Surface transitions of the semi-infinite Potts model II: the low bulk temperature regime

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Abstract: We consider the semi-infinite q–state Potts model. We prove, for lage q, the existence of a first order surface phase transition between the ordered phase and the the so-called "new low temperature phase" predicted in [25], in which the bulk is ordered whereas the surface is disordered.

Key words: Surface phase transitions, Semi-infinite lattice systems, Potts model, Random cluster model, Cluster expansion, Pirogov–Sinai theory, Alexander duality.

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1 Introduction and definitions

1.1 Introduction

This paper is the continuation of our study of surface phase transitions of the semi-infinite Potts model [9] (to be referred as paper I).

We consider the q–states Potts model on the half-infinite lattice with bulk coupling constant J and surface coupling constant K (see (1.1) below for the definition of the Hamiltonian).

In the many component limit $q \to \infty$, the mean field theory yields by looking at the behavior of a bulk and a surface order parameter, and after a suitable rescaling namely by taking the inverse temperature $\beta = \ln q$, the phase diagram shown in Figure [25].

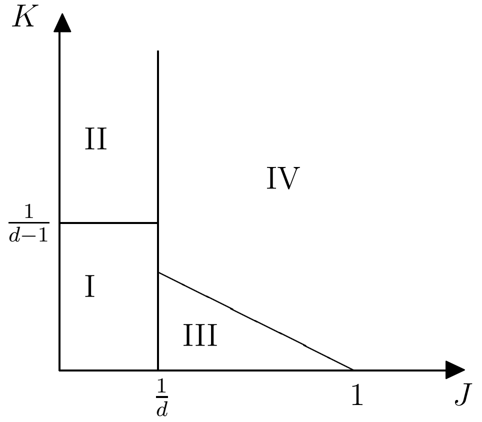

FIGURE 1: Mean field diagram borrowed from Ref. [25].

In region (I) (respectively (IV)) the bulk spins and the surface spins are disordered (respectively ordered). In region (II) the surface spins are ordered while the bulk spins are are disordered. The region (III) called new low temperature phase [25] corresponds to disordered surface spins and ordered bulk spins: this phase which is also predicted by renormalization group scheme, actually does not appear in the Ising case [15]. On the separating line between (I) and (IV) an ordinary transition occurs whereas so-called extraordinary phase transitions take place on the separating lines (I)-(III) and (II)-(IV). Finally, on the two remaining separation lines (I)-(II) and (II)-(IV), surface phase transitions arise.

In paper I, we studied the high bulk temperature regime showing that the first surface phase transition between a disordered and an ordered surface while the bulk is disordered actually holds whenever $e^{\beta J} - 1 < q^{1/d}$ and q is large enough. We consider here the situation $e^{\beta J} - 1 > q^{1/d}$ and prove the occurrence of the second surface transition, again for large values of q.

The results are based on the analysis of the induced effect of the bulk on the surface. Intuitively, this effect might be viewed as an external magnetic field. When the bulk is completely ordered (a situation that can be obtained by letting the coupling constant between bulk sites tends to infinity) the system reduces to Potts model in dimension $d-1$ with coupling constant K submitted to a magnetic field of strength J. Such a model is known to undergo a order-disordered phase transition near the line $\beta J(d-1) + \beta K = \ln q$ [4]. We control here this effect up to $e^{\beta J} - 1 > q^{1/d}$ by a suitable study of a surface free energy and its derivative with respect to the surface coupling constant, which contain the thermodynamic of the surface phase transition under consideration.

The technical tools involved in the analysis are the Fortuin-Kasteleyn representation [14], cluster-expansion [13, 18, 8, 26], Pirogov-Sinai theory [30], as already in paper I, but in addition Alexander's duality [1, 24, 22, 28].

The use of Fortuin-Kasteleyn representation is two-fold. It provides a uniform formulation of Ising/Potts/percolation models for which much (but not all) of the physical theory are best implemented (see [12] for a recent review). It can be defined for a wide class of model so that we think that results can be extended more easily (see e.g. [20, 28, 7]). This representation appears in Subsection 2.1 and at the beginning of Subsection 2.2 to express both partition functions (Z and Q) entering in the definition of the surface free energy in terms of random cluster model.

Alexander's duality is a transformation that associates to a subcomplex X of a cell–complex $\mathbb{K}$ the Poincar dual complex $[\mathbb{K} \setminus X]^*$of its complement. Alexander's Theorem provides dualities relations between the cells numbers and Betti numbers of X and those of $[\mathbb{K} \setminus X]^*$ (see e.g. [1, 24]). FK measures on lattices are usually expressed in terms of the above numbers for a suitably chosen cell-complex associated to the lattice under consideration. Alexander's duality provides thus a transformation on FK configurations (and FK measures) ([2]). In the case of the Ising/Potts models this transformation is in fact the counterpart of the Krammers-Wannier duality (or its generalizations [11, 21, 22]) : applying it after FK gives the same result than using first Krammers-Wannier duality and then taking FK representation [28, 5] . We use Alexander's duality first in Subsection 2.2. It allows to write the bulk partition function (Q) as a system of a gas of polymers interacting through hard-core exclusion potential. The important fact is that the activities of polymers can be controlled for the values of parameters under consideration. This partition function can then be exponentiated by standard cluster expansion. This duality appears again in Subsection 2.3 to obtain a suitable expression of the partition functions (Z).

Cluster expansion is used again in Subsection 2.3 to express the ratio Z/Q as a partition function of a system called Hydra model (different from that of paper I) invariant under horizontal translations .

Pirogov-Sinai theory, the well-known theory developed for translation invariant systems, is then implemented in Section 3 for the study of this system. Again cluster expansion enters in the game and the needed Peierls condition is proven in Appendix.

The above description gives the organization of the paper. We end this introduction with the main definitions and a statement about the surface phase transition.

1.2 Definitions

Consider a ferromagnetic Potts model on the semi-infinite lattice $\mathbb{L} = \mathbb{Z}^{d-1} \times \mathbb{Z}^+$ of dimension $d \geq 3$. At each site $i = \{i_1, ..., i_d\} \in \mathbb{L}$, with $i_\alpha \in \mathbb{Z}$ for $\alpha = 1, ..., d-1$ and $i_d \in \mathbb{Z}^+$, there is a spin variable σ_i taking its values in the set $\mathcal{Q} \equiv \{0, 1, \ldots, q-1\}$. We let $d(i,j) = \max_{\alpha=1,\ldots,d} |i_\alpha - j_\alpha|$ be the distance between two sites, $d(i,\Omega) = \min_{j\in\Omega} d(i,j)$ be the distance between the site i and a subset $\Omega \subset \mathbb{L}$, and $d(\Omega,\Omega') = \min_{i\in\Omega, j\in\Omega'} d(i,j)$ be the distance between two subsets of $\mathbb{L}$. The Hamiltonian of the system is given by

$$H \equiv -\sum_{\langle i,j\rangle} K_{ij}\delta(\sigma_i, \sigma_j) \tag{1.1}$$

where the sum runs over nearest neighbor pairs $\langle i,j\rangle$ (i.e. at Euclidean distance $d_{\mathrm{E}}(i,j) = 1$) of a finite subset $\Omega \subset \mathbb{L}$, and δ is the Kronecker symbol: $\delta(\sigma_i, \sigma_j) = 1$ if $\sigma_i = \sigma_j$, and 0 otherwise. The coupling constants K_{ij} can take two values according both i and j belong to the *boundary layer* $\mathbb{L}_0 \equiv \{i \in \mathbb{L} \mid i_d = 0\}$, or not:

$$K_{ij} = \begin{cases} K > 0 & \text{if} \quad \langle i,j\rangle \subset \mathbb{L}_0 \\ J > 0 & \text{otherwise} \end{cases} \tag{1.2}$$

The partition function is defined by:

$$Z^p(\Omega) \equiv \sum e^{-\beta H} \chi^p_\Omega \tag{1.3}$$

Here the sum is over configurations $\sigma_\Omega \in \mathcal{Q}^\Omega$, β is the inverse temperature, and χ^p_Ω is a characteristic function giving the boundary conditions. In particular, we will consider the following boundary conditions:

- the ordered boundary condition: $\chi^{\mathrm{o}}_\Omega = \prod_{i\in\partial\Omega} \delta(\sigma_i, 0)$, where the boundary of Ω is the set of sites of Ω at distance one to its complement $\partial\Omega = \{i \in \Omega : d(i, \mathbb{L} \setminus \Omega) = 1\}$.
- the ordered boundary condition in the bulk and free boundary condition on the surface: $\chi^{\mathrm{of}}_\Omega = \prod_{i\in\partial_b\Omega} \delta(\sigma_i, 0)$,where $\partial_b\Omega = \partial\Omega \cap (\mathbb{L} \setminus \mathbb{L}_0)$.

Let now consider the finite box

$$\Omega = \{i \in \mathbb{L} \mid \max_{\alpha=1,\ldots,d-1} |i_\alpha| \le L,\ 0 \le i_d \le M\}$$

its projection $\Sigma = \Omega \cap \mathbb{L}_0 = \{i \in \Omega \mid i_d = 0\}$ on the boundary layer and its bulk part $\Lambda = \Omega \backslash \Sigma = \{i \in \Omega \mid 1 \le i_d \le M\}$.

The *ordered surface free energy*, is defined by

$$g_{\mathrm{o}} = -\lim_{L\to\infty} \frac{1}{|\Sigma|} \lim_{M\to\infty} \ln \frac{Z^{\mathrm{o}}(\Omega)}{Q^{\mathrm{o}}(\Lambda)} \tag{1.4}$$

Here $|\Sigma| = (2L+1)^{d-1}$ is the number of lattice site in Σ, and $Q^{\mathrm{o}}(\Lambda)$ is the following bulk partition function:

$$Q^{\mathrm{o}}(\Lambda) = \sum \exp\Big\{\beta J \sum_{\langle i,j\rangle\subset\Lambda} \delta(\sigma_i, \sigma_j)\Big\} \prod_{i\in\partial\Lambda} \delta(\sigma_i, 0)$$

where the sum is over configurations $\sigma_\Lambda \in \mathcal{Q}^\Lambda$. The surface free energies do not depend on the boundary condition on the surface, in particular one can replace $Z^{\mathrm{o}}(\Omega)$ by $Z^{\mathrm{of}}(\Omega)$ in (1.4). The partial derivative of the surface free energy with respect to βK represents the mean surface energy. As a result of this paper we get for q large and $q^{1/d} < e^{\beta J} - 1 < q$ that the mean surface energy $\frac{\partial}{\partial \beta K} g_{\mathrm{o}}$ is discontinuous near $\beta K = \ln\left(1 + \left(\frac{q}{e^{\beta J}-1}\right)^{1/(d-1)}\right)$.

Namely, let $\langle\cdot\rangle^p$ denote the infinite volume expectation corresponding to the boundary condition p:

$$\langle f \rangle^p(\beta J, \beta K) = \lim_{L\to\infty, M\to\infty} \frac{1}{Z^p(\Omega)} \sum_{\sigma_\Omega \in \mathcal{Q}^\Omega} f\, e^{-\beta H} \chi^p_\Omega$$

defined for local observable f and let $e^{-\tau}$ be defined by (3.8) below. As a consequence of our main result (Theorem 3.5 in Section 3), we have the following

Corollary 1.1 *Assume that $q^{1/d} < e^{\beta J} - 1 < q$ and q is large enough, then there exists a unique value $K_t(\beta, J, q, d)$ such that for any n.n. pair ij of the surface or between the surface and the first layer*

$$\begin{aligned} \langle \delta(\sigma_i, \sigma_j)\rangle^{of}(\beta J, \beta K) &\le O(e^{-\tau}) && \textit{for} \quad K \le K_t \\ \langle \delta(\sigma_i, \sigma_j)\rangle^{o}(\beta J, \beta K) &\ge 1 - O(e^{-\tau}) && \textit{for} \quad K \ge K_t \end{aligned}$$

In that theorem the ratios of the partition functions entering in the definition of the surface free energy g_{o} (with both $Z^{\mathrm{o}}(\Omega)$ and $Z^{\mathrm{of}}(\Omega)$) are expressed in terms of partition functions of gas of polymers interacting through a two-body hard-core exclusion potential. For $q^{1/d} < e^{\beta J} - 1 < q$ and q large, the associated activities are small according the values of K namely for $K \ge K_t$ with the ordered boundary condition and for $K \le K_t$ with the ordered-free boundary condition. The system is then controlled by convergent cluster expansion.

2 Random cluster models and Hydra model

2.1 The Fortuin–Kasteleyn (FK) representation

By using the expansion $e^{\beta K_{ij}\delta(\sigma_i,\sigma_j)} = 1 + (e^{\beta K_{ij}} - 1)\delta(\sigma_i, \sigma_j)$, we obtain the Fortuin–Kasteleyn representation [14] of the partition function:

$$Z^p(\Omega) = \sum_{X \subset B(\Omega)} \prod_{\langle i,j\rangle \in X} (e^{\beta K_{ij}} - 1) q^{N^p_\Omega(X)} \tag{2.1}$$

where $B(\Omega) = \{\langle i, j\rangle : i \in \Omega, j \in \Omega\}$ is the set of bonds with both endpoints belonging to Ω, and $N^p_\Omega(X)$ is the number of connected components (regarding an isolated site $i \in \Omega$ as a component) of a given $X \subset B(\Omega)$. These numbers depend on the considered boundary condition; introducing $S(X)$ as the set of sites that belong to some bond of X and $C(X|V)$ as the number of connected components (single sites are not included) of X that do not intersect the set of sites V, they are given by:

$$N^{\rm o}_{\Omega}(X) = |\Omega| - |S(X)\cup\partial\Omega| + C(X|\partial\Omega)$$
$$N^{\rm of}_{\Omega}(X) = |\Omega| - |S(X)\cup\partial_b\Omega| + C(X|\partial_b\Omega)$$

Hereafter $|E|$ denotes the number of elements of the set E. We introduce the parameters

$$\begin{cases} \beta_s \equiv \frac{\ln(e^{\beta K}-1)}{\ln q} \\ \beta_b \equiv \frac{\ln(e^{\beta J}-1)}{\ln q} \end{cases} \tag{2.2}$$

and let $X_s = X\cap B(\mathbb{L}_0)$, $X_b = X\setminus X_s$, to get

$$Z^p(\Omega) = \sum_{X\subset B(\Omega)} q^{\beta_s|X_s|+\beta_b|X_b|+N^p_{\Omega}(X)} \tag{2.3}$$

The ground state diagram of this system is analogous to the diagram of Figure1, by replacing J by β_b and K by β_s (see paper I).

For the bulk partition function $Q^{\rm o}(\Lambda)$, one find that the FK representation reads

$$Q^{\rm o}(\Lambda) = \sum_{Y\subset B(\Lambda)} q^{\beta_b|Y|+N_\Lambda(Y|o)} = q^{\beta_b|B(\Lambda)|}\sum_{Y\subset B(\Lambda)} q^{-\beta_b|B(\Lambda)\setminus Y|+N_\Lambda(Y|o)} \tag{2.4}$$

where $N^{\rm o}_{\Lambda}(Y) = |\Lambda| - |S(X)\cup\partial\Lambda| + C(X|\partial\Lambda)$,

2.2 low temperature expansion of the bulk partition function

We give in this subsection an expansion of the partition function $Q^{\rm o}(\Lambda)$ at "temperature" $\beta_b > \frac{1}{d}$. The expansion is mainly based on a duality property and we first recall geometrical results on Poincaré and Alexander duality (see e.g. [24],[1],[11],[17]).

We first consider the lattice $\mathbb{Z}^d$ and the associated cell-complex $\mathbf{L}$ whose objects s_p are called p-cells ($0\le p\le d$): 0-cells are vertices, 1-cells are bonds, 2-cells are plaquettes etc...: a p-cell may be represented as $(x;\sigma_1 e_1,...,\sigma_p e_p)$

where $x \in \mathbb{Z}^d, (e_1, ..., e_d)$ is an orthonormal base of $\mathbb{R}^d$ and $\sigma_\alpha = \pm 1, \alpha = 1, ..., d$. Consider also the dual lattice

$$(\mathbb{Z}^d)^* = \left\{ x = (x_1 + \frac{1}{2}, ..., x_d + \frac{1}{2}) : x_\alpha \in \mathbb{Z}, \alpha = 1, ..., d \right\}$$

and the associated cell complex $\mathbf{L}^*$. There is a one to-one correspondence

$$s_p \leftrightarrow s^*_{d-p} \tag{2.5}$$

between p-cells of the complex $\mathbf{L}$ and the $d-p$-cells of $\mathbf{L}^*$. In particular to each bond s_1 corresponds the hypercube s^*_{d-1} that crosses s_1 in its middle. The dual E^* of a subset $E \subset \mathbf{L}$ is the subset of element of $\mathbf{L}^*$ that are in the one-to-one correspondence (2.5) with the elements of E.

We now turn to the Alexander duality in the particular case under consideration in this paper. Let $Y \subset B(\Lambda)$ be a set of bonds. We define the A-dual of Y as

$$\widehat{Y} = (B(\Lambda) \setminus Y)^* \tag{2.6}$$

As a property of Alexander duality one has

$$\left|\widehat{Y}\right| = |B(\Lambda) \setminus Y| \tag{2.7}$$

$$N_\Lambda(Y \mid o) = N_{\mathrm{cl}}(\widehat{Y}) \tag{2.8}$$

where $N_{\mathrm{cl}}(\widehat{Y})$ denote the number of independent closed $(d-1)$-surfaces of $\widehat{Y}$. We thus get

$$Q^{\mathrm{o}}(\Lambda) = q^{\beta_b |B(\Lambda)|} \sum_{\widehat{Y} \subset [B(\Lambda)]^*} q^{-\beta_b |\widehat{Y}| + N_{\mathrm{cl}}(\widehat{Y})} \tag{2.9}$$

This system can be described by a gas of polymers interacting through hard core exclusion potential. Indeed, we introduce polymers as connected subsets (in the $\mathbb{R}^d$ sense) of $(d-1)$-cells of $\mathbf{L}^*$ and let $\mathcal{P}(\Lambda)$ denote the set of polymers whose $(d-1)$-cells belong to $[B(\Lambda)]^*$. Two polymers γ_1 and γ_2 are compatible (we will write $\gamma_1 \sim \gamma_2$) if they do not intersect and incompatible otherwise (we will write $\gamma_1 \nsim \gamma_2$). A family of polymers is said compatible if any two polymers of the family are compatible and we will use $\mathbf{P}(\Lambda)$ to denote the set of compatible families of polymers $\gamma \in \mathcal{P}(\Lambda)$. Introducing the activity of polymers by

$$\varphi_{\mathrm{o}}(\gamma) = q^{-\beta_b |\gamma| + N_{\mathrm{cl}}(\gamma)} \tag{2.10}$$

one has:

$$Q^{\mathrm{o}}(\Lambda) = q^{\beta_b |B(\Lambda)|} \sum_{\widehat{Y} \in \widehat{\mathbf{P}}(\Lambda)} \prod_{\gamma \in \widehat{Y}} \varphi_{\mathrm{o}}(\gamma) \tag{2.11}$$

with the sum running over compatible families of polymers including the empty-set with weight equal to 1.

We will now introduce multi-indexes in order to write the logarithm of this partition function as a sum over these multi-indexes (see [26]). A multi-index C is a function from the set $\mathcal{P}(\Lambda)$ into the set of non negative integers, and we let $\operatorname{supp} C = \{\gamma \in \mathcal{P}(\Lambda) : C(\gamma) \geq 1\}$. We define the truncated functional

$$\Phi_0(C) = \frac{a(C)}{\prod_\gamma C(\gamma)!} \prod_\gamma \varphi_{\mathrm{o}}(\gamma)^{C(\gamma)} \tag{2.12}$$

where the factor $a(C)$ is a combinatoric factor defined in terms of the connectivity properties of the graph $G(C)$ with vertices corresponding to $\gamma \in \operatorname{supp} C$ (there are $C(\gamma)$ vertices for each $\gamma \in \operatorname{supp} C$) that are connected by an edge whenever the corresponding polymers are incompatible). Namely, $a(C) = 0$ and hence $\Phi_0(C) = 0$ unless $G(C)$ is a connected graph in which case C is called a *cluster* and

$$a(C) = \sum_{G \subset G(C)} (-1)^{|e(G)|} \tag{2.13}$$

Here the sum goes over connected subgraphs G whose vertices coincide with the vertices of $G(C)$ and $|e(G)|$ is the number of edges of the graph G. If the cluster C contains only one polymer, then $a(\gamma) = 1$. In other words, the set of all cells of polymers belonging to a cluster C is connected. The support of a cluster is thus a polymer and it is then convenient to define the following new truncated functional

$$\Phi(\gamma) = \sum_{C : \operatorname{supp} C = \gamma} \Phi_0(C) \tag{2.14}$$

As proven in paper I, we have the following

Theorem 2.1 *Assume that* $\beta_b > 1/d$ *and* $c_0 \nu_d q^{-\beta_b + \frac{1}{d}} \leq 1$, *where* $\nu_d = d^2 2^{4(d-1)}$, *and* $c_0 = \left[1 + 2^{d-2}(1 + \sqrt{1 + 2^{3-d}})\right] \exp\left[\frac{2}{1+\sqrt{1+2^{3-d}}}\right]$, *then*

$$Q^o(\Lambda) = e^{\beta_b |B(\Lambda)|} \exp\left\{ \sum_{\gamma \in \mathcal{P}(\Lambda)} \Phi(\gamma) \right\} \tag{2.15}$$

with a sum running over (non-empty) polymers, and the truncated functional Φ *satisfies the estimates*

$$|\Phi(\gamma)| \leq |\gamma| \left(c_0 \nu_d q^{-\beta_b + \frac{1}{d}} \right)^{|\gamma|} \tag{2.16}$$

The proof uses that the activities satisfy the bound $\varphi_{\mathrm{o}}(\gamma) \leq q^{-(\beta_b - 1/d)|\gamma|}$ (because $N_{\mathrm{cl}}(\gamma) \leq |\gamma|/d$) and the standard cluster expansion. The details are given in Ref. [9].

2.3 Hydra model

We now turn to the partition function $Z^p(\Omega)$. We will as in the previous subsection apply Alexander duality. It then turns out that the ratio $Z^p(\Omega)/Q^{\mathrm{o}}(\Lambda)$ of the partition functions partition entering in the definition (1.4) of the surface free energy g_{o} can be expressed as a partition function of geometrical objects to be called *hydras*.

Namely, we define the A-dual of a set of bonds $X \subset B(\Omega)$ as

$$\widehat{X} = (B(\Omega) \setminus X)^* \tag{2.17}$$

This transformation can be analogously define in terms of the occupation numbers

$$n_b = \begin{cases} 1 \text{ if} \quad b \in X \\ 0 \text{ otherwise} \end{cases} \tag{2.18}$$

For a configuration $n = \{n_b\}_{b \in B(\Omega)} \subset \{0,1\}^{B(\Omega)}$ we associate the configurations $\widehat{n} = \{\widehat{n}_s\}_{s \in [B(\Omega)]^*} \subset \{0,1\}^{[B(\Omega)]^*}$given by

$$\widehat{n}_{b^*} = 1 - n_b, \quad b \in B(\Omega) \tag{2.19}$$

where b^* is the $(d-1)$–cell dual of b under the correspondence (2.5).

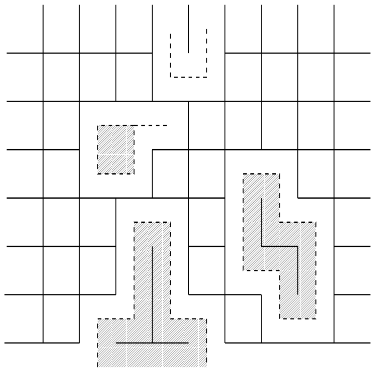

FIGURE 2: A configuration X (full lines) and its A-dual $\widehat{X}$ (dashed lines).

For any set of cells $\widehat{X}$ we will use the decomposition $\widehat{X} = \widehat{X}_s \cup \widehat{Z}_b \cup \widehat{Y}_b$ where $\widehat{X}_s$ is the set of cells whose dual are bonds with two endpoints on the boundary surface Σ, $\widehat{Z}_b$ is the set of cells whose dual are bonds with one endpoint on the boundary surface Σ and one endpoint in the bulk Λ and the remaining $\widehat{Y}_b$ is the set of cells whose dual are bonds with two endpoints in the bulk Λ. Thus, for the decomposition $X = X_s \cup X_b$ introduced above, we have

$$\begin{aligned} \left|\widehat{X}_s\right| &= |B(\Sigma) \setminus X_s| \\ \left|\widehat{Z}_b\right| + \left|\widehat{Y}_b\right| &= |B(\Omega) \setminus B(\Sigma)| - \left|\widehat{X}_b\right| \end{aligned}$$

We let B_0 be the set of bonds that have an endpoint on the boundary layer $\mathbb{L}_0$ and the other endpoint on the layer $\mathbb{L}_{-1} \equiv \{i \in \mathbb{L} \mid i_d = -1\}$ and let $\widetilde{N}_{\mathrm{cl}}(\widehat{X})$ be the number of independent closed surface of $\widehat{X} \cup B_0^*$: $\widetilde{N}_{\mathrm{cl}}(\widehat{X}) = N_{\mathrm{cl}}(\widehat{X} \cup B_0^*)$. As a result of Alexander duality, one has

$$N_\Omega^{\mathrm{o}}(X) = \widetilde{N}_{\mathrm{cl}}(\widehat{X})$$

Denoting by $B_1(\Omega)$ the set bonds that have an endpoint in $\partial_s\Omega$ the other endpoint in $\mathbb{L} \setminus \Omega$, we have furthermore

$$N_\Omega^{\mathrm{of}}(X) = \widetilde{N}_{\mathrm{cl}}(\widehat{X} \cup [B_1(\Omega)]^*)$$

These formula lead to the following expression for the partition function (2.3)

$$Z^p(\Omega) = q^{\beta_s|B(\Sigma)| + \beta_b|B(\Omega)\setminus B(\Sigma)|} \sum_{\widehat{X} \subset [B(\Omega)]^*} q^{-\beta_s|\widehat{X}_s| - \beta_b(\left|\widehat{Z}_b\right| + \left|\widehat{Y}_b\right|) + \widetilde{N}_{\mathrm{cl}}(\widehat{X}) + \widehat{\chi}_\Omega^p}$$

where $\widehat{\chi}_\Omega^{\rm o} = 0$ and $\widehat{\chi}_\Omega^{\rm of} = \widetilde{N}_{\rm cl}(\widehat{X} \cup [B_1(\Omega)]^*) - \widetilde{N}_{\rm cl}(\widehat{X})$. Notice that this Boltzmann weight equals the Boltzmann weight of the bulk partition function (2.9) for those $\widehat{X} \subset [B(\Lambda)]^*$ i.e. if $\widehat{X}_s = \varnothing$ and $\widehat{Z}_b = \varnothing$. They can thus be factorized in the ratio of the two partition function. Namely, let us first define hydras as components of $(d-1)$–cells non included in $[B(\Lambda)]^*$.

Definition 2.2 *A connected set of $(d-1)$–cells $\delta \subset [B(\Omega)]^*$ (in the $\mathbb{R}^d$ sense) is called* hydra *in Ω, if it contains a cell whose dual is a bond with at least one endpoint on the boundary surface Σ.*

Definition 2.3 *Given an hydra $\delta \subset [B(\Omega)]^*$, the components of δ included in $[B(\Sigma)]^*$ are called* legs *of the hydra, the components included in $[B(\Lambda)]^*$ are called* heads *of the hydra and the remaining components are called* bodies *of the hydra.*

The dual cells of bodies of hydras are bonds bewteen the boundary layer and the first layer $\mathbb{L}_1 \equiv \{i \in \mathbb{L} \mid i_d = 1\}$

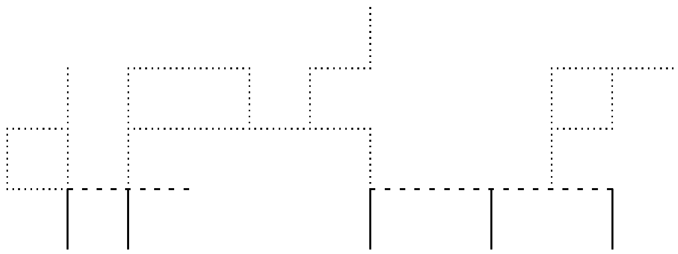

FIGURE 3: A hydra, in two dimensions (a dimension not considered in this paper), with 5 feet (components of full lines), 2 bodies (components of dashed lines), and 3 heads (components of dotted lines).

We let $\mathcal{H}(\Omega)$ denote the set of hydras in Ω. Two hydras δ_1 and δ_2 are said compatible (we will write $\delta_1 \sim \delta_2$) if they do not intersect. A family of hydras is said compatible if any two hydras of the family are compatible and we let $\mathbf{H}(\Omega)$ denote the set of compatible families of hydras $\delta \in \mathcal{H}(\Omega)$.

Clearly, a connected subset of cells included in $[B(\Omega)]^*$ is either a hydra $\delta \in \mathcal{H}(\Omega)$ or a polymer $\gamma \in \mathcal{P}(\Lambda)$ (defined in Subsection 2.2). Therefore any subset of $[B(\Omega)]^*$ is a disjoint union of a compatible family of hydras $\widehat{X} \in \mathbf{H}(\Omega)$ with a compatible family of polymers $\widehat{Y} \in \mathbf{P}(\Lambda)$.

The partition function $Z^p(\Omega)$ given by (2.1) reads thus:

$$Z^p(\Omega) = q^{\beta_s|B(\Sigma)|+\beta_b|B(\Omega)\setminus B(\Sigma)|} \tag{2.20}$$

$$\times \sum_{\widehat{X}\in\mathbf{H}(\Omega)} q^{-\beta_s|\widehat{X}_s|-\beta_b\left(\left|\widehat{Z}_b\right|+\left|\widehat{Y}_b\right|\right)+\widetilde{N}_{\mathrm{cl}}(\widehat{X})+\widehat{\chi}^p_\Omega} \sum_{\widehat{Y}\in\mathbf{P}(\Lambda):\widehat{Y}\sim\widehat{X}} \prod_{\gamma\in\widehat{Y}} \varphi_{\mathrm{o}}(\gamma) \tag{2.21}$$

where the compatibility $\widehat{Y} \sim \widehat{X}$ means no component of $\widehat{Y}$ is connected with a component of $\widehat{X}$.

According to Subsection 2.2, the last sum in the RHS of the above formula can be exponentiated as: $\exp\left\{\sum_{\gamma\in\mathcal{P}(\Lambda);\gamma\sim X}\Phi(\gamma)\right\}$. Hence dividing the above partition function by the partition function $Q^{\mathrm{o}}(\Lambda)$ we get by taking into account Theorem 2.1:

$$\Xi^p(\Omega) \equiv \frac{Z^p(\Omega)}{Q^{\mathrm{f}}(\Lambda)} = q^{\beta_s|B(\Sigma)|+\beta_b(|B(\Omega)|-|B(\Sigma)|-|B(\Lambda)|)} \tag{2.22}$$

$$\times \sum_{\widehat{X}\in\mathbf{H}(\Omega)} q^{-\beta_s|\widehat{X}_s|-\beta_b\left(\left|\widehat{Z}_b\right|+\left|\widehat{Y}_b\right|\right)+\widetilde{N}_{\mathrm{cl}}(\widehat{X})+\widehat{\chi}^p_\Omega} \exp\left\{-\sum_{\gamma\in\mathcal{P}(\Lambda);\gamma\nsim\widehat{X}}\Phi(\gamma)\right\}$$

Hereafter the incompatibility $\gamma \nsim X$ means that no component of $\widehat{X}$ is connected with δ.

$\Xi^p(\Omega)$ is thus the partition function of a gas of hydras $\widehat{X} = \{\delta_1,\ldots,\delta_n\}$ interacting through hard-core exclusion potential and through a long range interaction potential (decaying exponentially in the distance under the hypothesis of Theorem 2.1) defined on the polymers of the bulk.

If we neglect this long range potential, and if we moreover restrict to consider only hydras without head, the system of hydras will reduce itself to a $(d-1)$ Potts model with two-body interaction coupling K and magnetic field J (i.e. with formal Hamiltonan $H = -\sum_{\langle i,j\rangle\subset\mathbb{L}_0} K\delta(\sigma_i,\sigma_j) - \sum_{\langle i,k\rangle, i\in\mathbb{L}_0, k\in\mathbb{L}_1} J\delta(\sigma_i,0)$). This model undergoes a temperature driven first order phase transition, whenever q is large enough and $d \geq 3$ [4]. We will show that it is also the case for the hydra model (2.22) implementing the fact that the heads of hydras modify only weakly their activities and that the long range interaction potential decays exponentially (the needed assumptions are close to those of Theorem 2.1). To this end it is convenient to first rewrite this potential in terms of a model of *aggregates*. Let us introduce the (real-valued) functional

$$\Psi(\gamma) = e^{-\Phi(\gamma)} - 1 \tag{2.23}$$

defined on polymers $\gamma \in P(\Lambda)$. An aggregate A is a family of polymers whose support, $\operatorname{supp} A = \cup_{\gamma \in A} \gamma$, is connected. Two aggregates A_1 and A_2 are said compatible if $\operatorname{supp} A_1 \cap \operatorname{supp} A_2 = \emptyset$. A family of aggregates is said compatible if any two aggregates of the family are compatible and we will use $\mathbf{A}(\Lambda)$ to denote the set of compatible families of aggregates. Introducing the statistical weight of aggregates by

$$\omega(A) = \prod_{\gamma \in A} \Psi(\gamma) \tag{2.24}$$

we then get:

$$\begin{aligned} \exp\left\{ - \sum_{\gamma \in \mathcal{P}(\Lambda); \gamma \nsim X} \Phi(\gamma) \right\} &= \prod_{\gamma \in \mathcal{P}(\Lambda); \gamma \nsim X} (1 + \Psi(\gamma)) \\ &= \sum_{\mathcal{A} \in \mathbf{A}(\Lambda)} \prod_{A \in \mathcal{A}; A \nsim X} \omega(A) \end{aligned} \tag{2.25}$$

where $A \nsim X$ means that every polymer of the aggregate A is incompatible with X. Since the support of aggregates is a connected set of $(d-1)$–cells, i.e. a polymer, it is convenient (as it was done for clusters in Subsection 2.3) to sum the statistical weights (2.24) over aggregates with same support. We thus define the weight

$$\psi(\gamma) \equiv \sum_{A : \operatorname{supp} A = \gamma} \omega(A) \tag{2.26}$$

with $A \nsim X$, to get

$$\begin{aligned} \Xi^p(\Omega) = {} & q^{\beta_s |B(\Sigma)| + \beta_b (|B(\Omega)| - |B(\Sigma)| - |B(\Lambda)|)} \\ & \times \sum_{\widehat{X} \in \mathbf{H}(\Omega)} q^{-\beta_s |\widehat{X}_s| - \beta_b \left(\left| \widehat{Z}_b \right| + \left| \widehat{Y}_b \right| \right) + \widetilde{N}_{\mathrm{cl}}(\widehat{X}) + \widehat{\chi}^p_\Omega} \sum_{\widehat{Y} \in \mathbf{P}(\Lambda)} \prod_{\gamma \in \widehat{Y} \gamma \nsim \widehat{X}} \psi(\gamma) \end{aligned} \tag{2.27}$$

The system is thus described by two families: a compatible family of hydras (a subset of $[B(\mathbb{L})]^*$) and a compatible family of polymers (a subset of $[B(\mathbb{L} \diagdown \mathbb{L}_0)]^*$) each of these polymers being incompatible with the family of hydras.

We defined in the next subsection the diluted partition functions for our system. This partition function differs only from the "physical" partition function (2.27) by a boundary term and thus both partitions functions lead to the same free energy. The recurrence relations of Lemma 3.1 below allow to expand the diluted partition functions in term of matching signed contours interacting through hard-core exclusion potential.

2.4 Diluted partition functions

Note first that even though our model is defined on a d–dimensional box Ω it has a $(d-1)$-dimensional structure and the highest order of the logarithm of partition functions behaves like $O(|\Sigma|)$. It will be convenient to consider Ω as a set of lines and its dual Ω^* as a set of columns.

We let a line $L(x)$ be a cylinder set of sites of $\mathbb{L}$ whose projection on the boundary layer is the site x and whose height is less than a given number M: $L(x) = \{i \in \mathbb{L}\,(i_1, ..., i_{d-1}) = x \in \mathbb{L}_0, i_d \leq M\}$. We let $\mathbb{L}_M$ be the set of all such lines. The dual of a line is called column and is thus a set of d–cells of the complex $\mathbf{L}^*$. For $\Omega \subset \mathbb{L}_M$, we let $\Sigma = \Omega \cap \mathbb{L}_0$, be its projection on the boundary layer, $\Lambda = \Omega \setminus \Sigma$ and $\|\Omega^*\| = |\Sigma|$ be the number of columns of Ω^* (or of lines of Ω).

Consider a site $x \in \mathbb{L}$ and its dual d–cell x^*. We shall use $\mathcal{E}(x^*)$ to denote the set of $(d-1)$–cells of the boundary of x^* (there are the dual cells of the bonds whose x is an endpoint). For a set of d–cells D, we let $\mathcal{E}(D) = \cup_{x^* \in D} \mathcal{E}(x^*)$ be the union of the boundaries of the d–cells of D.

Next, it can easily be checked that the configuration $(\widehat{X}^{\mathrm{o}} = \varnothing, \widehat{Y} = \varnothing)$ and the configuration $(\widehat{X}^{\mathrm{of}} = [B(\mathbb{L}_0) \diagdown B(\mathbb{L} \diagdown \mathbb{L}_0)]^*, Y = \varnothing)$ are ground states of the system.

We will use $\mathbf{H}^p(\Omega)$ to denote the set of compatible families of hydras defined on $\mathcal{E}(\Omega^*) \cap [B(\mathbb{L})]^*$ that coincide with $\widehat{X}^p$ on $\mathcal{E}([\partial\Omega]^*)$, and use $\mathbf{P}^{\mathrm{dil}}(\Lambda)$ to denote the compatible families of polymers defined on $\mathcal{E}(\Omega^*) \diagdown (\mathcal{E}(\Sigma^*) \cup \mathcal{E}([\partial\Omega]^*)$.

For such configurations the Boltzmann weight in (2.27) reads

$$q^{-\beta_s|\widehat{X}_s| - \beta_b(|\widehat{Z}_b| + |\widehat{Y}_b|) + \widetilde{N}_{\mathrm{cl}}(\widehat{X})}$$

since for those $\widehat{X} \in \mathbf{H}^{\mathrm{of}}(\Omega)$ one has $\widetilde{N}_{\mathrm{cl}}(\widehat{X} \cup [B_1(\Omega)]^*) = \widetilde{N}_{\mathrm{cl}}(\widehat{X}))$.

We define, for (any) volume $\Omega \subset \mathbb{L}$, the diluted Hamiltonian of a configuration $\widehat{X} = \widehat{X}^p$ a.e., as:

$$H^{\mathrm{dil}}_{\Omega}(\widehat{X}) = \sum_{x^* \in \Omega^*} e_{x^*}(\widehat{X}) - \widetilde{N}_{\mathrm{cl}}(\widehat{X} \cap \mathcal{E}(\Omega^*) \tag{2.28}$$

where the energy per cell is defined by

$$e_{x^*}(\widehat{X}) = \tfrac{\beta_s}{2} \left| \widehat{X}_s \cap \mathcal{E}(x^*) \right| + \beta_b \left| \widehat{Z}_b \cap \mathcal{E}(x^*) \right| \text{ if } x \in \mathbb{L}_0$$

for the d–cells of the surface and by

$$e_{x^*}(\widehat{X}) = \tfrac{\beta_b}{2} \left| \widehat{Y}_b \cap \mathcal{E}(x^*) \right| \text{ if } x \in \mathbb{L} \diagdown \mathbb{L}_0$$

for the d–cells of the bulk. and

The diluted partition function is defined by

$$\Xi_p^{\mathrm{dil}}(\Omega) = \sum_{\widehat{X} \in \mathbf{H}^p(\Omega)} q^{-H_\Omega^{\mathrm{dil}}(X)} \sum_{Y \in \mathbf{P}^{\mathrm{dil}}(\Lambda)} \prod_{\gamma \in Y \gamma \nsim X} \psi(\gamma) \tag{2.29}$$

Up to a boundary term $O(\partial\Sigma)$ one has $\ln \Xi^p(\Omega) = [(d-1)\beta_s + \beta_b] \|\Omega\| \ln q + \ln \Xi_p^{\mathrm{dil}}(\Omega)$, hence

$$-\lim_{\Omega \uparrow \mathbb{L}} \frac{1}{\|\Omega\|} \ln \Xi_p^{\mathrm{dil}}(\Omega) = g_{\mathrm{o}} + [(d-1)\beta_s + \beta_b] \ln q \tag{2.30}$$

where $\Omega \uparrow \mathbb{L}$ means that we take first the limit $M \to \infty$ and then the limit $\Sigma \uparrow \mathbb{L}_0$ in the van-Hove or Fisher sense [29].

Notice that the diluted Hamiltonian on ground states reads on set of columns $\Omega^* \subset \mathbb{L}_M^*$:

$$H_\Omega^{\mathrm{dil}}(\widehat{X}^p) = e_p \|\Omega^*\| \tag{2.31}$$

with the specific energies

$$\begin{aligned} e_{\mathrm{o}} &= 0 \\ e_{\mathrm{of}} &= (d-1)\beta_s + \beta_b - 1 \end{aligned} \tag{2.32}$$

3 Surface transition in the bulk low temperature regime

3.1 Contours and Peierls estimates

We first define the contours of our system.

Let $\Omega \subset \mathbb{L}_M$, Ω^* its dual set and $(\widehat{X}, \widehat{Y})$ be a configuration of our system in Ω: $\widehat{X} \in \mathbf{H}^p(\Omega), \widehat{Y} \in \mathbf{P}^{\mathrm{dil}}(\Omega), Y \nsim X$.

A d–cell $x^* \in \Omega^*$ is called *p-correct*, if $\widehat{X}$ coincides with the ground state $\widehat{X}^p$ on the $(d-1)$–cells of the boundary $\mathcal{E}(x^*)$ of x^* and the intersection $\widehat{Y} \cap \mathcal{E}(x^*) = \varnothing$. A column is called p-correct if all the sites$d$–cells of the column are p-correct.

Columns and d–cells that are not p-correct are called *incorrect*.

The set of incorrect columns of a configuration $(\widehat{X}, \widehat{Y})$ is called *boundary* of the configuration $(\widehat{X}, \widehat{Y})$.

A triplet $\Gamma = \{\operatorname{supp}\Gamma, \widehat{X}(\Gamma), \widehat{Y}(\Gamma)\}$, where $\operatorname{supp}\Gamma$ is a maximal connected subset of the boundary of the configuration $(\widehat{X}, \widehat{Y})$ called support of Γ, $\widehat{X}(\Gamma)$ the restriction of $\widehat{X}$ to the boundary $\mathcal{E}(\operatorname{supp}\Gamma)$ of the support of Γ, and $\widehat{Y}(\Gamma)$ the restriction of $\widehat{Y}$ to $\mathcal{E}(\operatorname{supp}\Gamma)$, is called *contour* of the configuration (X, Y). Hereafter a set of d–cells is called connected if the graph that joins all the dual sites i, j of this set with $d(i,j) \leq 1$ is connected.

A triplet $\Gamma = \{\operatorname{supp}\Gamma, \widehat{X}(\Gamma), \widehat{Y}(\Gamma)\}$, where $\operatorname{supp}\Gamma$ is a connected set of columns is called *contour* if there exists a configuration $(\widehat{X}, \widehat{Y})$ such that Γ is a contour of $(\widehat{X}, \widehat{Y})$. We will use $|\Gamma|$ to denote the number of incorrect cells of $\operatorname{supp}\Gamma$ and $\|\Gamma\|$ to denote the number of columns of $\operatorname{supp}\Gamma$.

Consider the configuration having Γ as unique contour; it will be denoted $(\widehat{X}^\Gamma, \widehat{Y}^\Gamma)$. Let $L_p(\Gamma)$ be the set of p-correct columns of $\mathbb{L}_M^* \setminus \operatorname{supp}\Gamma$. Obviously, either a component of $L_{\mathrm{o}}(\Gamma)$ is infinite or a component of $L_{\mathrm{of}}(\Gamma)$ is infinite. In the first case Γ is called contour of the ordered class or o-contour and in the second case it is called of-contour. When Γ is a p-contour (we will let Γ^p denote such contours) we use $\operatorname{Ext}\Gamma$ to denote the unique infinite component of $L_p(\Gamma)$; this component is called *exterior* of the contour. The set of remaining components of $L_p(\Gamma)$ is denoted $\mathrm{Int}_p\Gamma$ and the set $L_{m\neq p}(\Gamma)$ is denoted $\mathrm{Int}_m\Gamma$. The union $\mathrm{Int}\Gamma = \mathrm{Int}_{\mathrm{f}}\Gamma \cup \mathrm{Int}_{\mathrm{fo}}\Gamma$ is called *interior* of the contour and $V(\Gamma) = \operatorname{supp}\Gamma \cup \mathrm{Int}\Gamma$.

Two contours Γ_1 and Γ_2 are said compatible if the union of their supports is not connected. They are mutually compatible external contours if $V(\Gamma_1) \subset \mathrm{Ext}\Gamma_2$ and $V(\Gamma_2) \subset \mathrm{Ext}\Gamma_1$.

We will use $G(\Gamma^p)$ to denote the set of configurations having Γ^p as unique external contour. The crystal partition function is then defined by :

$$\Xi^{\mathrm{cr}}(\Gamma^p) = \sum_{(\widehat{X},\widehat{Y}) \in G(\Gamma^p)} q^{-H^{\mathrm{dil}}_{V(\Gamma^p)}(\widehat{X})} \prod_{\gamma \in Y} \psi(\gamma) \tag{3.1}$$

The following set of recurrence equations holds :

Lemma 3.1

$$\Xi_p^{dil}(\Omega) = \sum_{\{\Gamma_1^p,\dots,\Gamma_n^p\}_{ext}} q^{-e_p\|Ext\|} \prod_{i=1}^{n} \Xi^{cr}(\Gamma_i^p) \tag{3.2}$$

Here the sum is over families $\{\Gamma_1^p,\dots,\Gamma_n^p\}_{ext}$ *of mutually compatible external contours in* Ω *(*$supp\,\Gamma_i^p \subset \Omega_{int} = \{i \in \Omega : d(i, \mathbb{L}_M \setminus \Omega) > 1\}$*),* $\|Ext\| = \|\Omega^*\| - \sum_i \|V(\Gamma_i^p)\|$ *where* $\|V(\Gamma_i^p)\|$ *is the number of columns of* $V(\Gamma_i^p)$*;*

$$\Xi^{cr}(\Gamma^p) = \varrho(\Gamma^p) \prod_{m\in\{o,of\}} \Xi_m^{dil}(Int_m \Gamma^p) \tag{3.3}$$

where:

$$\varrho(\Gamma^p) \equiv q^{-H^{dil}_{supp\,\Gamma^p}(X^{\Gamma^p})} \prod_{\gamma\in Y_{\Gamma^p}} \psi(\gamma) \tag{3.4}$$

Proof.

We have only to observe that for any $\widehat{X} \in \mathbf{H}^p(\Omega)$

$$H^{\mathrm{dil}}_{\Omega}(\widehat{X}) = \sum_{\Gamma} H^{\mathrm{dil}}_{\mathrm{supp}\,\Gamma}(\widehat{X}^{\Gamma}) + \sum_p e_p \left\| L_p(\widehat{X}) \cap \Omega^* \right\| \tag{3.5}$$

where the sum is over all contours of the boundary of the configuration $(\widehat{X}, \widehat{Y} = \emptyset)$ and $\left\| L_p(\widehat{X}) \cap \Omega^* \right\|$ is the number of p–correct columns inside Ω of this configuration.

■

Lemma 3.1 gives the following expansion for the partition function

$$\Xi_p^{\mathrm{dil}}(\Omega) = q^{-e_p\|\Omega\|} \sum_{\{\Gamma_1^p,\dots,\Gamma_n^p\}_{\mathrm{comp}}} \prod_{i=1}^{n} z(\Gamma_i^p) \tag{3.6}$$

where the sum is now over families of compatibles contours of the same class and

$$z(\Gamma_i^p) = \varrho(\Gamma^p) q^{e_p\|\Gamma^p\|} \frac{\Xi_m^{\mathrm{dil}}(\mathrm{Int}_m\Gamma^p)}{\Xi_p^{\mathrm{dil}}(\mathrm{Int}_m\Gamma^p)} \tag{3.7}$$

where $\|\Gamma^p\|$ is the number is the number of columns of $\mathrm{supp}\,\Gamma^p$ and $m \neq p$.

To control the behavior of our system, we need to show Peierls condition, that means that $\varrho(\Gamma^p) q^{e_p\|\Gamma^p\|}$ has good decaying properties with respect to the number of incorrect cells of supp Γ^p. We use in fact the modified Peierls condition introduced in Ref. [19] where $\varrho(\Gamma^p)q^{e_p\|\Gamma^p\|}$ is replaced by $\varrho(\Gamma^p)q^{\underline{e}\|\Gamma^p\|}$ with $\underline{e} = \min(e_{\mathrm{o}}, e_{\mathrm{of}})$.

Let

$$e^{-\tau} = \left(2^{(3d-2)}q^{-\frac{1-\beta_b}{2(d-1)}} + 3c2^{d+1}\nu_d^3 q^{\frac{1}{d}-\beta_b}\right)^{\|S\|} \frac{1}{1-6c\nu_d^3 q^{\frac{1}{d}-\beta_b}} \tag{3.8}$$

where $c = 8e(e-1)c_0$ and $\nu_d = d^2 2^{4(d-1)}$. We have the following

Proposition 3.2 *Let $S \subset \mathbb{L}_M^*$ be a finite connected set of columns, assume that $\frac{1}{d} < \beta_b < 1$ and $6c\nu_d^3 q^{-\frac{1}{d}+\beta_b} < 1$, then for all $\beta_s \in \mathbb{R}$:*

$$\sum_{\Gamma^p : supp\,\Gamma^p = S} |\varrho(\Gamma^p)|\, q^{\underline{e}\|\Gamma^p\|} \le e^{-\tau\|S\|} \tag{3.9}$$

where $\|S\|$ is the number of columns of S.

The proof is postponed to the Appendix.

The recurrence equations of Lemma 3.1 together with the Peierls estimates (3.9) allow to study the states invariant under horizontal translation (HTIS) of the hydra system as in paper I. This is the subject of next subsection.

3.2 Diagram of horizontal translation invariant states

To state our result, we first define the functional

$$K_p(S) = \sum_{\Gamma^p : \mathrm{supp}\,\Gamma^p = S} z(\Gamma^p) \tag{3.10}$$

Consider the partition function $\Xi_p^{\mathrm{dil}}(\Omega)$ (3.6) and for a compatible family $\{\Gamma_1^p, ..., \Gamma_n^p\}_{\mathrm{comp}}$ of p-contours, denote by $S_1, ..., S_n$ their respective supports. By summing over all contours with the same support this partition function can be written as the partition function of a gas of polymers S with activity $K_p(S) = \sum_{\Gamma^p : \mathrm{supp}\,\Gamma^p = S} z(\Gamma^p)$ interacting through hard-core exclusion potential:

$$\Xi_p^{\mathrm{dil}}(\Omega) = q^{-e_p\|\Omega\|} \sum_{\{S_1, ..., S_n\}_{\mathrm{comp}}} \prod_{i=1}^{n} K_p(S_i) \tag{3.11}$$

Here $\{S_1, ..., S_n\}_{\text{comp}}$ denotes compatible families of polymers, that is $d(S_i^*, S_j^*) > 1$ for every two S_i and S_j in the family: recall that by definitions of contours a polymer S is a set of columns whose graph that joins all the points of the dual of the columns of S at distance $d(i,j) \leq 1$ is connected.

Next, we introduce the so-called truncated contour models defined with the help of the following

Definition 3.3 *A truncated contour functional is defined as*

$$K_p'(S) = \begin{cases} K_p(S) & \text{if } \|K_p(S)\| \leq e^{-\alpha\|S\|} \\ 0 & \text{otherwise} \end{cases} \tag{3.12}$$

where $\|K_p(S)\| = \sum_{\Gamma^p : supp\,\Gamma^p = S} |z(\Gamma^p)|$ *, and* $\alpha > 0$ *is some positive parameter to be chosen later (see 3.5 below).*

Definition 3.4 *The collection* $\{S, p\}$ *of all p-contours* Γ^p *with support* $supp\,\Gamma^p = S$ *is called stable if*

$$\|K_p(S)\| \leq e^{-\alpha\|S\|} \tag{3.13}$$

i.e. if $K_p(S) = K_p'(S)$*.*

We define the truncated partition function $\Xi_p'(\Omega)$ as the partition function obtained from $\Xi_p^{\text{dil}}(\Omega)$ by leaving out unstable collections of contours, namely

$$\Xi_p'(\Omega) = q^{-e_p\|\Omega\|} \sum_{\{\Gamma_1^p,...,\Gamma_n^p\}_{\text{comp}}}{}^{'} \prod_{i=1}^{n} z(\Gamma_i^p) \tag{3.14}$$

$$= q^{-e_p\|\Omega\|} \sum_{\{S_1,...,S_n\}_{\text{comp}}} \prod_{i=1}^{n} K_p'(S_i) \tag{3.15}$$

Here the sum goes over compatible families of *stable collections of contours.* Let

$$h_p = -\lim_{\Omega \to L} \frac{1}{\|\Omega\|} \ln \Xi_p'(\Omega) \tag{3.16}$$

be the *metastable free energy* of the truncated partition function $\Xi_p'(\Omega)$.

For α large enough, the thermodynamic limit (3.16) can be controlled by a convergent cluster expansion. We conclude the existence of h_p, together with the bounds

$$e^{-\kappa e^{-\alpha}|\partial_s\Omega|} \leq \Xi'_p(\Omega)e^{h_p\|\Omega\|} \leq e^{\kappa e^{-\alpha}|\partial_s\Omega|} \tag{3.17}$$

$$|h_p - e_p \ln q| \leq \kappa e^{-\alpha} \tag{3.18}$$

where $\kappa = \kappa_{\rm cl}(\chi')^2$ where $\kappa_{\rm cl} = \frac{\sqrt{5}+3}{2}e^{\frac{2}{\sqrt{5}+1}}$ is the cluster constant [18] and $\kappa' = 3^{d-1} - 1$; $\partial_s\Omega = \partial\Omega \cap \mathbb{L}_0$ in the way defined in Subsection 1.2.

Theorem 3.5 *Assume that* $1/d < \beta_b < 1$ *and* q *is large enough so that* $e^{-\alpha} \equiv e^{-\tau+2\kappa'+3} < \frac{0.7}{\kappa\kappa'}$, *then there exists a unique* $\beta_s^t = \frac{1}{d-1}(1-\beta_b) + O(e^{-\tau})$ *such that :*

(i) *for* $\beta_s = \beta_s^t$

$$\Xi_p^{dil}(\Omega) = \Xi'_p(\Omega)$$

for both boundary conditions $p = o$ *and* $p = of$, *and the free energy of the hydra model is given by* $g_f + [(d-1)\beta_s + \beta_b]\ln q = h_o = h_{of}$

(ii) *for* $\beta_s > \beta_s^t$

$$\Xi_o^{dil}(\Omega) = \Xi'_o(\Omega)$$

and $g_o + [(d-1)\beta_s + \beta_b]\ln q = h_o < h_{of}$

(iii) *for* $\beta_s < \beta_s^t$

$$\Xi_{of}^{dil}(\Omega) = \Xi'_{of}(\Omega)$$

and $g_{of} + [(d-1)\beta_s + \beta_b]\ln q = h_{fo} < h_f$

Proof.

Starting from the Peierls estimates given in Proposition 3.2 and the definitions of this subsection the proof is the same as that of Theorem 3.5 in paper I. Let us only recall that to exponentiate the partition function $\mathcal{Z}_p(\Omega) = q^{e_p\|\Omega\|}\Xi'_p(\Omega)$ we define the truncated functional Φ^T associated to K'_p

$$\Phi^T(X) = \frac{a(X)}{\prod_\gamma X(S)!}\prod_S K'_p(S)^{X(S)} \tag{3.19}$$

defined on the multi-indexes X associated to the polymers (a multi-index being a function from the set of polymers into the set of non negative integers, and $a(X)$ is defined as in (2.13)). The number of polymers S with number of columns $\|S\| = n$ and containing a given column can be bounded by ν^n where $\nu = (3^{d-1} - 1)^2$ as in paper I: this is because the chosen definiton for connectedness of columns here is the same as that for connectedness of lines in paper I.

As a result of the standard cluster expansion [18, 26], we get for $\kappa e^{-\alpha} < 1$

$$\mathcal{Z}_p(\Omega) = \exp\left\{\sum_X \Phi_T(X)\right\}$$

where the sum is over multi-indexes whose support $\mathrm{supp}X = \{S : X(S) \geq 1\}$ belongs to Ω. The series $\sum_{X:\mathrm{supp}X \ni L} \left|\Phi^T(X)\right|$ is absolutely convergent and satisfies the bound

$$\sum_{X:\mathrm{supp}X \ni L} \left|\Phi^T(X)\right| \leq \kappa e^{-\alpha} \tag{3.20}$$

■

Let us introduce the Gibbs states $\langle \cdot \rangle^p$, associated to the boundary conditions $p \in \{\mathrm{f;of}\}$. Theorem above show also that at $\beta_s = \beta_s^t$

$$\langle\, n_b \,\rangle^{\mathrm{o}} = 1 - O(e^{-\tau}) \tag{3.21}$$

$$\langle\, n_b \,\rangle^{\mathrm{of}} = O(e^{-\tau}) \tag{3.22}$$

for any bond b of the boundary layer and any bond b between the boundary layer and the first layer.

Indeed by the correspondence (2.19) these equations are equivalent to

$$\langle\, \widehat{n}_{b^*} \,\rangle^{\mathrm{o}} = O(e^{-\tau}) \tag{3.23}$$

$$\langle\, \widehat{n}_{b^*} \,\rangle^{\mathrm{of}} = 1 - O(e^{-\tau}) \tag{3.24}$$

for the dual cells of the bonds under consideration. By definition of contours, with ordered (o) boundary conditions, any such cells are occupied only if there is an ordered contour surrounding it and that the correlation functions are controlled by the contour model cluster expansion. With ordered-free (of) boundary conditions, such cells are empty only if there is a of–contour surrounding it and again the correlations are controlled by cluster expansion. Obviously, the relation (3.23) holds true for any $\beta_s \geq \beta_s^t$ while the relation (3.24) hold true for any $\beta_s \leq \beta_s^t$.

This shows in particular that the derivative $\frac{\partial}{\partial K} g_{\mathrm{o}}$ of the free energy g_{o} with respect to the surface coupling constant K is discontinuous near $K = \beta^{-1} \ln \left(1 + \left(\frac{q}{e^{\beta J} - 1} \right)^{1/(d-1)} \right)$.

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Appendix: Proof of Proposition 3.2

We begin the proof by considering contours $\Gamma = \left\{\operatorname{supp}\Gamma, \widehat{X}^\Gamma, \widehat{Y}^\Gamma\right\}$ (where $\left\{\widehat{X}^\Gamma, \widehat{Y}^\Gamma\right\}$ is the configuration having Γ as unique contour) without polymers, i.e. $\widehat{Y}^\Gamma = \emptyset$. We have the decomposition $\widehat{X}^\Gamma = \widehat{X}_s^\Gamma \cup \widehat{Z}_b^\Gamma \cup \widehat{Y}_b^\Gamma$ where $\widehat{X}_s^\Gamma = \widehat{X}^\Gamma \cap [B(\mathbb{L}_0)]^*$, $\widehat{Y}_b^\Gamma = \widehat{X}^\Gamma \cap [B(\mathbb{L} \setminus \mathbb{L}_0)]^*$, and $\widehat{Z}_b^\Gamma = \widehat{X}^\Gamma \setminus (\widehat{X}_s^\Gamma \cup \widehat{Y}_b^\Gamma)$.

A d–cell $x^* \in \mathbb{L}_0^*$ will be called regular if the $(d-1)$–cells of its boundary that belong to the boundary layer are either all empty or all occupied. It will be called irregular otherwise. We denote by $R_0(\Gamma)$ the set of correct d–cells of the contour Γ: $R_0(\Gamma) = \left\{x^* \in \mathbb{L}_0 : \left|X^\Gamma \cap \mathcal{E}(x^*)\right| \text{ equals } 0 \text{ or } 2d-1\right\}$. We let $I_0(\Gamma)$ be the set of incorrect d–cells of the contour Γ: $I_0(\Gamma) = \left\{x^* \in \mathbb{L}_0 : 1 \leq \left|X^\Gamma \cap \mathcal{E}(x^*)\right| \leq 2(d-1)\right\}$.

Lemma A.1

$$\varrho(\Gamma) q^{\underline{e}\|\Gamma\|} \leq q^{-\frac{1-\beta_b}{2(d-1)}|I_0(\Gamma)| - \left(\beta_b - \frac{1}{d}\right)\left|Y_b^\Gamma\right|} \tag{A.1}$$

Proof.

By Lemma 3.1 and definition (2.28), one has

$$\varrho(\Gamma) = q^{\sum_{x^* \in R_0(\Gamma) \cup I_0(\Gamma)} e_{x^*}(\widehat{X}) - \beta_b \left|\widehat{Y}_b\right| + \widetilde{N}_{\rm cl}(\widehat{X})}$$

where $e_{x^*}(\widehat{X}) = -\frac{\beta_s}{2}\left|\widehat{X}_s \cap \mathcal{E}(x^*)\right| - \beta_b \left|\widehat{Z}_b \cap \mathcal{E}(x^*)\right|$ and to simplify formulae we put hereafter $\widehat{X}$, $\widehat{X}_s$, $\widehat{Z}_b$ and $\widehat{Y}_b$ instead of $\widehat{X}^\Gamma$, $\widehat{X}_s^\Gamma$, $\widehat{Z}_b^\Gamma$ and $\widehat{Y}_b^\Gamma$. We define

$$\overline{N}_{\rm cl}(\widehat{X}_s \cup \widehat{Z}_b) = \widetilde{N}_{\rm cl}(\widehat{X}_s \cup \widehat{Z}_b) - \sum_{x^* \in R_0(\Gamma)} \widetilde{N}_{\rm cl}(\widehat{X} \cap \mathcal{E}(x^*))$$

as the number of independent closed surfaces that are not boundaries of an occupied d–cell of the surface. This leads to the decomposition

$$\varrho(\Gamma) q^{\underline{e}\|\Gamma\|} = q^{-\mathbf{A}_s(\Gamma) - \mathbf{B}_s(\Gamma) - \mathbf{A}_b(\Gamma^p)} \tag{A.2}$$

where

$$\mathbf{A}_s(\Gamma) = \sum_{x^* \in R_0(\Gamma)} \left[e_{x^*}(\widehat{X}) - \underline{e} - \widetilde{N}_{\rm cl}(\widehat{X} \cap \mathcal{E}(x^*))\right] \tag{A.3}$$

$$\mathbf{B}_s(\Gamma) = \sum_{x^* \in I_0(\Gamma)} \left[e_{x^*}(\widehat{X}) - \underline{e}\right] - \overline{N}_{\rm cl}(\widehat{X}_s \cup \widehat{Z}_b) \tag{A.4}$$

$$\mathbf{A}_b(\Gamma) = \beta_b \left|\widehat{Y}_b\right| - \left[\widetilde{N}_{\rm cl}(\widehat{X}) - \widetilde{N}_{\rm cl}(\widehat{X}_s \cup \widehat{Z}_b)\right] \tag{A.5}$$

Clearly

$$\mathbf{A}_s(\Gamma) \geq 0 \tag{A.6}$$

Indeed the regular d–cells are either empty in which case the term inside brackets in (A.3) equals

$$-\underline{e} = -\min\left(e_{\mathrm{o}} = 0, e_{\mathrm{of}} = (d-1)\beta_s + \beta_b - 1\right)$$

or they are occupied in which case this term equals

$$(d-1)\beta_s + \beta_b - 1 - \min\left(e_{\mathrm{o}} = 0, e_{\mathrm{of}} = (d-1)\beta_s + \beta_b - 1\right)$$

Let us now bound $\mathbf{B}_s(\Gamma)$. We first notice that for incorrect d–cells x^* of the surface,

$$\begin{aligned} e_{x^*}(\widehat{X}) - \underline{e} &= \frac{\beta_s}{2}\left|\widehat{X}_s \cap \mathcal{E}(x^*)\right| + \beta_b\left|\widehat{Z}_b \cap \mathcal{E}(x^*)\right| - e_{\mathrm{of}}\chi\left(\beta_s \leq \frac{1-\beta_b}{d-1}\right) \\ &\geq \frac{1-\beta_b}{2(d-1)}\left|\widehat{X}_s \cap \mathcal{E}(x^*)\right| + \beta_b\left|\widehat{Z}_b \cap \mathcal{E}(x^*)\right| \end{aligned} \tag{A.7}$$

Furthermore the number $\overline{N}_{\mathrm{cl}}(\widehat{X}_s \cup \widehat{Z}_b)$ may be bounded as

$$\overline{N}_{\mathrm{cl}}(\widehat{X}_s \cup \widehat{Z}_b) \leq \sum_{\substack{x^*: d-1 \leq \left|\widehat{X} \cap \mathcal{E}(x^*)\right| \leq 2(d-1)-1 \\ \left|\widehat{Z}_b \cap \mathcal{E}(x^*)\right| = 1}} \frac{2^{\left|\widehat{X} \cap \mathcal{E}(x^*)\right|}}{2^{2(d-1)}} \leq \sum_{\substack{x^*: d-1 \leq \left|\widehat{X} \cap \mathcal{E}(x^*)\right| \leq 2(d-1)-1 \\ \left|\widehat{Z}_b \cap \mathcal{E}(x^*)\right| = 1}} \frac{1}{2} \tag{A.8}$$

If an incorrect site of the surface is such that $\left|\widehat{Z}_b \cap \mathcal{E}(x^*)\right| = 0$, then necessarily $\left|\widehat{X}_s \cap \mathcal{E}(x^*)\right|$ and thus such site gives a contribution

$$\frac{1-\beta_b}{2(d-1)}$$

to $\mathbf{B}_s(\Gamma)$. Let us now consider those incorrect d–cells for which $\left|\widehat{Z}_b \cap \mathcal{E}(x^*)\right| = 1$. Starting from (A.7) we get for such cells

$$
\begin{aligned}
e_{x^*}(\widehat{X}) - \underline{e} &\geq \beta_b + \frac{1-\beta_b}{2(d-1)}\left|\widehat{X}_s \cap \mathcal{E}(x^*)\right| \\
&= \frac{1-\beta_b}{2(d-1)} + \beta_b \frac{2(d-1)+1-\left|\widehat{X}_s \cap \mathcal{E}(x^*)\right|}{2(d-1)} + \frac{\left|\widehat{X}_s \cap \mathcal{E}(x^*)\right| - 1}{2(d-1)} \\
&\geq \frac{1-\beta_b}{2(d-1)} + \frac{1}{d}\frac{2(d-1)+1-\left|\widehat{X}_s \cap \mathcal{E}(x^*)\right|}{2(d-1)} + \frac{\left|\widehat{X}_s \cap \mathcal{E}(x^*)\right| - 1}{2(d-1)} \\
&= \frac{1-\beta_b}{2(d-1)} + \frac{\left|\widehat{X}_s \cap \mathcal{E}(x^*)\right| + 1}{2d}
\end{aligned}
$$

where for the second inequality we take into account that $\left|\widehat{X}_s \cap \mathcal{E}(x^*)\right| \leq 2(d-1)$ and $\beta_b \geq 1/d$. When furthermore $\left|\widehat{X}_s \cap \mathcal{E}(x^*)\right| \geq d-1$, one infer

$$
e_{x^*}(\widehat{X}) - \underline{e} \geq \frac{1-\beta_b}{2(d-1)} + \frac{1}{2}
$$

and it thus follows from (A.4)(A.8) and (A.7) that each incorrect cell with $\left|\widehat{Z}_b \cap \mathcal{E}(x^*)\right| = 1$, gives also a contribution at least $\frac{1-\beta_b}{2(d-1)}$ to $\mathbf{B}_s(\Gamma)$. Therefore

$$
q^{-\mathbf{B}_s(\Gamma)} \leq q^{-\frac{1-\beta_b}{2(d-1)}|I_0(\Gamma)|} \tag{A.9}
$$

Consider finally, the quantity $\mathbf{A}_b(\Gamma)$. We will prove the inequality

$$
\widetilde{N}_{\mathrm{cl}}(\widehat{X}) - \widetilde{N}_{\mathrm{cl}}(\widehat{X}_s \cup \widehat{Z}_b) \leq \frac{\left|\widehat{Y}_b\right|}{d} \tag{A.10}
$$

Notice first the obvious inequality

$$
\widetilde{N}_{\mathrm{cl}}(\widehat{X}) - \widetilde{N}_{\mathrm{cl}}(\widehat{X}_s \cup \widehat{Z}_b) \leq N_{\mathrm{cl}}(\widehat{Y}_b \cup B_{01}^*)
$$

where B_{01} is the set of bonds between the boundary layer and the first layer and $N_{\mathrm{cl}}(\widehat{Y}_b \cup B_{01}^*)$ is the number of independent closed surface of $\widehat{Y}_b \cup B_{01}^*$. The number $\left|\widehat{Y}_b\right|$ can be written

$$
\left|\widehat{Y}_b\right| = \sum_{x^* \in [\mathbb{L} \setminus \mathbb{L}_0]^* : \left|\widehat{Y}_b \cap \mathcal{E}(x^*)\right| \geq 1} \frac{\left|\widehat{Y}_b \cap \mathcal{E}(x^*)\right|}{2}
$$

(because each $(d-1)$–cell belongs to the boundary of two d–cells).

For the configurations $\widehat{Y}_b$ that do not hit B^*_{01} (meaning that there are no cell of $\widehat{Y}_b$ connected with B^*_{01} in the $\mathbb{R}^d$ sense) we get immediately (A.10) as already used in the proof of Theorem 2.1: closed surfaces of minimal area are d–cell and the number of $(d-1)$–cells in the boundary of d–cell equals $2d$.

For the configurations $\widehat{Y}_b$ that do hit B^*_{01}, notice first that $\left|\widehat{Y}_b\right|$ can be written

$$\left|\widehat{Y}_b\right| = \sum_{x^* \in [\mathbb{L}\setminus\mathbb{L}_0]^* : \left|\widehat{Y}_b \cap \mathcal{E}(x^*)\right| \geq 2} \frac{\left|\widehat{Y}_b \cap \mathcal{E}(x^*)\right|}{2} + \sum_{x^* \in [\mathbb{L}\setminus\mathbb{L}_0]^* : \left|\widehat{Y}_b \cap \mathcal{E}(x^*)\right| = 1} \frac{\left|\widehat{Y}_b \cap \mathcal{E}(x^*)\right|}{2} \tag{A.11}$$

Since $\widehat{Y}_b$ is finite, the set of cells with $\left|\widehat{Y}_b \cap \mathcal{E}(x^*)\right| = 1$ is non empty and furthermore for any $(d-1)$–cell of B^*_{01} there exists a d–cell x^*above it such that $\left|\widehat{Y}_b \cap \mathcal{E}(x^*)\right| = 1$ that can not contribute to $N_{\mathrm{cl}}(\widehat{Y}_b \cup B^*_{01})$. Now closed surfaces of minimal area are d–cells having in their boundary $2d-1$ cells of $\widehat{Y}_b$ and a cell of B^*_{01}. Thus for such surface we have a contribution $d-1/2$ coming from the first term of the RHS of (A.11) and a contribution $1/2$ coming from the second term.

This imply (A.10) giving

$$q^{-\mathbf{A}_b(\Gamma)} \leq q^{-\left(\beta_b - \frac{1}{d}\right)\left|\widehat{Y}_b\right|} \tag{A.12}$$

which in turn implies the lemma by taking into account (A.2), (A.6) and (A.9).

■

Considering still contours $\Gamma = \left\{\mathrm{supp}\Gamma, X^\Gamma, Y^\Gamma\right\}$ without polymers, i.e. $(Y^\Gamma = \emptyset)$ we have the

Lemma A.2 *Assume that $\beta_b > \frac{1}{d}$, and $2\nu_d q^{\frac{1}{d}-\beta_b} < 1$, then*

$$\sum_{\Gamma : supp\,\Gamma = S} \varrho(\Gamma) q^{\underline{e}\|\Gamma\|} \leq \left(2^{(3d-2)} q^{\frac{1-\beta_b}{2(d-1)}-} + 2^{d+1}\nu_d q^{\frac{1}{d}-\beta_b}\right)^{\|S\|} \frac{1}{1 - 2\nu_d q^{\frac{1}{d}-\beta_b}} \tag{A.13}$$

which shows that, whenever q is large enough, the Peierls condition holds true for the class of contours without polymers.

Proof.

First, observe that for contours Γ with support $\operatorname{supp}\Gamma = S$ and number of irregular cells of the boundary layer $|I_0(\Gamma)| = k$ one has $\left|\widehat{Y}_b\right| = |\delta_1| + ... + |\delta_m| \geq \|S\| - k$. Therefore,

$$\begin{aligned}
\sum_{\Gamma:\operatorname{supp}\Gamma=S} \varrho(\Gamma) q^{\underline{e}\|\Gamma\|} &\leq \sum_{0\leq k\leq\|S\|} \sum_{\Gamma^p:|I_0(\Gamma)|=k} q^{-\frac{1-\beta_b}{2(d-1)}k} q^{\left(\frac{1}{d}-\beta_b\right)\left|\widehat{Y}_b\right|} \\
&\leq \sum_{0\leq k\leq\|S\|} \binom{\|S\|}{k} 2^{(2d-1)k} 2^{\|S\|-k} q^{-\frac{1-\beta_b}{2(d-1)}k} \qquad \text{(A.14)} \\
&\quad \times \sum_{n\leq 2\|S\|} \sum_{\substack{\delta_1 \ni s_1,\dots,\delta_n \ni s_n \\ |\delta_1|+\dots+|\delta_n|\geq\|S\|-k}} \sum_{\substack{s_1,\dots,s_n \\ s_\alpha\in S;\, s_\alpha \not\sim B_{01}^*}} \prod_{j=1}^{m} q^{\left(\frac{1}{d}-\beta_b\right)|\delta_j|}
\end{aligned}$$

Here the binomial coefficient $\binom{\|S\|}{k}$ bounds the choice of irregular cells of the dual of the boundary layer while the factor $2^{(2d-1)k}2^{\|S\|-k}$ bounds the numbers of contours with $\|S\|$ columns and k irregular cells; the notation $s_\alpha \not\sim B_{01}^*$ means that a $(d-2)$–cell of the boundary of the $(d-1)$–cell s_α belongs to the boundary $\mathcal{E}(B_{01}^*)$. Then

$$\begin{aligned}
\sum_{\Gamma:\operatorname{supp}\Gamma=S} \varrho(\Gamma) q^{\underline{e}\|\Gamma\|} &\leq \sum_{0\leq k\leq\|S\|} \binom{\|S\|}{k} 2^{(2d-1)k} 2^{\|S\|-k} q^{-\frac{1-\beta_b}{2(d-1)}k} \\
&\quad \times \sum_{n\leq 2\|S\|} \binom{(d-1)\,\|S\|}{n} \sum_{m_1+\dots+m_n\geq\|S\|-k} \prod_{j=1}^{n} \left(\nu_d q^{\frac{1}{d}-\beta_b}\right)^{m_j} \qquad \text{(A.15)}
\end{aligned}$$

Here the binomial coefficient $\binom{(d-1)\|S\|}{n}$ bounds the choice for the components $\delta_1, ..., \delta_n$ of Y_b to hit the boundary layer at $s_1, ..., s_n$. The above inequality yields

$$
\begin{aligned}
\sum_{\Gamma:\operatorname{supp}\Gamma=S} \varrho(\Gamma) q^{\underline{e}\|\Gamma\|} \le & \sum_{0\le k\le \|S\|} \binom{\|S\|}{k} 2^{(2d-1)k} 2^{\|S\|-k} q^{-\frac{1-\beta_b}{2(d-1)}k} \\
& \times \sum_{n\le 2\|S\|} \binom{(d-1)\|S\|}{n} \sum_{m\ge \|S\|-k} \left(2\nu_d q^{\frac{1}{d}-\beta_b}\right)^m \\
\le \sum_{0\le k\le \|S\|} & \binom{\|S\|}{k} 2^{(2d-1)k} 2^{\|S\|-k} q^{-\frac{1-\beta_b}{2(d-1)}k} \\
& \times \left(2\nu_d q^{-\left(\frac{1}{d}-\beta_b\right)}\right)^{\|S\|-k} \frac{2^{(d-1)\|S\|}}{1-2\nu_d q^{\frac{1}{d}-\beta_b}} \qquad \text{(A.16)}
\end{aligned}
$$

that gives the inequality of the lemma.

■

We now turn to the general case of contours with non empty polymers and first give a bound on the activity $\psi(\gamma)$ of polymers.

Lemma A.3 *Assume that* $\beta_b > \frac{1}{d}$, *and* $c\nu_d^2 q^{-\frac{1}{d}-\beta_b} \le 1$ *with* $c = 8e(e-1)c_0$ *and* $\nu_d = (2d)^2$, *then*

$$
|\psi(\gamma)| \le \left(c\nu_d^2 q^{\frac{1}{d}-\beta_b}\right)^{|\gamma|} \qquad \text{(A.17)}
$$

Proof.

Let us first recall the definition (2.26): $\psi(\gamma) \equiv \sum_{A:\operatorname{supp} A=\gamma} \omega(A)$ where the weights of aggregates are defined by (see (2.23) and (2.24)): $\omega(A) = \prod_{\gamma\in A} e^{-\Phi(\gamma)} - 1$. By Theorem 2.1 we know that $|\Phi(\gamma)| \le \left(ec_0\nu_d q^{\frac{1}{d}-\beta_b}\right)^{|\gamma|}$ ($\le$ 1) for q large enough. Since for any $|x| \le 1$, $|e^{-x}-1| \le (e-1)|x|$, we have

$$
|\Psi(\gamma)| = \left|e^{-\Phi(\gamma)} - 1\right| \le (e-1)|\Phi(\gamma)| \le \left((e-1)ec_0\nu_d q^{\frac{1}{d}-\beta_b}\right)^{|\gamma|} \equiv e^{-\sigma|\gamma|} \qquad \text{(A.18)}
$$

Then,

$$\begin{aligned}
\sum_{A:\operatorname{supp} A=\gamma} |\omega(A)| &= \sum_{n\geq 1} \sum_{\gamma_1,\ldots,\gamma_n \operatorname{supp}\{\gamma_1,\ldots,\gamma_n\}=\gamma} \prod_{j=1}^{n} |\Psi(\gamma_j)| \\
&\leq \sum_{n\geq 1} 2^{|\gamma|} \sum_{\gamma_1 \ni s_1,\ldots,\gamma_n \ni s_n \operatorname{supp}\{\gamma_1,\ldots,\gamma_n\}=\gamma} \prod_{j=1}^{n} e^{-\sigma|\gamma_j|} \\
&\leq \sum_{n\geq 1} 2^{|\gamma|} \sum_{m_1,\ldots,m_n m_1+\ldots+m_n \geq |\gamma|} \prod_{j=1}^{n} \left(\nu_d e^{-\sigma}\right)^{m_j} \\
&\leq \sum_{n\geq 1} \sum_{m_1,\ldots,m_n m_1+\ldots+m_n \geq |\gamma|} \prod_{j=1}^{n} \left(2\nu_d e^{-\sigma}\right)^{m_j}
\end{aligned} \tag{A.19}$$

Here, we used as in the proof of Theorem\ref{T:CE} that the number of polymers of length m containing a given bond or a given vertex is less than ν_d^m; the term $2^{|\gamma|}$ bounds the combinatoric choice of the cells $s_j \in \gamma_j$, because γ being connected, it contains $n-1$ such intersecting cells (see [13]).

We put $k = m_1 + \ldots + m_n$ and notice that there are at most $\binom{k}{n-1}$ such numbers to get

$$\begin{aligned}
\sum_{A:\operatorname{supp} A=\gamma} |\omega(A)| &= \sum_{1\leq n\leq k} \sum_{k\geq|\gamma|} \binom{k}{n-1} \left(2\nu_d e^{-\sigma}\right)^{k} \\
&\leq \sum_{k\geq|\gamma|} \left(4\nu_d e^{-\sigma}\right)^{k} = \sum_{k\geq|\gamma|} \left(\frac{c}{2}\nu_d^2 q^{\frac{1}{d}-\beta_b}\right)^{k} \\
&\leq \frac{1}{1-\frac{c}{2}\nu_d^2 q^{-\frac{1}{d}+\beta_b}} \left(\frac{c}{2}\nu_d^2 q^{\frac{1}{d}-\beta_b}\right)^{|\gamma|}
\end{aligned} \tag{A.20}$$

provided that $\frac{c}{2}\nu_d^2 q^{\frac{1}{d}-\beta_b} < 1$. The lemma then follows by assuming that $\frac{c}{2}\nu_d^2 q^{\frac{1}{d}-\beta_b} \leq \frac{1}{2}$.

■

We finally turn to the

Proof of Proposition 3.2

Consider a contour $\Gamma = \left\{\operatorname{supp}\Gamma, \widehat{X}^\Gamma, \widehat{Y}^\Gamma\right\}$ and as above the decomposition $X^\Gamma = \widehat{X}_s^\Gamma \cup \widehat{Z}_b^\Gamma \cup \widehat{Y}_b^\Gamma$. Consider also the union $\widehat{T}_b = \widehat{Y}_b^\Gamma \cup \widehat{Y}^\Gamma$. Notice that the set $\widehat{T} = \widehat{X}_s^\Gamma \cup \widehat{Z}_b^\Gamma \cup \widehat{T}_b$ is a family of hydras and there are at most

$3^{|\widehat{T}_b|}$ contours corresponding to this family: this is because a $(d-1)$–cell in $\widehat{T}_b$ may be occupied either by $\widehat{Y}_b^\Gamma$ or by $\widehat{Y}^\Gamma$ or by both. Let

$$\widetilde{\varrho}(\widehat{T}) = \sum_{\Gamma:\widehat{Y}_b^\Gamma \cup \widehat{Y}^\Gamma = \widehat{T}} \left|\widetilde{\varrho}(\widehat{T})\right| \tag{A.21}$$

The above remark on the number of contours associated to $\widehat{T}$ and Lemma A.3 implies

$$\begin{aligned} \left|\widetilde{\varrho}(\widehat{T}) q^{\underline{e}\|\Gamma\|}\right| &\leq q^{-\frac{|I_0(\Gamma)|}{2(d-1)}} \left(3 \sup\left\{q^{\frac{1}{d}-\beta_b}, c\nu_d^2 q^{\frac{1}{d}-\beta_b}\right\}\right)^{|\widehat{T}_b|} \\ &\leq q^{-\frac{|I_0(\Gamma)|}{2(d-1)}} \left(3c\nu_d^2 q^{\frac{1}{d}-\beta_b}\right)^{|\widehat{T}_b|} \end{aligned} \tag{A.22}$$

The rest of the proof is then analog to that of Lemma A.2 starting from Lemma A.3 and replacing $q^{\frac{1}{d}-\beta_b}$ by $3c\nu_d^2 q^{\frac{1}{d}-\beta_b}$. It gives

$$\sum_{\Gamma:\operatorname{supp}\Gamma = S} |\varrho(\Gamma)|\, q^{\underline{e}\|\Gamma\|} \leq \left(2^{(3d-2)} q^{-\frac{1-\beta_b}{2(d-1)}} + 3c2^{d+1}\nu_d^3 q^{\frac{1}{d}-\beta_b}\right)^{\|S\|} \frac{1}{1-6c\nu_d^3 q^{\frac{1}{d}-\beta_b}} \tag{A.23}$$

provided $6c\nu_d^3 q^{\frac{1}{d}-\beta_b} < 1$ and ends the proof of the proposition. ■

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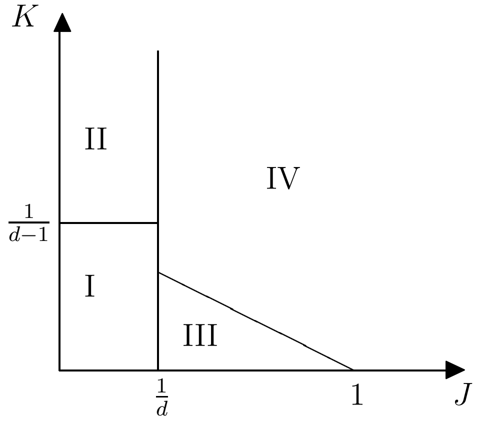
Figure captions

1. Mean field diagram borrowed from Ref. [25]

2. A configuration X (full lines) and its A-dual $\widehat{X}$ (dashed lines).

3. A hydra, in two dimensions (a dimension not considered in this paper), with 5 feet (components of full lines), 2 bodies (components of dashed lines), and 3 heads (components of dotted lines).



K
II
IV
$\frac{1}{d-1}$
I
III
$\frac{1}{d}$
1
J