$\mathcal{L}^r$-CONVERGENCE OF A RANDOM SCHRÖDINGER TO A LINEAR BOLTZMANN EVOLUTION

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Abstract. We study the macroscopic scaling and weak coupling limit for a random Schrödinger equation on $\mathbb{Z}^3$. We prove that the Wigner transforms of a large class of "macroscopic" solutions converge in r-th mean to solutions of a linear Boltzmann equation, for any finite value of $r \in \mathbb{R}_+$. This extends previous results where convergence in expectation was established.

1. Introduction

We consider the quantum mechanical dynamics of an electron against a background lattice of impurity ions exhibiting randomly distributed interaction strengths. Models of this type (Anderson model) are widely used to investigate qualitative features of technically highly relevant classes of materials that comprise semiconductors. Questions of key mathematical interest, treated intensively in the literature, address the emergence of electric conduction and insulation. While there exist landmark mathematical results explaining disorder-induced insulation at strong coupling (Anderson localization, [1, 6]), the weak coupling regime is at present far less understood. In the latter context, we shall here analyze issues regarding the derivation of macroscopic transport equations.

We study the macroscopic scaling and weak coupling limit of the quantum dynamics generated by the Hamiltonian

$$H_\omega = -\Delta + \lambda \sum_{y \in \mathbb{Z}^3} \omega_y \delta(x-y)$$

on $\ell^2(\mathbb{Z}^3)$. Here, Δ is the nearest neighbor discrete Laplacian, $0 < \lambda \ll 1$ is a small coupling constant that defines the disorder strength, and ω_y are independent, identically distributed Gaussian random variables.

Let $\phi_t \in \ell^2(\mathbb{Z}^3)$ be the solution of the Schrödinger equation

$$\left\{ \begin{array}{rcl} i\partial_t \phi_t & = & H_\omega \phi_t \\ \phi_0 & \in & \ell^2(\mathbb{Z}^3) \end{array} \right. , \tag{1}$$

with a non-random initial condition ϕ_0 which is supported on a region of diameter $O(\lambda^{-2})$. Let $W_{\phi_t}(x,v)$ denote its *Wigner transform*, where $x \in \mathbb{Z}^3$, and $v \in [0,1]^3$. We consider a scaling for small λ, in which $(T,X) := \lambda^2(t,x)$, $V := v$, are the macroscopic time, position, and velocity variables (while (t,x,v) are the microscopic ones). We then focus on a corresponding, appropriately rescaled version $W_\lambda^{resc}(T,X,V)$ of $W_{\phi_t}(x,v)$.

It was proved by Erdös and Yau for the continuum case, [4, 3], and by the author for the lattice case, [2], that for all test functions $J(X,V)$, and globally in macroscopic time T,

$$\lim_{\lambda\to 0}\mathbb{E}\Big[\int dXdV J(X,V)W_\lambda^{resc}(T,X,V)\Big] \\ = \int dXdV J(X,V)F(T,X,V)\;,$$

where $F(T,X,V)$ is the solution of a linear Boltzmann equation. The corresponding local in T result was achieved much earlier by Spohn, [8].

Our main result in this paper improves the mode of convergence in that we establish convergence in r-th mean for every finite $r\in\mathbb{R}_+$,

$$\lim_{\lambda\to 0}\mathbb{E}\Big[\Big|\int dXdV J(X,V)W_\lambda^{resc}(T,X,V) \\ -\int dXdV J(X,V)F(T,X,V)\Big|^r\Big] = 0\;.$$

Hence in particular, the variance of $\int dXdV J(X,V)W_\lambda^{resc}(T,X,V)$ vanishes in this limit. As an immediate corollary, one obtains convergence in probability, and other standard modes of convergence.

The proof comprises generalizations and extensions to the graph expansion method introduced by Erdös and Yau in [4, 3], and further elaborated on in [2]. The structure of graphs entering the problem at hand is significantly more complicated. But as it turns out, it is not very hard to show that the corresponding amplitudes are extremely small. The strategy then consists of balancing these small amplitudes against the very large number of graphs, which is similar to the approach in the previous works [4, 3, 2].

The present work, as well as [2] and [4, 3], address a time scale of order $O(\lambda^{-2})$, in which ballistic behavior is prevalent, because the average number of collisions experienced by the electron is finite. For this reason, the macroscopic dynamics is governed by a Boltzmann equation. Beyond this time scale, the average number of collisions is *infinite*, and the problem becomes much harder. Very recently, Erdös, Salmhofer and Yau have established that in a time scale of order $O(\lambda^{-2-\kappa})$ for an explicit numerical value of $\kappa>0$, the macroscopic dynamics in $d=3$ derived from the quantum dynamics is determined by a *diffusion equation*, [5]. This is a breakthrough result which is the first of its kind.

We note that control of the macroscopic dynamics up to a time scale $O(\lambda^{-2})$ allows to obtain lower bounds of a comparable scale on the localization lengths of eigenvectors of H_ω in 3 dimensions, for small λ, [2]. This extends recent results of Shubin, Schlag and Wolff, [7], who derived lower bounds of this type for the weakly disordered Anderson model in dimensions $d=1,2$ by use of techniques of harmonic analysis.

This work comprises a partial joint result with Laszlo Erdös (Lemma 5.1), to whom the author is deeply grateful for his support and generosity.

2. Definition of the model and statement of main results

We consider the discrete random Schrödinger operator

$$H_\omega = -\Delta + \lambda V_\omega \tag{2}$$

on $\ell^2(\mathbb{Z}^3)$, with potential function

$$V_\omega(x) = \sum_{y\in\mathbb{Z}^3} \omega_y \delta(x-y) \ , \tag{3}$$

where ω_y are independent, identically distributed Gaussian random variables satisfying $\mathbb{E}[\omega_x] = 0$, $\mathbb{E}[\omega_x^2] = 1$, for all $x \in \mathbb{Z}^3$. Expectations of higher powers of ω_x satisfy Wick's theorem, cf. [4], and our discussion below.

We use the convention

$$\hat{f}(k) = \sum_{x\in\mathbb{Z}^3} e^{-2\pi i k\cdot x} f(x) \quad , \quad \check{g}(x) = \int_{\mathbb{T}^3} dk\; g(k) e^{2\pi i k\cdot x} \tag{4}$$

for the Fourier transform and its inverse. In frequency space, the nearest neighbor lattice Laplacian defines the multiplication operator

$$(-\Delta f)\hat{}(k) = e(k)\hat{f}(k) \ , \tag{5}$$

where

$$\begin{aligned} e(k) &= 2\sum_{i=1}^{3} \big(1-\cos(2\pi k_i)\big) \\ &= 4\sum_{i=1}^{3} \sin^2(\pi k_i) \end{aligned} \tag{6}$$

is the quantum mechanical kinetic energy of the electron at momentum k.

Let $\phi_t \in \ell^2(\mathbb{Z}^3)$ denote the solution of the random Schrödinger equation

$$\begin{cases} i\partial_t \phi_t &= H_\omega \phi_t \\ \phi_0 &\in \ell^2(\mathbb{Z}^3) \ , \end{cases} \tag{7}$$

for a fixed realization of the random potential. Its (real, but not necessarily positive) Wigner transform $W_{\phi_t} : \mathbb{Z}^3 \times \mathbb{T}^3 \to \mathbb{R}$ is defined by

$$W_{\phi_t}(x,v) = \sum_y \overline{\phi_t(x+\frac{y}{2})}\phi_t(x-\frac{y}{2}) e^{2\pi i y v} \ , \tag{8}$$

and we note that Fourier transformation with respect to the variable x yields

$$\hat{W}_{\phi_t}(\xi,v) = \overline{\hat{\phi}_t(v+\frac{\xi}{2})}\hat{\phi}_t(v-\frac{\xi}{2}) \ , \tag{9}$$

for $\xi \in \mathbb{T}^3$.

The Wigner transform will serve as our key tool in the derivation of the macroscopic limit for the quantum dynamics described by (15). For this purpose, we introduce macroscopic variables $T := \epsilon t$, $X := \epsilon x$, $V := v$, and consider the rescaled Wigner transform

$$W^{\varepsilon}_{\phi_t}(X,V) := \varepsilon^{-3} W_{\phi_t}(\varepsilon^{-1}X, V) \tag{10}$$

for $T \geq 0$, $X \in (\varepsilon\mathbb{Z})^3$, and $V \in \mathbb{T}^3$.

For a Schwartz class function $J \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{T}^3)$, we write

$$\langle J, W^{\varepsilon}_{\phi_t} \rangle := \sum_{X \in (\varepsilon\mathbb{Z})^3} \int_{\mathbb{T}^3} dV \overline{J(X,V)} W^{\varepsilon}_{\phi_t}(X,V) \;, \tag{11}$$

where $J_\varepsilon(x,v) := J(\varepsilon x, v)$. Let $\hat{W}^{\varepsilon}_{\phi_t}$ be defined as in (9). Then,

$$\begin{aligned} (\ 11) &= \langle \hat{J}, \hat{W}^{\varepsilon}_{\phi_t} \rangle \\ &= \int_{\mathbb{T}^3 \times \mathbb{T}^3} d\xi dv \overline{\hat{J}_\varepsilon(\xi, v)} \hat{W}_{\phi_t}(\xi, v) \;, \end{aligned} \tag{12}$$

where we have defined

$$\hat{J}_\varepsilon(\xi, v) \;:=\; \varepsilon^{-3} \hat{J}(\xi/\varepsilon, v) \;, \tag{13}$$

which is a smooth delta peak of width ε with respect to the ξ-variable, and uniformly bounded with respect to ε in the v-variable.

The macroscopic limit under this scaling is determined by the linear Boltzmann equations, as was proven in [2] for the 3-dimensional lattice model, and non-Gaussian distributed random potentials (the Gaussian case follows also from [2]). The corresponding result for the 2- and 3-dimensional continuum model was proven in [4].

Theorem 2.1. *For $\varepsilon > 0$, we consider the initial data*

$$\phi^{\varepsilon}_0(x) := \varepsilon^{\frac{3}{2}} h(\varepsilon x) e^{\frac{iS(\varepsilon x)}{\varepsilon}} \;, \tag{14}$$

where h and S are Schwartz class functions on $\mathbb{R}^3$. Let ϕ^{ε}_t be the solution of the random Schrödinger equation

$$i\partial_t \phi^{\varepsilon}_t = H_\omega \phi^{\varepsilon}_t \tag{15}$$

with initial condition given by ϕ^{ε}_0, and let

$$W^{(\varepsilon)}_T(X,V) := W^{\varepsilon}_{\phi^{\varepsilon}_{\varepsilon^{-1}T}}(X,V) \tag{16}$$

denote its corresponding rescaled Wigner transform. If the scaling factor ε is set equal to $\varepsilon = \lambda^2$, where λ is the coupling constant in (2), it follows that

$$\lim_{\lambda \to 0} \mathbb{E}\big[\langle J, W^{(\lambda^2)}_T \rangle\big] = \langle J, F_T \rangle \;, \tag{17}$$

where $F_T(X,V)$ solves the linear Boltzmann equation

$$\begin{aligned}\partial_T F_T(X,V) + \sum_{j=1}^{3} (\sin 2\pi V_j)\partial_{X_j} F_T(X,V) \\ = \int_{\mathbb{T}^3} dU \sigma(U,V) \left[F_T(X,U) - F_T(X,V)\right] \ .\end{aligned} \tag{18}$$

The collision kernel is given by

$$\sigma(U,V) := 4\pi\delta(e(U) - e(V)) \ .$$

and the initial condition by

$$F_0(X,V) = |h(X)|^2 \delta(V - \nabla S(X)) \ , \tag{19}$$

which is the weak limit of $W^{\varepsilon}_{\phi^{\varepsilon}_0}$ as $\varepsilon \to 0$.

The present work aims at an extension of that result by significantly improving the mode of convergence. Our main result is the following theorem.

Theorem 2.2. *For any fixed, finite $r \in 2\mathbb{N}$, $T > 0$, and for any Schwartz class function J, the estimate*

$$\left(\mathbb{E}\left[\left| \left\langle J, W_T^{(\lambda^2)} \right\rangle - \mathbb{E}\left[\left\langle J, W_T^{(\lambda^2)} \right\rangle \right] \right|^r \right] \right)^{\frac{1}{r}} \leq c(r,T) \lambda^{\frac{1}{300r}} \tag{20}$$

is satisfied for λ sufficiently small, and a constant $c(r,T)$ that does not depend on λ. Consequently, convergence in r-th mean,

$$\lim_{\lambda \to 0} \mathbb{E}\left[\left| \left\langle J, W_T^{(\lambda^2)} \right\rangle - \left\langle J, F_T \right\rangle \right|^r \right] = 0 \ , \tag{21}$$

holds for any finite $r, T \in \mathbb{R}_+$.

We remark that in particular, the variance of $\left\langle J, W_T^{(\lambda^2)} \right\rangle$ vanishes in the limit $\lambda \to 0$. We shall here not list further standard modes of convergence implied by (21), which can be found in textbooks on probability theory, but only state the most important implication in the following corollary.

Corollary 2.1. *The rescaled Wigner transform $W_T^{(\lambda^2)}$ convergences in probability weakly to a solution of the linear Boltzmann equations, globally in $T > 0$, as $\lambda \to 0$. That is, for any Schwartz class function J, and any finite $T > 0$,*

$$\lim_{\nu \to 0} \mathbb{P}\left[\lim_{\lambda \to 0} \left| \left\langle J, W_T^{(\lambda^2)} \right\rangle - \left\langle J, F_T \right\rangle \right| > \nu \right] = 0 \ , \tag{22}$$

where F_T solves (18) with initial condition (19).

3. Proof of Theorem 2.2

To begin with, we expand ϕ_t into the truncated Duhamel series

$$\phi_t = \sum_{n=0}^{N} \phi_{n,t} + R_{N,t} \ , \tag{23}$$

where

$$\begin{aligned} \phi_{n,t} \ &:= \ (-i\lambda)^n \int_{\mathbb{R}_+^{n+1}} ds_0 \cdots ds_n \delta(\sum_{j=0}^{n} s_j - t) \\ &\quad \times e^{is_0\Delta} V_\omega e^{is_1\Delta} \cdots V_\omega e^{is_n\Delta} \phi_0 \end{aligned} \tag{24}$$

denotes the n-th Duhamel term, and where

$$R_{N,t} = -i\lambda \int_0^t ds e^{-i(t-s)H_\omega} V_\omega \phi_{N,s} \tag{25}$$

is the remainder term. The number N remains to be optimized. In frequency space,

$$\begin{aligned} \hat{\phi}_{n,t}(k_0) \ &= \ (-i\lambda)^n \int ds_0 \cdots ds_n \delta(\sum_{j=0}^{n} s_j - t) \\ &\quad \times \int_{(\mathbb{T}^3)^n} dk_1 \cdots dk_n e^{-is_0 e(k_0)} \hat{V}_\omega(k_1 - k_0) e^{-is_1 e(k_1)} \cdots \\ &\quad \cdots \hat{V}_\omega(k_n - k_{n-1}) e^{-is_n e(k_n)} \hat{\phi}_0(k_n) \ . \end{aligned} \tag{26}$$

Representing the delta distribution by an oscillatory integral, we find

$$\begin{aligned} \hat{\phi}_{n,t}(k_0) \ &= \ (-i\lambda)^n e^{\varepsilon t} \int_{\mathbb{R}} d\alpha e^{-it\alpha} \\ &\quad \times \int_{(\mathbb{T}^3)^n} dk_1 \cdots dk_n \frac{1}{e(k_0) - \alpha - i\varepsilon} \hat{V}_\omega(k_1 - k_0) \\ &\quad \cdots \hat{V}_\omega(k_n - k_{n-1}) \frac{1}{e(k_n) - \alpha - i\varepsilon} \hat{\phi}_0(k_n) \ . \end{aligned} \tag{27}$$

The function $\frac{1}{e(k)-\alpha-i\varepsilon}$ is referred to as a *particle propagator*, corresponding to the frequency space representation of the resolvent $\frac{1}{-\Delta-\alpha-i\varepsilon}$. Likewise, we note that (27) is equivalent to the n-th term in the resolvent expansion of

$$\phi_t = \frac{1}{2\pi i} \int_{-i\varepsilon+\mathbb{R}} dz e^{-itz} \frac{1}{H_\omega - z} \phi_0 \ . \tag{28}$$

By analyticity of the integrand in (27) with respect to the variable α, and due to its decay properties as $Im(z) \to -\infty$, the path of the α-integration can, for any fixed $n \in \mathbb{N}$, be deformed away from $\mathbb{R}$ into the closed contour

$$I = I_0 \cup I_1 \tag{29}$$

with

$$
\begin{aligned}
I_0 &:= [-1, 13] \\
I_1 &:= ([-1, 13] - i) \cup (-1 - i(0, 1]) \cup (13 - i(0, 1]) \; ,
\end{aligned}
$$

which encloses $\mathrm{spec}\big(-\Delta - i\varepsilon \big) = [0, 12] - i\varepsilon$.

The initial condition $\phi_0 \equiv \phi_0^\varepsilon$ in the random Schrödinger equation (15), as characterized in (14), satisfies

$$
|\hat{\phi}_0(k)| = \sqrt{\delta_\varepsilon(q - k)} \; , \tag{30}
$$

for some $q \in \mathbb{T}^3$, where δ_ε is a smooth bump function localized in a ball of radius ε, normalized by $\int_{\mathbb{T}^3} dk \, \delta_\varepsilon(k) = 1$. In particular,

$$
\begin{aligned}
\| f \hat{\phi}_0 \|_{L^1(\mathbb{T}^3)} &< c\varepsilon^{\frac{3}{2}} \| f \|_{L^\infty(\mathbb{T}^3)} \; , \\
\| \hat{\phi}_0 \|_{L^2(\mathbb{T}^3)} &= 1 \; ,
\end{aligned} \tag{31}
$$

for any $f \in L^\infty(\mathbb{T}^3)$.

Applying the partial time integration method introduced in [4], we choose $\kappa \in \mathbb{N}$ with $1 \ll \kappa \ll \varepsilon^{-1}$, and subdivide $[0, t]$ into κ subintervals bounded by the equidistant points $\theta_j = \frac{jt}{\kappa}$, where $j = 1, \dots, \kappa$. Then,

$$
R_{N,t} = -i\lambda \sum_{j=0}^{\kappa-1} e^{-i(t-\theta_{j+1})H_\omega} \int_{\theta_j}^{\theta_{j+1}} ds \, e^{-isH_\omega} V_\omega \phi_{N,s} \; . \tag{32}
$$

Let $\phi_{n,N,\theta}(s)$ denote the n-th Duhamel term conditioned on the requirement that the first N collisions occur in the time interval $[0, \theta]$, and all remaining $n - N$ collisions take place in the time interval $(\theta, s]$. Further expanding e^{-isH_ω} into a truncated Duhamel series with $3N$ terms, we find

$$
R_{N,t} = R_{N,t}^{(<4N)} + R_{N,t}^{(4N)} \; , \tag{33}
$$

where

$$
R_{N,t}^{(<4N)} = -i\lambda \sum_{n=N+1}^{4N-1} \tilde{\phi}_{n,N,t} \; , \tag{34}
$$

$$
\tilde{\phi}_{n,N,t} := -i\lambda \sum_{j=1}^{\kappa} e^{-i(t-\theta_j)H_\omega} V_\omega \phi_{n,N,\theta_{j-1}}(\theta_j) \tag{35}
$$

and

$$
R_{N,t}^{(4N)} = -i\lambda \sum_{j=1}^{\kappa} e^{-i(t-\theta_j)H_\omega} \int_{\theta_{j-1}}^{\theta_j} ds \; e^{-i(\theta_j - s)H_\omega} V_\omega \phi_{4N,N,\theta_{j-1}}(s) \; . \tag{36}
$$

Clearly, the Schwarz inequality implies that

$$
\| R_{N,t}^{(<4N)} \|_2 \leq (3N\kappa) \sup_{N<n<4N, 1\leq j\leq\kappa} \| \lambda V_\omega \phi_{n,N,\theta_{j-1}}(\theta_j) \|_2 \tag{37}
$$

and

$$\|R^{(4N)}_{N,t}\|_2 \leq t \sup_{1\leq j\leq\kappa} \sup_{s\in[\theta_{j-1},\theta_j]} \|\lambda V_\omega \phi_{4N,N,\theta_{j-1}}(s)\|_2 \; , \tag{38}$$

for every fixed realization of the random potential V_ω.

Let $r \in 2\mathbb{N}$, and let

$$W_{t;n_1,n_2}(x,v) := \sum_{y\in\mathbb{Z}^3} \overline{\psi_{n_2,t}(x+\frac{y}{2})}\psi_{n_1,t}(x-\frac{y}{2})e^{2\pi i y\cdot v} \tag{39}$$

denote the (n_1, n_2)-th term in the Wigner distribution, with

$$\psi_{n,t} := \begin{cases} \phi_{n,t} & \text{if } n \leq N \\ \tilde{\phi}_{n,N,t} & \text{if } N < n < 4N \\ R^{(4N)}_{N,t} & \text{if } n = 4N \; . \end{cases} \tag{40}$$

We note that Fourier transformation with respect to x yields

$$\hat{W}_{t;n_1,n_2}(\xi,v) = \overline{\hat{\psi}_{n_2,t}(v+\frac{\xi}{2})}\hat{\psi}_{n_1,t}(v-\frac{\xi}{2}) \; , \tag{41}$$

similarly as in (9). Then, clearly,

$$\begin{aligned} &\Big(\mathbb{E}\Big[\big(\langle \hat{J}_\varepsilon, \hat{W}_{\phi_t}\rangle - \mathbb{E}\langle \hat{J}_\varepsilon, \hat{W}_{\phi_t}\rangle\big)^r\Big]\Big)^{\frac{1}{r}} \\ &\leq \sum_{n_1,n_2=0}^{4N} \Big(\mathbb{E}\Big[\big|\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,n_2}\rangle - \mathbb{E}\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,n_2}\rangle\big|^r\Big]\Big)^{\frac{1}{r}} \; , \end{aligned} \tag{42}$$

and corresponding to $n_i \leq N$, $N < n_i < 4N$, $n_i = 4N$ for $i = 1, 2$, we distinguish the following cases, of which each will require a different method of estimation.

If $n_1, n_2 \leq N$, we note that

$$\begin{aligned} &\Big(\mathbb{E}\Big[\big|\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,n_2}\rangle - \mathbb{E}\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,n_2}\rangle\big|^r\Big]\Big)^{\frac{1}{r}} \\ &\quad = \Big(\mathbb{E}_{2-conn}\Big[\big|\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,n_2}\rangle\big|^r\Big]\Big)^{\frac{1}{r}} \\ &\quad = \Big(\mathbb{E}_{2-conn}\Big[\Big(\Big|\int d\xi dv \hat{J}_\varepsilon(\xi,v)\overline{\phi_{n_2,t}(v+\frac{\xi}{2})}\phi_{n_1,t}(v-\frac{\xi}{2})\Big|^2\Big)^{\frac{r}{2}}\Big]\Big)^{\frac{1}{r}} \; , \end{aligned} \tag{43}$$

where $\mathbb{E}_{2-conn}$ denotes the expectation based on *2-connected graphs*, cf. Definition 5.1 below.

If $N < n_i \leq 4N$ for at least one value of i, we use

$$\begin{aligned} |\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,n_2}\rangle| &= \Big|\int d\xi \, d\eta \hat{J}_\varepsilon(\xi,\eta)\overline{\psi_{n_2,t}(\xi-\frac{\eta}{2})}\psi_{n_1,t}(\xi+\frac{\eta}{2})\Big| \\ &\leq \Big(\int d\xi \sup_\eta |\hat{J}_\varepsilon(\xi,\eta)|\Big)\|\psi_{n_1,t}\|_2\|\psi_{n_2,t}\|_2 \; . \end{aligned} \tag{44}$$

Then, for a constant C which is independent of ε, the following estimates are straightforward.

If $n_{i_1} \leq N$ while $N < n_{i_2} \leq 4N$, we use the Schwarz inequality in the form

$$
\begin{aligned}
&\Big(\mathbb{E}\Big[\big|\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,n_2}\rangle - \mathbb{E}\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,n_2}\rangle\big|^r\Big]\Big)^{\frac{1}{r}} \\
&\quad \leq C\Big\{\Big(\mathbb{E}\Big[\|\psi_{n_1,t}\|_2^r\|\psi_{n_2,t}\|_2^r\Big]\Big)^{\frac{1}{r}} + \mathbb{E}\Big[\|\psi_{n_1,t}\|_2\|\psi_{n_2,t}\|_2\Big]\Big\} \\
&\quad \leq C\Big\{\Big(\mathbb{E}\Big[\|\psi_{n_1,t}\|_2^{2r}\Big]\Big)^{\frac{1}{2r}}\Big(\mathbb{E}\Big[\|\psi_{n_2,t}\|_2^{2r}\Big]\Big)^{\frac{1}{2r}} \\
&\qquad\qquad + \Big(\mathbb{E}\Big[\|\psi_{n_1,t}\|_2^2\Big]\mathbb{E}\Big[\|\psi_{n_2,t}\|_2^2\Big]\Big)^{\frac{1}{2}}\Big\}\,.
\end{aligned} \tag{45}
$$

Thus, in the case $n_1 \leq N$, $N < n_2 < 4N$,

$$
\begin{aligned}
&\Big(\mathbb{E}\Big[\big|\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,n_2}\rangle - \mathbb{E}\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,n_2}\rangle\big|^r\Big]\Big)^{\frac{1}{r}} \\
&\quad \leq C\kappa\Big\{\sup_j \Big(\mathbb{E}\Big[\|\phi_{n_1,t}\|_2^{2r}\Big)^{\frac{1}{2r}}\Big(\mathbb{E}\Big[\|\phi_{n_2,N,\theta_{j-1}}(\theta_j)\|_2^{2r}\Big]\Big)^{\frac{1}{2r}} \\
&\qquad\qquad + \sup_j \Big(\mathbb{E}\Big[\|\phi_{n_1,t}\|_2^2\Big]\mathbb{E}\Big[\|\phi_{n_2,N,\theta_{j-1}}(\theta_j)\|_2^2\Big]\Big)^{\frac{1}{2}}\Big\}\,.
\end{aligned} \tag{46}
$$

while for $n_1 \leq N$, $n_2 = 4N$,

$$
\begin{aligned}
&\Big(\mathbb{E}\Big[\big|\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,4N}\rangle - \mathbb{E}\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,4N}\rangle\big|^r\Big]\Big)^{\frac{1}{r}} \\
&\quad \leq Ct\Big\{\sup_j \sup_{s\in[\theta_{j-1},\theta_j]} \Big(\mathbb{E}\Big[\|\phi_{n_1,t}\|_2^{2r}\Big)^{\frac{1}{2r}}\Big(\mathbb{E}\Big[\|\phi_{4N,N,\theta_{j-1}}(s)\|_2^{2r}\Big]\Big)^{\frac{1}{2r}} \\
&\qquad\qquad + \sup_j \sup_{s\in[\theta_{j-1},\theta_j]} \Big(\mathbb{E}\Big[\|\phi_{n_1,t}\|_2^2\Big]\mathbb{E}\Big[\|\phi_{4N,N,\theta_{j-1}}(s)\|_2^2\Big]\Big)^{\frac{1}{2}}\Big\}\,.
\end{aligned} \tag{47}
$$

Next, if $N < n_1, n_2 \leq 4N$, we use the Schwarz inequality in the form

$$
\begin{aligned}
&\Big(\mathbb{E}\Big[\big|\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,n_2}\rangle - \mathbb{E}\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,n_2}\rangle\big|^r\Big]\Big)^{\frac{1}{r}} \\
&\quad \leq C\Big\{\Big(\mathbb{E}\Big[\Big(\|\psi_{n_1,t}\|_2 + \|\psi_{n_2,t}\|_2\Big)^{2r}\Big]\Big)^{\frac{1}{r}} + \mathbb{E}\Big[\|\psi_{n_1,t}\|_2\|\psi_{n_2,t}\|_2\Big]\Big\} \\
&\quad \leq C\sum_{j=1}^{2}\Big\{\Big(\mathbb{E}\Big[\|\psi_{n_j,t}\|_2^{2r}\Big]\Big)^{\frac{1}{r}} + \mathbb{E}\Big[\|\psi_{n_j,t}\|_2^2\Big]\Big\}\,.
\end{aligned} \tag{48}
$$

Hence, for $N < n_1, n_2 \leq 4N$,

$$\sum_{N<n_1,n_2\leq 4N} \Big(\mathbb{E}\Big[\big|\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,n_2}\rangle - \mathbb{E}\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,n_2}\rangle\big|^r\Big]\Big)^{\frac{1}{r}}$$
$$\leq C(N\kappa)^2\Big\{\sup_{n,j}\Big(\mathbb{E}\Big[\|\lambda V_\omega \phi_{n,N,\theta_{j-1}}(\theta_j)\|_2^{2r}\Big)^{\frac{1}{r}} + \sup_{n,j}\mathbb{E}\Big[\|\lambda V_\omega \phi_{n,N,\theta_{j-1}}(\theta_j)\|_2^2\Big]\Big\}$$
$$+Ct^2\Big\{\sup_j \sup_{s\in[\theta_{j-1},\theta_j]}\Big(\mathbb{E}\Big[\|\phi_{4N,N,\theta_{j-1}}(s)\|_2^{2r}\Big]\Big)^{\frac{1}{r}} + \sup_j \sup_{s\in[\theta_{j-1},\theta_j]}\mathbb{E}\Big[\|\phi_{4N,N,\theta_{j-1}}(s)\|_2^2\Big]\Big\}\ . \tag{49}$$

We shall next use Lemmata 4.1, 4.2, and 4.3 to bound the sums over the above contributions to $\langle \hat{J}_\varepsilon, \hat{W}_{\phi_t}\rangle$.

From Lemma 4.1, and $((nr)!)^{\frac{1}{r}} < n^n r^n$, one obtains

$$\sum_{n_1,n_2\leq N}\Big(\mathbb{E}_{2-conn}\Big[\big|\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,n_2}\rangle\big|^r\Big]\Big)^{\frac{1}{r}}$$
$$\leq C\varepsilon N^{N+2}(\log\frac{1}{\varepsilon})^3(cr\lambda^2\varepsilon^{-1}\log\frac{1}{\varepsilon})^N\ , \tag{50}$$

From (60) and Lemma 4.2,

$$\Big(\sum_{\substack{n_1\leq N\\ N<n_2<4N}} + \sum_{\substack{n_2\leq N\\ N<n_1<4N}}\Big)\Big(\mathbb{E}\Big[\big|\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,n_2}\rangle - \mathbb{E}\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,n_2}\rangle\big|^r\Big]\Big)^{\frac{1}{r}}$$
$$\leq \Big(C\kappa N^{N+3}(\log\frac{1}{\varepsilon})^3(cr\lambda^2\varepsilon^{-1}\log\frac{1}{\varepsilon})^N \times\Big(\frac{(c\lambda^2\varepsilon^{-1})^{4N}}{\sqrt{N!}} + (4N)^{4N}\Big(\varepsilon^{\frac{1}{6}}+\varepsilon^{\frac{3}{r}}\Big)(\log\frac{1}{\varepsilon})^3(cr\lambda^2\varepsilon^{-1}\log\frac{1}{\varepsilon})^{4N}\Big)\Big)^{\frac{1}{2}}\ . \tag{51}$$

From (60) and Lemma 4.3,

$$\Big(\sum_{\substack{n_1\leq N\\ n_2=4N}} + \sum_{\substack{n_2\leq N\\ n_1=4N}}\Big)\Big(\mathbb{E}\Big[\big|\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,4N}\rangle - \mathbb{E}\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,4N}\rangle\big|^r\Big]\Big)^{\frac{1}{r}}$$
$$\leq \Big(CtN^{5N+1}\kappa^{-2N}(\log\frac{1}{\varepsilon})^6(cr\lambda^2\varepsilon^{-1}\log\frac{1}{\varepsilon})^{5N}\Big)^{\frac{1}{2}}\ . \tag{52}$$

Finally, from Lemmata 4.2 and 4.3,

$$\begin{aligned}
&\sum_{N<n_1,n_2\leq 4N}\Big(\mathbb{E}\Big[\big|\langle\hat{J}_\varepsilon,\hat{W}_{t;n_1,n_2}\rangle-\mathbb{E}\langle\hat{J}_\varepsilon,\hat{W}_{t;n_1,n_2}\rangle\big|^r\Big]\Big)^{\frac{1}{r}}\\
&\quad\leq\ C(N\kappa)^2\frac{(c\lambda^2\varepsilon^{-1})^{4N}}{\sqrt{N!}}\\
&\quad\quad+C(N\kappa)^2(4N)^{4N}\Big(\varepsilon^{\frac{1}{6}}+\varepsilon^{\frac{3}{r}}\Big)(\log\frac{1}{\varepsilon})^3(c\lambda^2\varepsilon^{-1}\log\frac{1}{\varepsilon})^{4N}\\
&\quad\quad+C\varepsilon^{-2}\kappa^{-2N}(4N)^{4N}(\log\frac{1}{\varepsilon})^3(cr\lambda^2\varepsilon^{-1}\log\frac{1}{\varepsilon})^{4N}\ .
\end{aligned}\tag{53}$$

Consequently, for a choice of parameters

$$\begin{aligned}
\varepsilon &= \frac{1}{t}=\frac{\lambda^2}{T}\\
N &= \lfloor\frac{\log\frac{1}{\varepsilon}}{100r\log\log\frac{1}{\varepsilon}}\rfloor\\
\kappa &= \lceil(\log\frac{1}{\varepsilon})^{150r}\rceil\ ,
\end{aligned}\tag{54}$$

we find, for sufficiently small ε,

$$\begin{aligned}
\varepsilon^{-\frac{1}{70r}}<N^N &< \varepsilon^{-\frac{1}{100r}}\\
(4N)^{4N} &< \varepsilon^{-\frac{1}{20r}}\\
(cr\lambda^2\varepsilon^{-1}\log\frac{1}{\varepsilon})^{4N} &< \varepsilon^{-\frac{1}{50r}}\\
\frac{(c\lambda^2t)^{4N}}{\sqrt{N!}} &< \varepsilon^{\frac{1}{25r}}\\
\kappa^{-2N} &\leq \varepsilon^3\ .
\end{aligned}\tag{55}$$

It is hence easy to verify that for sufficiently small ε,

$$\begin{aligned}
(\ 50) &< (\log\frac{1}{\varepsilon})^{10}\varepsilon\,\varepsilon^{-\frac{1}{100r}}\varepsilon^{\frac{1}{50r}}<\varepsilon^{\frac{1}{2}}\\
(\ 51) &< \Big((\log\frac{1}{\varepsilon})^{10+150r}\varepsilon^{-\frac{1}{50r}}\Big(\varepsilon^{\frac{1}{25r}}+\varepsilon^{-\frac{1}{20r}}(\varepsilon^{\frac{1}{6}}+\varepsilon^{\frac{3}{r}})\varepsilon^{-\frac{1}{50r}}\Big)\Big)^{\frac{1}{2}}<\varepsilon^{\frac{1}{120r}}\\
(\ 52) &< \Big((\log\frac{1}{\varepsilon})^{10}\varepsilon^{-1-\frac{1}{20r}}\varepsilon^3\varepsilon^{-\frac{1}{50r}}\Big)^{\frac{1}{2}}<\varepsilon^{\frac{1}{2}}\\
(\ 53) &< (\log\frac{1}{\varepsilon})^{2+300r}\varepsilon^{\frac{1}{25r}}+(\log\frac{1}{\varepsilon})^{6+300r}\varepsilon^{-\frac{1}{20r}}(\varepsilon^{\frac{1}{6}}+\varepsilon^{\frac{3}{r}})\varepsilon^{-\frac{1}{50r}}\\
&\quad+(\log\frac{1}{\varepsilon})^3\varepsilon^{-2}\varepsilon^3\varepsilon^{-\frac{1}{20r}}\varepsilon^{-\frac{1}{50r}}<\varepsilon^{\frac{1}{30r}}\ .
\end{aligned}\tag{56}$$

Collecting all of the above,

$$\Big(\mathbb{E}\Big[\big|\langle\hat{J}_\varepsilon,\hat{W}_{\phi_t}\rangle-\mathbb{E}\langle\hat{J}_\varepsilon,\hat{W}_{\phi_t}\rangle\big|^r\Big]\Big)^{\frac{1}{r}}<\varepsilon^{\frac{1}{150r}}\ ,\tag{57}$$

for $\lambda = \sqrt{\varepsilon T}$. Hence in particular, using Theorem 2.1,

$$\begin{aligned}
&\lim_{\substack{\varepsilon\to 0\\ \lambda=\sqrt{\varepsilon T}}} \Big(\mathbb{E}\Big[\big(\langle \hat{J}, W_T^{\varepsilon}\rangle - \mathbb{E}\langle \hat{J}, F_T\rangle\big)^r\Big]\Big)^{\frac{1}{r}} \\
&\leq \lim_{\substack{\varepsilon\to 0\\ \lambda=\sqrt{\varepsilon T}}} \Big(\mathbb{E}\Big[\big(\langle \hat{J}_\varepsilon, \hat{W}_{\phi_t}\rangle - \mathbb{E}\langle \hat{J}_\varepsilon, \hat{W}_{\phi_t}\rangle\big)^r\Big]\Big)^{\frac{1}{r}} + \lim_{\substack{\varepsilon\to 0\\ \lambda=\sqrt{\varepsilon T}}} \Big|\langle J, W_T^{(\varepsilon)}\rangle - \langle J, F_T\rangle\Big| \\
&= 0 \,, \qquad (58)
\end{aligned}$$

for every fixed, finite value of $r \in 2\mathbb{N}$ and $T > 0$. This in turn implies that (58) holds for any fixed, finite $r \in \mathbb{R}_+$, globally in T. This proves Theorem 2.2.

4. Graph expansions and main technical lemmata

The key technical lemmata required to establish (57) are formulated this section. The method of proof is based on graph expansions and estimation of singular integrals in momentum space, generalizing the analysis in [4] and [2] to cover the case of convergence in r-th mean.

Lemma 4.1. *For any fixed $r \geq 2$ with $r \in 2\mathbb{N}$, and $\bar{n} := n_1 + n_2$, where $n_1, n_2 \leq N$,*

$$\Big(\mathbb{E}_{2-conn}\Big[\big|\langle \hat{J}_\varepsilon, \hat{W}_{t;n_1,n_2}\rangle\big|^r\Big]\Big)^{\frac{1}{r}} \leq \varepsilon\big((\frac{\bar{n}r}{2})!\big)^{\frac{1}{r}}(\log\frac{1}{\varepsilon})^3(c\lambda^2\varepsilon^{-1}\log\frac{1}{\varepsilon})^{\frac{\bar{n}}{2}} \,. \qquad (59)$$

Furthermore, for any fixed $r \geq 2$, $r \in 2\mathbb{N}$, and $n \leq N$,

$$\Big(\mathbb{E}\Big[\|\phi_{n,t}\|_2^{2r}\Big]\Big)^{\frac{1}{r}} \leq ((nr)!)^{\frac{1}{r}}(\log\frac{1}{\varepsilon})^3(c\lambda^2\varepsilon^{-1}\log\frac{1}{\varepsilon})^n \,. \qquad (60)$$

The estimate (59) is the key ingredient in our analysis. The central insight is that for every $r \geq 2$, the expectation over *2-connected graphs* (cf. Definition 5.1 below) is a factor ε^γ smaller than the a priori bound (60), for some $\gamma > 0$.

Lemma 4.2. *For any fixed $r \geq 2$, $r \in 2\mathbb{N}$, and $N < n < 4N$,*

$$\begin{aligned}
&\Big(\mathbb{E}\Big[\|\phi_{n,N,\theta_{j-1}}(\theta_j)\|_2^{2r}\Big]\Big)^{\frac{1}{r}} \qquad (61)\\
&\qquad \leq \frac{(c\lambda^2\varepsilon^{-1})^n}{\sqrt{n!}} + \Big((n!)\varepsilon^{\frac{1}{6}} + ((nr)!)^{\frac{1}{r}}\varepsilon^{\frac{3}{r}}\Big)(\log\frac{1}{\varepsilon})^3(c\lambda^2\varepsilon^{-1}\log\frac{1}{\varepsilon})^n \,.
\end{aligned}$$

The estimates (60) and (61) are obtained from a straightforward generalization of the corresponding results derived in [2, 4], combined with an adaptation of (59) for L^2-norms instead of Wigner distributions.

Lemma 4.3. *For any fixed $r \geq 2$, $r \in 2\mathbb{N}$,*

$$\Big(\mathbb{E}\Big[\|\phi_{4N,N,\theta_{j-1}}(\theta_j)\|_2^{2r}\Big]\Big)^{\frac{1}{r}} \leq \frac{((4Nr)!)^{\frac{1}{2r}}(\log\frac{1}{\varepsilon})^3(c\lambda^2\varepsilon^{-1}\log\frac{1}{\varepsilon})^{4N}}{\kappa^{2N}} \qquad (62)$$

This estimate on the error term again follows from a straightforward modification of results in [2, 4].

Let us to begin with state some of the key bounds which enter the proofs of these lemmata. For the particle propagators, we have the following estimates at hand, which were proved in [2, 4].

Lemma 4.4.

$$\sup_{\alpha\in I, p\in\mathbb{T}^3} \frac{1}{|e(p)-\alpha-i\varepsilon|} = \frac{1}{\varepsilon}$$

$$\int_{\mathbb{T}^3} \frac{dp}{|e(p)-\alpha-i\varepsilon|}\,,\ \int_I \frac{|d\alpha|}{|e(p)-\alpha-i\varepsilon|} < c\log\frac{1}{\varepsilon}\,. \tag{63}$$

Next, let us address the functions $\phi_{n,N,\theta_{j-1}}(\theta_j)$, and highlight a key property. From

$$\begin{aligned}\phi_{n,N,\theta}(\theta') &= (i\lambda)^n \int_{\mathbb{R}_+^{n-N+1}} ds_0\cdots ds_{n-N}\delta(\sum_{j=1}^{n-N} s_j' - (\theta'-\theta)) \\ &\quad\times e^{is_0\Delta}V_\omega\cdots V_\omega e^{is_{n-N}\Delta}V_\omega\phi_{N,\theta}\,,\end{aligned} \tag{64}$$

where $\theta'-\theta = \frac{t}{\kappa}$, we find

$$\begin{aligned}(\hat{\phi}_{n,N,\theta}(\theta'))(k_0) &= (i\lambda)^n e^{\frac{\varepsilon t}{\kappa}} \int_I d\alpha e^{-\frac{i\alpha t}{\kappa}} \int_{(\mathbb{T}^3)^{n-N+1}} dk_1\cdots dk_{n-N} \\ &\quad\times \frac{1}{e(k_0)-\alpha-i\kappa\varepsilon}\hat{V}_\omega(k_1-k_0)\cdots \\ &\qquad\cdots\frac{1}{e(k_{n-N})-\alpha-i\kappa\varepsilon}\hat{V}_\omega(k_{n-N+1}-k_{n-N}) \\ &\quad\times\hat{\phi}_{N,\theta}(k_{n-N+1})\,,\end{aligned} \tag{65}$$

with $\varepsilon = \frac{1}{t}$, as before, and

$$\begin{aligned}\hat{\phi}_{N,\theta}(k_{n-N+1}) &= (i\lambda)^N e^{\varepsilon\theta} \int_{(\mathbb{T}^3)^N} \prod_{j=n-N+1}^{n+1} dk_j \\ &\quad\times\frac{1}{e(k_{n-N+1})-\alpha-\frac{i}{\theta}}\hat{V}_\omega(k_{n-N+2}-k_{n-N+1})\cdots \\ &\qquad\cdots\hat{V}_\omega(k_{n+1}-k_n)\frac{1}{e(k_{n+1})-\alpha-\frac{i}{\theta}}\hat{\phi}_0(k_{n+1})\,.\end{aligned} \tag{66}$$

The key observation here is that there are $n-N+1$ propagators with imaginary part $-i\kappa\varepsilon$ in the denominator, where $\kappa\varepsilon \gg \varepsilon$, and $N+1$ propagators with denominators having an imaginary part $-\frac{i}{\theta}$, where $\frac{1}{\theta}$ can be comparable to ε in smallness. Therefore, we have a bound

$$\frac{1}{|e(p)-\alpha-i\kappa\varepsilon|} \le \frac{1}{\kappa\varepsilon} \ll \frac{1}{\varepsilon} \tag{67}$$

for $n-N+1$ propagators, which is much smaller than the bound $\frac{1}{|e(p)-\alpha-i\varepsilon|} \le \frac{1}{\varepsilon}$ considered earlier. This gain of a factor $\frac{1}{\kappa}$ as compared to (63) is precisely the benefit of the partial integration method. It is systematically exploited in the proof of Lemma 4.3, while it turns out not to be necessary for the proof of Lemma 4.2.

5. Proof of Lemma 4.1

We to begin with recall that

$$\begin{aligned}&\mathbb{E}_{2-conn}\Big[\big|\langle\hat{J}_\varepsilon,\hat{W}_{t;n_1,n_2}\rangle\big|^r\Big]\\ &\quad=\mathbb{E}_{2-conn}\Big[\Big(\big|\int dpdp'\hat{J}_\varepsilon(p-p',\frac{p+p'}{2})\overline{\hat{\phi}_{n_2,t}(p)}\hat{\phi}_{n_1,t}(p')\big|^2\Big)^{\frac{r}{2}}\Big]\,,\end{aligned}\tag{68}$$

where we remark that $\hat{J}_\varepsilon$ forces $|p-p'|<c\varepsilon$, while $|p+p'|$ is essentially unrestricted. Next, we introduce the following multiindex notation. Since n_1,n_2 remain fixed in the proof, let $n\equiv n_1$, and $\bar{n}\equiv n_1+n_2$. Then, for $j=1,\dots,r$, let

$$\begin{aligned}\underline{p}^{(j)}&:=(p_0^{(j)},\dots,p_{\bar{n}+1}^{(j)})\\ d\underline{p}^{(j)}&:=\prod_{\ell=0}^{\bar{n}+1}dp_\ell^{(j)}\\ d\underline{p}^{(j)}_{\hat{J}_\varepsilon}&:=\prod_{\ell=0}^{\bar{n}+1}dp_\ell^{(j)}\hat{J}_\varepsilon(p_n^{(j)}-p_{n+1}^{(j)},\frac{p_n^{(j)}+p_{n+1}^{(j)}}{2})\\ K^{(j)}[\underline{p}^{(j)},\alpha_j,\beta_j,\varepsilon]&:=\prod_{\ell=0}^{n}\frac{1}{e(p_\ell^{(j)})-\alpha_j-i\varepsilon_j}\prod_{\ell'=n+1}^{\bar{n}+1}\frac{1}{e(p_{\ell'}^{(j)})-\beta_j+i\varepsilon_j}\\ U^{(j)}[\underline{p}^{(j)}]&:=\prod_{\ell=1}^{n}\hat{V}_\omega(p_\ell^{(j)}-p_{\ell-1}^{(j)})\prod_{\ell'=n+2}^{\bar{n}+1}\hat{V}_\omega(p_{\ell'}^{(j)}-p_{\ell'-1}^{(j)})\,,\end{aligned}\tag{69}$$

where $\varepsilon_j:=(-1)^j\varepsilon$, $\alpha_j\in I$, and $\beta_j\in\bar{I}$ (the complex conjugate of I). With respect to the last line, we note that $\overline{\hat{V}_\omega(k)}=\hat{V}_\omega(-k)$.

Then,

$$\begin{aligned}(\,68)&=e^{2r\varepsilon t}\int_{(I\times\bar{I})^r}\prod_{j=1}^r d\alpha_j d\beta_j\,e^{-it\sum_{j=1}^r(-1)^j(\alpha_j-\beta_j)}\\ &\quad\times\int_{(\mathbb{T}^3)^{(\bar{n}+2)r}}\prod_{j=1}^r d\underline{p}^{(j)}_{\hat{J}_\varepsilon}\mathbb{E}_{2-conn}\Big[\prod_{j=1}^r U^{(j)}[\underline{p}^{(j)}]\Big]\\ &\quad\times\prod_{j=1}^r K^{(j)}[\underline{p}^{(j)},\alpha_j,\beta_j,\varepsilon]\hat{\phi}_0^{(j)}(p_0^{(j)})\overline{\hat{\phi}_0^{(j)}(p_{\bar{n}+1}^{(j)})}\,,\end{aligned}\tag{70}$$

where

$$\hat{\phi}_0^{(j)}:=\begin{cases}\bar{\hat{\phi}}_0 & \text{if } j\in2\mathbb{N}\\ \hat{\phi}_0 & \text{if } j\in2\mathbb{N}-1\,.\end{cases}\tag{71}$$

We may represent this integral graphically (i.e. in terms of *Feynman graphs*) as follows, cf. Fig. 1.

We draw r parallel, horizontal solid lines, which we refer to as *particle lines*, each containting $\bar{n}$ vertices enumerated starting from the left, accounting for copies of the random potential $\hat{V}_\omega$. Between the n-th and the $n+1$-th $\hat{V}_\omega$-vertex, we insert a distinguished vertex for the contraction with $\hat{J}_\varepsilon$ (henceforth referred to as the "$\hat{J}_\varepsilon$-vertex"). Then, the n edges on the left of the $\hat{J}_\varepsilon$-vertex correspond to the propagators in $\hat{\psi}_{n,t}$, while the $\bar{n}-n$ edges on the right correspond to those in $\overline{\hat{\psi}_{\bar{n}-n,t}}$. We shall refer to those edges as *propagator lines*. The expectation produces a sum over all possible products of $\frac{r\bar{n}}{2}$ delta distributions, each standing for one contraction between a pair of random potentials. We shall draw a dashed line connecting every pair of random potentials that are mutually contracted. We then identify the contraction type with the corresponding graph.

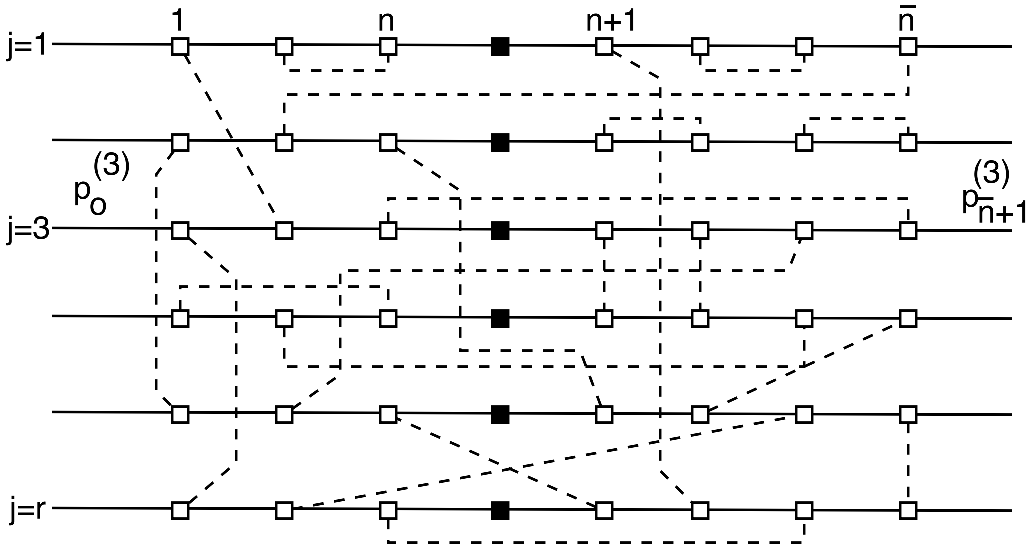

Figure 1. A (completely connected) contraction graph for the case $r=6$, $n=3$, $\bar{n}=7$. The $\hat{J}_\varepsilon$-vertices are drawn in black, while the $\hat{V}_\omega$-vertices are shown in white. The r particle lines are solid, while the lines corresponding to contractions of pairings of random potentials are dashed. For $j=3$ in the notation of (70), the momenta $p_0^{(3)}$ and $p_{\bar{n}+1}^{(3)}$ are written above the corresponding propagator lines.

It should be remarked that what is defined here as one particle line was considered as a pair of particle lines joined by a $\hat{J}_\varepsilon$-, or respectively, a δ-vertex in [2, 4]. Thus, according to the terminology of [2, 4], we would here be discussing the case of $2r$ particle lines. Due to a shift of emphasis in the work at hand, the convention presently introduced seems more convenient.

We particularly distinguish the class of *completely disconnected graphs*, in which random potentials are mutually contracted only if they are located on the same particle line. Clearly, all of its members possess r connectivity components. All other contraction types are referred to as *non-disconnected graphs*. On the other hand, contraction types are referred to as *2-connected graphs*, if they do not contain any connectivity component that comprises only one particle line, so that every connectivity component contains at least two particle lines. Accordingly, we may now provide the definition of the following notion which was already used in the preceding discussion.

Definition 5.1. *We refer to*

$$\mathbb{E}_{disc}\Big[\prod_{j=1}^{r} U^{(j)}[\underline{p}^{(j)}]\Big] := \prod_{j=1}^{r} \mathbb{E}\Big[U^{(j)}[\underline{p}^{(j)}]\Big] \,, \tag{72}$$

which includes contractions among random potentials $\hat{V}_\omega$ only if they lie on the same particle line, as the expectation based on completely disconnected graphs. Conversely, we let

$$\mathbb{E}_{n-d}\Big[\prod_{j=1}^{r} U^{(j)}[\underline{p}^{(j)}]\Big] := \mathbb{E}\Big[\prod_{j=1}^{r} U^{(j)}[\underline{p}^{(j)}]\Big] - \mathbb{E}_{disc}\Big[\prod_{j=1}^{r} U^{(j)}[\underline{p}^{(j)}]\Big] \,, \tag{73}$$

denote the expectation based on non-disconnected graphs, defined by the condition that there is at least one connectivity component comprising more than one particle line. Moreover, we refer to

$$\mathbb{E}_{2-conn}\Big[\prod_{j=1}^{r} U^{(j)}[\underline{p}^{(j)}]\Big] := \mathbb{E}\Big[\prod_{j=1}^{r} \Big(U^{(j)}[\underline{p}^{(j)}] - \mathbb{E}\Big[U^{(j)}[\underline{p}^{(j)}]\Big]\Big)\Big] \tag{74}$$

as the expectation based on 2-connected graphs. Those are defined by the requirement that any connectivity component comprises at least two particle lines.

For $r\bar{n} \in 2\mathbb{N}$, we let $\Pi^{(\hat{J}_\varepsilon)}_{r;\bar{n},n}$ denote the set of all graphs on $r \in \mathbb{N}$ particle lines, each containing $\bar{n}$ $\hat{V}_\omega$-vertices, and each with the $\hat{J}_\varepsilon$-vertex located between the n-th and $n+1$-th $\hat{V}_\omega$-vertex. Thus, clearly,

$$\mathbb{E}\Big[\prod_{j=1}^{r} U^{(j)}[\underline{p}^{(j)}]\Big] = \sum_{\pi \in \Pi^{(\hat{J}_\varepsilon)}_{r;\bar{n},n}} \delta_\pi(\underline{p}^{(1)}, \dots, \underline{p}^{(r)}) \,, \tag{75}$$

where $\delta_\pi(\underline{p}^{(1)}, \dots, \underline{p}^{(r)})$ is defined as follows. There are $\frac{r\bar{n}}{2}$ pairing contractions between $\hat{V}_\omega$-vertices in π. Every (dashed) contraction line connects a random potential $\hat{V}_\omega(p^{(j_1)}_{n_1+1} - p^{(j_1)}_{n_1})$ with a random potential $\hat{V}_\omega(p^{(j_2)}_{n_2+1} - p^{(j_2)}_{n_2})$, for some pair of multiindices $((j_1, n_1), (j_2, n_2))$ determined by π, for which

$$\begin{aligned}&\mathbb{E}\Big[\hat{V}_\omega(p^{(j_1)}_{n_1+1} - p^{(j_1)}_{n_1})\hat{V}_\omega(p^{(j_2)}_{n_2+1} - p^{(j_2)}_{n_2})\Big] \\ &\quad = \delta(p^{(j_1)}_{n_1+1} - p^{(j_1)}_{n_1} + p^{(j_2)}_{n_2+1} - p^{(j_2)}_{n_2}) \,.\end{aligned} \tag{76}$$

Then, $\delta_\pi(\underline{p}^{(1)}, \dots, \underline{p}^{(r)})$ is given by the product of delta distributions over all pairs of multiindices $((j_1, n_1), (j_2, n_2))$ determined by π.

Furthermore, we will refer to a graph with a single connectivity component as a *completely connected graph*. We denote the subset of $\Pi_{r;\bar{n},n}$ consisting of completely connected graphs by $\Pi^{(\hat{J}_\varepsilon)\,conn}_{r;\bar{n},n}$.

The following key lemma is partially a joint result with Laszlo Erdös.

Lemma 5.1. *Let $s \geq 1$, $s \in \mathbb{N}$, $s\bar{n} \in 2\mathbb{N}$, and let $\pi \in \Pi^{(\hat{J}_\varepsilon)\,conn}_{s;\bar{n},n}$ be a completely connected graph. Then,*

$$|\mathrm{Amp}_{\hat{J}_\varepsilon}(\pi)| \leq \varepsilon^{2(s-1)}(\log\frac{1}{\varepsilon})^{s+2}(c\lambda^2\varepsilon^{-1}\log\frac{1}{\varepsilon})^{\frac{s\bar{n}}{2}} \,. \tag{77}$$

Proof. By assumption, $\pi \in \Pi^{(\hat{J}_\varepsilon)\,conn}_{s;\bar{n},n}$ is a completely connected graph.

Let $\delta_\pi(\underline{p}^{(1)},\dots,\underline{p}^{(s)})$ denote the corresponding product of delta distributions. Then,

$$\begin{aligned}\mathrm{Amp}_{\hat{J}_\varepsilon}(\pi) \;=\;& \lambda^{2s\bar{n}}e^{2s\varepsilon t}\int_{(I\times\bar{I})^s}\prod_{j=1}^s d\alpha_j d\beta_j\, e^{-it\sum_{j=1}^s(-1)^j(\alpha_j-\beta_j)}\\ &\times\int_{(\mathbb{T}^3)^{(\bar{n}+2)s}}\prod_{j=1}^s d\underline{p}^{(j)}\delta_\pi(\underline{p}^{(1)},\dots,\underline{p}^{(s)})\\ &\times\int_{(\mathbb{T}^3)^s}\prod_{j=1}^s d\xi_j\hat{J}_\varepsilon(\xi_j,\frac{p_n^{(j)}+p_{n+1}^{(j)}}{2})\delta(p_n^{(j)}-p_{n+1}^{(j)}-\xi_j)\\ &\times\prod_{j=1}^s K^{(j)}[\underline{p}^{(j)},\alpha_j,\beta_j,\varepsilon]\hat{\phi}_0^{(j)}(p_0^{(j)})\overline{\hat{\phi}_0^{(j)}(p_{\bar{n}+1}^{(j)})}\,.\end{aligned} \tag{78}$$

Adding the arguments of all delta distributions contained in the product

$$\Big(\prod_{j=1}^s\delta(p_n^{(j)}-p_{n+1}^{(j)}-\xi_j)\Big)\delta_\pi(\underline{p}^{(1)},\dots,\underline{p}^{(s)})\,, \tag{79}$$

we find the condition

$$\sum_{j=1}^s(p_0^{(j)}-p_{\bar{n}+1}^{(j)}) = \sum_{j=1}^s\xi_j\,. \tag{80}$$

Due to the fact that π is completely connected, there cannot exist another linear conservation equation of this kind involving only a subset of $(p_0^{(j)},p_{\bar{n}+1}^{(j)})_{j=1}^s$.

By linear independence of the arguments of all individual delta distributions in this product, we can thus write

$$(\,79) = \delta(\sum_{j=1}^s(p_0^{(j)}-p_{\bar{n}+1}^{(j)})-\sum_{j=1}^s\xi_j)\delta'_{\pi,\xi}(\underline{p}^{(1)},\dots,\underline{p}^{(s)})\,, \tag{81}$$

where $\delta'_{\pi,\xi}$ is the product of $\frac{s\bar{n}}{2}+s-1$ delta distributions, of which $\frac{s\bar{n}}{2}$ correspond to $\hat{V}_\omega$-pair correlations. We require $\delta'_{\pi,\xi}$ to be of the form

$$\delta'_{\pi,\xi}(\underline{p}^{(1)},\dots,\underline{p}^{(s)}) := \Big(\prod_{j=1}^s\delta(p_n^{(j)}-p_{n+1}^{(j)}-\xi_j)\Big)\delta'_\pi(\underline{p}^{(1)},\dots,\underline{p}^{(s)})\,, \tag{82}$$

where δ'_π contains one less δ-distribution than δ_π. We will not make an a priori choice of the δ-distribution which is eliminated from δ_π to obtain δ'_π, but will determine it as we proceed with the constructions that follow below. Consequently, with $A_{n,\bar{n}} := \{1, \dots, n, n+2, \dots, \bar{n}\}$,

$$
\begin{aligned}
|\mathrm{Amp}_{\hat{J}_\varepsilon}(\pi)| \;\leq\; & \lambda^{2s\bar{n}} e^{2s\varepsilon t} \int_{(I\times\bar{I})^r} \prod_{j=1}^s |d\alpha_j|\,|d\beta_j| \int_{(\mathbb{T}^3)^{(\bar{n}+2)s}} \prod_{j=1}^s d\underline{p}^{(j)} \\
& \times \int_{(\mathbb{T}^3)^s} \prod_{j=1}^s d\xi_j |\hat{J}_\varepsilon(\xi_j, \frac{p_n^{(j)} + p_{n+1}^{(j)}}{2})| \\
& \times \delta'_{\pi,\xi}(\underline{p}^{(1)}, \dots, \underline{p}^{(s)}) \prod_{j=1}^s |K^{(j)}[\underline{p}^{(j)}, \alpha_j, \beta_j, \varepsilon]| \\
& \times \delta(\sum_{j=1}^s (p_0^{(j)} - p_{\bar{n}+1}^{(j)}) - \sum_{j=1}^s \xi_j) \prod_{j=1}^s |\hat{\phi}_0(p_0^{(j)})|\,|\hat{\phi}_0(p_{\bar{n}+1}^{(j)})| \\
= \;& \lambda^{2s\bar{n}} e^{2s\varepsilon t} \int_{(I\times\bar{I})^r} \prod_{j=1}^s |d\alpha_j|\,|d\beta_j| \int_{(\mathbb{T}^3)^s} \prod_{j=1}^s d\xi_j \\
& \times \int_{(\mathbb{T}^3)^{2s}} \prod_{j=1}^s dp_0^{(j)} dp_{\bar{n}+1}^{(j)} \delta(\sum_{j=1}^s (p_0^{(j)} - p_{\bar{n}+1}^{(j)}) - \sum_{j=1}^s \xi_j) \\
& \times \prod_{j=1}^s \frac{1}{|e(p_0^{(j)}) - \alpha_j - i\varepsilon_j|\,|e(p_{\bar{n}+1}^{(j)}) - \beta_j + i\varepsilon_j|} |\hat{\phi}_0(p_0^{(j)})|\,|\hat{\phi}_0(p_{\bar{n}+1}^{(j)})| \\
& \times \prod_{j=1}^s |\hat{J}_\varepsilon(\xi_j, \frac{p_n^{(j)} + p_{n+1}^{(j)}}{2})| \\
& \times S[(p_0^{(j)}, p_{n+1}^{(j)}, p_{\bar{n}+1}^{(j)}, \alpha_j, \beta_j)_{j=1}^s, \varepsilon]\,,
\end{aligned} \tag{83}
$$

where

$$
\begin{aligned}
& S[(p_0^{(j)}, p_{\bar{n}+1}^{(j)}, \alpha_j, \beta_j)_{j=1}^s, \varepsilon] \\
& \quad := \int_{(\mathbb{T}^3)^{s\bar{n}}} \prod_{j=1}^s \Big(\prod_{i\in A_{n,\bar{n}}} dp_i^{(j)} \Big) dp_{n+1}^{(j)} \\
& \qquad \times \prod_{j=1}^s |K_{A_{n,\bar{n}}}^{(j)}[(p_i^{(j)})_{i\in A_{n,\bar{n}}}, \alpha_j, \beta_j, \varepsilon]| \\
& \qquad \times \delta'_\pi(\underline{p}^{(1)}, \dots, \underline{p}^{(s)}) \prod_{j=1}^s \frac{\delta(p_{n+1}^{(j)} - p_n^{(j)} - \xi_j)}{|e(p_{n+1}^{(j)}) - \beta_j + i\varepsilon_j|}\,,
\end{aligned} \tag{84}
$$

with

$$K^{(j)}_{A_{n,\bar n}}[(p_i^{(j)})_{i\in A_{n,\bar n}},\alpha_j,\beta_j,\varepsilon] := \prod_{\substack{\ell\in A_{n,\bar n}\\ \ell\le n}} \frac{1}{e(p_\ell^{(j)})-\alpha_j-i\varepsilon_j} \prod_{\substack{\ell'\in A_{n,\bar n}\\ \ell'>n+1}} \frac{1}{e(p_{\ell'}^{(j)})-\beta_j+i\varepsilon_j}\,. \tag{85}$$

Clearly, it follows that

$$|\mathrm{Amp}_{\hat J_\varepsilon}(\pi)| \;\le\; \lambda^{2s\bar n}e^{2s\varepsilon t}\Big(\int_{\mathbb{T}^3} d\xi \sup_{p\in\mathbb{T}^3}|\hat J_\varepsilon(\xi,p)|\Big)^s\cdot(I)\cdot(II)\,, \tag{86}$$

where

$$\int_{\mathbb{T}^3} d\xi \sup_{p\in\mathbb{T}^3}|\hat J_\varepsilon(\xi,p)| < c \tag{87}$$

uniformly in ε, with

$$\begin{aligned}(I) \;:=\; &\sup_{\xi\in\mathbb{T}^3}\int_{(\mathbb{T}^3)^{2s-2}}\prod_{j=1}^{s-1}dp_0^{(j)}dp_{\bar n+1}^{(j)}|\hat\phi_0(p_0^{(j)})|\,|\hat\phi_0(p_{\bar n+1}^{(j)})|\\ &\times\int_{\mathbb{T}^3}dp_0^{(s)}|\hat\phi_0(p_0^{(s)})|\,|\hat\phi_0(\sum_{j=1}^{s-1}(p_0^{(j)}-p_{\bar n+1}^{(j)})+p_0^{(s)}-\xi)|\\ &\times\Big(\int_{(I\times\bar I)^r}\prod_{j=1}^{s}|d\alpha_j|\,|d\beta_j|\\ &\times\prod_{j=1}^{s-1}\frac{1}{|e(p_0^{(j)})-\alpha_j-i\varepsilon_j|\,|e(p_{\bar n+1}^{(j)})-\beta_j+i\varepsilon_j|}\\ &\times\frac{1}{|e(p_0^{(s)})-\alpha_j-i\varepsilon_j|\,|e(\sum_{j=1}^{s-1}(p_0^{(j)}-p_{\bar n+1}^{(j)})+p_0^{(s)}-\xi)-\beta_j+i\varepsilon_j|}\Big)\,,\end{aligned} \tag{88}$$

and

$$(II) \;:=\; \sup_{\alpha_1,\dots,\alpha_s}\sup_{\beta_1,\dots,\beta_s}\sup_{\xi_1,\dots,\xi_s}\sup_{(p_0^{(j)},p_{\bar n+1}^{(j)})_{j=1}^s} S'[(p_0^{(j)},p_{\bar n+1}^{(j)},\alpha_j,\beta_j,\xi_j)_{j=1}^s]\,, \tag{89}$$

where

$$\begin{aligned}&S'[(p_0^{(j)},p_{\bar n+1}^{(j)},\alpha_j,\beta_j,\xi_j)_{j=1}^s]\\ &\quad:=\varepsilon^{-s}\int_{(\mathbb{T}^3)^{s(\bar n-1)}}\prod_{j=1}^s\Big[\prod_{i\in A_{n,\bar n}}dp_i^{(j)}\Big]\\ &\qquad\times\delta'_\pi(\underline{p}^{(1)},\dots,\underline{p}^{(s)})\\ &\qquad\times\prod_{j=1}^s|K^{(j)}_{A_{n,\bar n}}[(p_i^{(j)})_{i\in A_{n,\bar n}},\alpha_j,\beta_j,\varepsilon]|\,\Big|_{\substack{p_{n+1}^{(j)}=p_n^{(j)}-\xi_j\\ j=1,\dots,s}}\,,\end{aligned} \tag{90}$$

from integrating our the variables $p_{n+1}^{(j)}$, for $j = 1, \dots, s$. Recalling the assumptions on the initial condition $\hat{\phi}_0$ in (31), and using

$$\sup_{p\in\mathbb{T}^3}\{\int_I |d\alpha|\frac{1}{|e(p)-\alpha-i\varepsilon|} \ , \ \int_{\bar{I}} |d\beta|\frac{1}{|e(p)-\beta+i\varepsilon|}\} \leq c\log\frac{1}{\varepsilon} \ , \tag{91}$$

cf. [2], we find

$$\begin{aligned}(I) \ &\leq \ \Big(\int_{\mathbb{T}^3}\prod_{j=1}^{s-1} dp|\hat{\phi}_0(p)|\Big)^{2s-2} \\ &\quad \times \sup_{p\in\mathbb{T}^3}\int_{\mathbb{T}^3} dp_0^{(s)}|\hat{\phi}_0(p_0^{(s)})|\,|\hat{\phi}_0(p+p_0^{(s)})| \\ &\quad \times \Big(\sup_{p\in\mathbb{T}^3}\int_{I^r}\frac{|d\alpha|}{|e(p)-\alpha-i\varepsilon|}\Big)^s\Big(\sup_{p\in\mathbb{T}^3}\int_{\bar{I}^r}\frac{|d\beta|}{|e(p)-\beta+i\varepsilon|}\Big)^s \\ &\leq \ c\varepsilon^{3(s-1)}(c\log\frac{1}{\varepsilon})^{2s} \ . \end{aligned}\tag{92}$$

On the other hand,

$$(II) \ \leq \ \varepsilon^{-\frac{s\bar{n}}{2}-(s-1)}(c\log\frac{1}{\varepsilon})^{\frac{s\bar{n}}{2}-s+1} \ , \tag{93}$$

from Lemma 5.2 below. We thus indeed infer that

$$\begin{aligned}|\mathrm{Amp}_{\hat{J}_\varepsilon}(\pi)| \ &\leq \ \lambda^{s\bar{n}}\varepsilon^{3(s-1)-\frac{s\bar{n}}{2}-(s-1)}(c\log\frac{1}{\varepsilon})^{\frac{s\bar{n}}{2}+s+2} \\ &= \ \varepsilon^{2(s-1)}(\log\frac{1}{\varepsilon})^{s+2}(c\varepsilon^{-1}\lambda^2\log\frac{1}{\varepsilon})^{\frac{s\bar{n}}{2}} \ , \end{aligned}\tag{94}$$

as claimed. □

Lemma 5.2. *Let*

$$\begin{aligned}&S'[(p_0^{(j)}, p_{\bar{n}+1}^{(j)}, \alpha_j, \beta_j, \xi_j)_{j=1}^s] \\ &\quad := \varepsilon^{-s}\int_{(\mathbb{T}^3)^{s(\bar{n}-1)}}\prod_{j=1}^s\Big[\prod_{i\in A_{n,\bar{n}}} dp_i^{(j)}\Big] \\ &\quad\quad \times \delta'_\pi(\underline{p}^{(1)}, \dots, \underline{p}^{(s)}) \\ &\quad\quad \times \prod_{j=1}^s |K_{A_{n,\bar{n}}}^{(j)}[(p_i^{(j)})_{i\in A_{n,\bar{n}}}, \alpha_j, \beta_j, \varepsilon]|\Big|_{\substack{p_{n+1}^{(j)}=p_n^{(j)}-\xi_j \\ j=1,\dots,s}} \ , \end{aligned}\tag{95}$$

cf. (90). Then,

$$\begin{aligned}&\sup_{\alpha_1,\dots,\alpha_s}\sup_{\beta_1,\dots,\beta_s}\sup_{\xi_1,\dots,\xi_s}\sup_{(p_0^{(j)},p_{\bar{n}+1}^{(j)})_{j=1}^s} S'[(p_0^{(j)}, p_{\bar{n}+1}^{(j)}, \alpha_j, \beta_j, \xi_j)_{j=1}^s] \\ &\quad \leq \varepsilon^{-\frac{s\bar{n}}{2}-(s-1)}(c\log\frac{1}{\varepsilon})^{\frac{s\bar{n}}{2}-s+1} \ . \end{aligned}\tag{96}$$

Proof. To bound the quantity (95), we to begin with introduce the *truncated contraction graph* π_{trunc} corresponding to the completely connected graph π of Lemma 5.1, which is obtained by eliminating all propagator lines in π corresponding to the momenta $(p_0^{(j)}, p_{\bar{n}+1}^{(j)})_{j=1}^s$, which we will refer to as *external momenta* (relative to π_{trunc}). To arrive at the graph corresponding to the integral (95), a single $\hat{V}_\omega$-contraction line must be removed. The algorithm outlined below will determine a convenient "a posteriori" choice for the latter.

We note that the dependence of (95) on the external momenta $(p_0^{(j)}, p_{\bar{n}+1}^{(j)})_{j=1}^s$ enters solely through the product of $\frac{r\bar{n}}{2} - 1$ delta distributions in $\delta'_\pi(\underline{p}^{(1)}, \dots, \underline{p}^{(s)})$.

Since the momenta $p_{n+1}^{(j)}$ have already been integrated out in the expression (95), pairs of edges adjacent to the $\hat{J}_\varepsilon$-vertices are counted only as a single edge. Thus, according to this convention, the total number of particle edges contained in π_{trunc} is $s\bar{n} - s$.

We construct a spanning tree T of π_{trunc} which contains all contraction lines among pairs of random potentials (dashed lines). This is easy to accomplish, since the subgraph of π consisting only of $\hat{V}_\omega$-contraction lines is a disjoint union of $\frac{s\bar{n}}{2}$ one-edge trees on pairs of vertices. All that is left to do is to arbitrarily add $\frac{s\bar{n}}{2} - 1$ edges corresponding to particle lines without generating a cycle (since T, being a tree on $s\bar{n}$ vertices, evidently comprises $s\bar{n} - 1$ edges). The particle momenta $p_\ell^{(j)}$ labelling the edges on T will be denoted *tree momenta*, while their $\frac{s\bar{n}}{2} - s + 1$ counterparts in the complement of T are referred to as *loop momenta* (since adding any one of the latter to T generates a cycle).

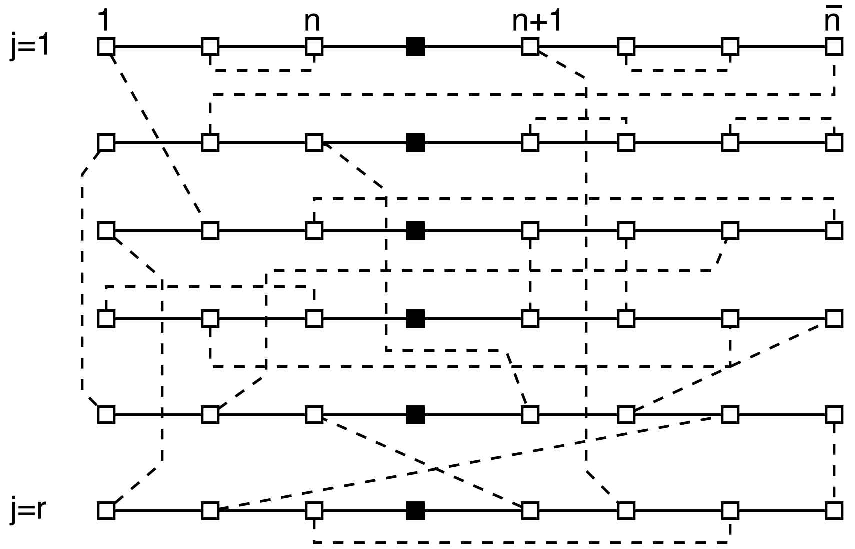

Figure 2. The truncated contraction graph π_{trunc} derived from the graph π given in Figure 1, for $s = r$.

We correspondingly have

$$
\begin{aligned}
&S'[(p_0^{(j)},p_{\bar n+1}^{(j)},\alpha_j,\beta_j,\xi_j)_{j=1}^s]\\
&\quad=\varepsilon^{-s}\int_{(\mathbb{T}^3)^{s(\bar n-1)}}\prod_{j=1}^s\Big[\prod_{i\in A_{n,\bar n}\cap T}dp_i^{(j)}\Big]\Big[\prod_{i'\in A_{n,\bar n}\cap T^c}dp_{i'}^{(j)}\Big]\\
&\qquad\times\delta'_\pi(\underline{p}^{(1)},\dots,\underline{p}^{(s)})\\
&\qquad\times\prod_{j=1}^s|K^{(j)}_{A_{n,\bar n}}[(p_i^{(j)})_{i\in A_{n,\bar n}\cap T},(p_{i'}^{(j)})_{i'\in A_{n,\bar n}\cap T^c},\alpha_j,\beta_j,\varepsilon]|\Big|_{\substack{p_{n+1}^{(j)}=p_n^{(j)}-\xi_j\\ j=1,\dots,s}}\,,
\end{aligned}
\tag{97}
$$

where T^c denotes the complement of T in π_{trunc}.

So far, we have not yet explicitly determined the delta distribution which is eliminated from δ_π to arrive at δ'_π, and shall do so next. This is accomplished as the *last* step of the following iteration. Let us pretend that instead of δ'_π, δ_π is given in the above integral. We then choose a delta distribution in δ_π, and integrate it against a suitable tree momentum $p_\ell^{(j)}\in T$, by which $p_\ell^{(j)}$ is replaced by a linear combination of the remaining tree and loop momenta. Subsequently, we integrate out a next delta distribution among the remaining ones in δ_π against a further tree momentum, and so on. Due to the linear independence of the arguments of the delta distributions in δ_π, this iteration terminates only when all $\frac{s\bar n}{2}-1$ tree momenta have been integrated out. In the end, one delta distribution contained in δ_π is left over, and we define it as the one that is eliminated from δ_π in order to arrive at δ'_π.

Therefore, the above iteration integrates out all delta distributions in δ'_π, and replaces the tree momenta by linear combinations of the loop momenta, and of the external momenta $(p_0^{(j)},p_{\bar n+1}^{(j)})_{j=1}^s$. Estimating all tree propagators by their L^∞-norms, we find

$$
\begin{aligned}
&S'[(p_0^{(j)},p_{\bar n+1}^{(j)},\alpha_j,\beta_j,\xi_j)_{j=1}^s]\\
&\quad\le\varepsilon^{-s}\int_{(\mathbb{T}^3)^{\frac{s\bar n}{2}-1}}\prod_{j=1}^s\Big[\prod_{i\in A_{n,\bar n}\cap T}dp_i^{(j)}\Big]\delta'_\pi(\underline{p}^{(1)},\dots,\underline{p}^{(s)})\\
&\qquad\times\int_{(\mathbb{T}^3)^{\frac{s\bar n}{2}-s+1}}\prod_{j=1}^s\Big[\prod_{i'\in A_{n,\bar n}\cap T^c}dp_{i'}^{(j)}\Big]\\
&\qquad\times\sup_{(p_i^{(j)})_{i\in A_{n,\bar n}\cap T}}\prod_{j=1}^s|K^{(j)}_{A_{n,\bar n}}[(p_i^{(j)})_{i\in A_{n,\bar n}\cap T},\\
&\qquad\qquad(p_{i'}^{(j)})_{i'\in A_{n,\bar n}\cap T^c},\alpha_j,\beta_j,\varepsilon]|\Big|_{\substack{p_{n+1}^{(j)}=p_n^{(j)}-\xi_j\\ j=1,\dots,s}}\,.
\end{aligned}
\tag{98}
$$

We then use

$$\int_{(\mathbb{T}^3)^{\frac{s\bar{n}}{2}-s+1}} \prod_{j=1}^{s} \Big[\prod_{i'\in A_{n,\bar{n}}\cap T^c} dp_{i'}^{(j)} \Big]$$
$$\times \sup_{(p_i^{(j)})_{i\in A_{n,\bar{n}}\cap T}} \prod_{j=1}^{s} |K_{A_{n,\bar{n}}}^{(j)}[(p_i^{(j)})_{i\in A_{n,\bar{n}}\cap T},$$
$$(p_{i'}^{(j)})_{i'\in A_{n,\bar{n}}\cap T^c}, \alpha_j, \beta_j, \varepsilon]| \Big|_{\substack{p_{n+1}^{(j)}=p_n^{(j)}-\xi_j \\ j=1,\dots,s}}$$
$$\leq \varepsilon^{-\frac{s\bar{n}}{2}+1} (c \log \frac{1}{\varepsilon})^{\frac{s\bar{n}}{2}-s+1} \ , \tag{99}$$

and

$$\sup_{\xi_1,\dots,\xi_s} \sup_{(p_0^{(j)}, p_{\bar{n}+1}^{(j)})_{j=1}^s} \int_{(\mathbb{T}^3)^{\frac{s\bar{n}}{2}-1}} \prod_{j=1}^{s} \Big[\prod_{i\in A_{n,\bar{n}}\cap T} dp_i^{(j)} \Big]$$
$$\times \delta'_\pi(\underline{p}^{(1)},\dots,\underline{p}^{(s)}) \Big|_{\substack{p_{n+1}^{(j)}=p_n^{(j)}-\xi_j \\ j=1,\dots,s}} < C \ . \tag{100}$$

We thus obtain

$$\sup_{\alpha_1,\dots,\alpha_s} \sup_{\beta_1,\dots,\beta_s} \sup_{\xi_1,\dots,\xi_s} \sup_{(p_0^{(j)}, p_{\bar{n}+1}^{(j)})_{j=1}^s} S'[(p_0^{(j)}, p_{\bar{n}+1}^{(j)}, \alpha_j, \beta_j, \xi_j)_{j=1}^s]$$
$$\leq \varepsilon^{-\frac{s\bar{n}}{2}-(s-1)} (c \log \frac{1}{\varepsilon})^{\frac{s\bar{n}}{2}-s+1} \ . \tag{101}$$

This implies the claim. □

The expectation (73) based on 2-connected graphs decomposes into a sum of products of delta distributions associated to completely connected graphs (Wick expansion), which all comprise at least two particle lines. There is thus a gain of a positive power of ε in the estimate on the Feynman amplitude of every graph, as stated in the following lemma, where we also provide a corresponding bound on non-disconnected graphs.

Lemma 5.3. *Let $r \in 2\mathbb{N}$, and $\Pi_{r;\bar{n},n}^{(\hat{J}_\varepsilon)\,2-conn}, \Pi_{r;\bar{n},n}^{(\hat{J}_\varepsilon)\,n-d} \subset \Pi_{r;\bar{n},n}^{(\hat{J}_\varepsilon)}$ denote the subclasses of 2-connected and non-disconnected graphs, respectively. Then,*

$$\sum_{\pi\in\Pi_{r;\bar{n},n}^{(\hat{J}_\varepsilon)\,2-conn}} |\mathrm{Amp}_{\hat{J}_\varepsilon}(\pi)| \leq (r\bar{n})! \varepsilon^r (\log \frac{1}{\varepsilon})^{3r-2} (c\lambda^2\varepsilon^{-1} \log \frac{1}{\varepsilon})^{\frac{r\bar{n}}{2}} \ . \tag{102}$$

and

$$\sum_{\pi\in\Pi_{r;\bar{n},n}^{(\hat{J}_\varepsilon)\,n-d}} |\mathrm{Amp}_{\hat{J}_\varepsilon}(\pi)| \leq (r\bar{n})! \varepsilon^2 (\log \frac{1}{\varepsilon})^{3r-2} (c\lambda^2\varepsilon^{-1} \log \frac{1}{\varepsilon})^{\frac{r\bar{n}}{2}} \ . \tag{103}$$

Proof. Let us assume that π has m connectivity components, comprising $s_1,\dots,s_m$ particle lines, with $\sum_{l=1}^m s_l = r$. Then, by the previous lemma,

$$|\mathrm{Amp}_{\hat{J}_\varepsilon}(\pi)| \leq \varepsilon^{2\sum_{l=1}^m(s_l-1)}(\log\frac{1}{\varepsilon})^{\sum_{l=1}^m(s_l+2)}(c\varepsilon^{-1}\lambda^2\log\frac{1}{\varepsilon})^{\frac{\bar{n}\sum_{l=1}^m s_l}{2}}$$

$$= \varepsilon^{2(r-m)}(\log\frac{1}{\varepsilon})^{r+2m}(c\varepsilon^{-1}\lambda^2\log\frac{1}{\varepsilon})^{\frac{r\bar{n}}{2}} \ . \tag{104}$$

This bound is maximal among non-disconnected graphs if $m = r-1$, such that $s_l = 1$ for all l except for one index l^*, for which $s_{l^*} = 2$ (the case $m = r$ would correspond to a completely disconnected graph, and is by assumption excluded). Among 2-connected graphs, it is maximal if $m = \frac{r}{2}$. Thus, clearly, the amplitude of any non-disconnected graph can be bounded by

$$|\mathrm{Amp}_{\hat{J}_\varepsilon}(\pi)| \leq \varepsilon^2(\log\frac{1}{\varepsilon})^{3r-2}(c\varepsilon^{-1}\lambda^2\log\frac{1}{\varepsilon})^{\frac{r\bar{n}}{2}} \ . \tag{105}$$

For 2-connected graphs, the factor ε^2 is replaced by ε^r. The fact that there are no more than $(r\bar{n})!2^{r\bar{n}}$ graphs in $\Pi^{(\hat{J}_\varepsilon)}_{r;\bar{n},n}$ is clear. $\square$

Lemma 5.4. *For any fixed $r \geq 2$, $r \in 2\mathbb{N}$, and $n \leq N$,*

$$\Big(\mathbb{E}\Big[\|\phi_{n,t}\|_2^{2r}\Big]\Big)^{\frac{1}{r}} \leq ((2nr)!)^{\frac{1}{r}}(\log\frac{1}{\varepsilon})^3(c\lambda^2\varepsilon^{-1}\log\frac{1}{\varepsilon})^n \ . \tag{106}$$

Proof. One obtains the left hand side of (106) from the quantity studied in Lemma 5.1 by restriction to the special case $n = n_1 = n_2$, and by replacing $\hat{J}_\varepsilon(\xi,\eta)$ by $\delta(\xi)$. The assertion follows from a straightforward adaptation of the arguments applied in the proof of Lemma 5.1. The only significant difference is that here, the dominant terms are due to the completely disconnected graphs included in the full expectation. The bound on disconnected graphs is obtained from Lemma 5.1 by setting $s = 1$, and the r-th power (the latter is compensated by the $\frac{1}{r}$ power in evaluating the L^r-norm with respect to the randomness). Correspondingly, there is no factor ε^γ in this bound, for any $\gamma > 0$. $\square$

6. Proof of Lemma 4.2

Based on the previous discussion, is straightforward to see that

$$\begin{aligned}
&\mathbb{E}\big[\|\phi_{n,N,\theta_{j-1}}(\theta_j)\|_2^{2r}\big] \\
&= e^{2r\varepsilon\theta_j}\int_{(I\times\bar{I})^r}\prod_{j=1}^r d\alpha_j d\beta_j\, e^{-i\theta_j\sum_{j=1}^r(\alpha_j-\beta_j)} \\
&\quad\times\int_{(\mathbb{T}^3)^{(\bar{n}+2)r}}\prod_{j=1}^r d\underline{p}^{(j)}\delta(p_n^{(j)}-p_{n+1}^{(j)})\mathbb{E}\Big[\prod_{j=1}^r U^{(j)}[\underline{p}^{(j)}]\Big] \\
&\quad\times\prod_{j=1}^r K^{(j)}_{n,N,\theta_{j-1},\kappa}[\underline{p}^{(j)},\alpha_j,\beta_j,\varepsilon]\hat{\phi}_0(p_0^{(j)})\overline{\hat{\phi}_0(p_{\bar{n}+1}^{(j)})} \ .
\end{aligned}\tag{107}$$

The key differences between this expression and the integrals (70) considered above, are that now, we only study the special case $n = n_1 = n_2$, that we replace the distinguished vertex

$\hat{J}_\varepsilon(\xi, v)$ by $\delta(\xi)$, and that we are now considering the full instead of the non-disconnected expectation. We will refer to $\delta(\xi)$ as the "*L^2-delta*", since it is responsible for the L^2-inner product on the left hand side of (107). Furthermore,

$$
\begin{aligned}
&K^{(j)}_{n,N,\theta_{j-1},\kappa}[\underline{p}^{(j)}, \alpha_j, \beta_j, \varepsilon] \\
&:= \prod_{\ell_1=0}^{n-N} \frac{1}{e(p^{(j)}_{\ell_1}) - \alpha_j - i\kappa\varepsilon} \prod_{\ell_2=n-N+1}^{n} \frac{1}{e(p^{(j)}_{\ell_2}) - \alpha_j - \frac{i}{\theta_{j-1}}} \\
&\quad \times \prod_{\ell_3=n+1}^{n+N} \frac{1}{e(p^{(j)}_{\ell_3}) - \beta_j + \frac{i}{\theta_{j-1}}} \prod_{\ell_4=n+N+1}^{2n+1} \frac{1}{e(p^{(j)}_{\ell_4}) - \beta_j + i\kappa\varepsilon} ,
\end{aligned}
\tag{108}
$$

cf. the discussion following (64).

The expectation in (107) again decomposes into a sum of products of delta distributions, and the corresponding contributions to (107) can be represented by Feynman graphs. Referring to the notational conventions introduced after (70), we have $\bar{n} = 2n$. Correspondingly, let $\Pi_{r;\bar{n},n}$ denote the set of graphs on r particle lines, each containing $\bar{n}$ vertices from copies of the random potential $\hat{V}_\omega$, and with the L^2-delta located between the n-th and the $n+1$-st $\hat{V}_\omega$-vertex. For $\pi \in \Pi_{r;n,\bar{n}}$, let $\mathrm{Amp}_\delta(\pi)$ denote the amplitude corresponding to the graph π, given by the integral obtained from replacing $\mathbb{E}\big[\prod_{j=1}^r U^{(j)}[\underline{p}^{(j)}]\big]$ in (107) by $\delta_\pi(\underline{p}^{(1)}, \dots, \underline{p}^{(r)})$ (the product of delta distributions corresponding to the contraction graph π). The subscript in Amp_δ implies that instead of $\hat{J}_\varepsilon$ as before, we now have the L^2-delta at the distinguished vertex.

Let $\Pi^{conn}_{r;\bar{n},n}$ denote the subclass of $\Pi_{r;\bar{n},n}$ comprising completely connected graphs. Then, the following estimate holds.

Lemma 6.1. *Let $s \geq 1$, $s \in \mathbb{N}$, and let $\pi \in \Pi^{conn}_{s;2n,n}$ (that is, $\bar{n} = 2n$) be a completely connected graph. Then,*

$$
|\mathrm{Amp}_\delta(\pi)| \leq \varepsilon^{2(s-1)} (\log \frac{1}{\varepsilon})^{s+2} (c\lambda^2 \varepsilon^{-1} \log \frac{1}{\varepsilon})^{sn} . \tag{109}
$$

Proof. The proof is completely analogous to the one given for Lemma 5.1 (using $\theta_{j-1}, \frac{1}{\kappa\varepsilon} \leq \frac{1}{\varepsilon}$), and shall not be reiterated here. All steps taken there can be adapted to the proof of Lemma 6.1 with minor modifications. $\square$

In contrast to the situation in the context of Lemma 5.1, the expectation in (107) does not exclude completely disconnected graphs. We recall that the sum over all amplitudes of completely disconnected graphs on r particle lines can be estimated by

$$
\sum_{\pi \in \Pi^{disc}_{r;\bar{n},n}} |\mathrm{Amp}_\delta(\pi)| \leq \Big(\sum_{\pi \in \Pi^{conn}_{1;\bar{n},n}} |\mathrm{Amp}_\delta(\pi)| \Big)^r . \tag{110}
$$

The bound for $s = 1$ in Lemma 6.1, however, is not good enough, since there is no factor ε^γ for any $\gamma > 0$. Using Lemma 6.1 for $s = 1$, one will not be able to compensate the large number $\sim 2^{nr}(n!)^r$ of disconnected graphs. We shall hence recall another result from [2] (the continuum version is proved in [4]), formulated in the following lemma.

Lemma 6.2. *Let $\bar{n} = 2n$. Then,*

$$\sum_{\pi \in \Pi^{conn}_{1;\bar{n},n}} |\mathrm{Amp}_\delta(\pi)| \leq \frac{(c\lambda^2\varepsilon^{-1})^n}{\sqrt{n!}} + (n!)\varepsilon^{\frac{1}{6}}(\log\frac{1}{\varepsilon})^3(c\lambda^2\varepsilon^{-1}\log\frac{1}{\varepsilon})^n \ . \tag{111}$$

The term $\frac{(c\lambda^2\varepsilon^{-1})^n}{\sqrt{n!}}$ bounds the so-called *ladder contribution*, while the last term carries an additional $\varepsilon^{\frac{1}{6}}$-factor relative to the bound provided by Lemma 5.1 for $s = 1$. It is obtained from *crossing* and *nesting type subgraphs* that appear in all non-ladder graphs. This issue is discussed in much detail in [2], and will not be recapitulated here. Since the number of non-ladder graphs is bounded by $n!2^n$, there is a factor $n!$.

The sum over non-disconnected graphs can be estimated by the same bound as in Lemma 5.3. The result is formulated in the following lemma.

Lemma 6.3. *Let $r \in 2\mathbb{N}$, $\bar{n} = 2n$, and $\Pi^{n-d}_{r;\bar{n},n} \subset \Pi_{r;\bar{n},n}$ denote the subclass of non-disconnected graphs. Then,*

$$\sum_{\pi \in \Pi^{n-d}_{r;\bar{n},n}} |\mathrm{Amp}_\delta(\pi)| \leq (r\bar{n})!\varepsilon^2(\log\frac{1}{\varepsilon})^{3r-2}(c\lambda^2\varepsilon^{-1}\log\frac{1}{\varepsilon})^{\frac{r\bar{n}}{2}} \ . \tag{112}$$

Combining (111) with Lemma 6.3, and applying the Minkowski inequality, the statement of Lemma 4.2 follows straightforwardly.

7. Proof of Lemma 4.3

We have, for $r \in 2\mathbb{N}$, $s \in [\theta_{j-1}, \theta_j]$, $n = 4N$, and $\bar{n} = 8N$,

$$\begin{aligned}
&\mathbb{E}\big[\|\phi_{4N,N,\theta_{j-1}}(s)\|_2^{2r}\big] \\
&\quad = e^{2r\varepsilon s}\int_{(I\times\bar{I})^r}\prod_{j=1}^r d\alpha_j d\beta_j\, e^{-is\sum_{j=1}^r(\alpha_j-\beta_j)} \\
&\qquad \times \int_{(\mathbb{T}^3)^{(\bar{n}+2)r}}\prod_{j=1}^r d\underline{p}^{(j)}\delta(p^{(j)}_{4N} - p^{(j)}_{4N+1})\mathbb{E}\Big[\prod_{j=1}^r U^{(j)}[\underline{p}^{(j)}]\Big] \\
&\qquad \times \prod_{j=1}^r K^{(j)}_{4N,N,\theta_{j-1},\kappa}[\underline{p}^{(j)},\alpha_j,\beta_j,\varepsilon]\hat{\phi}_0(p^{(j)}_0)\overline{\hat{\phi}_0(p^{(j)}_{8N+1})} \ ,
\end{aligned} \tag{113}$$

where again, $\varepsilon = \frac{1}{t}$. All notations are the same as in the proof of Lemma 4.2, in particlar, cf. (108) for the definition of $K^{(j)}_{4N,N,\theta_{j-1},\kappa}$. Let $\Pi^{conn}_{s;8N,4N}$ denote the subset of $\Pi_{s;8N,4N}$ of completely connected graphs.

Lemma 7.1. *Let $s \geq 1$, $s \in \mathbb{N}$, and let $\pi \in \Pi^{conn}_{s;8N,4N}$ be a completely connected graph. Then,*

$$|\mathrm{Amp}_\delta(\pi)| \leq \kappa^{-2rN}\varepsilon^{2(s-1)}(\log\frac{1}{\varepsilon})^{s+2}(c\lambda^2\varepsilon^{-1}\log\frac{1}{\varepsilon})^{\frac{s\bar{n}}{2}} \ . \tag{114}$$

Proof. We adapt the proof given for Lemma 5.1 in the following manner. First of all, we observe that (107) contains $r(6N+2)$ propagators with imaginary part $\pm i\kappa\varepsilon$, and $2rN$ propagators with imaginary part $\pm\frac{i}{\theta_{j-1}}$. In the proof of Lemma 5.1, $4rN$ out of all propagators were estimated in L^∞-norm, while the remaining ones were estimated in L^1.

Carrying out the same arguments line by line, we shall also estimate $4rN$ out of all propagators in (113) in L^∞. Since there are in total only $2rN$ propagators whose denominators carry an imaginary part $\pm\frac{i}{\theta_{j-1}}$, it follows that at least $2rN$ propagators bounded in L^∞ exhibit a denominator with an imaginary part $\pm i\kappa\varepsilon$. Correspondingly, one obtains an improvement of the upper bound by a factor κ^{-1} for each of the latter, in comparison to the L^∞-bound of $\frac{1}{\varepsilon}$ used throughout the proof of Lemma 5.1. Hence, there is in total a gain of a factor of at least κ^{-2rN} above the estimate derived for Lemma 5.1. □

Lemma 7.2. *Let $r \in 2\mathbb{N}$. Then,*

$$\sum_{\pi\in\Pi_{r;8N,4N}} |\mathrm{Amp}_\delta(\pi)| \leq (4rN)!\kappa^{-2rN}(\log\frac{1}{\varepsilon})^{3r}(c\lambda^2\varepsilon^{-1}\log\frac{1}{\varepsilon})^{4rN}\,. \tag{115}$$

Proof. Given a fixed $\pi \in \Pi_{r;8N,4N}$ with m connectivity components, let us assume that π comprises $s_1, \dots, s_m$ particle lines, where $\sum_{l=1}^m s_l = r$. Then,

$$\begin{aligned}|\mathrm{Amp}_\delta(\pi)| &\leq \kappa^{-2N\sum_{l=1}^m s_l}\varepsilon^{2\sum_{l=1}^m(s_l-1)} \\ &\quad\times(\log\frac{1}{\varepsilon})^{\sum_{l=1}^m(s_l+2)}(c\varepsilon^{-1}\lambda^2\log\frac{1}{\varepsilon})^{\frac{\bar{n}\sum_{l=1}^m s_l}{2}} \\ &\leq \kappa^{-2rN}(\log\frac{1}{\varepsilon})^{3r}(c\varepsilon^{-1}\lambda^2\log\frac{1}{\varepsilon})^{4rN}\,. \qquad (116)\end{aligned}$$

Furthermore, $\Pi_{r;8N,4N}$ contains no more than $(4rN)!2^{4rN}$ elements. □

The corresponding sum over disconnected graphs can be estimated by the same bound as in Lemma 6.3. This proves Lemma 4.3.

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