Hamiltonian stability and subanalytic geometry

Laurent Niederman (*)

Abstract: In the 70's, Nekhorochev proved that for an analytic nearly integrable Hamiltonian system, the action variables of the unperturbed Hamiltonian remain nearly constant over an exponentially long time with respect to the size of the perturbation, provided that the unperturbed Hamiltonian satisfies some generic transversality condition known as *steepness*. Using theorems of real subanalytic geometry, we derive a geometric criterion for steepness: a numerical function h which is real analytic around a compact set in $\mathbb{R}^n$ is steep if and only if its restriction to any proper affine subspace of $\mathbb{R}^n$ admits only isolated critical points. Moreover, we obtain sharp results of exponential stability under the previous assumption.

We also state a necessary condition for exponential stability, which is close to steepness.

Finally, we give methods to compute the steepness indices for an arbitrary steep function.

Key words: Hamiltonian systems – Stability – Subanalytic Geometry – Curve Selection Lemma – Lojasiewicz's inequalities.

AMS classification numbers: 14P15, 32B20, 32S05, 37J40, 70J08, 70J09, 70J14.

I Introduction:

I.1 Set-up

One of the main problem in Hamiltonian dynamic is the stability of motions in nearly-integrable systems (for example: the n-body planetary problem). The main tool of investigation is the construction of normal forms (see Giorgilli [8] for an introduction and a survey about these topics).

This yields two kinds of theorems:

i) Results of stability over infinite times provided by K.A.M. theory which are valid for solutions with initial conditions in a Cantor set of large measure but no information is given on the other trajectories.

ii) On the other hand, global results of stability over open sets which are valid only over exponentially long times with respect to the size of the perturbation.

Here, we focus our attention on the integrable Hamiltonians which satisfy the following property:

(*) *Université Paris XI — Topologie et Dynamique — UMR 8628 du CNRS — Bât. 425, 91405 ORSAY Cedex France;*

E-mail: Laurent.Niederman@math.u-psud.fr

and

IMCCE — Astronomie et Systèmes Dynamiques — UMR 8028 du CNRS — 77 avenue Denfert-Rochereau, 75014 PARIS France.

Definition I.1. (exponential stability)

Consider an open set $\mathcal{P} \subset \mathbb{R}^n$*, an analytic integrable Hamiltonian* $h : \mathcal{P} \longrightarrow \mathbb{R}$ *and action-angle variables* $(I, \varphi) \in \mathcal{P} \times \mathbb{T}^n$ *with* $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

For an arbitrary $\rho > 0$*, let* $\mathcal{O}_\rho$ *be the space of analytic functions over a complex neighbourhood* $\mathcal{P}_\rho \subset \mathbb{C}^{2n}$ *of size* ρ *around* $\mathcal{P} \times \mathbb{T}^n$ *equipped with the supremum norm* $||.||_\rho$ *over* $\mathcal{P}_\rho$.

We say that the Hamiltonian h *is exponentially stable if there exist positive constants* ρ, C_1, C_2, a, b *and* ε_0 *which depend only on* h *and such that:*

i) $h \in \mathcal{O}_\rho$.

ii) For any function $\mathcal{H} \in \mathcal{O}_\rho$ *such that* $||\mathcal{H} - h||_\rho = \varepsilon < \varepsilon_0$*, an arbitrary solution* $(I(t), \varphi(t))$ *of the Hamiltonian system associated to* $\mathcal{H}$ *with an initial action* $I(t_0)$ *not too close from the boundary of* $\mathcal{P}$ *satisfies:*

$$||I(t) - I(t_0)|| \leq C_1 \varepsilon^b \text{ for } |t - t_0| \leq \exp(C_2/\varepsilon^a)$$

Rk: Along the same lines, the previous definition can be extended to an integrable Hamiltonian in the Gevrey class.

In the seventies, Nekhorochev ([18], [19]) introduced the class of steep functions in order to get a sufficient condition for exponential stability.

Definition I.2. (steepness)

Consider an open set $\mathcal{P}$ *in* $\mathbb{R}^n$. *A real analytic function* $h : \mathcal{P} \longrightarrow \mathbb{R}$ *is said to be steep at a point* $I \in \mathcal{P}$ *along an affine subspace* Λ *which contains* I *if there are constants* $C > 0$, $\delta > 0$ *and* $p > 0$ *such that along any analytic regular curve* γ *in* Λ *connecting* I *and a point at a distance* $d < \delta$*, the norm of the projection of the gradient* $\nabla f(x)$ *onto the direction of* Λ *is greater than* Cd^p *at some point;* (C, δ) *and* p *are respectively called the steepness coefficients and the steepness index.*

Under the previous assumptions, the function h *is said to be steep at the point* $I \in \mathcal{P}$ *if* I *is not a critical point for* h *and if, for every* $k \in \{1, \dots, n-1\}$*, there exist positive constants* C_k, δ_k *and* p_k *such that* h *is steep at* I *along any affine subspace of dimension* k *containing* I *uniformly with respect to the coefficients* (C_k, δ_k) *and the index* p_k.

Finally, a real analytic function h *is steep over a domain* $\mathcal{P} \subseteq \mathbb{R}^n$ *with the steepness coefficients* $(C_1, \dots, C_{n-1}, \delta_1, \dots, \delta_{n-1})$ *and the steepness indices* $(p_1, \dots, p_{n-1})$ *if there are no critical points for* h *in* $\mathcal{P}$ *and* h *is steep at any point* $I \in \mathcal{P}$ *uniformly with respect to these coefficients and indices.*

For instance, convex functions are steep with all the steepness indices equal to *one*. On the other hand, $f(x, y) = x^2 - y^2$ is a typical non steep function but by adding a third order term (e.g. y^3) we recover steepness. Moreover, this definition is minimal since a function can be steep along all subspaces of dimension lower than or equal to $k < n-1$ and not steep for a subspace of dimension l greater than k (consider the function $f(x, y, z) = (x^2 - y)^2 + z$ at $(0, 0, 0)$ along all the lines and along the plane $z = 0$).

Actually, these definitions look slightly less restrictive than the initial one given by Nekhorochev. But they retain the key property needed to derive estimates of stability. We will actually prove in paragraph III.2 that they are equivalent to the original one.

In this setting, Nekhorochev proved the following:

Theorem I.3. ([19], [20])
If h is real analytic, non-degenerate ($|\nabla^2 h(I)| \neq 0$ for any $I \in \mathcal{P}$) and steep then h is exponentially stable.

Rk: I conjecture that the degeneracy condition can be removed if one uses a global resonant normal form instead of a local one (see [22]).

If h is quasi-convex, Lochak ([14], [15]) and Pöschel ([24]) have proved the previous estimates with the exponents $a = b = (2n)^{-1}$. This result has been generalized to the steep case by Niederman ([22]) with the values $a = b = (2np_1 \dots p_{n-1})^{-1}$ ($= 1/2n$ if h is convex).

Recently, Marco and Sauzin ([17]), following an idea of Herman showed that if h is quasi-convex and the total Hamiltonian $\mathcal{H}$ is Gevrey of order α (i.e. $\mathcal{H}$ is infinitely differentiable and $|\partial^k \mathcal{H}| \leq C^k (k!)^\alpha$) then the previous estimates are valid with $a = b = (2n\alpha)^{-1}$. Indeed these exponential bounds come from the Gevrey character of the normalizing transformations involved in the proof. Moreover, in the same setting (h quasi-convex and $\mathcal{H}$ Gevrey of order $\alpha > 1$, i.e. $\mathcal{H}$ *non analytic*), Marco and Sauzin ([17]) build examples of nearly integrable systems where an important instability of the action variables occurs for arbitrary small perturbations over times of order $\exp(1/\varepsilon^{a_*})$ with $a_* = (2(n-2)\alpha)^{-1}$. Hence, the times of stability in these estimates are nearly optimal (and actually optimal in the neighbourhood of resonances, see [17]) and the Gevrey character of the perturbation is a close to minimal *regularity* condition for exponential stability.

I.2 Geometric results

Here, we study a minimal *non-degeneracy* condition on the unperturbed Hamiltonian needed to derive exponential stability results and give a *geometric* criterion equivalent to steepness.

Using tools of real subanalytic geometry (see [3], [4], [16]): the curve selection lemma and the Lojasiewicz's inequalities for continuous subanalytic functions, we prove the following theorems:

Theorem I.4.
Let h be a numerical function which is real analytic in the vicinity of the closed ball $\overline{\mathrm{B}}_R$ of radius $R > 0$ in $\mathbb{R}^n$ and has no critical points ($\nabla h(x) \neq 0$ for all $x \in \overline{\mathrm{B}}_R$). Then h is steep if and only if its restriction $h_{|\Lambda}$ to any proper affine subspace $\Lambda \subset \mathbb{R}^n$ admits only isolated critical points.

Theorem I.5.
Consider an integrable Hamiltonian h which is real analytic in the vicinity of the closed ball $\overline{\mathrm{B}}_R \subset \mathbb{R}^n$ (which is here the action space). If h is exponentially stable then its restriction to any proper affine subspace whose direction is generated by vectors with integer components admits only isolated critical points.

This last statement is proved thanks to a result of Nekhorochev ([20]) about sufficient conditions on an integrable Hamiltonian which ensure the existence of arbitrary small perturbations giving rise to solutions with a drift of the action variables over linear times with respect to the size of the perturbation ("systems with fast drift"). The same problem has been studied in the realm of KAM theory by Michael Herman [11] who exhibited nearly integrable Hamiltonian systems with a dense Cantor set of invariant tori together with orbits which drift away to infinity.

We see that a gap subsists between the sufficient geometric condition for exponential stability given in theorem I.4. and the necessary condition derived in theorem I.5. Nevertheless, steepness is only a sufficient condition for exponential stability but the converse is not true. For instance: $h(I_1, I_2) = I_1^2 - I_2^2$ is not steep and the perturbed Hamiltonian $\mathcal{H}(I_1, I_2, \varphi_1, \varphi_2) = h(I_1, I_2) + \varepsilon f(\varphi_1, \varphi_2)$ with $f(\varphi_1, \varphi_2) = \sin(\varphi_1 + \varphi_2)$ admits the special solution $I(t) = (\varepsilon t, \varepsilon t)$, $\varphi(t) = (-\varepsilon t^2, \varepsilon t^2)$ hence $\|I(t) - I(0)\| = \sqrt{2}\varepsilon t$ and we have a drift over polynomial times (even linear times). On the other hand, the Hamiltonian $h(I_1, I_2) = I_1^2 - 2I_2^2$ is not steep but is exponentially stable (its isotropic direction is the line directed by $(1, \sqrt{2})$). More generally, the Hamiltonian $h(I_1, I_2) = I_1^2 - \delta I_2^2$ for $\delta > 0$ is not steep but it is difficult to determine if it is exponentially stable (for instance when δ is the square of a Liouville number).

In the context of KAM theory, the usual non-degeneracy condition is the invertibility of the gradient map associated to the unperturbed Hamiltonian. But the minimal condition needed for the existence of invariant tori in the perturbed system is the Rüssmann condition (see [5]): the image of the gradient map should not be included in a hyperplane. This last property is much easier to check for an arbitrary integrable Hamiltonian. Especially in the n-body problem, the unperturbed system given by uncoupled Kepler problems is strongly degenerate and Michel Herman showed that the use of Rüssman's condition significantly simplifies the proof of the existence of quasi-periodic planetary motions. A complete proof of this latter result has been given recently by Féjoz ([6]). Over exponentially long times, our condition should be useful to prove stability results in the secular planetary problem (see also [21]). In the same way, Benettin, Fasso, Guzzo ([2]) and Guzzo, Morbidelli ([9]) have also studied stability properties of problems in celestial mechanics by reducing them to a perturbed steep, non-convex, integrable Hamiltonian system.

I.3 Effective computation of the steepness indices

In order to get quantitative estimates, we have to compute the steepness indices of an integrable steep Hamiltonian.

Under the assumptions of theorem I.2, the steepness indices can be seen as the Lojasiewicz exponents of two functions according to the following:

Definition I.6. (Lojasiewicz's exponent [3], [4], [16])

(i) Let M be a real analytic manifold, K a compact subset of M and f, g two vector-valued functions continuous over K, we set:

$$\mathcal{E}_K(f,g) = \{\alpha \in \mathbb{R}_+ \text{ such that there exists a constant } C \textit{ with } ||f(u)||^\alpha \leq C||g(u)||, \forall u \in K\}$$

and $\alpha_K(f,g) = \mathrm{Inf}\{\mathcal{E}_K(f,g)\}$ *with* $\mathrm{Inf}\{\varnothing\}$ *defined as* $+\infty$*;* $\alpha_K(f,g)$ *is called the Lojasiewicz's exponent of* f *with respect to* g *over* K*.*

(ii) We will also consider the case where f is defined on a compact subset of $\mathbb{R}^n$ and admits an isolated zero at x, then we set:

$$\alpha_x(f) = \mathrm{Inf}\left\{\alpha \in \mathbb{R}_+ \text{ such that } \exists C > 0, R > 0 \text{ with } ||u-x||^\alpha \leq C||f(u)|| \text{ if } ||u-x|| \leq R\right\}$$

hence $\alpha_x(f) = \alpha_K(f, \mathrm{dist}(., x))$.

For $k \in \{1, \ldots, n-1\}$, we denote by $\mathrm{Graff}_R(k,n)$ the k-dimensional affine Grassmannian over $\overline{\mathrm{B}}_R$ (i.e.: the set of affine subspaces of dimension k in $\mathbb{R}^n$ which intersect $\overline{\mathrm{B}}_R$).

With theses definitions, we prove the following:

Theorem I.7.
Consider an integrable Hamiltonian h which satisfies our assumptions of theorem (I.4). For $\Lambda \in \mathrm{Graff}_R(k,n)$, we consider the set of critical points of the restriction $h_{|\Lambda}$ to Λ:

$$Z_\Lambda = \left\{x \in \Lambda \text{ such that } \nabla h_{|\Lambda}(x) = 0\right\} = \left\{x \in \Lambda \text{ such that } \mathrm{Proj}_{\overrightarrow{\Lambda}}(\nabla h(x)) = 0\right\}$$

(since the gradient of $h_{|\Lambda}$ is the projection ∇h on the direction $\overrightarrow{\Lambda}$ of Λ).
Then the steepness index of order k satisfies:

$$p_k = \mathrm{Sup}_{\Lambda \in \mathrm{Graff}_R(k,n)} \left(\mathrm{Sup}_{x \in Z_\Lambda} \left(\alpha_x \left(||\nabla h_{|\Lambda}||\right)\right)\right).$$

The point of this refinement lies in the fact that the steepness indices are obtained as the maximum of a family of Lojasiewicz's exponents at an isolated zero of a real-analytic function. The latter quantities can be computed along the lines of a theorem of Gwozdziewicz ([10]).

For $\Lambda \in \mathrm{Graff}_R(k,n)$, let f_Λ be the numerical function real analytic around $\overline{\mathrm{B}}_R \cap \Lambda$ defined by $f_\Lambda(u) = \left||\nabla h_{|\Lambda}(u)|\right|^2$ for $u \in \Lambda$. With $(\vec{e}_1, \ldots, \vec{e}_k)$ the canonical basis of $\mathbb{R}^k$, we consider the set (called polar curve):

$$\mathcal{P}_\Lambda^{(j)} = \nabla f_\Lambda^{-1}(\mathbb{R}\vec{e}_j) = \left\{u \in \overline{\mathrm{B}}_R \cap \Lambda \text{ such that } \nabla f_\Lambda(u) = \lambda \vec{e}_j \text{ for} \lambda \in \mathbb{R}\right\}.$$

Consider $x \in Z_\Lambda$ an isolated zero of f_Λ and, for $j \in \{1, \ldots, k\}$, we define the partial exponent:

$$\alpha_x^{(j)}(f_\Lambda) = \mathrm{Inf}\left\{\alpha \in \mathbb{R}_+ \text{ such that } \exists C > 0, R > 0 \text{ with } ||u-x||^\alpha \leq C|f(u)| \text{ if } \begin{array}{c} ||u-x|| \leq R \\ \text{and } u \in \mathcal{P}_\Lambda^{(j)} \end{array}\right\}$$

then Gwozdziewicz's theorem ensures that $\alpha_x(f_\Lambda) = \mathrm{Max}_{1 \leq j \leq k}\left(\alpha_x^{(j)}(f_\Lambda)\right)$ and finally $2\alpha_x\left(h_{|\Lambda}\right) = \alpha_x(f_\Lambda)$ with our definition of f_Λ.

Summarizing, in order to compute the Lojasiewicz's exponent of a real analytic function at an isolated zero, one has only to estimate the growth of the function along one of the polar curves which is usually an analytic set of dimension one.

Actually, a geometric criterion which ensures steepness has already been proved by Ilyashenko ([13]). He showed that a complex-valued holomorphic function on a domain of $\mathbb{C}^n$ whose restriction to any (complex) affine subspace admits only $\mathbb{C}$-isolated critical points is steep on $\mathbb{C}^n$ (with a generalization of our definition in the complex field). Hence, Ilyashenko considers a stronger property than steepness in the real case, moreover his estimates on the steepness indices are very rough.

Indeed, the key estimate is a lower bound on the growth of an holomorphic function with respect to the distance to the zero set $Z_f = \{x \text{ such that } ||f(x)|| = 0\}$. More specifically, consider an holomorphic function on an open set in $\mathbb{C}^n$ which admits a finite number of zeros, then $||f(x)|| \geq C Dist(x, Z_f)^p$ where p is the number of zeros counted with their multiplicity (or Milnor number, see [1, p. 30]) which is the number of zeros obtained by bifurcation ($\sharp\{x \text{ such that } ||f(x)|| = \varepsilon\}$ for ε close to 0). For instance, the function $f(z_1, z_2) = (z_1^2, z_2^3)$ admits only one zero over $\mathbb{C}^2$ with the multiplicity $p_f(0,0) = 6$ while the Lojasiewicz's exponent is $\alpha_f(0,0) = 3$, hence the previous lemma gives only $||f(z_1, z_2)|| \geq C||(z_1, z_2)||^6$ in the vicinity of $(0,0)$ instead of $||f(z_1, z_2)|| \geq C||(z_1, z_2)||^3$ which is the best lower bound. The same phenomena occurs in a general setting (see [23]) and our estimates on the steepness indices are sharp.

Going back to Hamiltonian dynamic in case of steepness and non-degeneracy, according to the results of [22], the exponential estimates of stability for any small enough analytic perturbation of the considered Hamiltonian are valid with $a = b = 1/(2np_1 \dots p_{n-1})$.

II Essential results of subanalytic geometry.

In order to get a self-consistent paper, we introduce the theorems which will be used in our proof. The definitions come from [3] :

Definition II.1.

Let M be a real analytic manifold. If U is an open set in M, let $\mathcal{O}(U)$ denote the ring of real analytic functions on U. A subset $X \subset M$ is semianalytic if each $a \in M$ has a neighbourhood U such that $X \cap U = \cup_{i=1}^{p} \cap_{j=1}^{q} X_{ij}$ where each X_{ij} is either a set defined as $\{f_{ij} = 0\}$ or $\{f_{ij} > 0\}$ for some $f_{ij} \in \mathcal{O}(U)$ (we say that X is described by $\{f_{ij}\}$).

Definition II.2.

A subset X of a real analytic manifold M is subanalytic if each point of M admits a neighbourhood U such that $X \cap U$ is a projection of a relatively compact semianalytic set A (i.e. : there is a real analytic manifold N and a relatively compact semianalytic set $A \subset M \times N$ such that $X \cap U = \Pi(A)$ with the canonical projection Π from $M \times N$ to N).

Definition II.3.

Let $X \subset M$ and let N be a real analytic manifold. A mapping $f : X \longrightarrow N$ is subanalytic if its graph is subanalytic in $M \times N$

Theorem II.4.

(i) The intersection or the union of a finite collection of subanalytic sets is subanalytic.

(ii) The closure of a subanalytic set remains subanalytic.

(iii) The complement of a subanalytic set is subanalytic.

(iv) The image of a relatively compact subanalytic set by a subanalytic mapping remains subanalytic.

Two examples of subanalytic functions ([3, p. 19])

a) Let X be a subanalytic set of $\mathbb{R}^n$, the distance function $\delta_X(x) = \mathrm{Min}_{x' \in \overline{X}}(||x - x'||)$ is continuous subanalytic (while δ_A is not analytic even if A is analytic).

For instance, the norm is subanalytic.

b) Let M and N be real analytic manifolds and X (resp. T) be subanalytic subsets of M (resp. N), where T is compact. If $f : X \times T \longrightarrow \mathbb{R}$ is a continuous subanalytic function, then $g(x) = \mathrm{Min}_{t \in T}(f(x,t))$ is continuous subanalytic.

The key ingredient for our proofs are the following two theorems:

Theorem II.5. (Curve selection lemma, [12, 16])

Let X be a subanalytic subset of a real analytic manifold M. For any point x in the closure $\overline{X}$ there exists an analytic arc $\gamma :]-1, +1[\longrightarrow M$ with $\gamma(0) = x$ and $\gamma(]0,1[) \subset X$.

Theorem II.6. (Lojasiewicz's inequalities, [3, 4])

Let f and g be two vector-valued continuous subanalytic functions over a compact set K in a real analytic manifold M such that their zero sets satisfy $\varnothing \neq Z_g \subset Z_f$, then $\mathcal{E}_K(f,g)$ is non-empty, $\alpha_K(f,g) \in \mathbb{Q}$ and $\alpha_K(f,g) \in \mathcal{E}_K(f,g)$.

Rk: Specifically, if g is a vector-valued continuous subanalytic function over a compact set $K \subset \mathbb{R}^n$, if $X = Z_g$ and $f(x) = \delta_X(x)$ then for all $x \in K$ we get $\left(\delta_{Z_g}(x)\right)^{\alpha} \leq C||g(y)||$.

III Proofs of the theorems.

III.1 Preliminaries.

Even if they remain true in a wider setting, we prove the following results of subanalytic geometry under the specific hypotheses which are satisfied by the numerical real-analytic functions whose restriction to any affine subspace admits only isolated critical points (the assumption of our theorem I.4).

Proposition III.1.

Let $0 < r \leq R$ and $\overline{\mathrm{B}}_R$ (resp. $\mathcal{S}_r$) be the closed ball (resp. the sphere) of radius R (resp. r) centered at zero in $\mathbb{R}^n$. We consider a function $f : \overline{\mathrm{B}}_R \times K \longrightarrow \mathbb{R}$ continuous subanalytic over $\overline{\mathrm{B}}_R \times K$ where K is a real-analytic compact manifold.

With these notations and $\overline{\mathrm{B}}^*_R = \overline{\mathrm{B}}_R \backslash \{0\}$, the set M^* consisting of minima of f along each fiber of the foliation $\overline{\mathrm{B}}^*_R \times K = \bigcup_{0<r\leq R} \mathcal{S}_r \times K$:

$$M^* = \left\{(u,y) \in \overline{\mathrm{B}}^*_R \times K \text{ such that } ||x|| = ||u|| \Longrightarrow f(x,y) \geq f(u,y)\right\},$$

is a subanalytic relatively compact set in $\overline{\mathrm{B}}_R \times K$.

Proof: Consider the sets

$$\mathcal{A} = \left\{(x,u,y) \in \overline{\mathrm{B}}^*_R \times \overline{\mathrm{B}}^*_R \times K \text{ such that } f(x,y) \geq f(u,y)\right\}$$

and $\mathcal{B} = \left\{(x,u,y) \in \overline{\mathrm{B}}^*_R \times \overline{\mathrm{B}}^*_R \times K \text{ such that } ||x|| = ||u||\right\}$.

They are compact semianalytic sets of $\overline{\mathrm{B}}_R \times \overline{\mathrm{B}}_R \times K$ since $\mathcal{A}$ and $\mathcal{B}$ are defined with analytic equalities and inequalities, hence $\mathcal{B} \backslash \mathcal{A}$ is also semianalytic.

Consider the projection $\Pi : \mathbb{R}^n \times \mathbb{R}^n \times K \longrightarrow \mathbb{R}^n \times K$ defined by $\Pi(x,u,y) = (u,y)$. By definition the set $\Pi(\mathcal{B}\backslash\mathcal{A})$ is subanalytic in $\overline{\mathrm{B}}_R \times K$, and so is its complement $M^* = \overline{\mathrm{B}}^*_R \times K \backslash (\Pi(\mathcal{B}\backslash\mathcal{A}))$.▶

Corollary III.2.
For any $y \in K$, we consider the projection $M_y = \left\{u \in \overline{\mathrm{B}}_R \text{ such that } (u,y) \in M^*\right\}$. There exist a regular real-analytic arc $\gamma :]-1,1[\longrightarrow M_y$ such that $\gamma(0) = 0$.

Proof: For any $y \in K$, the set $M^*_y = M^* \cap \left(\overline{\mathrm{B}}_R \times \{y\}\right)$ is a subanalytic set as an intersection of subanalytic sets and M_y is the projection of M^*_y on its first component. Finally, the origin 0 is in the closure of M_y and the curve selection lemma (theorem II.5.) yields a non trivial analytic arc γ included in M_y with $\gamma(0) = 0$.▶

Theorem III.3.
With the notations of the previous proposition:
i) The function $m(r,y) = \mathrm{Min}\,(f\,(\mathcal{S}_r \times \{y\}))$ for $0 \le r \le R$ and $y \in K$ is a continuous subanalytic function over $[0,R] \times K$.
ii) The function $\mathcal{M}(r,y) = \mathrm{Max}_{t\in[0,r]}(m(t,y))$ is also continuous subanalytic.

Proof: The continuity of m and $\mathcal{M}$ are proved by abstract nonsense.
The function $\mathcal{F}(u,y) = (||u||, y, f(u,y))$ has subanalytic components hence $\mathcal{F}$ is a continuous subanalytic function from $\overline{\mathrm{B}}_R \times K$ to $[0,R] \times K \times \mathbb{R}$. The graph of m is given by the image $\mathcal{F}(M^*)$ which is a subanalytic set since M^* is subanalytic. Hence m too is subanalytic. This could also be proved with our second example of subanalytic function $\mathrm{Min}_{t\in K}\,(f(x,t))$.
For the second claim of the theorem, we consider the set:

$$A = \{(r,u,y) \in [0,R] \times \mathbb{R} \times K \text{ such that } u \le \mathcal{M}(r,y)\}\,.$$

Then $A = \Pi(B)$ where Π is the projection $\Pi(r,u,y,t) = (r,u,y)$ and

$$B = \{(r,u,y,t) \in [0,R] \times \mathbb{R} \times K \times [0,R] \text{ such that } 0 \le t \le r \text{ and } u \le m(t,y)\}$$

since $A = \{(r,u,y) \in [0,R] \times \mathbb{R} \times K \text{ such that } \exists t \in [0,r] \text{ with } u \le m(t,y)\}$.
Hence, A is subanalytic as a projection of a subanalytic set and the graph of $\mathcal{M}$ is given by the border ∂A which is also subanalytic. This proves the desired claim.▶

III.2 Proof of the geometric criterion for steepness (Theorem I.4.)

We consider a numerical function f real analytic around the closed ball $\overline{\mathrm{B}}_R$ of radius R in $\mathbb{R}^n$ and introduce the following functions $\widetilde{\mathcal{M}}_1, \dots, \widetilde{\mathcal{M}}_{n-1}$ which measure the steepness of f at a point $x_0 \in K$ along $\mathrm{Graff}_{x_0}(k,n)$, the set of all affine subspaces of dimension k which contain x_0.

For any $\widetilde{\Lambda}_k \in \mathrm{Graff}_{x_0}(k,n)$ and any $d > 0$ small enough so that the closed ball of radius d centered at x_0 is included in the domain of analyticity of f, we define $\widetilde{\mathcal{M}}_k$ as

$$\widetilde{\mathcal{M}}_k\left(d, \widetilde{\Lambda}_k\right) = \mathrm{Max}_{0\le\xi\le d}\left(\mathrm{Min}_{x\in\mathcal{S}_\xi(x_0)\cap\widetilde{\Lambda}_k}\left|\left|\nabla f_{|_{\widetilde{\Lambda}_k}}(x)\right|\right|\right)$$

where $\mathcal{S}_\xi(x_0)$ is the sphere of radius ξ centered at x_0 and $f_{|\widetilde{\Lambda}_k}$ is the restriction of f to $\widetilde{\Lambda}_k$. Hence $\nabla f_{|\widetilde{\Lambda}_k}(x) = \mathrm{Proj}_{\Lambda_k}(\nabla f(x))$ is the projection of $\nabla f(x)$ onto the direction of $\widetilde{\Lambda}_k$ which is denoted Λ_k.

Theorem III.4. (Nekhorochev's definition of steepness)
With the previous notations, f is steep at the point $x_0 \in K$ if and only if for each $k \in \{1, \dots, n-1\}$, there exist two positive coefficients (C_k, δ_k) and an index p_k such that:

$$\forall d \in [0, \delta_k[\,;\ \forall \widetilde{\Lambda}_k \in \mathrm{Graff}_R(k,n) \textit{ with } x_0 \in \widetilde{\Lambda}_k \textit{ we have } \widetilde{\mathcal{M}}_k\left(d, \widetilde{\Lambda}_k\right) \geq C_k d^{p_k}$$

Proof: If the previous estimate is satisfied then, for any affine subspace $\widetilde{\Lambda}_k$ which contains x_0, an analytic regular curve γ in $\widetilde{\Lambda}_k$ connecting x_0 and a point at a distance $d < \delta_k$ cross a sphere of radius $0 \leq \xi \leq d$ where the norm of the projection of the gradient ∇f onto Λ_k is greater than $C_k d^{p_k}$ at any point. Hence, our initial definition of steepness is satisfied.

Conversely, the function

$$f_{\widetilde{\Lambda}_k}(x) = ||\mathrm{Proj}_{\Lambda_k}(\nabla f(x_0 + x))|| \text{ for } x \in \Lambda_k$$

is continuous subanalytic around 0. Hence, the application of corollary III.2 yields a regular analytic curve $\gamma(t)$ consisting of minima of $f_{\widetilde{\Lambda}_k}$ on each sphere $\mathcal{S}_d(x_0)$ of radius d around x_0 for d small enough with $\gamma(0) = x_0$. Finally, if $d < \delta_k$ our initial definition of steepness yields a point $\gamma(t)$ with $||\gamma(t)|| = \xi \leq d$ such that $||\mathrm{Proj}_{\Lambda_k}(\nabla f(\gamma(t)))|| > C_k d^{p_k}$.

Hence, by definition of γ, we find a sphere of radius $0 < \xi \leq d$ for all $d < \delta_k$ where the norm of the projection of the gradient ∇f onto Λ_k is greater than $C_k d^{p_k}$ at any point. This proves the theorem.▶

In the sequel, for the sake of simplicity we will consider a function f which is real analytic around the closed ball $\overline{\mathrm{B}}_{2R}$ for some $R > 0$.

Following [25, p.400], we consider the Stiefel manifold $V_k^0(\mathbb{R}^n)$ composed of all orthonormal families in $\mathbb{R}^n$ of cardinality k and the k-dimensional Grassmanian $G_k(\mathbb{R}^n)$ which is the set of all vectorial subspaces of dimension k in $\mathbb{R}^n$.

$G_k(\mathbb{R}^n)$ is isomorphic to the quotient $V_k^0(\mathbb{R}^n)/\left(\mathcal{O}(k) \times \mathcal{O}(n-k)\right)$; the latter component is the stabilizer of a subspace of dimension k under the action of the orthonormal group $\mathcal{O}(n)$.

Hence, around any subspace in $\mathbb{R}^n$, there exist a local section of $V_k^0(\mathbb{R}^n)$ over $G_k(\mathbb{R}^n)$. Moreover, since all the previous manifolds are real analytic, these sections can be real analytic.

Summarizing, we can find real analytic applications $\mathcal{T}$ from open subsets $\Omega_k \subset G_k(\mathbb{R}^n)$ to $(\mathbb{R}^n)^k$ such that $\mathcal{T}(\Lambda_k) = (\mathcal{T}_1(\Lambda_k), \dots, \mathcal{T}_k(\Lambda_k))$ is an orthonormal basis of the k-dimensional subspace $\Lambda_k \subset \mathbb{R}^n$.

Finally, the function $\mathcal{Q}_k(X, X_0, \Lambda_k) = \text{Proj}_{\Lambda_k}(\nabla f(X_0 + x_1\mathcal{T}_1(\Lambda_k) + \ldots + x_k\mathcal{T}_k(\Lambda_k)))$, with $X = (x_1, \ldots, x_k)$, is real analytic over $\overline{\mathrm{B}}_R^{(k)} \times \overline{\mathrm{B}}_R \times \Omega_k \subset \mathbb{R}^k \times \mathbb{R}^n \times G_k(\mathbb{R}^n)$ where $\overline{\mathrm{B}}_R^{(k)}$ is the closed ball of radius R in $\mathbb{R}^k$.

Theorem III.6.
Under the previous assumptions, for $k \in \{1, \ldots, n-1\}$:
i) The function

$$\mathcal{M}_k(r, X_0, \Lambda_k) = \text{Max}_{t\in[0,r]}\left(\text{Min}_{||X||=t}\left(||\mathcal{Q}_k(X, X_0, \Lambda_k)||\right)\right)$$

is continuous subanalytic over $[0, R] \times \overline{\mathrm{B}}_R \times G_k(\mathbb{R}^n)$.
ii) If the restriction of f on any affine subspace $\Lambda_k \in \text{Graff}_R(k, n)$ admit only isolated critical points then the zero set of $\mathcal{M}_k$ satisfies $Z_{\mathcal{M}_k} \subset \{0\} \times \overline{\mathrm{B}}_R \times G_k(\mathbb{R}^n)$.

Proof: $||\mathcal{Q}_k(X, X_0, \Lambda_k)||$ is subanalytic on its domain of definition as the modulus of a real analytic function.

Hence, the local subanalyticity of $\mathcal{M}_k$ comes from theorem III.3.

Finally, $\mathcal{M}_k$ is univalent and globally defined over $[0, R] \times \overline{\mathrm{B}}_R \times G_k(\mathbb{R}^n)$ since it does not depend of the choice of the function $\mathcal{T}$.

For the second point, the function $\mathcal{M}_k(., X_0, \Lambda_k)$ is an increasing function over $[0, R]$ and $\mathcal{M}_k(0, X_0, \Lambda_k) \neq 0$ implies $\mathcal{M}_k(r, X_0, \Lambda_k) \neq 0$ for all $r \in [0, R]$.

Conversely, $\mathcal{M}_k(0, X_0, \Lambda_k) = 0$ implies that X_0 is a critical point of $f_{|\widetilde{\Lambda}_k}$ where $\widetilde{\Lambda}_k$ is the affine subspace $X_0 + \Lambda_k$. Under our assumption, such a critical point is isolated and $||\mathcal{Q}_k(X, X_0, \Lambda_k)|| \neq 0$ for any X close to 0 and, by monotonicity, $\mathcal{M}_k(r, X_0, \Lambda_k) \neq 0$ for any $r \in]0, R]$.▶

End of the proof of the theorem I.4.

The function $\mathcal{N}(r, X_0, \Lambda_k) = r$ is continuous subanalytic over $[0, R] \times \overline{\mathrm{B}}_R \times G_k(\mathbb{R}^n)$ with $Z_{\mathcal{M}_k} \subset \{0\} \times \overline{\mathrm{B}}_R \times G_k(\mathbb{R}^n) = Z_{\mathcal{N}}$ and, for $k \in \{1, \ldots, n-1\}$, the existence of the Lojasiewicz's exponent (II.6.) on the compact real analytic manifold $[0, R] \times \overline{\mathrm{B}}_R \times G_k(\mathbb{R}^n)$ implies:

$$\exists C_k > 0; \exists \alpha_k > 0 \text{ such that } \mathcal{M}_k(r, X_0, \Lambda_k) \geq C_k r^{\alpha_k} \text{ for all } (r, X_0, \Lambda_k) \in [0, R] \times \overline{\mathrm{B}}_R \times G_k(\mathbb{R}^n)$$

Finally, $\widetilde{\mathcal{M}}_k\left(r, \widetilde{\Lambda}_k\right) = \mathcal{M}_k(r, X_0, \Lambda_k)$ for any affine subspace $\widetilde{\Lambda}_k = X_0 + \Lambda_k$, hence

$$\exists C_k > 0; \exists \alpha_k > 0 \text{ such that } \widetilde{\mathcal{M}}_k\left(r, \widetilde{\Lambda}_k\right) \geq C_k r^{\alpha_k} \text{ for all } \left(r, \widetilde{\Lambda}_k\right) \in [0, R] \times \text{Graff}_R(k, n).$$

Hence f is steep if our assumption in theorem I.2 is satisfied and we prove the converse in the sequel.▶

III.3 Proof of our necessary condition for exponential stability (Theo. I.5.)

We prove this theorem (I.5.) by abstract nonsense, Nekhorochev ([20, section 4]) considered the following class of functions:

Definition III.7.
Let $\mathcal{F}$ be the class of functions $f : \mathcal{P} \longrightarrow \mathbb{R}$ real analytic over a domain $\mathcal{P} \subseteq \mathbb{R}^n$ such that there exist an affine subspace $\widetilde{\Lambda}$ whose direction Λ is generated by vectors with integer components and a regular analytic curve $\gamma_f : [0,1] \longrightarrow \widetilde{\Lambda} \cap \mathcal{P}$ where

$$\mathrm{Proj}_\Lambda\left(\nabla f(\gamma_f(t))\right) = 0 \text{ for all } t \in [0,1].$$

In this setting, we have:

Theorem III.8. (Systems with fast drift [20])
For any Hamiltonian $h \in \mathcal{F}$ (defined above) and any $\varepsilon > 0$, there exists a nearly-integrable Hamiltonian system deriving from $\mathcal{H}(I,\varphi) = h(I) + \varepsilon f(I,\varphi)$ in the action-angle variables $(I,\varphi) \in \mathcal{P} \times \mathbb{T}^n$ which admits a solution $(I(t),\varphi(t))$ defined over $[0,1/\varepsilon]$ such that $I(0) = \gamma_h(0)$ and $I(1/\varepsilon) = \gamma_h(1)$.
Hence, we have a drift along a curve with a length independant of ε over a polynomial time $1/\varepsilon$

Rk: This is the strongest possible drift with a perturbation of magnitude ε.

Proof of Theo. I.5.: Here, we consider an integrable Hamiltonian h with an affine subspace $\widetilde{\Lambda}$ whose direction Λ is generated by vectors with integer components such that the zero set $Z_g \cap \widetilde{\Lambda}$ of the real analytic function $g = \mathrm{Proj}_\Lambda\left(\nabla h(x)\right)$ admits an accumulation point. Applying corollary III.2 to the restriction of g to $\widetilde{\Lambda}$, we find a regular analytic arc γ_h in $\widetilde{\Lambda}$ with an accumulation of critical points. Hence, γ_h is included in $Z_g \cap \widetilde{\Lambda}$ and satisfies Nekhorochev's conditions for systems with fast drift (III.7). Consequently, there exists an analytic perturbation of h where the action variables drift over linear times along γ_h.▶

III.4 Proof of our estimates on the steepness indices (Theorem I.7.)

The application of the following lemma to the function $\mathcal{M}_k\left(r, X_0, \Lambda_k\right)$ defined over $[0,R] \times \overline{\mathrm{B}}_R \times G_k(\mathbb{R}^n)$ in the proof of the theorem I.4. gives exactly the desired claim.

Lemma III.9. (Lojasiewicz's exponent for a family of subanalytic functions)
Consider a numerical subanalytic function f defined on a product set $M = [0,R] \times K$ for some $R > 0$ and a real analytic compact manifold K with the zero set $Z_f \subset \{0\} \times K$, consequently if $g(x,r) = r$ over M then $Z_f \subset Z_g$.
We denote by f_x and g_x the restrictions of f and g on the fibers $M_x = [0,R] \times \{x\}$ for an arbitrary $x \in K$ and $\alpha = \alpha_M(f,g)$, $\alpha_x = \alpha_{M_x}(f_x, g_x)$ are the Lojasiewicz's exponents of these functions on their domain of definition, then $\alpha = \mathrm{Sup}_{x\in K}\left(\alpha_x\right)$.

Proof: We slightly extend our definition of the Lojasiewicz's exponent with the following:

Definition III.10. ([7])
Let M be a real analytic compact manifold and f, g two vector-valued functions continuous over M, we set:

$$\mathcal{E}_M(f,g) = \left\{\alpha \in \mathbb{R}_+ \text{ such that } \exists h \in \mathcal{C}^0(M) \ \ \textit{with } ||f(x)||^\alpha \leq h(x)||g(x)||, \forall x \in M\right\}$$

and $\alpha_M(f,g) = \mathrm{Inf}\,(\mathcal{E}_M(f,g))$ *with* $\mathrm{Inf}\{\varnothing\}$ *defined as* $+\infty$.

Rk : With the compactness of M, this is equivalent to our initial definition II.6.

Now, since each fiber $M_x = [0,R] \times \{x\}$ is included in M, by definition we have $\alpha \geq \alpha_x$ for any $x \in K$ and $\alpha \geq \widetilde{\alpha} = \mathrm{Sup}_{x\in K}\,(\alpha_x)$.

Conversely, if $\beta = \widetilde{\alpha} + \varepsilon$ for some $\varepsilon > 0$ then $h(r,x) = \dfrac{r^\beta}{|f(r,x)|}$ is continuous over $]0,R] \times K$ and $\lim_{x\to 0} h(r,x) = 0$ for any $x \in K$ since $\alpha \geq \widetilde{\alpha}$.

The sequence $h_n(x) = \mathrm{Sup}_{0\leq r\leq 1/n}(h(r,x))$ is a decreasing sequence of continuous functions over the compact K with $\lim_{n\to\infty} h_n(x) = 0$ for any $x \in K$.

Dini's theorem ensures the uniform convergence of the sequence h_n to 0 over K and $h(r,x)$ can be extended to a continuous function over M with $h(0,x) = 0$ for all $x \in K$.

Hence, $\alpha \leq \widetilde{\alpha} + \varepsilon$ for all $\varepsilon > 0$ and $\alpha \leq \widetilde{\alpha}$.▶

Acknowledgments

I wish to thank (in alphabetical order) Jacques Chaumat, Jacques Féjoz, Pierre Lochak, Jean-Pierre Marco and David Sauzin for helpful discussions.

References

[1] Arnold, V. I., Gusein-Zade, S. M., Varchenko, A.N. : 1988, Singularities of differentiable maps. Volume II : Monodromy and asymptotics of integrals, *Monographs in Mathematics* **83**, Birkhauser-Verlag, Boston.

[2] Benettin, G., Fasso, F., Guzzo, M. : 1998, Nekhorochev stability of L4 and L5 in the spatial restricted three body problem, *Regul. Chaotic Dyn.* **3** (3), pp. 56-72.

[3] Bierstone, E., Milman, P.D. : 1988, Semianalytic and subanalytic sets, *Publ. Math. I.H.E.S.* **67**, pp. 5-42.

[4] Bochnak, J., Risler, J.J. : 1975, Sur les exposants de Lojasiewicz, *Comment. Math. Helvetici* **50**, pp. 493-507.

[5] Broer, H.W., Huitema, G.B., Sevryuk, M.B. : 1996, Quasi-periodicity in families of dynamical systems : order amidst chaos, LNM **1645**, Springer Verlag, New-York.

[6] Féjoz, J. : 2003, Démonstration du théorème d'Arnold sur la stabilité du Système solaire (d'après Herman), submitted to *Ergodic Theory and Dynamical Systems.*

[7] Fekak, A. : 1992, Exposants de Lojasiewicz pour les fonctions semi-algébriques, *Ann. Pol. Math.* **56** (2), pp. 123-131.

[8] Giorgilli, A. : 1998, On the problem of stability for near to integrable Hamiltonian systems, Proceedings of the International Congress of Mathematicians Berlin 1998, *Documenta Mathematica*, Vol. III, extra volume ICM 1998, pp. 143–152.

[9] Guzzo, M., Morbidelli, A. : 1997, Construction of a Nekhorochev like result for the asteroid belt dynamical system, *Celestial Mechanics* **66**, pp. 255-292.

[10] Gwozdziewicz, J. : 1999, The Lojasiewicz exponent of an analytic function at an isolated zero, *Comment. Math. Helv.* **74** (3), pp. 364-375.

[11] Herman, M.: 1992, Dynamics connected with indefinite normal torsion, in *Twists Mapping and Their Applications*, McGehee, R., Meyer, K. (Eds), I.M.A. conference proceedings series **44**, Springer-Verlag, New-York, pp. 153-182.

[12] Hironaka, H.: 1973, Subanalytic sets, in *Number Theory, Algebraic Geometry and Commutative Algebra*, volume in honor of A. Akizuki, Kinokunya, Tokyo, pp. 453-493.

[13] Ilyashenko, I.S.: 1986, A steepness test for analytic functions, *Russian Math. Surveys* **41**, pp. 229-230.

[14] Lochak, P.: 1992, Canonical perturbation theory via simultaneous approximation, *Russian Math. Surveys* **47**, pp. 57-133.

[15] Lochak, P., Neishtadt, A.I.: 1992, Estimates of stability time for nearly integrable systems with a quasiconvex Hamiltonian, *Chaos* **2** (4), pp. 495-499.

[16] Lojasiewicz, S.: 1993, Sur la géométrie semi- et sous-analytique. (On semi- and subanalytic geometry), *Ann. Inst. Fourier* **43** (5), pp. 1575-1595.

[17] Marco J.P., Sauzin D.: 2003, Stability and instability for Gevrey quasi-convex near-integrable Hamiltonian Systems, *Publ. Math. Inst. Hautes Etudes Sci.* **96**, pp. 199-275.

[18] Nekhorochev, N.N.: 1973, Stable lower estimates for smooth mappings and for gradients of smooth functions, *Math. USSR Sbornik* **19**(3), pp. 425-467.

[19] Nekhorochev, N.N.: 1977, An exponential estimate of the time of stability of nearly integrable Hamiltonian systems, *Russian Math. Surveys* **32**, pp. 1-65.

[20] Nekhorochev, N.N.: 1979, An exponential estimate of the time of stability of nearly integrable Hamiltonian systems 2, *Trudy Sem. Petrovs* **5**, pp. 5-50, translated in Oleinik, O.A. (Ed.), Topics in Modern Mathematics, Petrovskii Semin. (**5**), New York: Consultant Bureau 1985.

[21] Niederman, L.: 1996, Stability over exponentially long times in the planetary problem, *Nonlinearity* **9** (6), pp. 1703-1751.

[22] Niederman, L.: 2001, Exponential stability for small perturbations of steep integrable Hamiltonian systems, to appear in *Erg. Theo. Dyn. Syst.*.

[23] Ploski, A.: 1988, Multiplicity and the Lojasiewicz exponent, *Banach Cent. Publ.* **20**, pp. 353-364.

[24] Pöschel, J.: 1993, Nekhorochev estimates for quasi-convex Hamiltonian systems, *Math. Z.* **213**, pp. 187-217.

[25] Spivak, M.: 1979, A comprehensive introduction to differential geometry. Vol. V, Publish or Perish, Berkeley.