact that $\sqrt{-\Delta_x}\,\overline{\psi} \in L^{2,\,t}(\mathbb{R}^3)$ for any $t < 5/2$. The right hand side of (8.3) equals

$$\begin{aligned}\int e^{ix\cdot k}\,\overline{\sqrt{-\Delta}\,\overline{\psi(x)}}\,dx &= (2\pi)^{3/2}\,\overline{\mathcal{F}[\sqrt{-\Delta}\,\overline{\psi}\,](k)} \\ &= (2\pi)^{3/2}\,|k|\,\overline{\mathcal{F}[\,\overline{\psi}\,](k)}.\end{aligned}$$

Noting that

$$\overline{\mathcal{F}[\,\overline{\psi}\,](k)} = (2\pi)^{-3/2}\int \varphi_0(x,\,k)\,\psi(x)\,dx,$$

we obtain

$$\begin{aligned}(\varphi_0(\cdot,\,k),\,\sqrt{-\Delta_x}\,\overline{\psi})_{s,-s} &= \int |k|\,\varphi_0(x,\,k)\,\psi(x)\,dx \\ &= \langle |k|\,\varphi_0(\cdot,\,k),\,\psi\rangle. \qquad (8.4)\end{aligned}$$

Combining (8.4) with (8.3) gives the lemma. □

Following Agmon[1], we define two families of generalized eigenfunctions of $\sqrt{-\Delta} + V(x)$ on $\mathbb{R}^3$ by

$$\varphi^{\pm}(x,\,k) := \varphi_0(x,\,k) - R^{\mp}(|k|)\{V(\cdot)\varphi_0(\cdot,\,k)\}(x) \qquad (8.5)$$

for k with $|k| \in (0,\,+\infty) \setminus \sigma_p(H)$. Note that the second terms on the right hand side of (8.5) make sense, provided that $|V(x)| \le C\langle x\rangle^{-\sigma}$, $\sigma > 2$. In fact, $V(\cdot)\varphi(\cdot,\,k) \in L^{2,s}(\mathbb{R}^3)$ for all s with $1/2 < s < \sigma - 3/2$.

Theorem 8.1 *Let $\sigma > 2$. If $|k| \in (0,\,+\infty) \setminus \sigma_p(H)$, then both $\varphi^{\pm}(x,\,k)$ satisfy the equation*

$$(\sqrt{-\Delta_x} + V(x))u = |k|u \quad \text{in } \mathcal{S}'(\mathbb{R}^3_x). \qquad (8.6)$$

Proof. As remarked just before the theorem, we see that $V(\cdot)\varphi_0(\cdot,\,k)$ belongs to $L^{2,s}(\mathbb{R}^3)$ for all s with $1/2 < s < \sigma - 3/2$. Hence, by Theorem 7.2(ii), we get

$$\begin{aligned}&(\sqrt{-\Delta_x} + V(x) - |k|)\big[\,R^{\mp}(|k|)\{V(\cdot)\varphi_0(\cdot,\,k)\}\,\big](x) \\ &\qquad\qquad = V(\cdot)\varphi_0(\cdot,\,k) \quad \text{in } \mathcal{S}'(\mathbb{R}^3_x), \qquad (8.7)\end{aligned}$$

which, together with Lemma 8.1, implies that

$$
\begin{aligned}
&(\sqrt{-\Delta_x} + V(x))\varphi^{\pm}(x,\,k) \\
&\qquad = (\sqrt{-\Delta_x} + V(x))\varphi_0(x,\,k) \\
&\qquad\qquad - (\sqrt{-\Delta_x} + V(x))\big[R^{\mp}(|k|)\{V(\cdot)\varphi_0(\cdot,\,k)\}\big](x) \\
&\qquad = |k|\varphi_0(x,\,k) + V(x)\varphi_0(x,\,k) \\
&\qquad\qquad - V(x)\varphi_0(x,\,k) - |k|\,\big[R^{\mp}(|k|)\{V(\cdot)\varphi_0(\cdot,\,k)\}\big](x) \\
&\qquad = |k|\big[\,\varphi_0(x,\,k) - R^{\mp}(|k|)\{V(\cdot)\varphi_0(\cdot,\,k)\}(x)\,\big].
\end{aligned}
$$

By the definition (8.5), this gives the theorem. □

Remark. For each k with $|k| \in (0,\,+\infty) \setminus \sigma_p(H)$, the generalized eigenfunctions $\varphi^{\pm}(x,\,k)$ are unique in the following sense: If $\tilde{\varphi}^{+}(x,\,k)$ (resp. $\tilde{\varphi}^{-}(x,\,k)$) satisfies the equation (8.6), and in addition, $\tilde{\varphi}^{+}(x,\,k) - \varphi_0(x,\,k)$ (resp. $\tilde{\varphi}^{-}(x,\,k) - \varphi_0(x,\,k)$) belongs to $L^{2,-s}(\mathbb{R}^3) \cap H^1_{\mathrm{loc}}(\mathbb{R}^3)$, $1/2 < s < \min(\sigma/2, 1)$, and satisfies the incoming radiation condition (6.3) (resp. the outgoing radiation condition (6.2)), then $\tilde{\varphi}^{+}(x,\,k) = \varphi^{+}(x,\,k)$ (resp. $\tilde{\varphi}^{-}(x,\,k) = \varphi^{-}(x,\,k)$). This is a direct consequence of assertion (i) of Theorem 7.2.

We are in a position to show that the generalized eigenfunctions $\varphi^{+}(x,\,k)$ and $\varphi^{-}(x,\,k)$, defined by (8.5), are characterized as the unique solutions to the integral equations

$$
\varphi(x) = \varphi_0(x,\,k) - \int_{\mathbb{R}^3} g^{-}_{|k|}(x-y)\,V(y)\,\varphi(y)\,dy \tag{8.8}
$$

and

$$
\varphi(x) = \varphi_0(x,\,k) - \int_{\mathbb{R}^3} g^{+}_{|k|}(x-y)\,V(y)\,\varphi(y)\,dy \tag{8.9}
$$

respectively. (Recall that $g^{\pm}_{\lambda}(x-y)$ are the integral kernels of the boundary values $R^{\pm}_0(\lambda)$. See Theorem 4.1.) We call (8.8) and (8.9) the modified Lippmann-Schwinger equations, because the leading terms of $g^{\pm}_{\lambda}(x-y)$ are the same, up to a constant, as the integral kernels of the Lippmann-Schwinger equations, namely,

$$
g^{\pm}_{\lambda}(x-y) = \frac{\lambda}{2\pi}\cdot\frac{e^{\pm i\lambda|x-y|}}{|x-y|} + O(|x-y|^{-2}) \quad \text{as } |x-y| \to +\infty.
$$

(Recall (4.6) and (5.17).)

Our generalized eigenfunctions $\varphi^{\pm}(x,\,k)$ are expected to behave like the plane wave $\varphi_0(x,\,k)$, which belongs to $L^{2,-s}(\mathbb{R}^3)$ only for $s > 3/2$. Thus it is natural to take $L^{2,-s}(\mathbb{R}^3)$, with $s > 3/2$, to be the space of functions in which we deal with the integral equations (8.8) and (8.9). It is evident from Theorem 4.1 that (8.8) and (8.9) can be formally rewritten in the forms $\big(I + R_0^-(|k|)V\big)\varphi = \varphi_0(\cdot,\, k)$ and $\big(I + R_0^+(|k|)V\big)\varphi = \varphi_0(\cdot,\, k)$ respectively. For these reasons, we prepare the following lemma, which is a variant of Lemma 7.3. The only difference between Lemmas 7.3 and 8.2 lies in their assumptions. In Lemma 8.2, s is allowed to be greater than 3/2.

Lemma 8.2 *Let $\sigma > 2$, and suppose that $1/2 < s < \sigma - 1/2$. Then the conclusions of Lemma 7.3 hold.*

Proof. We only give the proof of (7.7) in the case where the superscripts are the plus sign. The proof of (7.7) in the other case and the proof of (7.8) are similar.

It is obvious that we shall follow the line of the proof of Lemma 7.3. By assumption, we can choose t so that

$$1/2 < t < \min{(s,\, \sigma - s)}. \tag{8.10}$$

We note that $R_0^+(z)$ and $R^+(z)$ can be regarded as $\mathbf{B}(L^{2,t}, L^{2,-t})$-valued continuous functions on $\mathbb{C}^+ \cup (0,\, +\infty)$ and $\mathbb{C}^+ \cup \{(0,\, +\infty) \setminus \sigma_p(H)\}$ respectively, as mentioned in the proof of Lemma 7.1. From this fact, we can deduce that $R_0^+(z)$ and $R^+(z)$ are $\mathbf{B}(L^{2,t}, L^{2,-s})$-valued continuous functions on $\mathbb{C}^+ \cup (0,\, +\infty)$ and $\mathbb{C}^+ \cup \{(0,\, +\infty) \setminus \sigma_p(H)\}$ respectively, since $-s < -t$. In view of (8.10) we have $V \in \mathbf{B}(L^{2,-s}, L^{2,t})$. Therefore, $R_0^+(z)V$ and $R^+(z)V$ are $\mathbf{B}(L^{2,-s})$-valued continuous functions on $\mathbb{C}^+ \cup (0,\, +\infty)$ and $\mathbb{C}^+ \cup \{(0,\, +\infty) \setminus \sigma_p(H)\}$ respectively. Recalling (7.9), which was shown to be valid for all $z \in \mathbb{C}^+$, we conclude that (7.7) in the case where the superscripts are "+" holds. □

Theorem 8.2 *Let $\sigma > 2$, and suppose that $3/2 < s < \sigma - 1/2$. If $|k| \in (0,\, +\infty) \setminus \sigma_p(H)$, then $\varphi^+(x,\, k)$ and $\varphi^-(x,\, k)$ are the unique solution of the modified Lippmann-Schwinger equations (8.8) and (8.9) in $L^{2,-s}(\mathbb{R}^3_x)$ respectively.*

Proof. We shall give the proof only for $\varphi^+(x,\, k)$.

It follows from the definition (8.5) that

$$\varphi^+(\cdot,\, k) = \big(I - R^-(|k|)V\big)\varphi_0(\cdot,\, k), \tag{8.11}$$

where we regard $\varphi_0(\cdot,\, k)$ as a function belonging to $L^{2,-s}(\mathbb{R}^3_x)$. Combining (8.11) with (7.8), we have

$$\big(I + R_0^-(|k|)V\big)\varphi^+(\cdot,\, k) = \varphi_0(\cdot,\, k), \tag{8.12}$$

from which we obtain

$$\varphi^+(\cdot,\, k) = \varphi_0(\cdot,\, k) - R_0^-(|k|)V\varphi^+(\cdot,\, k). \tag{8.13}$$

Since the integral kernel of $R_0^-(|k|)$ is given by $g_{|k|}^-(x-y)$, we conclude from (8.13) that $\varphi^+(x,\, k)$ satisfies the modified Lippmann-Schwinger equation (8.8). Uniqueness follows from (8.12) and (7.7). $\square$

9 Continuity of the generalized eigenfunctions

The aim of this section is to prove the following result.

Theorem 9.1 *Let $\sigma > 2$. Then the generalized eigenfunctions $\varphi^{\pm}(x, k)$ defined by* (8.5) *have the following properties:*

(i) *For each interval $[a, b] \subset (0, +\infty) \setminus \sigma_p(H)$, there exists a constant C_{ab}, depending on a and b, such that*

$$|\varphi^{\pm}(x, k)| \le C_{ab} \tag{9.1}$$

for all $(x, k) \in \mathbb{R}^3 \times \{ k \,|\, a \le |k| \le b \}$.

(ii) *$\varphi^{\pm}(x, k)$ are continuous functions on $\mathbb{R}^3_x \times \{ k \mid |k| \in (0, +\infty) \setminus \sigma_p(H) \}$.*

We shall give a proof of Theorem 9.1 by means of a series of lemmas. Hence, throughout the present section we shall assume that

$$\sigma > 2$$

without saying so every time. We shall first prepare a few lemmas and then prove assertion (i) of Theorem 9.1. We shall next show a few lemmas, of which combination directly gives a proof of assertion (ii) of Theorem 9.1. The estimate (9.1) will be useful in the discussions for the proof of assertion (ii).

Lemma 9.1 *If $s > 3/2$, then $\varphi^{\pm}(\cdot, k)$ are $L^{2,-s}(\mathbb{R}^3_x)$-valued continuous functions on $\{ k \mid |k| \in (0, +\infty) \setminus \sigma_p(H) \}$.*

Proof. We note that $\varphi_0(\cdot, k)$ is $L^{2,-s}(\mathbb{R}^3_x)$-valued continuous function on $\mathbb{R}^3_k$. On the other hand, for any t with $1/2 < t < \sigma - 3/2$, $V(\cdot)\varphi_0(\cdot, k)$ is $L^{2,t}(\mathbb{R}^3_x)$-valued continuous function on $\mathbb{R}^3_k$ (see the assumption (1.6)). This fact, together with Theorem 7.1 (iii), implies that $R^{\mp}(|k|)\{V(\cdot)\varphi_0(\cdot, k)\}$ are $L^{2,-t}(\mathbb{R}^3_x)$-valued continuous functions on $\{ k \mid |k| \in (0, +\infty) \setminus \sigma_p(H) \}$. Since t can be taken to be less than s, it follows that $R^{\mp}(|k|)\{V(\cdot)\varphi_0(\cdot, k)\}$ are $L^{2,-s}(\mathbb{R}^3_x)$-valued continuous functions on $\{ k \mid |k| \in (0, +\infty) \setminus \sigma_p(H) \}$. In view of the definition (8.5), we have proved the lemma. □

Lemma 9.2 *If $s > 3/2$, then $V(\cdot)\varphi^{\pm}(\cdot, k)$ are $L^{2,\sigma-s}(\mathbb{R}^3_x)$-valued continuous functions on $\{ k \mid |k| \in (0, +\infty) \setminus \sigma_p(H) \}$.*

Proof. The lemma is a direct consequence of Lemma 9.1 and the assumption (1.6). □

In the rest of this section, we assume that s satisfies the inequalities

$$\frac{3}{2} < s < \sigma - \frac{1}{2} \tag{9.2}$$

In order to prove assertion (i) of Theorem 9.1, we need intermediate estimates, which only assure that, for each k, $\varphi^{\pm}(x,\, k)$ are sums of bounded functions of x and functions of x belonging to $L^6(\mathbb{R}^3) \cap L^{2,\,-t}(\mathbb{R}^3)$ for all $t > 3/2$. To derive the intermediate estimates mentioned above, we appeal to Theorem 8.2; assuming that $|k| \in (0,\, +\infty) \setminus \sigma_p(H)$, we have

$$\varphi^{\pm}(x,\, k) = \varphi_0(x,\, k) - G^{\mp}_{|k|}\big(\, V(\cdot)\varphi^{\pm}(\cdot,\, k)\big)(x), \tag{9.3}$$

(see (4.13) and (4.14) for the notation $G^{\mp}_{|k|}$). According to the identities (5.3), we then decompose $\varphi^{\pm}(x,\, k)$ into two parts:

$$\varphi^{\pm}(x,\, k) = \psi^{\pm}_0(x,\, k) + \psi^{\pm}_1(x,\, k), \tag{9.4}$$

where

$$\begin{aligned} \psi^{\pm}_0(x,\, k) \;&:=\; \varphi_0(x,\, k) - K^{\mp}_{|k|}\big(\, V(\cdot)\varphi^{\pm}(\cdot,\, k)\big)(x) \\ &\qquad\qquad - M_{|k|}\big(\, V(\cdot)\varphi^{\pm}(\cdot,\, k)\big)(x), \end{aligned} \tag{9.5}$$

$$\psi^{\pm}_1(x,\, k) \;:=\; -G_0\big(\, V(\cdot)\varphi^{\pm}(\cdot,\, k)\big)(x). \tag{9.6}$$

Lemma 9.3 *Suppose that $[a,\, b] \subset (0,\, +\infty) \setminus \sigma_p(H)$. Then there exists a constant C_{ab}, depending on a and b, such that*

$$|\psi^{\pm}_0(x,\, k)| \le C_{ab} \tag{9.7}$$

for all $(x,\, k) \in \mathbb{R}^3 \times \{\, k \,|\, a \le |k| \le b\,\}$.

Proof. Let k satisfy $a \le |k| \le b$. Appealing to the definition (9.5), we have

$$\begin{aligned} |\psi^{\pm}_0(x,\, k)| \;&\le\; 1 + |\, K^{\mp}_{|k|}\big(\, V(\cdot)\varphi^{\pm}(\cdot,\, k)\big)(x)| \\ &\qquad\qquad + |\, M_{|k|}\big(\, V(\cdot)\varphi^{\pm}(\cdot,\, k)\big)(x)| \\ &\le\; 1 + C\, b\, \|V(\cdot)\varphi^{\pm}(\cdot,\, k)\|_{L^{2,\,\sigma - s}} \\ &\qquad\qquad + C_{ab}\, \|V(\cdot)\varphi^{\pm}(\cdot,\, k)\|_{L^2}, \end{aligned} \tag{9.8}$$

where we have used Lemmas 5.7 and 5.4 (note that $\sigma - s > 1/2$ by (9.2)). Here we note that the constants C and C_{ab} in (9.8) are independent of k with $a \le |k| \le b$. Lemma 9.2, together with (9.8), implies the lemma. □

Lemma 9.4 *Let $|k| \in (0, +\infty) \setminus \sigma_p(H)$. Then we have*

$$\psi_1^{\pm}(\cdot,\, k) \in L^6(\mathbb{R}^3) \cap L^{2,\,-t}(\mathbb{R}^3)$$

for every $t > 3/2$. Moreover, for each compact interval $[a,\, b] \subset (0, +\infty) \setminus \sigma_p(H)$ and each $t > 3/2$, there corresponds a positive constant C_{tab} such that

$$\|\psi_1^{\pm}(\cdot,\, k)\|_{L^6} + \|\psi_1^{\pm}(\cdot,\, k)\|_{L^{2,-t}} \le C_{tab}$$

for all k with $a \le |k| \le b$.

Proof. Since $\sigma - s > 1/2$ by (9.2), it follows from Lemma 9.2 that $V(\cdot)\varphi^{\pm}(\cdot,\, k) \in L^2(\mathbb{R}^3)$. Then the definition of $\psi_1^{\pm}$ and the inequality (5.4) show that

$$\|\psi_1^{\pm}(\cdot,\, k)\|_{L^6} \le C\, \|V(\cdot)\varphi^{\pm}(\cdot,\, k)\|_{L^2}, \tag{9.9}$$

where C is a constant independent of k. Similarly, the definition of $\psi_1^{\pm}$ and Lemma 5.1(ii) give

$$\|\psi_1^{\pm}(\cdot,\, k)\|_{L^{2,-t}} \le C_t\, \|V(\cdot)\varphi^{\pm}(\cdot,\, k)\|_{L^2} \tag{9.10}$$

for every $t > 3/2$, where the constant C_t is dependent on t but independent of k. The assertions of the lemma now follow from (9.9), (9.10) and Lemma 9.2. $\square$

Proof of assertion(i) of Theorem 9.1 In view of (9.4) and Lemma 9.3, it is sufficient to show that there exists a constant C_{ab} such that

$$|\psi_1^{\pm}(x,\, k)| \le C_{ab} \tag{9.11}$$

for all $(x,\, k) \in \mathbb{R}^3 \times \{\, k \,|\, a \le |k| \le b\,\}$.

It follows from (9.4) and (9.6) that

$$\psi_1^{\pm}(x,\, k) = -G_0\big(V(\cdot)\psi_0^{\pm}(\cdot,\, k)\big)(x) - G_0\big(V(\cdot)\psi_1^{\pm}(\cdot,\, k)\big)(x). \tag{9.12}$$

We apply Lemma 9.3 to the first term on the right hand side of (9.12) and appeal to the definition (2.5) of G_0, and obtain

$$\big|G_0\big(V(\cdot)\psi_0^{\pm}(\cdot,\, k)\big)(x)\big| \le \frac{\|V(\cdot)\langle\cdot\rangle^{\sigma}\|_{L^\infty}}{2\pi^2} \int_{\mathbb{R}^3} \frac{C_{ab}}{|x-y|^2\langle y\rangle^{\sigma}}\, dy, \tag{9.13}$$

where the constant C_{ab} is the same as in (9.7), and is independent of k with $a \le |k| \le b$. By virtue of Lemma A.1 in Appendix, the function of x defined by the integral on the right hand side of (9.13) is bounded on $\mathbb{R}^3$. Thus the first term on the right hand side of (9.12) possesses the desired estimate. To handle the second term on the right hand side of (9.12), we decompose it into two parts:

$$
\begin{aligned}
G_0\big(V(\cdot)\psi_1^{\pm}(\cdot,\,k)\big)(x) &= \frac{1}{2\pi^2}\int_{|x-y|\le 1}\frac{V(y)\psi_1^{\pm}(y,\,k)}{|x-y|^2}\,dy \\
&\quad + \frac{1}{2\pi^2}\int_{|x-y|\ge 1}\frac{V(y)\psi_1^{\pm}(y,\,k)}{|x-y|^2}\,dy \\
&=: I^{\pm}(x,\,k) + I\!I^{\pm}(x,\,k). \qquad (9.14)
\end{aligned}
$$

We apply the Hölder inequality to $I^{\pm}(x,\,k)$, and get

$$
\begin{aligned}
|I^{\pm}(x,\,k)| &\le \frac{1}{2\pi^2}\Big\{\int_{|x-y|\le 1}\big(\frac{1}{|x-y|^2}\big)^{6/5}\,dy\Big\}^{5/6} \\
&\qquad \times\Big\{\int_{|x-y|\le 1}\big|V(y)\psi_1^{\pm}(y,\,k)\big|^6\,dy\Big\}^{1/6} \\
&\le \frac{1}{2\pi^2}\Big\{\int_{|y|\le 1}|y|^{-12/5}\,dy\Big\}^{5/6}\|V\|_{L^\infty}\|\psi_1^{\pm}(\cdot,\,k)\|_{L^6}. \qquad (9.15)
\end{aligned}
$$

Since $-12/5 > -3$, Lemma 9.4 and (9.15) imply that $I^{\pm}(x,\,k)$ satisfy the desired estimate. Similarly, we apply the Schwarz inequality to $I\!I^{\pm}(x,\,k)$, and we obtain

$$
\begin{aligned}
|I\!I^{\pm}(x,\,k)| &\le \frac{1}{2\pi^2}\Big\{\int_{|x-y|\ge 1}\big(\frac{1}{|x-y|^2}\big)^{2}\,dy\Big\}^{1/2} \\
&\qquad \times\Big\{\int_{|x-y|\ge 1}\big|V(y)\psi_1^{\pm}(y,\,k)\big|^2\,dy\Big\}^{1/2} \\
&\le \frac{1}{2\pi^2}\Big\{\int_{|y|\ge 1}|y|^{-4}\,dy\Big\}^{2} \\
&\qquad \times\|V(\cdot)\langle\cdot\rangle^{\sigma}\|_{L^\infty}\|\psi_1^{\pm}(\cdot,\,k)\|_{L^{2,-\sigma}}. \qquad (9.16)
\end{aligned}
$$

Since $-4 < -3$ and $\sigma > 3/2$, Lemma 9.4, together with (9.16), implies that $I\!I^{\pm}(x,\,k)$ have the desired estimate. Summing up, we have shown that (9.11) holds for all $(x,\,k)$ in $\mathbb{R}^3\times\{\,k\,|\,a\le|k|\le b\,\}$. □

In order to prepare lemmas, of which combination will directly give the proof of assertion (ii) of Theorem 9.1, it is convenient to write

$$
\begin{aligned}
\psi_{0\kappa}^{\pm}(x,\,k) &:= -K_{|k|}^{\mp}\big(\,V(\cdot)\varphi^{\pm}(\cdot,\,k)\big)(x), \qquad (9.17)\\
\psi_{0\mu}^{\pm}(x,\,k) &:= -M_{|k|}\big(\,V(\cdot)\varphi^{\pm}(\cdot,\,k)\big)(x). \qquad (9.18)
\end{aligned}
$$

According to (9.4)–(9.6), we then have

$$\varphi^{\pm}(x,\,k) = \varphi_0(x,\,k) + \psi^{\pm}_{0\kappa}(x,\,k) + \psi^{\pm}_{0\mu}(x,\,k) + \psi^{\pm}_{1}(x,\,k). \tag{9.19}$$

Lemma 9.5 $\psi^{\pm}_{0\kappa}(x,\,k)$ *are continuous on* $\mathbb{R}^3_x \times \{\, k \mid |k| \in (0,\,+\infty)\backslash\sigma_p(H)\,\}$.

Proof. Let $(x_0,\,k_0)$ be an arbitrary point in $\mathbb{R}^3_x \times \{\, k \mid |k| \in (0,\,+\infty)\setminus \sigma_p(H)\,\}$. We shall show that

$$\psi^{\pm}_{0\kappa}(x,\,k) \to \psi^{\pm}_{0\kappa}(x_0,\,k_0) \quad \text{as } (x,\,k) \to (x_0,\,k_0). \tag{9.20}$$

Let $\varepsilon > 0$ be given. One can then choose $r > 0$ so that

$$\frac{|k_0|}{2\pi}\,\|V\|_{L^\infty}\Big\{\sup_{\substack{y\in\mathbb{R}^3\\ |k-k_0|\le r}} |\varphi^{\pm}(y,\,k)|\Big\}\int_{|y|\le 2r}\frac{1}{|y|}\,dy < \varepsilon. \tag{9.21}$$

Note that, by virtue of assertion (i) of Theorem 9.1, the supremum in (9.21) is finite. To show (9.20), we write

$$\begin{aligned} & \psi^{\pm}_{0\kappa}(x,\,k) - \psi^{\pm}_{0\kappa}(x_0,\,k_0) \\ = \;& \{\psi^{\pm}_{0\kappa}(x,\,k) - \psi^{\pm}_{0\kappa}(x_0,\,k)\} + \{\psi^{\pm}_{0\kappa}(x_0,\,k) - \psi^{\pm}_{0\kappa}(x_0,\,k_0)\} \\ =: \;& I^{\pm}_{0\kappa}(x,\,k) + II^{\pm}_{0\kappa}(k). \end{aligned} \tag{9.22}$$

If $|x - x_0| \le r$ and $|k - k_0| \le r$, we then have, appealing to the definition (5.1),

$$\begin{aligned} |I^{\pm}_{0\kappa}(x,\,k)| \le 2\frac{|k|}{|k_0|}\varepsilon + \frac{|k|}{2\pi}\Big\{\sup_{\substack{y\in\mathbb{R}^3\\ |k-k_0|\le r}} |\varphi^{\pm}(y,\,k)|\Big\} \times \\ \times \int_{\mathbb{R}^3}\Big|\Big\{1_{E(x,\,2r)}(y)\frac{e^{\pm i|k||x-y|}}{|x-y|} - 1_{E(x_0,\,2r)}(y)\frac{e^{\pm i|k||x_0-y|}}{|x_0-y|}\Big\}V(y)\Big|\,dy, \end{aligned} \tag{9.23}$$

where $E(x,\,2r) = \{\, y \,|\, |x-y| > 2r\,\}$ and we have used (9.21). We note here that

$$1_{E(x,\,2r)}(y)\frac{1}{|x-y|} \le \frac{3}{2} \times 1_{E(x_0,\,r)}(y)\frac{1}{|x_0-y|} \tag{9.24}$$

whenever $|x - x_0| \le r$. Hence, the integrand in (9.23) is bounded, for all $(x,\,k)$ with $|x - x_0| \le r$, by the function

$$\frac{5}{2} \times 1_{E(x_0,\,r)}(y)\frac{1}{|x_0-y|}\,|V(y)| \tag{9.25}$$

which is in $L^1(\mathbb{R}^3_y)$ (recall that we made the assumption (1.6) with $\sigma > 2$). Hence we can apply the Lebesgue dominated convergence theorem to the integral in (9.23), and deduce that

$$\limsup_{(x,k)\to(x_0,k_0)} |I^{\pm}_{0\kappa}(x,\, k)| \le 2\varepsilon. \tag{9.26}$$

In a similar fashion to (9.23), if $|k - k_0| \le r$, we have

$$\begin{aligned}
|I\!I^{\pm}_{0\kappa}(k)| &\le |K^{\mp}_{|k|}\big(V(\cdot)\varphi^{\pm}(\cdot,\, k)\big)(x_0) - K^{\mp}_{|k_0|}\big(V(\cdot)\varphi^{\pm}(\cdot,\, k)\big)(x_0)| \\
&\quad + |K^{\mp}_{|k_0|}\big(V(\cdot)\varphi^{\pm}(\cdot,\, k)\big)(x_0) - K^{\mp}_{|k_0|}\big(V(\cdot)\varphi^{\pm}(\cdot,\, k_0)\big)(x_0)| \\
&\le \Big(\frac{|k|}{|k_0|} + 3\Big)\varepsilon + \frac{1}{2\pi}\int_{\mathbb{R}^3} \Big| 1_{E(x_0,\, 2r)}(y) \frac{1}{|x_0 - y|} \times \\
&\qquad \Big| |k| e^{\mp i|k||x_0-y|} - |k_0| e^{\mp i|k_0||x_0-y|} \Big| \big| V(y)\varphi^{\pm}(y,\, k)\big|\, dy \qquad (9.27) \\
&\quad + \frac{|k_0|}{2\pi}\int_{\mathbb{R}^3} \Big| 1_{E(x_0,\, 2r)}(y) \frac{e^{\mp i|k_0||x_0-y|}}{|x_0 - y|} V(y) \Big| \times \\
&\qquad |\varphi^{\pm}(y,\, k) - \varphi^{\pm}(y,\, k_0)\big|\, dy. \qquad (9.28)
\end{aligned}$$

The integral in (9.27) is estimated by

$$\Big\{ \int_{\mathbb{R}^3} \Big| 1_{E(x_0,\, 2r)}(y)\, \frac{|V(y)|^2 \langle y\rangle^{2s}}{|x_0 - y|^2} \times \Big| |k| e^{\mp i|k||x_0-y|} - |k_0| e^{\mp i|k_0||x_0-y|} \Big|^2 dy \Big\}^{1/2} \|\varphi^{\pm}(\cdot,\, k)\|_{L^{2,-s}} \tag{9.29}$$

In view of (9.2), it follows that

$$1_{E(x_0,\, 2r)}(y)\, \frac{|V(y)|^2 \langle y\rangle^{2s}}{|x_0 - y|^2} \in L^1(\mathbb{R}^3_y).$$

Therefore, applying the Lebesgue dominated convergence theorem to the integral in (9.29) and appealing to Lemma 9.1, we see that the integral in (9.27) tends to 0 as k approaches k_0. Also, the integral in (9.28) is estimated by

$$\Big\{ \int_{\mathbb{R}^3} \Big| 1_{E(x_0,\, 2r)}(y)\, \frac{|V(y)|^2 \langle y\rangle^{2s}}{|x_0 - y|^2} \Big|^2 dy \Big\}^{1/2} \times \|\varphi^{\pm}(\cdot,\, k) - \varphi^{\pm}(\cdot,\, k_0)\|_{L^{2,-s}},$$

which tends to 0, by Lemma 9.1, as k approaches k_0. Thus, we have shown that

$$\limsup_{(x,k)\to(x_0,k_0)} |I\!I^{\pm}_{0\kappa}(k)| \le 4\varepsilon. \tag{9.30}$$

Combining (9.22), (9.26) and (9.30), we deduce that

$$\limsup_{(x,k)\to(x_0,k_0)} |\psi^\pm_{0\kappa}(x,\, k) - \psi^\pm_{0\kappa}(x_0,\, k_0)| \le 6\varepsilon. \tag{9.31}$$

Since ε is arbitrary, (9.31) implies (9.20). $\square$

Lemma 9.6 $\psi^\pm_{0\mu}(x,\, k)$ *are continuous on* $\mathbb{R}^3_x \times \{\ k \mid |k| \in (0,\, +\infty) \backslash \sigma_p(H)\ \}$.

Proof. The proof is similar to that of Lemma 9.5.

Let $(x_0,\, k_0)$ be an arbitrary point in $\mathbb{R}^3_x \times \{\ k \mid |k| \in (0,\, +\infty) \setminus \sigma_p(H)\ \}$. We shall show that

$$\psi^\pm_{0\mu}(x,\, k) \to \psi^\pm_{0\mu}(x_0,\, k_0) \quad \text{as } (x,\, k) \to (x_0,\, k_0). \tag{9.32}$$

To show this, we first need to appeal to the definition (4.5) of $m_\lambda(x)$ and the inequality (5.16). We then have

$$|m_\lambda(x)| \le \frac{\text{const.}}{2\pi^2} \cdot \frac{\lambda}{|x|}(1+\lambda|x|)^{-1} \le \text{const.}' \frac{\lambda}{|x|}, \tag{9.33}$$

where const. is the same as in (5.16) and const.$' :=$ const.$/2\pi^2$.

Let $\varepsilon > 0$ be given. We choose $r > 0$ so that

$$\text{const.}'\, |k_0|\, \|V\|_{L^\infty} \Big\{ \sup_{\substack{y\in\mathbb{R}^3 \\ |k-k_0|\le r}} |\varphi^\pm(y,\, k)| \Big\} \int_{|y|\le 2r} \frac{1}{|y|}\, dy < \varepsilon. \tag{9.34}$$

Similarly to (9.22), we write

$$\begin{aligned} & \psi^\pm_{0\mu}(x,\, k) - \psi^\pm_{0\mu}(x_0,\, k_0) \\ = \ & \{\psi^\pm_{0\mu}(x,\, k) - \psi^\pm_{0\mu}(x_0,\, k)\} + \{\psi^\pm_{0\mu}(x_0,\, k) - \psi^\pm_{0\mu}(x_0,\, k_0)\} \\ =: \ & I^\pm_{0\mu}(x,\, k) + II^\pm_{0\mu}(k). \end{aligned} \tag{9.35}$$

If $|x - x_0| \le r$ and $|k - k_0| \le r$, then it follows from the definition (5.2) and (9.33), (9.34) that

$$\begin{aligned} |I^\pm_{0\mu}(x,\, k)| \le 2\frac{|k|}{|k_0|}\varepsilon + \Big\{ \sup_{\substack{y\in\mathbb{R}^3 \\ |k-k_0|\le r}} |\varphi^\pm(y,\, k)| \Big\} \times \\ \times \int_{\mathbb{R}^3} \Big| \Big\{ 1_{E(x,\, 2r)}(y)\, m_{|k|}(x-y) - 1_{E(x_0,\, 2r)}(y)\, m_{|k|}(x_0-y) \Big\} V(y) \Big|\, dy. \end{aligned} \tag{9.36}$$

Noting (9.33) and (9.24), we find that the integrand in (9.36) is bounded by the function

$$\frac{5}{2} \times 1_{E(x_0,\, r)}(y) \frac{\text{const.}'(|k_0| + r)}{|x_0 - y|}\, |V(y)| \in L^1(\mathbb{R}^3_y) \tag{9.37}$$

for all (x, k) with $|x - x_0| \le r$, $|k - k_0| \le r$. Therefore, the Lebesgue dominated convergence theorem applied to the integral in (9.36) gives

$$\limsup_{(x,k)\to(x_0,k_0)} |I^{\pm}_{0\mu}(x, k)| \le 2\varepsilon. \tag{9.38}$$

Here we have used the fact that $m_{|k|}(x)$ is continuous on $\{\, \mathbb{R}^3_x \setminus \{0\}\} \times \mathbb{R}^3_k$. In a similar manner to (9.36), if $|x - x_0| \le r$ and $|k - k_0| \le r$, then we have

$$|II^{\pm}_{0\mu}(k)| \le \Big(\frac{|k|}{|k_0|} + 3\Big)\varepsilon$$

$$+ \int_{\mathbb{R}^3} \Big| 1_{E(x_0,\,2r)}(y)\big(m_{|k|}(x_0 - y) - m_{|k_0|}(x_0 - y)\big)\Big| \big|V(y)\varphi^{\pm}(y,\, k)\big|\, dy \tag{9.39}$$

$$+ \int_{\mathbb{R}^3} \Big| 1_{E(x_0,\,2r)}(y)\, m_{|k_0|}(x_0 - y)V(y)\Big| \big|\varphi^{\pm}(y,\, k) - \varphi^{\pm}(y,\, k_0)\big|\, dy. \tag{9.40}$$

The integral in (9.39) is estimated by

$$\Big\{ \int_{\mathbb{R}^3} 1_{E(x_0,\,2r)}(y) \,\big|\big(m_{|k|}(x_0 - y) - m_{|k_0|}(x_0 - y)\big)\, V(y)\langle y\rangle^s \,\big|^2\, dy\Big\}^{1/2} \times$$

$$\times \|\varphi^{\pm}(\cdot,\, k)\|_{L^{2,-s}}. \tag{9.41}$$

In view of the inequality (5.17) and the continuity of $m_{|k|}(x)$, as mentioned after (9.38), we can apply the Lebesgue dominated convergence theorem to the integral in (9.41), and deduce that the integral in (9.39) tends to 0 as k approaches k_0. Also, the integral in (9.40) is estimated by

$$\Big\{ \int_{\mathbb{R}^3} 1_{E(x_0,\,2r)}(y) \,\big|m_{|k_0|}(x_0 - y)\, V(y)\langle y\rangle^s\big|^2\, dy\Big\}^{1/2} \times$$

$$\times \|\varphi^{\pm}(\cdot,\, k) - \varphi^{\pm}(\cdot,\, k_0)\|_{L^{2,-s}}, \tag{9.42}$$

which tends to 0, by Lemma 9.1 and the inequality (5.17), as k approaches k_0. Thus we have shown that

$$\limsup_{(x,k)\to(x_0,k_0)} |II^{\pm}_{0\mu}(x, k)| \le 4\varepsilon.$$

By the same arguments as in the end of the proof of Lemma 9.5, we conclude that (9.32) is verified. □

Lemma 9.7 *$\psi^{\pm}_1(x, k)$ are continuous on $\mathbb{R}^3_x \times \{\, k \mid |k| \in (0, +\infty)\backslash\sigma_p(H) \,\}$.*

The proof of Lemma 9.7 is similar to those of Lemmas 9.5 and 9.6. Actually it is much easier because the integral kernel of the operator G_0 is independent of the variable k (recall the definitions (2.5) and (9.6)). For this reason, we omit the proof of Lemma 9.7.

Proof of assertion(ii) of Theorem 9.1 Assertion(ii) is a direct consequence of (9.19) and Lemmas 9.5, 9.6 and 9.7. □

10 Asymptotic behaviors of the generalized eigenfunctions

We shall first show that the generalized eigenfunctions $\varphi^{\pm}(x, k)$, defined by (8.5), are distorted plane waves, and give estimates of the differences between $\varphi^{\pm}(x, k)$ and the plane wave $\varphi_0(x, k) = e^{ix\cdot k}$ (Theorem 10.1). We shall next prove that $\varphi^{\pm}(x, k)$ are asymptotically equal to the sums of the plane wave and the spherical waves $e^{\mp i|x||k|}/|x|$ under the assumption that $\sigma > 3$, and shall give estimates of the differences between $\varphi^{\pm}(x, k)$ and the sums mentioned above (Theorem 10.2).

In view of the definition (8.5) and Theorem 7.2(ii), it is clear that $\varphi^{-}(x, k)$ (resp. $\varphi^{+}(x, k)$) is the sum of the plane wave $e^{ix\cdot k}$ and the solution of the equation (7.2) with the outgoing radiation condition (6.2) (resp. the incoming radiation condition (6.3)). However, the radiation conditions (6.2) and (6.3) are generalizations of the radiation condition mentioned in the beginning of Section 6, and this generalization makes it unclear that

$$R^{\mp}(|k|)\{V(\cdot)\varphi_0(\cdot, k)\}(x)$$

behave as $e^{\mp i|x||k|}/|x|$ at infinity. Theorem 10.2 shows that this is indeed the case if $\sigma > 3$.

Theorem 10.1 *Let $\sigma > 2$. If $|k| \in (0, +\infty) \setminus \sigma_p(H)$, then*

$$\left|\varphi^{\pm}(x, k) - e^{x\cdot k}\right| \le C_k \begin{cases} \langle x\rangle^{-(\sigma-2)} & \text{if } \ 2 < \sigma < 3, \\ \langle x\rangle^{-1}\log(1+\langle x\rangle) & \text{if } \ \sigma = 3, \\ \langle x\rangle^{-1} & \text{if } \ \sigma > 3, \end{cases} \tag{10.1}$$

where the constant C_k is uniform for k in any compact subset of $\{\, k \mid |k| \in (0, +\infty) \setminus \sigma_p(H) \,\}$.

Proof. In view of (9.19), it is sufficient to show that all of $\psi^{\pm}_{0\kappa}(x, k)$, $\psi^{\pm}_{0\mu}(x, k)$ and $\psi^{\pm}_{1}(x, k)$ satisfy the estimates (10.1).

By assertion(i) of Theorem 9.1 and the definitions (9.17) and (5.1), we have

$$|\psi^{\pm}_{0\kappa}(x, k)| \le \frac{|k|}{2\pi}\|\langle\cdot\rangle^{\sigma}V(\cdot)\varphi^{\pm}(\cdot, k)\|_{L^\infty}\int_{\mathbb{R}^3}\frac{1}{|x-y|\langle y\rangle^{\sigma}}\,dy. \tag{10.2}$$

If we apply Lemma A.1 in Appendix, with $n = 3$, $\beta = 1$ and $\gamma = \sigma$, to the integral on the right hand side of (10.2), we can deduce from assertion(i) of

Theorem 9.1 and(10.2) that $\psi^{\pm}_{0\kappa}(x,\,k)$ satisfy the desired estimates. By the definitions (9.18), (5.2) and the inequality (5.17), we get

$$|\psi^{\pm}_{0\mu}(x,\,k)| \le C_{|k|} \|\langle\cdot\rangle^{\sigma} V(\cdot)\varphi^{\pm}(\cdot,\,k)\|_{L^{\infty}} \int_{\mathbb{R}^3} \frac{1}{|x-y|^2\langle y\rangle^{\sigma}}\,dy, \tag{10.3}$$

where the constant $C_{|k|}$ is the one specified in (5.17). Similarly, by the definition (2.5), we obtain

$$|\psi^{\pm}_{1}(x,\,k)| \le \frac{1}{2\pi^2} \|\langle\cdot\rangle^{\sigma} V(\cdot)\varphi^{\pm}(\cdot,\,k)\|_{L^{\infty}} \int_{\mathbb{R}^3} \frac{1}{|x-y|^2\langle y\rangle^{\sigma}}\,dy. \tag{10.4}$$

Lemma A.1 with $n=3$, $\beta=2$ and $\gamma=\sigma$ now gives

$$|\psi^{\pm}_{0\mu}(x,\,k)| + |\psi^{\pm}_{1}(x,\,k)| \le C'_k \begin{cases} \langle x\rangle^{-(\sigma-1)} & \text{if} \quad 2<\sigma<3, \\ \langle x\rangle^{-2}\log(1+\langle x\rangle) & \text{if} \quad \sigma=3, \\ \langle x\rangle^{-2} & \text{if} \quad \sigma>3, \end{cases} \tag{10.5}$$

where the constant C'_k is uniform for k in any compact subset of $\{\,k\,|\,|k| \in (0,\,+\infty)\setminus\sigma_p(H)\,\}$. □

Theorem 10.2 *Let $\sigma>3$, and suppose that $|k|\in(0,\,+\infty)\setminus\sigma_p(H)$. Then for $|x|\ge 1$ we have*

$$\begin{aligned} &\Big|\varphi^{\pm}(x,\,k) - \Big(e^{ix\cdot k} + \frac{e^{\mp i|k|\,|x|}}{|x|}\,f^{\pm}(|k|,\,\omega_x,\,\omega_k)\Big)\Big| \\ &\le C_k \begin{cases} |x|^{-(\sigma-1)/2} & \textit{if } 3<\sigma<5, \\ |x|^{-2}\log(1+|x|) & \textit{if } \sigma=5, \\ |x|^{-2} & \textit{if } \sigma>5, \end{cases} \end{aligned} \tag{10.6}$$

where $\omega_x = x/|x|$, $\omega_k = k/|k|$,

$$f^{\pm}(\lambda,\,\omega_x,\,\omega_k) = -\frac{\lambda}{2\pi}\int_{\mathbb{R}^3} e^{\pm i\lambda\omega_x\cdot y}\,V(y)\,\varphi^{\pm}(y,\,\lambda\omega_k)\,dy, \tag{10.7}$$

and the constant C_k is uniform for k in any compact subset of $\{\,k\,|\,|k| \in (0,\,+\infty)\setminus\sigma_p(H)\,\}$.

We shall give a proof of Theorem 10.2 by means of a series of lemmas.

Lemma 10.1 *Let $\sigma > 3$. Then*

$$\left| \varphi^{\pm}(x,\, k) - \left(e^{ix\cdot k} + \psi^{\pm}_{0\kappa}(x,\, k)\right) \right| \le C_k \langle x\rangle^{-2},$$

where C_k is a constant uniform for k in any compact subset of $\{\, k \,|\, |k| \in (0,\, +\infty) \setminus \sigma_p(H)\,\}$.

Proof. The lemma is a direct consequence of (10.5) and (9.19). □

In view of Lemma 10.1, it is apparent that we need to evaluate the differences

$$\psi^{\pm}_{0\kappa}(x,\, k) - \frac{e^{\mp i|k|\,|x|}}{|x|} f^{\pm}(|k|,\, \omega_x,\, \omega_k), \tag{10.8}$$

which are equal to

$$\frac{|k|}{2\pi} \int_{\mathbb{R}^3} \left\{ \frac{e^{\mp i|k|(|x|-\omega_x\cdot y)}}{|x|} - \frac{e^{\mp i|k||x-y|}}{|x-y|} \right\} V(y)\, \varphi^{\pm}(y,\, |k|\omega_k)\, dy \tag{10.9}$$

by (9.17), (5.1) and (10.7). Thus we are led to consider the following integrals:

$$\frac{1}{|x|} \int_{\mathbb{R}^3} e^{ia(|x|-\omega_x\cdot y)} u(y)\, dy, \tag{10.10}$$

$$\int_{\mathbb{R}^3} \frac{e^{ia|x-y|}}{|x-y|}\, u(y)\, dy, \tag{10.11}$$

and their difference. The same integrals as in (10.10) and (10.11) were discussed in Ikebe[7, §3], though our arguments below are slightly different from those of [7], and our estimates are slight refinements of those of [7].

Lemma 10.2 *Let $a \in \mathbb{R}$ and let u satisfy*

$$|u(x)| \le C\langle x\rangle^{-\sigma}, \quad \sigma > 3. \tag{10.12}$$

Then for $|x| \ge 1$ we have

$$\left| \int_{|y|\ge\sqrt{|x|}} e^{ia(|x|-\omega_x\cdot y)} u(y)\, dy \right| \le C_1 \, \|\langle\cdot\rangle^{\sigma} u\|_{L^\infty}\, |x|^{-(\sigma-3)/2}, \tag{10.13}$$

$$\left| \int_{|y|\ge\sqrt{|x|}} \frac{e^{ia|x-y|}}{|x-y|}\, u(y)\, dy \right| \le C_2 \, \|\langle\cdot\rangle^{\sigma} u\|_{L^\infty}\, |x|^{-(\sigma-1)/2}, \tag{10.14}$$

where the constants C_1 and C_2 are independent of a.

Proof. It follows that

$$\Big|\int_{|y|\geq\sqrt{|x|}} e^{ia(|x|-\omega_x\cdot y)}u(y)\,dy\Big| \;\leq\; \int_{|y|\geq\sqrt{|x|}} \|\langle\cdot\rangle^{\sigma}u\|_{L^\infty}\;|y|^{-\sigma}\,dy$$
$$\leq\; C_1\|\langle\cdot\rangle^{\sigma}u\|_{L^\infty}\;|x|^{-(\sigma-3)/2}. \quad (10.15)$$

To show (10.14), we decompose the integral in (10.14) into two parts:

$$\int_{|y|\geq\sqrt{|x|}} \frac{e^{ia|x-y|}}{|x-y|}\,u(y)\,dy =$$
$$\Big\{\int_{F_0(x)}+\int_{F_1(x)}\Big\}\frac{e^{ia|x-y|}}{|x-y|}\,u(y)\,dy, \quad (10.16)$$

where

$$F_0(x) := \{\, y\in\mathbb{R}^3 \mid |y|\geq\sqrt{|x|},\;\; |x-y|\leq\frac{|x|}{2}\,\},$$
$$F_1(x) := \{\, y\in\mathbb{R}^3 \mid |y|\geq\sqrt{|x|},\;\; |x-y|\geq\frac{|x|}{2}\,\}.$$

If $y\in F_0(x)$, then

$$|y| = |x-(x-y)| \geq |x|-|x-y| \geq \frac{|x|}{2},$$

hence we have

$$\begin{aligned}\Big|\int_{F_0(x)} &\frac{e^{ia|x-y|}}{|x-y|}\,u(y)\,dy\Big| \\ &\leq \int_{F_0(x)} \frac{1}{|x-y|}\,\|\langle\cdot\rangle^{\sigma}u\|_{L^\infty}\,|y|^{-\sigma}\,dy \\ &\leq \|\langle\cdot\rangle^{\sigma}u\|_{L^\infty}\,2^{\sigma}|x|^{-\sigma}\int_{|x-y|\leq|x|/2}\frac{1}{|x-y|}\,dy \\ &= C'\,\|\langle\cdot\rangle^{\sigma}u\|_{L^\infty}\,|x|^{-(\sigma-2)}. \end{aligned} \quad (10.17)$$

If $y\in F_1(x)$, then $|x-y|\geq|x|/2$, therefore we get

$$\begin{aligned}\Big|\int_{F_1(x)} &\frac{e^{ia|x-y|}}{|x-y|}\,u(y)\,dy\Big| \\ &\leq \int_{|y|\geq\sqrt{|x|}} \frac{2}{|x|}\,\|\langle\cdot\rangle^{\sigma}u\|_{L^\infty}\,|y|^{-\sigma}\,dy \\ &\leq C''\,\|\langle\cdot\rangle^{\sigma}u\|_{L^\infty}\,|x|^{-(\sigma-1)/2}. \end{aligned} \quad (10.18)$$

Since $\sigma - 2 > (\sigma - 1)/2$, we conclude from (10.16)–(10.18) that the inequality (10.14) holds. $\square$

In view of (10.10), (10.11) and Lemma 10.2, we now need to consider the integral

$$\int_{|y| \le \sqrt{|x|}} \Big(\frac{1}{|x|} \, e^{ia(|x| - \omega_x \cdot y)} - \frac{e^{ia|x-y|}}{|x-y|} \Big) u(y) \, dy. \tag{10.19}$$

To get an estimate on the integral (10.19), we split it into two parts:

$$\frac{1}{|x|} \int_{|y| \le \sqrt{|x|}} \Big(e^{ia(|x| - \omega_x \cdot y)} - e^{ia|x-y|} \Big) \, u(y) \, dy \tag{10.20}$$

$$+ \int_{|y| \le \sqrt{|x|}} \Big(\frac{1}{|x|} - \frac{1}{|x-y|} \Big) e^{ia|x-y|} \, u(y) \, dy, \tag{10.21}$$

and evaluate these two integrals separately.

Lemma 10.3 *Under the same assumptions as in Lemma 10.2, we have*

$$\Big| \frac{1}{|x|} \int_{|y| \le \sqrt{|x|}} \Big(e^{ia(|x| - \omega_x \cdot y)} - e^{ia|x-y|} \Big) \, u(y) \, dy \Big|$$

$$\le C_3 \, |a| \, \| \langle \cdot \rangle^\sigma u \|_{L^\infty} \begin{cases} |x|^{-(\sigma-1)/2} & \text{if } 3 < \sigma < 5, \\ |x|^{-2} \log(1 + |x|) & \text{if } \sigma = 5, \\ |x|^{-2} & \text{if } \sigma > 5 \end{cases} \tag{10.22}$$

for $|x| \ge 1$, where the constant C_3 is independent of a.

Proof. We start with simple remarks that

$$|x - y| = |x| \Big(1 - 2 \, \frac{\omega_x \cdot y}{|x|} + \frac{|y|^2}{|x|^2} \Big)^{1/2} \tag{10.23}$$

and

$$\big| (1 + \rho)^{1/2} - (1 + \frac{\rho}{2}) \big| \le \frac{\sqrt{2}}{2} \, \rho^2, \qquad \rho \ge -\frac{1}{2}. \tag{10.24}$$

It is easy to see that

$$\big| -2 \, \frac{\omega_x \cdot y}{|x|} + \frac{|y|^2}{|x|^2} \big| \le \frac{1}{2} \tag{10.25}$$

if $\sqrt{|x|} \geq 5$ and $|y| \leq \sqrt{|x|}$. Hence, it follows from (10.23)–(10.25) that

$$\big||x-y| - (|x| - \omega_x \cdot y)\big| \leq 3\sqrt{2}\,\frac{|y|^2}{|x|} \tag{10.26}$$

when $\sqrt{|x|} \geq 5$ and $|y| \leq \sqrt{|x|}$. Using the inequality

$$|e^{i\alpha} - e^{i\alpha}| \leq |\alpha - \beta|, \qquad \alpha,\, \beta \in \mathbb{R},$$

we have

$$\begin{aligned}
&\Big|\int_{|y|\leq\sqrt{|x|}} \big(e^{ia(|x|-\omega_x\cdot y)} - e^{ia|x-y|}\big)\, u(y)\, dy\Big| \\
&\leq \int_{|y|\leq\sqrt{|x|}} \big|a(|x|-\omega_x\cdot y) - a|x-y|\big|\, \|\langle\cdot\rangle^\sigma u\|_{L^\infty}\, \langle y\rangle^{-\sigma}\, dy \qquad (10.27)\\
&\leq 3\sqrt{2}\, |a|\, \|\langle\cdot\rangle^\sigma u\|_{L^\infty}\, \frac{1}{|x|} \int_{|y|\leq\sqrt{|x|}} |y|^2\, \langle y\rangle^{-\sigma}\, dy \qquad (10.28)
\end{aligned}$$

when $\sqrt{|x|} \geq 5$. Here we have used (10.26) in the second inequality (10.28). Now we have

$$\begin{aligned}
&\int_{|y|\leq\sqrt{|x|}} |y|^2\, \langle y\rangle^{-\sigma}\, dy \\
&\leq 2^{\sigma/2} \int_{|y|\leq\sqrt{|x|}} (1+|y|)^{2-\sigma}\, dy \qquad \Big(\because\ \langle y\rangle \geq \frac{1}{\sqrt{2}}\,(1+|y|)\,\Big)\\
&\qquad = 2^{\sigma/2} \times 4\pi \int_0^{\sqrt{|x|}} (1+r)^{-\sigma+4}\, dr \qquad (10.29)\\
&\qquad\leq 2^{(\sigma+4)/2}\pi \times \begin{cases} \dfrac{|x|^{-(\sigma-5)/2}}{5-\sigma} & \text{if } 3 < \sigma < 5, \\[2ex] \log(1+|x|) & \text{if } \sigma = 5, \\[2ex] \dfrac{1}{\sigma-5} & \text{if } \sigma > 5, \end{cases} \qquad (10.30)
\end{aligned}$$

where we have used spherical polar coordinates in (10.29). Combining (10.30) with (10.28) yields the desired inequalities. $\square$

Lemma 10.4 *Under the same assumptions as in Lemma 10.2, we have*

$$\Big| \int_{|y| \le \sqrt{|x|}} \Big(\frac{1}{|x|} - \frac{1}{|x-y|} \Big) e^{ia|x-y|} \, u(y) \, dy \Big|$$
$$\le C_4 \, \| \langle \cdot \rangle^\sigma u \|_{L^\infty} \begin{cases} |x|^{-\sigma/2} & \text{if } 3 < \sigma < 4, \\ |x|^{-2} \log(1+|x|) & \text{if } \sigma = 4, \\ |x|^{-2} & \text{if } \sigma > 4 \end{cases} \tag{10.31}$$

for $|x| \ge 1$, where the constant C_4 is independent of a.

Proof. If $\sqrt{|x|} \ge 5$ and $|y| \le \sqrt{|x|}$, then the inequality (10.26) implies

$$\big| \, |x-y| - |x| \, \big| \le |y| + 3\sqrt{2} \frac{|y|^2}{|x|}.$$

Also, if $\sqrt{|x|} \ge 5$ and $|y| \le \sqrt{|x|}$, we then have

$$|x-y| \ge |x| - |y| \ge |x| - \frac{|x|}{5} = \frac{4}{5} \, |x|.$$

Using these two inequalities, we arrive at

$$\Big| \int_{|y| \le \sqrt{|x|}} \Big(\frac{1}{|x|} - \frac{1}{|x-y|} \Big) e^{ia|x-y|} \, u(y) \, dy \Big|$$
$$\le \int_{|y| \le \sqrt{|x|}} \frac{5}{4} \cdot \frac{1}{|x|^2} \, \big(|y| + 3\sqrt{2} \frac{|y|^2}{|x|} \, \big) \, \| \langle \cdot \rangle^\sigma u \|_{L^\infty} \, \langle y \rangle^{-\sigma} \, dy$$
$$= \frac{5 \, \| \langle \cdot \rangle^\sigma u \|_{L^\infty} \, 2^{\sigma/2}}{4} \Big(\frac{1}{|x|^2} \int_{|y| \le \sqrt{|x|}} (1+|y|)^{1-\sigma} \, dy$$
$$+ 3\sqrt{2} \frac{1}{|x|^3} \int_{|y| \le \sqrt{|x|}} (1+|y|)^{2-\sigma} \, dy \Big) \tag{10.32}$$

provided that $\sqrt{|x|} \ge 5$. By introducing spherical polar coordinates, we obtain

$$\int_{|y| \le \sqrt{|x|}} (1+|y|)^{1-\sigma} \, dy$$
$$\le 4\pi \begin{cases} \dfrac{|x|^{-(\sigma-4)/2}}{4-\sigma} & \text{if } \ 3 < \sigma < 4, \\ \log(1+|x|) & \text{if } \ \sigma = 4, \\ \dfrac{1}{\sigma - 4} & \text{if } \ \sigma > 4. \end{cases} \tag{10.33}$$

Combining (10.32) with (10.33) and (10.30), we conclude that the desired inequalities are verified. $\square$

Proof of Theorem 10.2 We write

$$
\begin{aligned}
&\varphi^{\pm}(x,\,k) - \Big(e^{ix\cdot k} + \frac{e^{\mp i|k|\,|x|}}{|x|}\, f^{\pm}(|k|,\,\omega_x,\,\omega_k)\Big)\\
&\quad = \varphi^{\pm}(x,\,k) - \big(e^{ix\cdot k} + \psi^{\pm}_{0\kappa}(x,\,k)\big)\\
&\qquad\quad + \Big(\psi^{\pm}_{0\kappa}(x,\,k) - \frac{e^{\mp i|k|\,|x|}}{|x|}\, f^{\pm}(|k|,\,\omega_x,\,\omega_k)\Big)\\
&\quad = \varphi^{\pm}(x,\,k) - \big(e^{ix\cdot k} + \psi^{\pm}_{0\kappa}(x,\,k)\big)\\
&\qquad + \frac{|k|}{2\pi}\cdot\frac{1}{|x|}\int_{|y|\ge\sqrt{|x|}} e^{\mp i|k|(|x|-\omega_x\cdot y)}\, V(y)\,\varphi^{\pm}(y,\,|k|\omega_k)\,dy\\
&\qquad - \frac{|k|}{2\pi}\int_{|y|\ge\sqrt{|x|}} \frac{e^{\mp i|k||x-y|}}{|x-y|} V(y)\,\varphi^{\pm}(y,\,|k|\omega_k)\,dy\\
&+ \frac{|k|}{2\pi}\cdot\frac{1}{|x|}\int_{|y|\le\sqrt{|x|}} \big(e^{\mp i|k|(|x|-\omega_x\cdot y)} - e^{\mp i|k||x-y|}\big)\, V(y)\,\varphi^{\pm}(y,\,|k|\omega_k)\,dy\\
&\quad + \frac{|k|}{2\pi}\int_{|y|\le\sqrt{|x|}} \Big\{\frac{1}{|x|} - \frac{1}{|x-y|}\Big\}\, e^{\mp i|k||x-y|}\, V(y)\,\varphi^{\pm}(y,\,|k|\omega_k)\,dy,
\end{aligned}
$$

where we have used the fact that (10.8) equals (10.9), and decomposed the integral in (10.9) into four parts. Now the conclusion of the theorem follows from assertion(i) of Theorem 9.1 and Lemmas 10.1–10.4. $\square$

11 Appendix

In this appendix we shall derive a few formulae and estimates concerning the cosine integral and the sine integral functions for the reader's convenience, the formulae and estimates which seem not to be found in the literature. We begin with the definitions of these functions and some basic facts (cf. [5] and [6]).

A.1. The cosine integral function. The definition is

$$\mathrm{ci}(\rho) = -\mathrm{Ci}(\rho) = \int_{\rho}^{+\infty} \frac{\cos t}{t}\, dt, \quad \rho > 0$$

(cf. [5, p. 386]). We have

$$|\mathrm{ci}(\rho)| \leq \text{const.} \begin{cases} \rho^{-1} & \text{if } \rho \geq 1, \\ \\ 1 + |\log \rho| & \text{if } 0 < \rho \leq 1. \end{cases}$$

The estimate for $\rho \geq 1$ follows from

$$\mathrm{ci}(\rho) = -\frac{\sin \rho}{\rho} + \frac{\cos \rho}{\rho^2} - 2\int_{\rho}^{+\infty} \frac{\cos t}{t^3}\, dt,$$

which can be shown by repeated use of integration by parts. The estimate for $0 < \rho \leq 1$ follows from [6, p. 145, Formula(6)].

The cosine integral function $\mathrm{ci}(\rho)$ has an analytic continuation $\mathrm{ci}(z)$, which is a many-valued function with a logarithmic branch-point at $z = 0$ (see [6, p.145]). In this paper, we choose the principal branch:

$$\mathrm{ci}(z) = -\gamma - \mathrm{Log}\, z - \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)!2m} z^{2m}, \quad z \in \mathbf{C} \setminus (-\infty, 0], \tag{A.1}$$

where γ is Euler's constant and $|\mathrm{Im}\, \mathrm{Log}\, z| < \pi$. Note that the power series

$$h_e(z) := \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m)!2m} z^{2m}$$

on the right hand side of (A.1) is an entire function and satisfies that $h_e(-z) = h_e(z)$, i.e., $h_e(z)$ is an even function.

A.2. The sine integral function. The definition is

$$\mathrm{si}(\rho) = -\int_{\rho}^{+\infty} \frac{\sin t}{t}\, dt, \quad \rho > 0$$

(cf. [5, p. 386]). Since

$$\mathrm{si}(\rho) = -\frac{\pi}{2} + \int_0^{\rho} \frac{\sin t}{t}\, dt,$$

we can show, by integration by parts, that

$$|\mathrm{si}(\rho)| \le \mathrm{const.}(1 + |\rho|\,)^{-1}.$$

Moreover, we see that $\mathrm{si}(\rho)$ has an analytic continuation $\mathrm{si}(z)$:

$$\mathrm{si}(z) = -\frac{\pi}{2} + \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!\,(2m+1)}\, z^{2m+1}. \tag{A.2}$$

It follows from (A.2) that $\mathrm{si}(z)$ is an entire function and satisfies that

$$\mathrm{si}(-z) = -\pi - \mathrm{si}(z) \tag{A.3}$$

(cf. [6, p.145]).

A.3. Laplace transforms. In computing the resolvent kernel of $\sqrt{-\Delta}$ in Section 2, we applied the following formula

$$\int_0^{+\infty} e^{-pt} \frac{1}{t^2 + a^2}\, dt = -\frac{1}{a} \{\, \mathrm{ci}(ap)\sin(ap) + \mathrm{si}(ap)\cos(ap)\, \}, \tag{A.4}$$

where $\mathrm{Re}\; p > 0,\ a > 0$ (cf. [5, p. 269, Formula(46)]).

For the purpose of applications in Section 2, it is convenient to replace p in (A.4) with $-z$. We thus have the function

$$\begin{aligned} &-\{\, \mathrm{ci}(-z)\sin(-z) + \mathrm{si}(-z)\cos(-z)\} \\ &\quad = \ \sin(z)\,\mathrm{ci}(-z) - \cos(z)\,\mathrm{si}(-z), \end{aligned}$$

which is holomorphic in $\mathbb{C} \setminus [0, +\infty)$.

A.4. Estimates of a convolution. We have often encountered the convolution of the form

$$\Phi(x) := \int_{\mathbb{R}^n} \frac{1}{|x-y|^{\beta} \langle y \rangle^{\gamma}}\, dy$$

in the previous sections, and used Lemma A.1 below several times. Although the results exhibited in Lemma A.1 are well-known, it appears neither in a convenient form for our purpose (see Ikebe [7]) nor in an accessible form (see Kuroda [13] which is written in Japanese) in the literature. For this reason, we reproduce the results here for the reader's convenience.

Lemma A.1. *If $0 < \beta < n$ and $\beta + \gamma > n$, then $\Phi(x)$ is a bounded continuous function satisfying*

$$|\Phi(x)| \le C_{\beta\gamma n} \begin{cases} \langle x \rangle^{-(\beta+\gamma-n)} & \text{if } 0 < \gamma < n, \\ \langle x \rangle^{-\beta} \log(1 + \langle x \rangle) & \text{if } \gamma = n, \\ \langle x \rangle^{-\beta} & \text{if } \gamma > n, \end{cases}$$

where $C_{\beta\gamma n}$ is a constant depending on β, γ and n.

We shall divide the proof into four steps.

Step 1. $\Phi(x)$ *is a continuous function on* $\mathbb{R}^n$.

Proof. Let x_0 be an arbitrary point in $\mathbb{R}^n$, and let $\varepsilon > 0$ be given. Since $0 < \beta < n$, we can choose $r > 0$ so that

$$\int_{|y| \le 2r} \frac{1}{|y|^\beta}\, dy < \varepsilon. \tag{A.5}$$

We then decompose $\Phi(x)$ into two parts:

$$\Phi(x) = \Big(\int_{B(x,\,2r)} + \int_{E(x,\,2r)} \Big) \frac{1}{|x-y|^\beta \langle y \rangle^\gamma}\, dy =: \Phi_b(x) + \Phi_e(x), \tag{A.6}$$

where $B(x,\,2r) = \{\, y \,|\, |x-y| \le 2r \,\}$ and $E(x,\,2r)$ is the same as in the proof of Lemma 9.5. By (A.5), we get

$$0 < \Phi_b(x) < \varepsilon \tag{A.7}$$

for all $x \in \mathbb{R}^n$. It follows from the definition of $\Phi_e(x)$ that

$$\begin{aligned} &\Phi_e(x) - \Phi_e(x_0) \\ &= \int_{\mathbb{R}^n} \Big\{ 1_{E(x,\,2r)}(y) \frac{1}{|x-y|^\beta \langle y \rangle^\gamma} - 1_{E(x_0,\,2r)}(y) \frac{1}{|x_0-y|^\beta \langle y \rangle^\gamma} \Big\}\, dy. \end{aligned} \tag{A.8}$$

Note that the inequality (9.24) implies that

$$1_{E(x,\,2r)}(y) \frac{1}{|x-y|^\beta} \le \big(\frac{3}{2}\big)^\beta \times 1_{E(x_0,\,2r)}(y) \frac{1}{|x_0-y|^\beta}$$

whenever $|x - x_0| \le r$. Hence, the integrand in (A.8) is bounded by

$$\Big\{ \big(\frac{3}{2}\big)^\beta + 1 \Big\} \times 1_{E(x_0,\,2r)}(y) \frac{1}{|x_0-y|^\beta \langle y \rangle^\gamma}, \tag{A.9}$$

in absolute value, for all x with $|x-x_0| \le r$. Since $\beta+\gamma > n$, by assumption of the lemma, we see that the function in (A.9) belongs to $L^1(\mathbb{R}^n)$. Therefore, the Lebesgue dominated convergence theorem is applicable to the right hand side of (A.8) and shows that

$$\lim_{x \to x_0} \big(\Phi_e(x) - \Phi_e(x_0)\big) = 0.$$

Combining this with (A.6) and (A.7), we deduce that

$$\limsup_{x \to x_0} \big|\Phi(x) - \Phi(x_0)\big| \le 2\varepsilon.$$

Since ε was arbitrary, this completes the proof of the step 1. $\square$

To establish the desired inequalities, we make another decomposition of $\Phi(x)$:

$$\Phi(x) = \Phi_1(x) + \Phi_2(x) + \Phi_3(x),$$

where

$$\begin{aligned}
\Phi_1(x) &= \int_{|y| \le |x|/2} \frac{1}{|x-y|^\beta \langle y \rangle^\gamma}\, dy, \\
\Phi_2(x) &= \int_{|x|/2 < |y| \le 2|x|} \frac{1}{|x-y|^\beta \langle y \rangle^\gamma}\, dy, \\
\Phi_3(x) &= \int_{2|x| < |y|} \frac{1}{|x-y|^\beta \langle y \rangle^\gamma}\, dy.
\end{aligned}$$

Since $\Phi(x)$ is bounded on each compact subset of $\mathbb{R}^n$ by continuity of $\Phi(x)$, it is sufficient to get estimates of Φ_i's for $|x| \ge 1$.

Step 2. For $|x| \ge 1$, we have

$$|\Phi_1(x)| \le C_{\beta\gamma n} \begin{cases} |x|^{-(\beta+\gamma-n)} & \text{if } 0 < \gamma < n, \\ |x|^{-\beta} \log(1+\langle x \rangle) & \text{if } \gamma = n, \\ |x|^{-\beta} & \text{if } \gamma > n. \end{cases}$$

Proof. Note that $|x-y| \ge |x|/2$ if $|y| \le |x|/2$. This fact implies that

$$\Phi_1(x) \le 2^\beta\, |x|^{-\beta} \int_{|y| \le |x|/2} \frac{1}{\langle y \rangle^\gamma}\, dy.$$

If $0 < \gamma < n$, then we get, using spherical polar coordinates,

$$\int_{|y|\le|x|/2} \frac{1}{\langle y\rangle^\gamma}\,dy \le \omega_n \int_0^{|x|/2} r^{-\gamma+n-1}\,dr = \frac{\omega_n\, 2^{-\gamma+n}}{n-\gamma}|x|^{-\gamma+n},$$

where ω_n denotes the area of the unit sphere in $\mathbb{R}^n$. Similarly, if $\gamma = n$, we then have

$$\begin{aligned}
\int_{|y|\le|x|/2} \frac{1}{\langle y\rangle^\gamma}\,dy &\le \omega_n \int_0^{|x|/2} \Big(\frac{1+r}{\sqrt{2}}\Big)^{-\gamma} r^{n-1}\,dr \\
&\le \omega_n\, 2^{\gamma/2} \int_0^{|x|/2} (1+r)^{-1}\,dr \\
&\le \omega_n\, 2^{\gamma/2} \log\big(1+\frac{|x|}{2}\big),
\end{aligned}$$

where we have used the inequality $\langle y\rangle \ge (1+|y|)/\sqrt{2}$. If $\gamma > n$, we evidently have

$$\int_{|y|\le|x|/2} \frac{1}{\langle y\rangle^\gamma}\,dy \le \int_{\mathbb{R}^n} \frac{1}{\langle y\rangle^\gamma}\,dy < +\infty.$$

Summing up, we conclude that the desired inequalities for $\Phi_1(x)$ hold. □

Step 3. *For* $|x| \ge 1$*, we have*

$$|\Phi_2(x)| \le C_{\beta\gamma n}\,|x|^{-(\beta+\gamma-n)}.$$

Proof. Let $B^*(x)$ and $E^*(x)$ be the sets defined by

$$\begin{aligned}
B^*(x) &:= \{\, y \in \mathbb{R}^n \mid |x-y| < \frac{|x|}{2} \,\}, \\
E^*(x) &:= \{\, y \in \mathbb{R}^n \mid \frac{|x|}{2} < |y| \le 2|x|,\ |x-y| \ge \frac{|x|}{2} \,\}.
\end{aligned}$$

Then we have

$$\Phi_2(x) = \int_{B^*(x)} \frac{1}{|x-y|^\beta \langle y\rangle^\gamma}\,dy + \int_{E^*(x)} \frac{1}{|x-y|^\beta \langle y\rangle^\gamma}\,dy.$$

Since $B^*(x)$ is a subset of the annulus $\{\, y \mid |x|/2 < |y| \le 2|x| \,\}$, it follows that

$$\langle y\rangle \ge \frac{1}{\sqrt{2}}(1+|y|) \ge \frac{1}{2\sqrt{2}}|x| \qquad (\forall y \in B^*(x)\,),$$

which gives

$$\int_{B^*(x)} \frac{1}{|x-y|^\beta \langle y\rangle^\gamma}\,dy \;\le\; 2^{3\gamma/2}\,|x|^{-\gamma}\int_{|x-y|<|x|/2}\frac{1}{|x-y|^\beta}\,dy = \frac{2^{3\gamma/2+\beta-n}\,\omega_n}{n-\beta}\,|x|^{-\gamma-\beta+n}.$$

If $y\in E^*(x)$, then $|x-y|\ge |y|/4$, therefore

$$\int_{E^*(x)} \frac{1}{|x-y|^\beta \langle y\rangle^\gamma}\,dy \;\le\; 4^\beta\int_{|x|/2<|y|\le 2|x|}\frac{1}{|y|^{\beta+\gamma}}\,dy = \frac{4^\beta\,\omega_n}{\beta+\gamma-n}\,(2^{\beta+\gamma-n}-2^{-\beta-\gamma+n})\,|x|^{-\beta-\gamma+n}.$$

Summing up, we obtain the desired inequality for $\Phi_2(x)$. □

Step 4. For $|x|\ge 1$, $\Phi_3(x)$ satisfies the same inequality as $\Phi_2(x)$:

$$|\Phi_3(x)|\le C_{\beta\gamma n}\,|x|^{-(\beta+\gamma-n)}.$$

Proof. If $2|x|<|y|$, then it follows that

$$|x-y|\ge |y|-|x|\ge \frac{|y|}{2}.$$

Hence we have

$$\Phi_3(x)\le 2^\beta\int_{2|x|<|y|}\frac{1}{|y|^{\beta+\gamma}}\,dy = \frac{2^{-\gamma+n}\,\omega_n}{\beta+\gamma-n}\,|x|^{-\beta-\gamma+n}.$$

This completes the proof. □

It is evident that Lemma A.1 follows from the steps 1–4.

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