The Fundamental Theorem of Calculus in $\mathbb{R}^n$

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It is shown that there is a more direct extension of the well-known Fundamental Theorem of Calculus (FTC) to the case of several variables than the one provided by the General Stokes' Theorem.

1 The Theorem

Let $C(\mathbb{R}^n)$ denote the set of contiuous real-valued functions of n real variables. For $f \in C(\mathbb{R}^n)$ and $a, b \in \mathbb{R}^n$ we define

$$\int_a^b f(x)\,\mathrm{d}x := \int_{a_n}^{b_n} \big(\ldots \big(\int_{a_1}^{b_1} f(x_1, \ldots, x_n)\,\mathrm{d}x_1 \big) \ldots \big)\,\mathrm{d}x_n \;,$$

where it is not assumed that $a_i \leq b_i$ and it is understood that reversal of boundaries changes the sign of the integral. Further we define

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) := f'(x) := \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} f(x_1, \ldots, x_n)$$

if the objects on the right hand side (i.e. the limits involved) exist and depend on x in such a way that $f' \in C(\mathbb{R}^n)$. Let $\tilde{C}(\mathbb{R}^n)$ denote the set of fs for which f' is defined in this sense. Finally we define that F is an antiderivative of $f \in C(\mathbb{R}^n)$ iff $F \in \tilde{C}(\mathbb{R}^n)$ and $F' = f$.

The *First Fundamental Theorem of Calculus* states that

$$\int_a^b f(x)\,\mathrm{d}x = F\big|_a^b \,,$$

for all $a, b \in \mathbb{R}^n$, where F is any antiderivative of f and

$$F\big|_a^b := \sum_{\varepsilon_1, \ldots, \varepsilon_n = 0}^{1} (-1)^{\varepsilon_1 + \ldots + \varepsilon_n} F(\varepsilon_1 a_1 + \bar{\varepsilon}_1 b_1, \ldots, \varepsilon_n a_n + \bar{\varepsilon}_n b_n) \;, \quad \bar{\varepsilon}_i := 1 - \varepsilon_i \;. \tag{1}$$

This sum of 2^n terms reads for $n \in 1,2,3$ as follows

$$
\begin{aligned}
&F(b_1) - F(a_1)\,,\\
&F(b_1,b_2) - F(b_1,a_2) - F(a_1,b_2) + F(a_1,a_2)\,,\\
&F(b_1,b_2,b_3) - F(b_1,a_2,b_3) - F(a_1,b_2,b_3) + F(a_1,a_2,b_3)\\
&\quad - F(b_1,b_2,a_3) + F(b_1,a_2,a_3) + F(a_1,b_2,a_3) - F(a_1,a_2,a_3)\,.
\end{aligned}
$$

The *Second Fundamental Theorem of Calculus* says that for any $a \in \mathbb{R}^n$

$$F(x) := \int_a^x f(y)\,\mathrm{d}y$$

defines an antiderivative F of f and that any antiderivative of f deviates from F at most by a sum of functions each which depending only on a strict subset of the n variables under consideration.

The differential operator $\frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n}$ is of first order with respect to each variable individually, so it is a very specific instance of a differential operator of order n. From the previously stated facts one obtains a formula for $f'(x)$ which involves only a single limit and is also useful for numerical computation

$$f'(x) = \lim{}_{h\to 0} \frac{1}{(2h)^n} f\big|_{x-(h,\ldots,h)}^{x+(h,\ldots,h)}\,.$$

2 Discussion

It is only a technical matter to proof these statements by induction, starting from their known truth for $n = 1$. A more direct evidence comes from observing that the $n = 1$ case implies all previous statements for functions of the type

$$f(x_1,\ldots,x_n) = f_1(x_1)\cdots f_n(x_n)$$

and, thus, for all finite linear combinations of such functions.

The present theorem contrasts with an opinion which is inherent in most teaching on calculus and which is comprehensively stated in

http:/mathforum.org/library/drmath/view/53755.html

as follows:

> 'In higher dimensions, there is no Fundamental Theorem of Calculus connecting multiple integrals with partial derivatives, so there isn't an "antidifferentiation" process for functions of several variables. The closest correspondence would probably [be?] the Divergence Theorem or Stokes Theorem, which connects integrals of certain ...
>
> ...Multiple integrals can often be evaluated as iterated (or repeated) one-dimensional integrals so that the usual techniques can be used, but this doesn't always work.'

Most textbooks avoid such direct statements but create the same impression by their selection of material.

The reader will probably not escape the feeling that the stated theorem is nearly trivial: In a sense, the normal FTC is simply applied to repeated one-dimensional integrals (see the citation above). However, it is not completely trivial to find the notions that allow to give this ‘repeated FTC’ the form and logical structure of the FTC with n as a variable. That this works suggests that the antiderivative in n dimensions, as defined above, is a more natural construct than it is apparently recognized to be. For instance, it is at least a nice change in perspective do understand the area of a rectangle as resulting from the antiderivative $F(x,y) = xy$ of the constant function $f(x,y) = 1$ via equation (1). Notice that F can be selected as not containing redundant constants (although $F(x,y) = xy + 137x$ would also be a valid antiderivative of 1). The repeated integral approach, by contrast, would inject problem-related constants (edge coordinates of the rectangle) into the result of the first integration. If expressions are more complicated, such constants can create in the next integration step excessively complex expressions for which todays computer algebra systems are unable to find the fitting simplifications. Especially, the programming-friendly property of equation (1) to represent the result by evaluations of the same function for varying values of the arguments will get lost in the proposed simplifications.

For a domain of integration which is a union of adjacent axis-parallel rectangular parallelepipeds, which is a useful approximation in computational physics, equation (1), when applied to each of the parallelepipeds, can easily seen to express the integral of a function as a sum of evaluations of any antiderivative at surface points. So we have a simple connection between an integral of a function over such a domain and a sum of values that the antiderivative takes on the surface. This is an exact relation that could also be obtained from the General Stokes’ Theorem. However, due to the edgy nature of our surface such an application could not be justified directly by the student’s version of the theorem which is for smooth surfaces.

Most of the considerations needed to prove this theorem appear in [1], p. 205–207, in a study of absolutely continuous functions of several variables. Another connection to well-known facts can be found in [2], p. 3–4 in a study of the partial differential equation $u_{xy} = f(x,y)$.

I came to consider the matter by the practical need (for the work [3]) to have an efficient explicit formulas for the electrostatic potential of a rectangular patch of uniform surface charge. Here, the computer algebra system *Mathematica* allowed to find Integrate[Integrate[1/r,x],y] (for $\mathrm{r} := \sqrt{\mathrm{x}^2 + \mathrm{y}^2 + \mathrm{z}^2}$, z is treated as a parameter here) and thus an antiderivative of 1/r. Then the integral of the Coulomb potential over a rectangular region of the x-y-plane follows from the ’First FTC for 2 variables’ in a form that, due to the repetitive structure of equation (1), can be programmed compactly. That the required antiderivative can also be requested in the form Integrate[1/r,x,y] (which seems to be an un-documented feature of *Mathematica* so far) shows that the author of the program is aware of the concept of an antiderivative with respect to several variables. Since there seems to be no automatic way for *Mathematica* to produce a purely real result for the antiderivative of 1/r, one has to interfere manually which could introduce errors. Here it helps that the correctness of an expression for the antiderivative can easily be checked by differentiation.

References

[1] W.L. Smirnow: Lehrgang der Höheren Mathematik Teil V, Deutscher Verlag der Wissenschaften, Berlin 1962

[2] R. Courant and D. Hilbert: Methods of Mathematical Physics, Volume II Partial Differential Equations, Interscience Publishers, New York and London 1962

[3] U. Mutze, E. Stelter, T. Dera: Simulation of Electrophotographic Development, Final Program and Proceedings of IS& T's NIP19: International Conference on Digital Printing Technologies September 28 - October 3, 2003 (IS& T: The Society for Imaging Science and Technology, Springfield VA, 2003) p.57.