Forced vibrations of wave equations with non-monotone nonlinearities

Massimiliano Berti, Luca Biasco *

Abstract

We prove existence and regularity of periodic in time solutions of completely resonant nonlinear forced wave equations with Dirichlet boundary conditions for a large class of non-monotone forcing terms. Our approach is based on a variational Lyapunov-Schmidt reduction. It turns out that the infinite dimensional bifurcation equation exhibits an intrinsic lack of compactness. We solve it via a minimization argument and a-priori estimate methods inspired to regularity theory of [R67].

Key words: Wave Equation, Periodic Solutions, Variational Methods, A-priori Estimates, Lyapunov-Schmidt Reduction.

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Contents

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1 Introduction

In this paper we consider the problem of finding nontrivial time-periodic solutions of the completely resonant nonlinear *forced* wave equation

$$\Box u = \varepsilon f(t, x, u; \varepsilon) \tag{1.1}$$

with Dirichlet boundary conditions

$$u(t, 0) = u(t, \pi) = 0 \tag{1.2}$$

where $\Box := \partial_{tt} - \partial_{xx}$ is the D'Alembertian operator, ε is a small parameter and the nonlinear forcing term $f(t, x, u; \varepsilon)$ is T-periodic in time. We consider the case when T is a rational multiple of 2π and, for simplicity of exposition, we shall assume

$$T = 2\pi \ .$$

We look for nontrivial 2π-periodic in time solutions $u(t, x)$ of (1.1)-(1.2), i.e. satisfying

$$u(t + 2\pi, x) = u(t, x) \ . \tag{1.3}$$

For $\varepsilon = 0$, (1.1)-(1.2) reduces to the linear homogeneous wave equation

$$\begin{cases} \Box u = 0 \\ u(t, 0) = u(t, \pi) = 0 \end{cases} \tag{1.4}$$

which possesses an *infinite* dimensional space of solutions which are 2π-periodic in time and of the form $v(t, x) = \hat{v}(t + x) - \hat{v}(t - x)$ for any 2π-periodic function $\hat{v}(\cdot)$. For this reason equation (1.1)-(1.2) is called completely resonant.

The main difficulty for proving existence of solutions of (1.1)-(1.2)-(1.3) for $\varepsilon \neq 0$ is to find from which periodic orbits of the linear equation (1.4) the solutions of the nonlinear equation (1.1) branch off. This requires to solve an infinite dimensional bifurcation equation (also called kernel equation) with an intrinsic *lack of compactness*.

The first breakthrough regarding problem (1.1)-(1.2)-(1.3) was achieved by Rabinowitz in [R67] where existence and regularity of solutions was proved for nonlinearities satisfying the strongly monotone assumption $(\partial_u f)(t, x, u) \geq \beta > 0$. Using methods inspired by the theory of elliptic regularity, [R67] proved the existence of a unique curve of smooth solutions for ε small. Other existence results of weak and classical solutions have been obtained, still in the strongly monotone case, in [DST68]-[L69]-[BN78].

Subsequently, Rabinowitz [R71] was able to prove existence of weak solutions of (1.1)-(1.2)-(1.3) for a class of weakly monotone nonlinearities like $f(t,x,u) = u^{2k+1} + G(t,x,u)$ where $G(t,x,u_2) \geq G(t,x,u_1)$ if $u_2 \geq u_1$. Actually, in [R71] bifurcation of a global continuum branch of weak solutions is proved. For other local existence results in the weakly monotone case we mention [T69]-[H70].

In all the quoted papers the monotonicity assumption (strong or weak) is the key property for overcoming the lack of compactness in the infinite dimensional kernel equation.

We underline that, in general, the weak solutions obtained in [R71] are only continuous functions. Concerning regularity, Brezis and Nirenberg [BN78] proved -but only for strongly monotone nonlinearities- that any L^∞-solution of (1.1)-(1.2)-(1.3) is smooth, even in the nonperturbative case $\varepsilon = 1$, whenever the nonlinearity f is smooth.

On the other hand, very little is known about existence and regularity of solutions if we drop the monotonicity assumption on the forcing term f. Willem [W81], Hofer [H82] and Coron [C83] have considered the class of equations (1.1)-(1.2) where $f(t,x,u) = g(u) + h(t,x)$, $\varepsilon = 1$, and $g(u)$ satisfies suitable linear growth conditions. Existence of weak solutions is proved, in [W81]-[H82], for a set of h dense in L^2, although explicit criteria that characterize such h are not provided. The infinite dimensional bifurcation problem is overcome by assuming non-resonance hypothesys between the asymptotic behaviour of $g(u)$ and the spectrum of $\Box$. On the other side, Coron [C83] finds weak solutions assuming the additional symmetry $h(t,x) = h(t+\pi, \pi - x)$ and restricting to the space of functions satisfying $u(t,x) = u(t+\pi, \pi - x)$, where the Kernel of the d'Alembertian operator $\Box$ reduces to 0. For some more recent results see for example [BDL99].

In the present paper we prove existence and regularity of solutions of (1.1)-(1.2)-(1.3) for a large class of NONMONOTONE forcing terms $f(t,x,u)$, including, for example,

$f(t,x,u) = \pm u^{2k} + h(t,x)$, see *Theorem 1* ;

$f(t,x,u) = \pm u^{2k} + u^{2k+1} + h(t,x)$, see *Theorem 2* ;

$f(t,x,u) = \pm u^{2k} + \widetilde{f}(t,x,u)$ with $(\partial_u \widetilde{f})(t,x,u) \geq \beta > 0$, see *Theorem 3* .

The precise results will be stated in the next subsection 1.1, see *Theorems 1, 2* and *3*. Their proof is based on a variational Lyapunov-Schmidt reduction, minimization arguments and a-priori estimate methods inspired to regularity theory of [R67]. We anticipate that our approach -explained in subsection 1.2- is not merely a sharpening of the ideas of [R67]-[R71], which, to deal with non monotone nonlinearities, require a significant change of prospective.

We mention that in the last years several results on bifurcation of free vibrations for completely resonant autonomous wave equations have been proved in [B99]-[BP01]-[BB03]-[BB04a]-[BB04b]-[GMP04]. The main differences with respect to the present case are that: a "small divisor" problem in solving the "range equation" appears (here no small divisor problem is present due to the assumption $T = 2\pi$, see remark 1.6), but the infinite dimensional "bifurcation equation" -whose solution is the main problem of the present paper- gains crucial compactness properties, see remark 1.7.

1.1 Main Results

We look for solutions $u : \Omega \to \mathbb{R}$ of (1.1) in the Banach space

$$E := H^1(\Omega) \cap C_0^{1/2}(\overline{\Omega}), \qquad \Omega := \mathbb{T} \times (0, \pi)$$

where $H^1(\Omega)$ is the usual Sobolev space and $C_0^{1/2}(\overline{\Omega})$ is the space of all the 1/2-Hölder continuous functions $u : \overline{\Omega} \longrightarrow \mathbb{R}$ satisfying (1.2), endowed with norm[1]

$$\|u\|_E := \|u\|_{H^1(\Omega)} + \|u\|_{C^{1/2}(\overline{\Omega})} \ .$$

Critical points of the Lagrangian action functional $\Psi \in C^1(E, \mathbb{R})$

$$\Psi(u) := \Psi(u, \varepsilon) := \int_\Omega \Big[\frac{u_t^2}{2} - \frac{u_x^2}{2} + \varepsilon F(t, x, u; \varepsilon) \Big] \, dtdx \ , \tag{1.5}$$

where $F(t, x, u; \varepsilon) := \int_0^u f(t, x, \xi; \varepsilon) d\xi$, are weak solutions of (1.1)-(1.2)-(1.3).

For $\varepsilon = 0$, the critical points of Ψ in E reduce to the solutions of the linear equation (1.4) and form the subspace $V := N \cap H^1(\Omega)$ where

$$N := \left\{ \ v(t, x) = \hat{v}(t + x) - \hat{v}(t - x) \ \Big| \ \hat{v} \in L^2(\mathbb{T}) \text{ and } \int_0^{2\pi} \hat{v}(s) \, ds = 0 \ \right\}. \tag{1.6}$$

Note that $V := N \cap H^1(\Omega) = \{ v(t, x) = \hat{v}(t + x) - \hat{v}(t - x) \in N \mid \hat{v} \in H^1(\mathbb{T}) \} \subset E$, since any function $\hat{v} \in H^1(\mathbb{T})$ is 1/2-Hölder continuous.

Let $N^\perp := \{ h \in L^2(\Omega) \mid \int_\Omega hv = 0, \ \forall v \in N \}$ denote the $L^2(\Omega)$-orthogonal of N.

We prove the following Theorem:

Theorem 1 *Let $f(t, x, u) = \beta u^{2k} + h(t, x)$ and $h \in N^\perp$ satisfies $h(t, x) \geq \text{const} > 0$ (or $h(t, x) \leq \text{const} < 0$) a.e. in Ω. Then, for ε small enough, there exists at least one weak solution $u \in E$ of (1.1)-(1.2)-(1.3) with $\|u\|_E \leq C|\varepsilon|$. If, moreover, $h \in H^j(\Omega) \cap C^{j-1}(\overline{\Omega})$, $j \geq 1$, then $u \in H^{j+1}(\Omega) \cap C_0^j(\overline{\Omega})$ with $\|u\|_{H^{j+1}(\Omega)} + \|u\|_{C^j(\overline{\Omega})} \leq C|\varepsilon|$ and therefore, for $j \geq 2$, u is a classical solution.*

Theorem 1 is a Corollary of the following more general result which enables to deal with non-monotone nonlinearities like, for example, $f(t, x, u) = \pm(\sin x) \ u^{2k} + h(t, x)$, $f(t, x, u) = \pm \ u^{2k} + u^{2k+1} + h(t, x)$.

[1]Here $\|u\|_{H^1(\Omega)}^2 := \|u\|_{L^2(\Omega)}^2 + \|u_x\|_{L^2(\Omega)}^2 + \|u_t\|_{L^2(\Omega)}^2$ and

$$\|u\|_{C^{1/2}(\overline{\Omega})} := \|u\|_{C^0(\Omega)} + \sup_{(t,x) \neq (t_1, x_1)} \frac{|u(t, x) - u(t_1, x_1)|}{(|t - t_1| + |x - x_1|)^{1/2}} \ .$$

Theorem 2 *Let* $f(t,x,u) = g(t,x,u) + h(t,x)$, $h(t,x) \in N^{\perp}$ *and*

$$g(t,x,u) := \beta(x)u^{2k} + \mathcal{R}(t,x,u)$$

where $\mathcal{R}$, $\partial_t \mathcal{R}$, $\partial_u \mathcal{R} \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ *satisfy*[2]

$$\|\mathcal{R}(\cdot,u)\|_{C(\overline{\Omega})} = o(u^{2k}), \quad \|\partial_t \mathcal{R}(\cdot,u)\|_{C(\overline{\Omega})} = O(u^{2k}), \quad \|\partial_u \mathcal{R}(\cdot,u)\|_{C(\overline{\Omega})} = o(u^{2k-1}), \quad (1.7)$$

and $\beta \in C([0,\pi],\mathbb{R})$ *verifies, for* $x \in (0,\pi)$, $\beta(x) > 0$ (*or* $\beta(x) < 0$) *and* $\beta(\pi - x) = \beta(x)$.

(**i**) (**Existence**) *Assume there exists a weak solution* $H \in E$ *of* $\Box H = h$ *such that*

$$H(t,x) > 0 \qquad (\text{or } H(t,x) < 0) \qquad \forall (t,x) \in \Omega \ . \qquad (1.8)$$

Then, for ε *small enough, there exists at least one weak solution* $u \in E$ *of* (1.1)-(1.2)-(1.3) *satisfying* $\|u\|_E \le C|\varepsilon|$.

(**ii**) (**Regularity**) *If, moreover,* $h \in H^j(\Omega) \cap C^{j-1}(\overline{\Omega})$, $\beta \in H^j\big((0,\pi)\big)$, $\mathcal{R}$, $\partial_t \mathcal{R}$, $\partial_u \mathcal{R}$ $\in C^j(\overline{\Omega} \times \mathbb{R})$, $j \ge 1$, *then* $u \in H^{j+1}(\Omega) \cap C^j_0(\overline{\Omega})$ *and, for* $j \ge 2$, u *is a classical solution.*

Note that Theorem 2 does not require any growth condition on g at infinity. In particular it applies for any analytic function $g(u)$ satisfying $g(0) = g'(0) = \ldots = g^{2k-1}(0) = 0$ and $g^{2k}(0) \neq 0$.

We now collect some comments on the previous results.

Remark 1.1 *The assumption* $h \in N^{\perp}$ *is not of technical nature both in Theorem 1 and in Theorem 2* (*at least if* $g = g(x,u) = g(x,-u) = g(\pi - x, u)$). *Indeed, if* $h \notin N^{\perp}$, *periodic solutions of problem* (1.1)-(1.2)-(1.3) *do not exist in any fixed ball* $\{\|u\|_{L^\infty} \le R\}$, $R > 0$, *for* ε *small; see remark 4.4.*

Remark 1.2 *In Theorem 2 hypothesys* (1.8) *and* $\beta > 0$ (*or* $\beta < 0$) *are assumed to prove the existence of a minimum of the "reduced action functional"* Φ, *see* (1.17). *A sufficient condition implying* (1.8) *is* $h \ge \text{const} > 0$ *a.e. in* Ω, *see the "maximum principle" Proposition 4.10. This is also the key step to derive Theorem 1 from Theorem 2.*
We remark that hypothesis (1.8) *can be weakened requiring only*[3] $H \ge 0$ *in* Ω *and*

$$H > 0 \quad \text{in} \quad \Omega^k := \Big\{(t,x) \in \Omega \ \Big| \ \beta_k \le x \le \pi - \beta_k\Big\} \qquad (1.9)$$

where $\beta_1 := \frac{\pi}{8}$, $\beta_k := \frac{\pi}{4(2k+1)}$ *for* $k \ge 2$, *see remark 4.3.*
Analogously, in Theorem 1, we can require only $h \ge 0$ *a.e. in* Ω *and* $h \ge \text{const} > 0$ *a.e. in some open set containing* Ω^k *(see again Proposition 4.10). In particular, if* h *is continuous, it is sufficient to require* $h > 0$ *in* Ω.

[2]The notation $f(z) = o(z^p)$, $p \in \mathbb{N}$, means that $f(z)/|z|^p \to 0$ as $z \to 0$. $f(z) = O(z^p)$ means that there exists a constant $C > 0$ such that $|f(z)| \le C|z|^p$ for all z in a neighboorhood of 0.

[3]Or $H \le 0$ in Ω and $H < 0$ in Ω_k.

Remark 1.3 (Regularity) *It is quite surprising that the weak solution u of Theorems 1, 2 is actually smooth. Indeed, while regularity always holds true for strictly monotone nonlinearities (see [R67]-[BN78]), yet for weakly monotone f it is not proved in general, unless the weak solution u verifies $\|\Pi_N u\|_{L^2} \geq C > 0$ (see [R71]). Note, on the contrary, that the weak solution u of Theorem 2 satisfies $\|\Pi_N u\|_{L^2} = O(\varepsilon)$.*

Moreover, assuming

$$\|\partial_t^l \partial_x^m \partial_u^n \mathcal{R}\|_{C(\overline{\Omega})} = O(u^{2k-n}), \quad \forall\, 0 \leq l, n \leq j+1\,, 0 \leq m \leq j\,, l+m+n \leq j+1 \tag{1.10}$$

we can also prove the estimate (see remark 4.8)

$$\|u\|_{H^{j+1}(\Omega)} + \|u\|_{C^j(\overline{\Omega})} \leq C|\varepsilon|\,. \tag{1.11}$$

Remark 1.4 (Multiplicity) *For nonmonotone nonlinearities f one can* NOT *in general expect unicity of the solutions. Actually, for $f(t,x,u) = g(x,u) + h(t,x)$ with $g(x,u) = g(x,-u)$, $g(\pi - x, u) = g(x,u)$, there exist infinitely many $h \in N^\perp$ for which problem* (1.1)-(1.2)-(1.3) *has (at least)* 3 *solutions, see remark 4.5.*

Finally, we extend the result of [R67] proving existence of periodic solutions for non-monotone nonlinearities $f(t,x,u)$ obtained adding to a nonlinearity $\widetilde{f}(t,x,u)$ as in [R67] (i.e. $\partial_u \widetilde{f} \geq \beta > 0$) any nonmonotone term $a(x,u)$ satisfying

$$a(x,-u) = a(x,u)\,, \qquad a(\pi - x, u) = a(x,u) \tag{1.12}$$

or

$$a(x,-u) = -a(x,u)\,, \qquad a(\pi - x, u) = -a(x,u)\,. \tag{1.13}$$

A prototype nonlinearity is $f(t,x,u) = \pm u^{2k} + \widetilde{f}(t,x,u)$ with $\partial_u \widetilde{f} \geq \beta > 0$.

Theorem 3 *Let $f(t,x,u) = \widetilde{f}(t,x,u) + a(x,u)$ where f, $\partial_t f$, $\partial_u f$ are continuous, $\partial_u \widetilde{f} \geq \beta > 0$ and $a(x,u)$ satisfy* (1.12) *or* (1.13). *Then, for ε small enough,* (1.1)-(1.2)-(1.3) *has at least one weak solution $u \in E$. If moreover f, $\partial_t f$, $\partial_u f \in C^j(\overline{\Omega} \times \mathbb{R})$, $j \geq 1$, then $u \in H^{j+1}(\Omega) \cap C_0^j(\overline{\Omega})$.*

In the next subsection we describe our method of proof.

1.2 Scheme of the Proof

In order to find critical points of the Lagrangian action functional $\Psi : E \to \mathbb{R}$ defined in (1.5) we perform a variational Lyapunov-Schmidt reduction, decomposing the space $E := H^1(\Omega) \cap C_0^{1/2}(\overline{\Omega})$ as

$$E = V \oplus W$$

where

$$V := N \cap H^1(\Omega) \qquad \text{and} \qquad W := N^\perp \cap H^1(\Omega) \cap C_0^{1/2}(\overline{\Omega})\,.$$

Setting $u = v + w$ with $v \in V$, $w \in W$ and denoting by Π_N and $\Pi_{N^\perp}$ the projectors from $L^2(\Omega)$ onto N and $N^\perp$ respectively, problem (1.1)-(1.2)-(1.3) is equivalent to solve the *kernel equation*

$$\Pi_N \, f(v + w, \varepsilon) = 0 \tag{1.14}$$

and the *range equation*

$$w = \varepsilon \Box^{-1} \, \Pi_{N^\perp} \, f(v + w, \varepsilon) \tag{1.15}$$

where $\Box^{-1} : N^\perp \to N^\perp$ is the inverse of $\Box$ and $f(u, \varepsilon)$ denotes the Nemitski operator associated to f, namely

$$[f(u, \varepsilon)](t, x) := f(t, x, u, \varepsilon).$$

Remark 1.5 *The usual approach (see [R67]-[DST68]-[T69]-[R71]) is to find, first, by the monotonicity of f, the unique solution $v = v(w)$ of the kernel equation* (1.14) *and, next, to solve the range equation* (1.15). *On the other hand, for nonmonotone forcing terms, one can not in general solve uniquely the kernel equation –recall by remark 1.4 that in general unicity of solutions does not hold. Therefore we must solve first the range equation and thereafter the kernel equation.*

We solve, first, the range equation by means of a quantitative version of the Implicit Function Theorem, finding a solution $w := w(v, \varepsilon) \in W$ of (1.15) with $\|w(v, \varepsilon)\|_E = O(\varepsilon)$, see Proposition 3.2. Here no serious difficulties arise since $\Box^{-1}$ acting on W is a compact operator, due to the assumption $T = 2\pi$, see (2.2).

Remark 1.6 *More in general, $\Box^{-1}$ is compact on the orthogonal complement of ker$(\Box)$ whenever T is a rational multiple of 2π. On the contrary, if T is an irrational multiple of 2π, then $\Box^{-1}$ is, in general, unbounded* (*a "small divisor" problem appears*), *but the kernel of $\Box$ reduces to* 0 (*there is no bifurcation equation*). *For existence of periodic solutions in the case $T/2\pi$ is irrational see [PY89].*

Once the range equation (1.15) has been solved by $w(v, \varepsilon) \in W$ it remains the infinite dimensional kernel equation (also called bifurcation equation)

$$\Pi_N f(v + w(v, \varepsilon), \varepsilon) = 0 \; . \tag{1.16}$$

We note (see Lemma 3.3) that (1.16) is the Euler-Lagrange equation of the *reduced Lagrangian action functional*

$$\Phi : V \longrightarrow \mathbb{R} \qquad\qquad \Phi(v) := \Phi(v, \varepsilon) := \Psi(v + w(v, \varepsilon), \varepsilon) \; . \tag{1.17}$$

Φ lacks compactness properties and, therefore, to find critical points of Φ, we can not rely on critical point theory:

Remark 1.7 *Implementing an analogue Lyapunov-Schmidt reduction in the autonomous case (see [BB03]) it turns out that, in the corresponding reduced Lagrangian action functional, a further term proportional to $\|v\|^2_{H^1}$ is present. Therefore it is possible to apply critical point theory (e.g. the Mountain Pass Theorem) to find existence and multiplicity of solutions, see [BB04a]. The elliptic term $\|v\|^2_{H^1}$ helps also in proving regularity results for the solutions.*

We attempt to minimize Φ.

We do not try to apply the direct methods of the calculus of variations. Indeed Φ, even though it could possess some coercivity property, will not be convex (being f non monotone). Moreover, without assuming any growth condition on the nonlinearity f, the functional Φ could neither be well defined on any L^p-space.

Therefore we minimize Φ in any $\overline{B_R} := \big\{ v \in V, \|v\|_{H^1} \leq R \big\}$, $\forall R > 0$, as in [R67]. By standard compactness arguments Φ attains minimum at, say, $\bar{v} \in \overline{B_R}$. Since $\bar{v}$ could belong to the boundary $\partial \overline{B_R}$, $\bar{v}$ could not be a solution of (1.16) and we can only conclude the variational inequality

$$D_v \Phi(\bar{v})[\varphi] = \int_\Omega f(\bar{v} + w(\bar{v}, \varepsilon), \varepsilon)\varphi \leq 0 \tag{1.18}$$

for any *admissible variation* $\varphi \in V$, i.e. if $\bar{v} + \theta\varphi \in \overline{B_R}$, $\forall \theta < 0$ sufficiently small.

The heart of the existence proof of the weak solution u of Theorem 1, Theorem 2 and Theorem 3 is to obtain, choosing suitable admissible variations like in [R67], the A-PRIORI ESTIMATE $\|\bar{v}\|_{H^1} < R$ for some $R > 0$, i.e. to show that $\bar{v}$ is an *inner* minimum point of Φ in B_R.

The strong monotonicity assumption $(\partial_u f)(t, x, u) \geq \beta > 0$ would allow here to get such a-priori estimates by arguments similar to [R67]. On the contrary, the main difficulty for proving Theorems 1, 2 and 3 which deal with non-monotone nonlinearities is to obtain such a priori-estimates for $\bar{v}$.

The most difficult cases are the proof of Theorems 1 and 2. To understand the problem, let consider the particular nonlinearity $f(t, x, u) = u^{2k} + h(t, x)$ of Theorem 1. The even term u^{2k} does not give any contribution into the variational inequality (1.18) at the 0^{th}-order in ε, since the right hand side of (1.18) reduces, for $\varepsilon = 0$, to

$$\int_\Omega \Big(\bar{v}^{2k} + h(t, x) \Big) \varphi = 0, \qquad \forall \varphi \in V$$

by (2.18) and $h \in N^\perp$.

Therefore, for deriving, if ever possible, the required a-priori estimates, we have to develop the variational inequality (1.18) at higher orders in ε. We obtain

$$0 \geq \int_\Omega 2k\bar{v}^{2k-1}\varphi \; w(\varepsilon, \bar{v}) + O(w^2(\varepsilon, \bar{v})) = \int_\Omega \varepsilon \; 2k\bar{v}^{2k-1}\varphi \, \square^{-1}(h + \bar{v}^{2k}) + O(\varepsilon^2) \tag{1.19}$$

because $w(\bar{v},\varepsilon) = \varepsilon\Box^{-1}(\bar{v}^{2k} + h) + o(\varepsilon)$ (recall that $\bar{v}^{2k}$, $h \in N^{\perp}$).

We now sketch how the ε-order term in the variational inequality (1.19) allows to prove an L^{2k}-estimate for $\bar{v}$. Inserting the admissible variation $\varphi := \bar{v}$ in (1.19) we get

$$\int_{\Omega} H\bar{v}^{2k} + \bar{v}^{2k}\Box^{-1}\bar{v}^{2k} \leq O(\varepsilon) \tag{1.20}$$

where H is a weak solution of $\Box H = h$ which verifies $H(t,x) > 0$ in Ω (H exists by the "maximum principle" Proposition 4.10). The crucial fact is that the first term in (1.20) satisfies the coercivity inequality

$$\int_{\Omega} Hv^{2k} \geq c(H)\int_{\Omega} v^{2k}, \qquad \forall\, v \in V \tag{1.21}$$

for some constant $c(H) > 0$, see Proposition 4.1. The second term $\int_{\Omega}\bar{v}^{2k}\Box^{-1}\bar{v}^{2k}$ will be negligible, ε-close to the origin, with respect to $\int_{\Omega} Hv^{2k}$ and (1.20), (1.21) will provide the L^{2k}-estimate for $\bar{v}$.

We remark that the inequality (1.21) is not trivial because H vanishes at the boundary ($H(t,0) = H(t,\pi) = 0$). Actually, the proof of (1.21) relies on the form $v(t,x) = \hat{v}(t+x) - \hat{v}(t-x)$ of the functions of V.

Next, we can obtain, choosing further admissible variations φ in (1.19) and using inequalities similar to (1.21), an L^{∞}-estimate for $\bar{v}$ and, finally, the required H^1-estimate, proving the existence of a weak solution $u \in E$, see section 4.

Moreover, using similar techniques inspired to regularity theory and further suitable variations, we can also obtain a-priori estimates for the L^{∞}-norm of the higher order derivatives of $\bar{v}$ and for its H^j-Sobolev norms. In this way we can prove the regularity of the solution u -fact quite surprising for non-monotone nonlinearities-, see subsection 4.5. Theorem 2 is proved developing such ideas and a careful analysis of the further term $\mathcal{R}$.

The proof of Theorem 3 is easier than for Theorems 1 and 2. Indeed the additional term $a(x,u)$ does not contribute into the variational inequality (1.18) at the 0^{th}-order in ε, because $\int_{\Omega} a(x,\bar{v})\varphi \equiv 0$, $\forall \varphi \in V$, by (2.19). Therefore the dominant term in the variational inequality (1.18) is provided by the monotone forcing term $\widetilde{f}$ and the required a-priori estimates are obtained with arguments similar to [R67], see section 5.

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Notations: $\Omega := \mathbb{T} \times (0,\pi)$ where $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. We denote by $C^j(\overline{\Omega})$ the Banach space of functions $u : \overline{\Omega} \longrightarrow \mathbb{R}$ with j derivatives in Ω continuous up to the boundary $\partial\Omega$, endowed with the standard norm $\| \ \|_{C^j}$. $C_0^j(\overline{\Omega}) := C^j(\overline{\Omega}) \cap C_0(\overline{\Omega})$ where $C_0(\overline{\Omega})$ is the space of real valued continuous functions satisfying $u(t,0) = u(t,\pi) = 0$. Moreover $H^j(\Omega) := W^{j,2}(\Omega)$

are the usual Sobolev spaces with scalar product $\langle \cdot, \cdot \rangle_{H^j}$ and norm $\| \ \|^2_{H^j(\Omega)}$. Here $C^j(\mathbb{T})$ denotes the Banach space of periodic functions $u : \mathbb{T} \to \mathbb{R}$ with j continuous derivatives. Finally, $H^j(\mathbb{T})$ is the usual Sobolev space of 2π-periodic functions.

2 Preliminaries

We first collect some important properties on the D'Alembertian operator $\Box$.

Definition 2.1 *Given $f(t,x) \in L^2(\Omega)$, a function $u \in L^2(\Omega)$ is said to be a weak solution of $\Box u = f$ in Ω satisfying the boundary conditions $u(t,0) = u(t,\pi) = 0$, iff*

$$\int_\Omega u \Box \varphi = \int_\Omega f \varphi \qquad \forall \varphi \in C^2_0(\overline{\Omega}).$$

It is easily verified that, if $u \in C^2(\overline{\Omega})$ is a weak solution of $\Box u = f$ according to definition 2.1, then u is a classical solution and $u(t,0) = u(t,\pi) = 0$.

The kernel $N \subset L^2(\Omega)$ of the D'Alembertian operator $\Box$, i.e. the space of weak solutions of the homogeneous linear equation $\Box v = 0$ verifying the Dirichlet boundary conditions $v(t,0) = v(t,\pi) = 0$, is the subspace N defined in (1.6). N coincides with the closure in $L^2(\Omega)$ of the classical solution of $\Box v = 0$ verifying Dirichlet boundary conditions which, as well known, are of the form $v(t,x) = \hat{v}(t+x) - \hat{v}(t-x)$, $\hat{v} \in C^2(\mathbb{T})$.

Using Fourier series we can also characterize N as

$$N = \left\{ v(t,x) \in L^2(\Omega) \mid v(t,x) = \sum_{j \in \mathbb{Z}} a_j e^{\mathrm{i} jt} \sin jx \qquad \text{with} \qquad \sum_{j\in\mathbb{Z}} |a_j|^2 < \infty \right\}.$$

The range of $\Box$ in $L^2(\Omega)$ is

$$\begin{aligned} N^\perp &:= \left\{ f \in L^2(\Omega) \;\Big|\; \int_\Omega fv = 0, \ \forall v \in N \right\}, \\ &= \left\{ f(t,x) \in L^2(\Omega) \;\Big|\; f(t,x) = \sum_{\substack{l\in\mathbb{Z}, j\geq 1 \\ j \neq |l|}} f_{lj} e^{\mathrm{i}lt} \sin jx \quad \text{with} \quad \sum_{\substack{l\in\mathbb{Z}, j\geq 1 \\ j \neq |l|}} |f_{lj}|^2 < \infty \right\}. \end{aligned}$$

i.e. $\forall f(t,x) \in N^\perp$ there exists a unique weak solution $u = \Box^{-1} f \in N^\perp$ of $\Box u = f$.

Furthermore $\Box^{-1}$ is a bounded operator such that

$$\Box^{-1} \ : \ N^\perp \longrightarrow N^\perp \cap H^1(\Omega) \cap C^{1/2}_0(\overline{\Omega}) \tag{2.1}$$

i.e. there exists a suitable constant $\underline{c} \geq 1$ such that

$$\left\| \Box^{-1} f \right\|_E \leq \underline{c} \, \|f\|_{L^2} \qquad \forall f \in L^2(\Omega) \tag{2.2}$$

where $\|u\|_E := \|u\|_{H^1} + \|u\|_{C^{1/2}}$. By (2.2) and the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, the operator $\Box^{-1} : N^\perp \to N^\perp$ is compact.

These assertions follow easily from the Fourier series representation (see e.g. [BN78])

$$f(t,x) := \sum_{j\geq 1, j\neq |l|} f_{lj} e^{\mathrm{i}lt} \sin jx \quad \Rightarrow \quad u = \Box^{-1} f := \sum_{j\geq 1, j\neq |l|} \frac{f_{lj}}{-l^2 + j^2} e^{\mathrm{i}lt} \sin jx$$

noting that u is a weak solution of (1.1) (according to definition 2.1) iff $u_{lj} = f_{lj}/(-l^2+j^2)$ $\forall l \in \mathbf{Z}$, $j \geq 1$, see e.g. [DT68], [H82].

To continue, $\Box^{-1}$ is a bounded operator also between the spaces

$$L^\infty(\Omega) \longrightarrow C^{0,1}(\overline{\Omega}) , \qquad H^k(\Omega) \longrightarrow H^{k+1}(\Omega) , \qquad C^k(\overline{\Omega}) \longrightarrow C^{k+1}(\overline{\Omega}) \tag{2.3}$$

as follows by the integral formula for $u = \Box^{-1} f = \Pi_{N^\perp} \psi$ where (see e.g. [L69], [BCN80])

$$\psi(t,x) := -\frac{1}{2} \int_x^\pi \int_{t+x-\xi}^{t-x+\xi} f(\xi,\tau) \; d\tau d\xi + c\frac{(\pi - x)}{\pi} ,$$

$\Pi_{N^\perp} : L^2(\Omega) \to N^\perp$ is the orthogonal projector onto $N^\perp$ and

$$c := \frac{1}{2} \int_0^\pi \int_{t-\xi}^{t+\xi} f(\xi,\tau) \; d\tau d\xi \equiv \text{const} \tag{2.4}$$

is a constant independent of t, because[4] $f \in N^\perp$.

We also have, since $\partial_t^j H$ is a weak solution of $\Box(\partial_t^j H) = \partial_t^j h$ and (2.1) applies,

$$h \in H^j(\Omega) \qquad \Longrightarrow \qquad \partial_t^j H \in C_0^{1/2}(\overline{\Omega}) \; . \tag{2.5}$$

Finally, the projector $\Pi_N : L^2(\Omega) \to N$ can be written as $\Pi_N u = p(t+x) - p(t-x)$ where

$$p(y) := \frac{1}{2\pi} \int_0^\pi \Big[u(y-s,s) - u(y+s,s) \Big] ds$$

[4]We have that $2c = \int_{\mathcal{T}(t)} f = \lim_{n\to\infty} \int_{\mathcal{T}(t)} f_n$ where $\mathcal{T}(t) := \{(\tau,\xi) \in \Omega \text{ s.t. } t-\xi < \tau < t+\xi,$ $0 < \xi < \pi\}$ and

$$f_n(t,x) := \sum_{\substack{|l|,j\leq n \\ j\neq |l|}} f_{l,j} e^{\mathrm{i}lt} \sin jx \xrightarrow{L^2} \sum_{j\neq |l|} f_{l,j} e^{\mathrm{i}lt} \sin jx = f(t,x) \in N^\perp \; .$$

The claim follows since $\int_{\mathcal{T}(t)} f_n$ is, for any n, independent on t :

$$\begin{aligned} \int_{\mathcal{T}(t)} f_n &= \int_0^\pi \sum_{\substack{|l|,j\leq n \\ j\neq |l|}} f_{l,j} \sin j\xi \int_{t-\xi}^{t+\xi} e^{\mathrm{i}lt} d\tau d\xi \\ &= \sum_{1\leq j\leq n} f_{0,j} \int_0^\pi 2\xi \sin j\xi \, d\xi + \sum_{\substack{|l|,j\leq n \\ j\neq |l|, l\neq 0}} 2\frac{f_{l,j}}{l} e^{\mathrm{i}lt} \int_0^\pi \sin j\xi \sin l\xi \, d\xi = \sum_{1\leq j\leq n} f_{0,j} \int_0^\pi 2\xi \sin j\xi \, d\xi \; . \end{aligned}$$

and therefore, since $u \in C^j(\overline{\Omega}) \Rightarrow p \in C^j(\mathbb{T})$ and $u \in H^j(\Omega) \Rightarrow p \in H^j(\mathbb{T})$,

$$\Pi_N, \ \Pi_{N\perp} : C^j(\overline{\Omega}) \to C^j(\overline{\Omega}) \qquad \text{are bounded} \tag{2.6}$$

$$\Pi_N, \ \Pi_{N\perp} : H^j(\Omega) \to H^j(\Omega) \qquad \text{are bounded.} \tag{2.7}$$

Kernel properties and technical Lemmata

Let define, for $0 \le \alpha < 1/2$,

$$\Omega_\alpha := \mathbb{T} \times (\alpha\pi, \pi - \alpha\pi) \subseteq \Omega \ . \tag{2.8}$$

Lemma 2.2 *Let $a \in L^1(\Omega)$.*

$$\int_{\Omega_\alpha} a(t,x)\, dt\, dx = \frac{1}{2} \int_0^{2\pi} \int_{-2\pi+s_+ +2\alpha\pi}^{s_+ - 2\alpha\pi} a\left(\frac{s_+ + s_-}{2}, \frac{s_+ - s_-}{2}\right) ds_-\, ds_+ \ . \tag{2.9}$$

In particular for $p, q \in L^1(\mathbb{T})$

$$\int_{\Omega_\alpha} p(t+x)q(t-x)\, dt\, dx = \frac{1}{2}\int_0^{2\pi} p(s) ds \int_0^{2\pi} q(s) ds - \frac{1}{2}\int_{-2\alpha\pi}^{2\alpha\pi}\int_0^{2\pi} p(y)q(z+y) dy dz \tag{2.10}$$

and

$$\int_{\Omega_\alpha} p(t+x)\, dt\, dx = \int_{\Omega_\alpha} p(t-x)\, dt\, dx = \pi(1-2\alpha)\int_0^{2\pi} p(s) ds \ . \tag{2.11}$$

Moreover, given $f, g : \mathbb{R} \to \mathbb{R}$ continuous,

$$\int_{\Omega_\alpha} f(p(t+x))g(p(t-x))\, dt\, dx = \int_{\Omega_\alpha} f(p(t-x))g(p(t+x))\, dt\, dx \ . \tag{2.12}$$

Proof: In the Appendix. □

Lemma 2.3 *For any $v = \hat{v}(t+x) - \hat{v}(t-x) := v_+ - v_- \in N$*

$$\|v\|^2_{L^2(\Omega)} = 2\pi \|\hat{v}\|^2_{L^2(\mathbb{T})} = 2\pi \int_0^{2\pi} \hat{v}^2 \ . \tag{2.13}$$

Moreover

$$\|v_t\|^2_{L^2(\Omega)} = \|v_x\|^2_{L^2(\Omega)} = 2\pi\|\hat{v}'\|^2_{L^2(\mathbb{T})} \qquad \forall v \in N \cap H^1(\Omega) \tag{2.14}$$

$$\|\hat{v}\|_{L^\infty(\mathbb{T})} \le \|v\|_{L^\infty(\Omega)} \le 2\|\hat{v}\|_{L^\infty(\mathbb{T})} \qquad \forall v \in N \cap L^\infty(\Omega) \tag{2.15}$$

$$\|v\|_{L^\infty(\Omega)} \le \|v\|_{H^1(\Omega)} \qquad \forall v \in N \cap H^1(\Omega) \ . \tag{2.16}$$

PROOF: In the Appendix. $\square$

Lemma 2.4 *Let* $\varphi_1, \dots, \varphi_{2k+1} \in N$ *and* $\varphi_1 \cdot \ldots \cdot \varphi_{2k+1} \in L^1(\Omega)$. *Then*

$$\int_{\Omega_\alpha} \varphi_1 \cdot \ldots \cdot \varphi_{2k+1} = 0 \ . \tag{2.17}$$

In particular $\varphi_1 \cdot \ldots \cdot \varphi_{2k} \in N^\perp$.

Moreover, if $a : \Omega \longrightarrow \mathbb{R}$ *satisfies (i)* $a(x,u) = a(\pi - x, u)$ *and* $a(x,u) = a(x,-u)$ *or (ii)* $a(x,u) = -a(\pi - x, u)$ *and* $a(x,u) = -a(x,-u)$, *then*

$$\int_{\Omega_\alpha} a(x,v)\, \varphi = 0 \qquad \forall\, v \in N \cap L^\infty \ , \ \varphi \in N \tag{2.18}$$

and

$$\int_{\Omega_\alpha} (\partial_u a)(x,v)\, \varphi_1 \varphi_2 = 0 \qquad \forall\, v \in N \cap L^\infty \ , \ \varphi_1, \varphi_2 \in N \ . \tag{2.19}$$

PROOF: In the Appendix. $\square$

Lemma 2.5 *The following inequalities hold:*

$$(a-b)^{2k} \le 2^{2k-1}(a^{2k} + b^{2k}) \qquad \forall\ a, b \in \mathbb{R}\ , \tag{2.20}$$

$$(a-b)^{2k} \ge a^{2k} + b^{2k} - 2k(a^{2k-1}b + ab^{2k-1}) \qquad \forall\ a, b \in \mathbb{R}, \quad k \in \mathbb{N}, k \ge 2 \tag{2.21}$$

$$(a+b)^{2k-1} - a^{2k-1} \ge 4^{1-k}\, b^{2k-1} \qquad \forall\ a \in \mathbb{R}\ ,\ b > 0\ ,\ k \in \mathbb{N}^+\ . \tag{2.22}$$

PROOF: In the Appendix. $\square$

Lemma 2.6 *Let* $v \in N \cap L^{2k}(\Omega)$ *and* $k \in \mathbb{N}^+$. *Then*

$$\int_\Omega v^{2k} \le \pi 4^k \int_0^{2\pi} \hat{v}^{2k} \ . \tag{2.23}$$

Moreover, for $k = 1$

$$\int_{\Omega_\alpha} v^2 \ge 2\pi \Big[1 - 4\alpha\Big] \int_0^{2\pi} \hat{v}^2 \ge \pi \int_0^{2\pi} \hat{v}^2$$

if $0 \le \alpha \le 1/8$. *For* $k \ge 2$,

$$\int_{\Omega_\alpha} v^{2k} \ge 2\pi \Big[1 - 2(1+2k)\alpha\Big] \int_0^{2\pi} \hat{v}^{2k} \ge \pi \int_0^{2\pi} \hat{v}^{2k}$$

if $0 \le \alpha \le 1/4(1+2k)$.

PROOF: In the Appendix. $\square$

Generalities about the difference quotients

For $f \in L^2(\Omega)$ we define the difference quotient of size $h \in \mathbb{R} \setminus \{0\}$

$$(D_h f)(t,x) := \frac{f(t+h,x) - f(t,x)}{h}$$

and the h-translation

$$(T_h f)(t,x) := f(t+h,x)$$

with respect to time.

The following Lemma collects some elementary properties of the difference quotient.

Lemma 2.7 *Let $f, g \in L^2(\Omega)$, $h \in \mathbb{R} \setminus \{0\}$. The following holds*

(i) Leibniz rule:

$$\begin{aligned} D_h(fg) &= (D_h f)g + T_h f D_h g \ , \qquad (2.24) \\ D_h f^m &= (D_h f) \sum_{j=0}^{m-1} f^{m-j-1} T_h f^j \\ &= m(D_h f) f^{m-1} + (D_h f) \sum_{j=0}^{m-1} f^{m-j-1}(T_h f^j - f^j) \ , \qquad (2.25) \\ \int_\Omega D_h(fg) &= \int_\Omega (D_h f) g + f(D_{-h} g) \ ; \qquad (2.26) \end{aligned}$$

(ii) integration by parts:

$$\int_\Omega f(D_{-h} g) = -\int_\Omega (D_h f) g \ ; \qquad (2.27)$$

(iii) weak derivative: If there exists a constant $C > 0$ such that $\forall h$ small

$$\|D_h f\|_{L^2} \leq C \quad \Rightarrow \quad \text{then } f \text{ has a weak derivative } f_t \text{ and } \|f_t\|_{L^2} \leq C \qquad (2.28)$$

Moreover, if f has a weak derivative $f_t \in L^2(\Omega)$, then

(iv) estimate on the difference quotient:

$$\|D_h f\|_{L^2(\Omega)} \leq \|f_t\|_{L^2(\Omega)} \ ; \qquad (2.29)$$

(v) convergence:

$$D_h f \xrightarrow{L^2} f_t \qquad as \quad h \longrightarrow 0 \ . \qquad (2.30)$$

PROOF. In the Appendix. $\square$

3 The Lyapunov-Schmidt decomposition

3.1 The Range equation

We first solve the range equation (1.15) applying the following quantitative version of the Implicit Function Theorem, whose standard proof is omitted.

Proposition 3.1 *Let X, Y, Z be Banach spaces and $x_0 \in X$, $y_0 \in Y$. Fix $r, \rho > 0$ and define $X_r := \{x \in X \text{ s.t. } \|x - x_0\|_X < r\}$ and $Y_\rho := \{y \in Y \text{ s.t. } \|y - y_0\|_Y < \rho\}$. Let $\mathcal{F} \in \mathcal{C}^1(U, Z)$ where $\overline{Y_\rho} \times \overline{X_r} \subset U \subset Y \times X$ is an open set. Suppose that*

$$\mathcal{F}(y_0, x_0) = 0 \tag{3.1}$$

and that $D_y\mathcal{F}(y_0, x_0) \in \mathcal{L}(Y, Z)$ is invertible. Let $T := \Big(D_y\mathcal{F}(y_0, x_0)\Big)^{-1}$ and $\|T\| := \|T\|_{\mathcal{L}(Z,Y)}$ be its norm. If

$$\begin{aligned} \sup_{\overline{X_r}} \|\mathcal{F}(y_0, x)\|_Z &\leq \frac{\rho}{2\|T\|}, & (3.2)\\ \sup_{\overline{Y_\rho}\times\overline{X_r}} \Big\| \mathrm{Id}_Y - T D_y\mathcal{F}(y, x)\Big\|_{\mathcal{L}(Y,Y)} &\leq \frac{1}{2}, & (3.3) \end{aligned}$$

then there exists $y \in \mathcal{C}^1(X_r, \overline{Y_\rho})$ such that $\mathcal{F}(y(x), x) \equiv 0$.

Appling Proposition 3.1 to the range equation (1.15) we derive:

Proposition 3.2 *Let $f = f(t, x, u, \varepsilon)$, $f_u := \partial_u f$ and $\varepsilon\partial_\varepsilon f$ be continuous on $\overline{\Omega} \times \mathbb{R} \times [-1, 1]$. Then $\forall\, R > 0$ there exists a unique function*

$$w = w(v, \varepsilon) \in \mathcal{C}^1\bigg(\{\|v\|_{L^\infty} < 2R\} \times \{|\varepsilon| < \varepsilon_0(R)\}\,,\, \Big\{\|w\|_E \leq C_0(R)|\varepsilon|\Big\}\bigg) \tag{3.4}$$

solving the range equation (1.15), where $\varepsilon_0(R) := 1/2C_0(R)$ and[5]

$$C_0(R) := 1 + \sqrt{2}\pi\underline{c} \max_{\overline{\Omega}\times\{|u|\leq 3R+1\}\times\{|\varepsilon|\leq 1\}} \Big[|f(t, x, u, \varepsilon)| + |f_u(t, x, u, \varepsilon)|\Big]\,. \tag{3.5}$$

Moreover, the following continuity property holds

$$v_n\,,\ \bar v \in V\,,\ \ \|v_n\|_{L^\infty}\,,\ \|\bar v\|_{L^\infty} \leq R\,,\ \ v_n \xrightarrow{L^2} \bar v \quad \Longrightarrow \quad w(v_n) \xrightarrow{E} w(\bar v)\,. \tag{3.6}$$

[5] $\underline{c}$ is defined in (2.2).

PROOF: Let $X := V \times \mathbb{R}$, $Y = Z := W$ (namely $x := (v,\varepsilon)$ and $y := w$) and $\|x\|_X := \|v\|_{L^\infty} + \frac{R}{\varepsilon_0(R)}|\varepsilon|$. Let also $x_0 := (0,0)$, $y_0 := 0$, $r := 3R$, $\rho := 1$, $\mathcal{F}(y,x) := \mathcal{F}(w,v,\varepsilon) := w - \varepsilon\Box^{-1}\Pi_{N^\perp} f(v+w,\varepsilon)$ and $U := W \times V \times (-1,1)$. Note that $\mathcal{F}(\cdot,\cdot) \in C^1$ since the Nemitski operator $\varepsilon f \in C^1(E \times (-1,1), L^2(\Omega))$ and (2.1) holds. Moreover (3.1) holds and $T = \mathrm{Id}_W$ (hence $\|T\| = 1$).

If $\|v\|_{L^\infty} \le 3R$ and $\|w\|_{L^\infty} \le \|w\|_E \le 1$, then

$$|v(t,x) + w(t,x)| \le 3R + 1\,, \qquad \forall\,(t,x) \in \overline{\Omega}\,. \tag{3.7}$$

Using (2.2), Bessel inequality $\|\Pi_{N^\perp} f\|_{L^2} \le \|f\|_{L^2}$, $\|f\|_{L^2(\Omega)} \le \sqrt{2}\pi\|f\|_{L^\infty(\Omega)}$ and estimate (3.7), we obtain, $\forall|\varepsilon| \le \varepsilon_0(R)$ and $\|v\|_{L^\infty} \le 3R$,

$$\|\mathcal{F}(0,v,\varepsilon)\|_E \le |\varepsilon|\ \underline{c}\|f(v,\varepsilon)\|_{L^2} \le |\varepsilon|\ \sqrt{2}\pi\underline{c}\|f(v,\varepsilon)\|_{L^\infty} \le C_0(R)|\varepsilon| \le \frac{1}{2} \tag{3.8}$$

where $C_0(R)$ is defined in (3.5). Hence (3.2) follows from (3.8).
Since

$$D_w\mathcal{F}(w,v,\varepsilon)[\widetilde{w}] = \widetilde{w} - \varepsilon\Box^{-1}\Pi_{N^\perp}\Big(f_u(v+w,\varepsilon)\widetilde{w}\Big) \qquad \forall\,\widetilde{w} \in W\,,$$

we deduce, arguing as before, $\forall\|v\|_{L^\infty} \le 3R$, $\forall\|w\|_E \le 1$, $\forall|\varepsilon| \le \varepsilon_0$,

$$\sup_{\|\widetilde{w}\|_E = 1}\Big\|\widetilde{w} - D_w\mathcal{F}(w,v,\varepsilon)[\widetilde{w}]\Big\|_E \le |\varepsilon|\ \sqrt{2}\pi\underline{c}\|f_u(v+w,\varepsilon)\|_{L^\infty} \le \varepsilon_0(R)\,C_0(R) = \frac{1}{2} \tag{3.9}$$

and (3.3) follows. Now we can apply Proposition 3.1 finding a function $w = w(v,\varepsilon) \in C^1(\{\|x\|_X < r\}, \overline{W}_1)$ satisfying the range equation (1.15). Finally, note that

$$\{\|v\|_{L^\infty} < 2R\} \times \{|\varepsilon| < \varepsilon_0(R)\} \subset \{\|x\|_X < r = 3R\}$$

and, arguing as above,

$$\Big\|w(v,\varepsilon)\Big\|_E = \Big\|\varepsilon\ \Box^{-1}\Pi_{N^\perp} f(v + w(v,\varepsilon);\varepsilon)\Big\|_E \le |\varepsilon|\sqrt{2}\pi\underline{c}\|f(v+w;\varepsilon)\|_{L^\infty} \le |\varepsilon| C_0(R)\ ,$$

whence (3.4) follows.
We now prove (3.6). Let $w_n := w(v_n,\varepsilon)$ and $\bar{w} := w(\bar{v},\varepsilon)$ denote, for brevity, the solutions of $w_n = \varepsilon\Box^{-1}\Pi_{N^\perp} f(v_n + w_n)$ and $\bar{w} = \varepsilon\Box^{-1}\Pi_{N^\perp} f(\bar{v} + \bar{w})$. By (2.2), using that $\partial_u f$ is continuous in $\overline{\Omega} \times \mathbb{R} \times [-1,1]$, recalling the definition of $C_0(R)$ in (3.5) and that $\varepsilon_0(R)C_0(R) = 1/2$

$$\begin{aligned}
\|w_n - \bar{w}\|_E &\le |\varepsilon|\ \underline{c}\Big\|f(v_n + w_n,\varepsilon) - f(\bar{v} + \bar{w},\varepsilon)\Big\|_{L^2} \\
&\le \varepsilon_0(R)\,\underline{c}\ \|(v_n - \bar{v}) + (w_n - \bar{w})\|_{L^2} \max_{\overline{\Omega}\times\{|u|\le 3R+1\}\times\{|\varepsilon|\le 1\}} |f_u(t,x,u,\varepsilon)| \\
&\le \varepsilon_0(R)\,C_0(R)\|v_n - \bar{v}\|_{L^2} + \varepsilon_0(R)\,C_0(R)\|w_n - \bar{w}\|_{L^\infty} \\
&= \frac{1}{2}\|v_n - \bar{v}\|_{L^2} + \frac{1}{2}\|w_n - \bar{w}\|_E\,,
\end{aligned}$$

whence $\|w_n - \bar{w}\|_E \le \|v_n - \bar{v}\|_{L^2}$ and (3.6) follows. $\square$

3.2 The Kernel equation

Once the range equation (1.15) has been solved by $w(v,\varepsilon) \in W$ there remains the infinite dimensional kernel equation (1.16).

Since V is dense in N with the L^2-norm, equation (1.16) is equivalent to

$$\int_\Omega f(v + w(v,\varepsilon),\varepsilon)\varphi = 0 \qquad \forall\, \varphi \in V \tag{3.10}$$

which is the Euler-Lagrange equation of the *reduced Lagrangian action functional* $\Phi : V \longrightarrow \mathbb{R}$, $\Phi(v) := \Phi(v,\varepsilon) := \Psi(v + w(v,\varepsilon),\varepsilon)$, defined in (1.17). Indeed:

Lemma 3.3 $\Phi \in \mathcal{C}^1(\{\|v\|_{H^1} < 2R\},\mathbb{R})$ *and a critical point* $\bar{v}$ *of* Φ *is a weak solution of the kernel equation* (1.16)*. Moreover* Φ *can be written as*

$$\Phi(v) = \varepsilon \int_\Omega \Big[F(v + w(v);\varepsilon) - \frac{1}{2} f(v + w(v);\varepsilon)w(v)\Big]\, dtdx\ . \tag{3.11}$$

PROOF. Since $\Psi(\cdot,\varepsilon) \in C^1(E,\mathbb{R})$ and, by Proposition 3.2, $w(\cdot,\varepsilon) \in C^1(\{\|v\|_{H^1} < 2R\},\mathbb{R})$ (note that $\{\|v\|_{H^1} < 2R\} \subset \{\|v\|_{L^\infty} < 2R\}$ by (2.16)), then $\Phi \in \mathcal{C}^1(\{\|v\|_{H^1} < 2R\},\mathbb{R})$ and

$$D\Phi(v)[\varphi] = D\Psi(v + w(v))\Big[\varphi + dw(v)[\varphi]\Big] \qquad \forall\, \varphi \in V\ . \tag{3.12}$$

We claim that, since $w = w(v) \in E$ is a weak solution of the range equation (1.15) and $\widetilde{w} := dw(v)[\varphi] \in W$, then

$$D\Psi(v + w(v))\Big[dw(v)[\varphi]\Big] = 0\ . \tag{3.13}$$

Indeed, since $v_t, v_x \in N$ and $\widetilde{w}_t$, $\widetilde{w}_x \in N^\perp$,

$$\begin{aligned} D\Psi(v + w)[\widetilde{w}] &= \int_\Omega (v + w)_t\widetilde{w}_t - (v + w)_x\widetilde{w}_x + \varepsilon f(v + w,\varepsilon)\widetilde{w} \\ &= \int_\Omega w_t\widetilde{w}_t - w_x\widetilde{w}_x + \varepsilon\Pi_{N^\perp} f(v + w,\varepsilon)\widetilde{w} = 0 \end{aligned} \tag{3.14}$$

because $w \in E$ is a weak solution of the range equation $\Box w = \varepsilon\Pi_{N^\perp} f(v + w,\varepsilon)$ and $w(t,0) = w(t,\pi) = 0$.

By (3.12), (3.13) and since w_t, $w_x \in N^\perp$ and φ_t, $\varphi_x \in N$

$$\begin{aligned} D\Phi(v)[\varphi] &= D\Psi(v + w)[\varphi] = \int_\Omega (v + w)_t\varphi_t - (v + w)_x\varphi_x + \varepsilon f(v + w,\varepsilon)\varphi \\ &= \int_\Omega v_t\varphi_t - v_x\varphi_x + \varepsilon f(v + w,\varepsilon)\varphi \\ &= \varepsilon \int_\Omega f(v + w,\varepsilon)\varphi\ = \varepsilon \int_\Omega \Pi_{N^\perp} f(v + w,\varepsilon)\varphi \end{aligned} \tag{3.15}$$

where in (3.15) we used $\int_\Omega v_t\varphi_t - v_x\varphi_x = 0$ since $v, \varphi \in V$.

Finally we prove (3.11) as in [BB03]. Since $v_t, v_x \in N$, $w_t, w_x \in N^\perp$ and (2.14)

$$\begin{aligned}\Phi(v) &= \int_\Omega \frac{(v+w(v))_t^2}{2} - \frac{(v+w(v))_x^2}{2} + \varepsilon F(v+w(v);\varepsilon) \\ &= \int_\Omega \frac{(w(v))_t^2}{2} - \frac{(w(v))_x^2}{2} + \varepsilon F(v+w(v);\varepsilon)\end{aligned}$$

and since $\int_\Omega (w(v))_t^2 - (w(v))_x^2 = -\int_\Omega \varepsilon f(v+w(v);\varepsilon)w(v)$ we deduce (3.11). □

The next Lemma proves a L^2-continuity property for Φ.

Lemma 3.4 *Let $R > 0$ and $|\varepsilon| \le \varepsilon_0(R)$ (where $\varepsilon_0(R) < 1$ is defined in Proposition 3.2). Then*

$$v_n,\, \bar v \in V\ , \quad \|v_n\|_{H^1}, \|\bar v\|_{H^1} \le R\ , \quad v_n \stackrel{L^2}{\longrightarrow} \bar v \qquad \Longrightarrow \qquad \Phi(v_n) \longrightarrow \Phi(\bar v)\ . \tag{3.16}$$

Proof: Setting $w_n := w(v_n, \varepsilon)$ and $\bar w := w(\bar v, \varepsilon)$, we have

$$\begin{aligned}\left|\int_\Omega F(v_n+w_n) - F(\bar v + \bar w)\right| &\le \max_{\overline{\Omega}\times\{|u|\le R+1\}\times\{|\varepsilon|\le 1\}} |f(t,x,u,\varepsilon)| \int_\Omega |v_n - \bar v + w_n - \bar w| \\ &\le C_0(R)\Big(\|v_n - \bar v\|_{L^1} + \|w_n - \bar w\|_{L^1}\Big) \longrightarrow 0\end{aligned}$$

as $n \to \infty$, by (3.6) and the fact that $\|v_n - \bar v\|_{L^2} \to 0$. An analogous estimate holds for the second term in the integral in (3.11) proving the Lemma. □

By standard compactness argument the functional Φ attains minimum (resp. maximum) in $\overline{B_R} := \{v \in V, \|v\|_{H^1} \le R\}$, $\forall R > 0$. Indeed, let $v_n \in \overline{B_R}$ be a minimazing (resp. maximizing) sequence $\Phi(v_n) \to \inf_{\overline{B_R}} \Phi$. Since $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $N \cap H^1$, up to a subsequence $v_n \stackrel{H^1}{\rightharpoonup} \bar v$ for some $\bar v = \bar v(R,\varepsilon) \in \overline{B_R}$. By the Rellich Theorem $v_n \stackrel{L^2}{\to} \bar v$ and therefore, by Lemma 3.4, $\bar v$ is a minimum (resp. maximum) point of Φ restricted to $\overline{B_R}$.

Since $\bar v$ could belong to the boundary $\partial\overline{B_R}$ we only have the variational inequality (1.18) for any *admissible variation* $\varphi \in V$, namely for any $\varphi \in V$ such that $\bar v + \theta\varphi \in \overline{B_R}$, $\forall \theta < 0$ sufficiently small. As proved by Rabinowitz [R67], a sufficient condition for $\varphi \in V$ to be an admissible variation is the positivity of the scalar product

$$\langle \bar v, \varphi\rangle_{H^1} > 0\ . \tag{3.17}$$

The heart of the existence proof of Theorems 1, 2 and 3 is to obtain, choosing suitable admissible variations, the *a-priori estimate* $\|\bar v\|_{H^1} < R$ for some $R > 0$. i.e. to show that $\bar v$ is an *inner* minimum (resp. maximum) point of Φ in B_R.

4 Proof of Theorems 1 and 2

The main difficulty for proving Theorems 1 and 2 is to obtain the fore mentioned a priori-estimate for $\bar{v}$.

4.1 Proof of Theorem 2

We look for small amplitude solutions of (1.1) with forcing term $f(t,x,u) = g(t,x,u) + h(t,x)$ where $g(t,x,u) = \beta(x)u^{2k} + \mathcal{R}(t,x,u)$, $h(t,x) \in N^{\perp}$ and $\mathcal{R}(t,x,u)$ satisfies (1.7). Perform the change of variables $u = \varepsilon(H + \widetilde{u})$ and set $\widetilde{\varepsilon} := \varepsilon^{2k}$

$$\begin{aligned} \Box\widetilde{u} = g\big(t,x,\varepsilon(H+\widetilde{u})\big) &= \varepsilon^{2k}\Big[\beta(x)(H+\widetilde{u})^{2k} + \varepsilon^{-2k}\mathcal{R}\big(t,x,\varepsilon(H+\widetilde{u})\big)\Big] \\ &= \widetilde{\varepsilon}\Big[\beta(x)(H+\widetilde{u})^{2k} + \widetilde{\varepsilon}^{-1}\mathcal{R}\big(t,x,\widetilde{\varepsilon}^{\frac{1}{2k}}(H+\widetilde{u})\big)\Big] . \end{aligned}$$

Recalling $\widetilde{u} \to u$, $\widetilde{\varepsilon} \to \varepsilon$, we look for solutions of the problem

$$\begin{cases} \Box u = \varepsilon f(t,x,u;\varepsilon) \\ u(t,0) = u(t,\pi) = 0 \\ u(t+2\pi,x) = u(t,x) \end{cases} \tag{4.1}$$

where the nonlinear forcing term is

$$f(t,x,u;\varepsilon) := \beta(x)(H+u)^{2k} + \mathcal{R}^*(t,x,u;\varepsilon) \tag{4.2}$$

and

$$\mathcal{R}^*(t,x,u;\varepsilon) := \varepsilon^{-1}\mathcal{R}\Big(t,x,\varepsilon^{\frac{1}{2k}}\big(H(t,x)+u\big)\Big) . \tag{4.3}$$

Moreover, eventually substituting $\varepsilon \to -\varepsilon$ and $\beta \to -\beta$ we can always suppose

$$\beta(x)H(t,x) > 0 , \qquad \forall\,(t,x) \in \Omega . \tag{4.4}$$

By (1.7), $\mathcal{R}^*$, $\partial_u\mathcal{R}^*$, $\varepsilon\partial_\varepsilon\mathcal{R}^*$ are continuous in $\overline{\Omega} \times \mathbb{R} \times [-1,1]$ (recall that $H \in E$) and

$$\forall\, R_0 > 0 , \qquad \|\mathcal{R}^*(\cdot;\varepsilon)\|_{C(\overline{\Omega}\times\{|u|\le R_0\})} , \|\partial_u\mathcal{R}^*(\cdot;\varepsilon)\|_{C(\overline{\Omega}\times\{|u|\le R_0\})} \overset{\varepsilon\to 0}{\longrightarrow} 0 . \tag{4.5}$$

Moreover, since $H \in H^1(\Omega)$, $\partial_t\mathcal{R}^* = \varepsilon^{-1}\partial_t\mathcal{R} + \varepsilon^{\frac{1}{2k}-1}\partial_u\mathcal{R}\ H_t$ and (1.7), then

$$\Big\|\partial_t\mathcal{R}^*(\cdot,u(\cdot);\varepsilon)\Big\|_{L^2(\Omega)} \le C^*(\|u\|_{L^\infty(\Omega)}) \qquad \forall\, u \in L^\infty(\Omega) ,$$

for a suitable increasing function $C^*(\cdot)$.

In order to find solutions of problem (4.1) we perform the Lyapunov-Schmidt reduction of the previous section. We fix $R > 0$ to be chosen later (large enough!). Since $f, \partial_u f, \varepsilon\partial_\varepsilon f$

are continuous on $\overline{\Omega}\times\mathbb{R}\times[-1,1]$, using Proposition 3.2 we solve the range equation (1.15) finding $w = w(v,\varepsilon)$ for $\|v\|_{L^\infty} < 2R$ and $|\varepsilon| < \varepsilon_0 := \varepsilon_0(R)$. Now we look for minimum or maximum points of the corresponding reduced action functional Φ in $\overline{B_R}$ according to whether $\varepsilon > 0$ or $\varepsilon < 0$. Since Φ attains minimum or maximum at some point $\bar v := \bar v(\varepsilon) := \bar v(t,x;\varepsilon)$ in $\overline{B_R}$, to conclude the existence proof of Theorem 2-(i) we need to show that $\bar v$ is an interior point in B_R, i.e. $\|\bar v\|_{H^1} < R$, for a suitable choice of R large enough. Let $\bar w := \bar w(t,x;\varepsilon) := w(\bar v(\varepsilon),\varepsilon)(t,x) \in E$ and $\bar u := \bar u(t,x;\varepsilon) := \bar v + \bar w \in E$. By (3.4) and the definitions of $C_0(\cdot)$, $\varepsilon_0(\cdot)$ given in Proposition 3.2, we have

$$\|\bar w\|_E \le C_0(R)|\varepsilon| \le \frac{1}{2}, \qquad \|\bar u\|_{C(\overline{\Omega})} \le R + \frac{1}{2}, \tag{4.6}$$

since, by (2.16), $\|\bar v\|_{C(\overline{\Omega})} \le \|\bar v\|_{H^1(\Omega)} \le R$. Let

$$\mathcal{R}_*(t,x;\varepsilon) := \mathcal{R}^*(t,x,\bar u(t,x;\varepsilon);\varepsilon). \tag{4.7}$$

We have $\mathcal{R}_* \in C(\overline{\Omega})$ and, choosing $R_0 := R + 1/2$ in (4.5),

$$\|\mathcal{R}_*(\cdot;\varepsilon)\|_{C(\overline{\Omega})} \overset{\varepsilon\to 0}{\longrightarrow} 0. \tag{4.8}$$

Moreover, since $\partial_t\mathcal{R}_* = \partial_t\mathcal{R}^*(\bar u) + \partial_u\mathcal{R}^*(\bar u)\ (\bar v_t + \bar w_t)$, we have $\partial_t\mathcal{R}_* \in L^2(\Omega)$ with

$$\Big\|\partial_t\mathcal{R}_*(\cdot;\varepsilon)\Big\|_{L^2(\Omega)} \le C_*\big(\|\bar v\|_{L^\infty(\Omega)}\big) + o(1)\|\bar v_t\|_{L^2(\Omega)}, \tag{4.9}$$

for a suitable increasing function $C_*(\cdot)$. By (4.2), (4.3) and (4.7), the variational inequality (1.18) yields, for any admissible variation $\varphi \in V$,

$$0 \ge \varepsilon \int_\Omega \beta(x)\big(H + \bar v + \bar w\big)^{2k}\varphi + \mathcal{R}_*\varphi \qquad \forall\,\varepsilon > 0$$

if $\bar v$ is a minimum point, respectively

$$0 \le \varepsilon \int_\Omega \beta(x)\big(H + \bar v + \bar w\big)^{2k}\varphi + \mathcal{R}_*\varphi \qquad \forall\,\varepsilon < 0$$

if $\bar v$ is a maximum point. However, in both cases we get, dividing by ε,

$$\int_\Omega \beta(x)\big(H + \bar v + \bar w\big)^{2k}\varphi \le \int_\Omega -\mathcal{R}_*\varphi. \tag{4.10}$$

The required a-priori estimate for the H^1-norm of $\bar v$ will be proved in several steps inserting into the variational inequality (4.10) suitable admissible variations. We shall derive, first, an L^{2k}-estimate for $\bar v$ (it is needed at least when $k \ge 2$), see (4.15), next, an L^∞-estimate, see (4.29), and, finally, the H^1-estimate, see (4.41).

The following key estimate will be heavily exploited.

Proposition 4.1 *Let $k \in \mathbb{N}^+$ and $B \in C(\overline{\Omega})$ with $B \geq 0$ in Ω. Define*

$$c_k(B) := \frac{1}{4^k} \min_{\bar{\Omega}_{\alpha_k}} B \qquad \textit{where} \quad \alpha_1 := \frac{1}{8}\,, \quad \alpha_k := \frac{1}{4(1+2k)} \quad \textit{for} \ \ k \geq 2\,. \tag{4.11}$$

Then $\forall\, v \in N \cap L^{2k}(\Omega)$

$$\int_\Omega B v^{2k} \geq c_k(B) \int_\Omega v^{2k}\,. \tag{4.12}$$

Proof: Since $B \geq 0$ in Ω and using Lemma 2.6

$$\int_\Omega B v^{2k} \geq \min_{\bar{\Omega}_{\alpha_k}} B \int_{\Omega_{\alpha_k}} v^{2k} \geq \pi \min_{\bar{\Omega}_{\alpha_k}} B \int_0^{2\pi} \hat{v}^{2k} \geq \frac{1}{4^k} \min_{\bar{\Omega}_{\alpha_k}} B \int_\Omega v^{2k}\,.$$

□

Remark 4.2 *We stress that estimate* (4.12) *is not trivial* (*in the case* $c_k(B) > 0$) *since* B *could vanish on* $\partial\Omega$ (*in particular, in this case,* (4.12) *does not hold true in the whole* $L^{2k}(\Omega)$).

Remark 4.3 *If* (1.9) *holds and* $\beta > 0$ *then the quantity* $c_k(\beta H)$*, defined in* (4.11)*, is strictly positive* (*because* $\Omega_{\alpha_k} = \Omega^k$). *For this reason we can prove Theorem 2 assuming only* (1.9).

In the following κ_i will denote positive constants depending only on H, β, $\mathcal{R}$, k but not on R, ε. We also recall the notation $o(1)$ for a function tending to 0 as $\varepsilon \to 0$ uniformly.

4.2 The L^{2k}-estimate.

Take $\varphi := \bar{v}$ in the variational inequality (4.10); φ is an admissible variation since $\langle \bar{v}, \varphi \rangle_{H^1} = \|\bar{v}\|_{H^1}^2 > 0$.
By (4.10), $\|w(\bar{v}, \varepsilon)\|_E = O(\varepsilon)$ (recall (4.6)) and (4.8), there exists $0 < \varepsilon_1 \leq \varepsilon_0$ such that

$$\int_\Omega \beta(\bar{v} + H)^{2k} \bar{v} \leq 1 \qquad \text{for} \quad |\varepsilon| \leq \varepsilon_1\,, \tag{4.13}$$

since

$$\int_\Omega \beta(\bar{v} + H)^{2k} \bar{v} \leq \int_\Omega |\mathcal{R}_* \bar{v}| + \Big|\beta(H + \bar{v} + \bar{w})^{2k} - \beta(H + \bar{v})^{2k}\Big| |\bar{v}| \leq o(1) \|\bar{v}\|_{L^1} \leq o(1) R.$$

Noting that, $\int_\Omega \beta \bar{v}^{2k+1} = 0$ by (2.18) with $a(x,u) = \beta(x) u^{2k}$ and $\beta(\pi - x) = \beta(x)$, we derive

$$\begin{aligned} \int_\Omega \beta(\bar{v} + H)^{2k} \bar{v} \ &= \ \int_\Omega \beta \Big[(\bar{v} + H)^{2k} - \bar{v}^{2k} \Big] \bar{v} \\ &= \ \int_\Omega 2k\beta H \bar{v}^{2k} + \beta \sum_{j=0}^{2k-2} \binom{2k}{j} \bar{v}^{j+1} H^{2k-j} \\ &\geq \ 2k c_k(\beta H) \|\bar{v}\|_{L^{2k}}^{2k} - \kappa_1 \|\bar{v}\|_{L^{2k}}^{2k-1} - \kappa_2 \|\bar{v}\|_{L^{2k}}\,, \end{aligned} \tag{4.14}$$

where $c_k(\beta H) > 0$ was defined in (4.11) (recall (4.4)) and we have used Proposition 4.1 and Hölder inequality to estimate $\|\bar{v}\|_{L^i} \leq C_{i,k}\|\bar{v}\|_{L^{2k}}$ $(i \leq 2k-1)$.

Finally, by (4.13) and (4.14) we deduce

$$\|\bar{v}\|_{L^{2k}} \leq \kappa_3 \qquad \text{for} \quad |\varepsilon| \leq \varepsilon_1 \, . \tag{4.15}$$

4.3 The L^∞-estimate.

To obtain the L^∞-estimate for $\bar{v}$ we consider an admissible variation φ, which is a nonlinear function of $\bar{v}$, and it is constructed as in [R67].

Let, for $M > 0$,

$$q(\lambda) := q_M(\lambda) := \begin{cases} 0 \, , & \text{if} \quad |\lambda| \leq M \\ \lambda - M \, , & \text{if} \quad \lambda \geq M \\ \lambda + M \, , & \text{if} \quad \lambda \leq M \, . \end{cases} \tag{4.16}$$

For $\bar{v}(t,x) = \bar{v}_+(t,x) - \bar{v}_-(t,x) = \hat{\bar{v}}(t+x) - \hat{\bar{v}}(t-x)$, we define

$$\varphi := q_+ - q_- := q(\bar{v}_+) - q(\bar{v}_-) \in V \, .$$

We take

$$M := \frac{1}{2}\|\hat{\bar{v}}\|_{L^\infty(\mathbb{T})}$$

and we can assume $M > 0$, i.e. $\bar{v}$ is not identically zero.

In [R67] it is proved that such φ is an admissible variation. We report the proof for completeness. By (3.17), it is sufficient to prove that $\langle \bar{v}, \varphi \rangle_{H^1} > 0$.

Using (2.10), (2.11) and (2.12)

$$\langle \bar{v}_+ - \bar{v}_-, q(\bar{v}_+) - q(\bar{v}_-) \rangle_{H^1} = \int_\Omega \bar{v}_+ q(\bar{v}_+) + \bar{v}_- q(\bar{v}_-) + 2\int_\Omega q'(\bar{v}_+)\Big[(\bar{v}_+)_t^2 + (\bar{v}_+)_x^2\Big] \tag{4.17}$$

Since q is a monotone odd function of its argument and by our choice of M, $\bar{v}_\pm q(\bar{v}_\pm) > 0$ in a positive measure set, and, since $q' \geq 0$, the second term in (4.17) is non-negative.

We also have, since q is a monotone function,

$$\bar{v}\varphi = (\bar{v}_+ - \bar{v}_-)(q_+ - q_-) = (\bar{v}_+ - \bar{v}_-)(q(\bar{v}_+) - q(\bar{v}_-)) \geq 0 \, . \tag{4.18}$$

Insert such φ in the variational inequality (4.10). Here the dominant term is $\int \beta(\bar{v}+H)^{2k}\varphi$, in the sense that, by $\|w(v,\varepsilon)\|_E = O(\varepsilon)$ (recall (4.6)), (4.8) and $\|\bar{v}\|_{L^\infty} \leq R$, we obtain that there exists $0 < \varepsilon_2 \leq \varepsilon_1$ such that

$$\int_\Omega \beta(\bar{v}+H)^{2k}\varphi \leq \|\varphi\|_{L^1} \qquad \text{for} \quad |\varepsilon| \leq \varepsilon_2 \, . \tag{4.19}$$

Since $\int_\Omega \beta \bar v^{2k}\varphi = 0$ by (2.18) and $\beta(\pi - x) = \beta(x)$, we have (recall $\beta > 0$)

$$\int_\Omega \beta(\bar v + H)^{2k}\varphi = \int_\Omega \beta\Big[(\bar v + H)^{2k} - \bar v^{2k}\Big]\varphi \geq \int_\Omega 2k\beta H \bar v^{2k-1}\varphi - \kappa_4\big(\|\bar v\|_{L^\infty}^{2k-2} + 1\big)\|\varphi\|_{L^1}\,. \tag{4.20}$$

We now estimate the dominant term $2k\int_\Omega \beta H \bar v^{2k-1}\varphi$. Since $\bar v\varphi \geq 0$ and $\min_{\Omega_{1/4}} \beta H > 0$ (by (4.4))

$$\int_\Omega 2k\bar v^{2k-1}\beta H\varphi = 2k\int_\Omega \beta H(\bar v\varphi)\bar v^{2k-2} \geq \kappa_5 \int_{\Omega_{1/4}} (\bar v\varphi)\bar v^{2k-2}\,. \tag{4.21}$$

By (4.19), (4.20) and (4.21), we have

$$\int_{\Omega_{1/4}} \bar v^{2k-1}\varphi \leq \kappa_6\big(\|\bar v\|_{L^\infty(\Omega)}^{2k-2} + 1\big)\|\varphi\|_{L^1(\Omega)}\,. \tag{4.22}$$

We have to give a lower bound of the positive integral $\int_{\Omega_{1/4}} \bar v^{2k-1}\varphi = \int_{\Omega_{1/4}} (\bar v\varphi)\bar v^{2k-2}$ $= \int_{\Omega_{1/4}} (\bar v\varphi)(\bar v_+ - \bar v_-)^{2k-2}$.

We first consider the (more difficult) case $k \geq 2$, in which the L^{2k}-estimate for $\bar v$ obtained in the previous subsection is needed, the (simpler) case $k = 1$ will be treated later.

Using (2.21) we obtain

$$\begin{aligned}
\int_{\Omega_{1/4}} \bar v^{2k-1}\varphi \;&\geq\; \int_{\Omega_{1/4}} \bar v\varphi\Big[\bar v_+^{2k-2} + \bar v_-^{2k-2} - (2k-2)(\bar v_+^{2k-3}\bar v_- + \bar v_+\bar v_-^{2k-3})\Big] \\
&=\; 2\int_{\Omega_{1/4}} \bar v_+^{2k-1}q_+ - \bar v_+^{2k-1}q_- + \bar v_+^{2k-2}\bar v_- q_- - \bar v_+^{2k-2}\bar v_- q_+ \\
&\qquad + (2k-2)[-\bar v_+^{2k-2}\bar v_- q_+ + \bar v_+^{2k-2}\bar v_- q_- - \bar v_+^{2k-3}\bar v_-^2 q_- + \bar v_+^{2k-3}\bar v_-^2 q_+] \\
&\geq\; 2\int_{\Omega_{1/4}} \bar v_+^{2k-1}q_+ && (4.23) \\
&\quad -2\int_{\Omega_{1/4}} \bar v_+^{2k-1}q_- + (2k-1)\bar v_+^{2k-2}\bar v_- q_+ + (2k-2)\bar v_+^{2k-3}\bar v_-^2 q_- && (4.24)
\end{aligned}$$

where in the equality we have used (2.12) and in the last inequality the fact that $\bar v_+ q_+$, $\bar v_- q_- \geq 0$ (since $\lambda q(\lambda) \geq 0$).

The dominant term is (4.23). Since $\lambda^{2k-1}q(\lambda) \geq M^{2k-1}|q(\lambda)|$, by (2.11) we obtain

$$2\int_{\Omega_{1/4}} \bar v_+^{2k-1}q_+ = 2\pi(1 - \frac{2}{4})\int_0^{2\pi} \hat{\bar v}^{2k-1}(s)q(\hat{\bar v}(s))ds \geq \pi M^{2k-1}\|q(\hat{\bar v})\|_{L^1(\mathbb{T})}\,. \tag{4.25}$$

We now give an upper estimate of the three terms in (4.24). By (2.10)

$$\begin{aligned}
\left| 2\int_{\Omega_{1/4}} \bar v_+^{2k-1} q_- \right| &\leq 2\int_{\Omega} \left|\bar v_+^{2k-1}\right| |q_-| \leq \|\hat{\bar v}\|^{2k-1}_{L^{2k-1}(\mathbb{T})} \|q(\hat{\bar v})\|_{L^1(\mathbb{T})}\,, \\
\left| 2\int_{\Omega_{1/4}} \left(\bar v_+^{2k-2} q_+\right) \bar v_- \right| &\leq \|\hat{\bar v}^{2k-2} q(\hat{\bar v})\|_{L^1(\mathbb{T})} \|\hat{\bar v}\|_{L^1(\mathbb{T})} \leq (2M)^{2k-2} \|q(\hat{\bar v})\|_{L^1(\mathbb{T})} \|\hat{\bar v}\|_{L^1(\mathbb{T})} \\
\left| 2\int_{\Omega_{1/4}} \bar v_+^{2k-3} \left(\bar v_-^2 q_-\right) \right| &\leq \|\hat{\bar v}^{2k-3}\|_{L^1(\mathbb{T})} \|\hat{\bar v}^2 q(\hat{\bar v})\|_{L^1(\mathbb{T})} \leq (2M)^2 \|\hat{\bar v}\|^{2k-3}_{L^{2k-3}(\mathbb{T})} \|q(\hat{\bar v})\|_{L^1(\mathbb{T})}\,.
\end{aligned}$$

By the previous inequalities, (4.25), Hölder inequality[6] and (4.15), we finally have

$$\int_{\Omega_{1/4}} \bar v^{2k-1} \varphi \geq \pi M^{2k-1} \|q(\hat{\bar v})\|_{L^1(\mathbb{T})} - \kappa_7 (M^{2k-2}+1) \|q(\hat{\bar v})\|_{L^1(\mathbb{T})}\,. \tag{4.26}$$

Now we note that by (2.11)

$$\int_{\Omega} |\varphi| \leq \int_{\Omega} |q(v_+)| + |q(v_-)| \ = 2\pi \|q(\hat{\bar v})\|_{L^1(\mathbb{T})}\,. \tag{4.27}$$

We collect (4.22) and (4.26) using (4.27) in order to obtain

$$M^{2k-1} \|q(\hat{\bar v})\|_{L^1(\mathbb{T})} \leq \kappa_8 \Big(\|\bar v\|^{2k-2}_{L^\infty(\Omega)} + M^{2k-2} + 1 \Big) \|q(\hat{\bar v})\|_{L^1(\mathbb{T})}\,. \tag{4.28}$$

Since $M := \|\hat{\bar v}\|_{L^\infty(\mathbb{T})}/2$ hence $\|q(\hat{\bar v})\|_{L^\infty(\mathbb{T})} = M$, $\|\bar v\|_{L^\infty(\Omega)} \leq 2\|\hat{\bar v}\|_{L^\infty(\mathbb{T})} = 4M$ and $\|\hat{\bar v}\|_{L^\infty(\mathbb{T})} \neq 0$. Hence, by (4.28),

$$M^{2k-1} \|q(\hat{\bar v})\|_{L^1(\mathbb{T})} \leq \kappa_9 \Big(M^{2k-2} + 1 \Big) \|q(\hat{\bar v})\|_{L^1(\mathbb{T})}$$

and, dividing by $\|q(\hat{\bar v})\|_{L^1(\mathbb{T})} \neq 0$, we finally obtain $M^{2k-1} \leq \kappa_9 (M^{2k-2}+1)$.

By our choice of M the L^∞-estimate follows for $k \geq 2$,

$$\|\bar v\|_{L^\infty} \leq \kappa_{10} \qquad \text{for} \quad |\varepsilon| \leq \varepsilon_2\,. \tag{4.29}$$

We now briefly discuss the case $k = 1$, which is simpler and where a previous L^2-estimate for $\bar v$ is not necessary to obtain (4.29). In fact by (4.19) and (4.20) (with $k = 1$), we obtain

$$\int_{\Omega} \beta H \bar v \varphi \leq \kappa_{11} \|\varphi\|_{L^1}\,. \tag{4.30}$$

For $0 < \alpha < 1/2$ to be chosen later, we have

$$\int_{\Omega} \beta H \bar v \varphi \geq \min_{\overline{\Omega}_\alpha} (\beta H) \int_{\Omega_\alpha} \bar v \varphi\,. \tag{4.31}$$

[6] To estimate $\|\hat{\bar v}\|_{L^j(\mathbb{T})} \leq C_{j,k} \|\hat{\bar v}\|_{L^{2k}(\mathbb{T})}$ for $j < 2k$.

We have to give a lower bound of

$$\int_{\Omega_\alpha} \bar v\varphi = \int_{\Omega_\alpha} \bar v_+ q_+ + \bar v_- q_- - \int_{\Omega_\alpha} \bar v_+ q_- + \bar v_- q_+ \,. \tag{4.32}$$

By (2.11) and $\lambda q(\lambda) \geq M|q(\lambda)|$

$$\int_{\Omega_\alpha} \bar v_+ q_+ = \int_{\Omega_\alpha} \bar v_- q_- = \pi(1-2\alpha)\int_0^{2\pi} \hat{\bar v}(s) q(\hat{\bar v}(s))ds \geq \pi(1-2\alpha)M\|q(\hat{\bar v})\|_{L^1(\mathbb{T})}\,. \tag{4.33}$$

Moreover, since $\hat{\bar v}$ has zero average, by (2.10), we have

$$\left|\int_{\Omega_\alpha} \bar v_+ q_-\right| = \left|\int_{\Omega_\alpha} \bar v_- q_+\right| \leq \frac{1}{2}\int_{-2\alpha\pi}^{2\alpha\pi}\int_0^{2\pi} |q(\hat{\bar v}(y))\hat{\bar v}(z+y)|dydz \leq 2\alpha\pi M\|q(\hat{\bar v})\|_{L^1(\mathbb{T})}\,. \tag{4.34}$$

Collecting (4.32), (4.33) and (4.34) we obtain

$$\int_{\Omega_\alpha} \bar v\varphi \geq 2\pi(1-6\alpha)M\|q(\hat{\bar v})\|_{L^1(\mathbb{T})} \geq \frac{1}{2}\pi M\|q(\hat{\bar v})\|_{L^1(\mathbb{T})}\,, \tag{4.35}$$

choosing $\alpha := 1/8$. Collecting (4.30), (4.31) and (4.35), we obtain

$$M\|q(\hat{\bar v})\|_{L^1(\mathbb{T})} \leq \kappa_{12}\|\varphi\|_{L^1(\Omega)}\,.$$

Using (4.27) and dividing by $\|q(\hat{\bar v})\|_{L^1(\mathbb{T})} \neq 0$ in the previous inequality we finally obtain (4.29) also in the case $k=1$.

4.4 The H^1-estimate.

We note that $\varphi := -D_{-h}D_h\bar v$ is an admissible variation, since using (2.27),

$$\langle -D_{-h}D_h\bar v, \bar v\rangle_{H^1} = \langle D_h\bar v, D_h\bar v\rangle_{H^1} > 0.$$

Since $\partial_t[\beta(H+\bar v+\bar w)^{2k}]$, $\partial_t\mathcal{R}_* \in L^2(\Omega)$ (see (4.9)), we have, as $h\to 0$,

$$\int_\Omega \beta\big(H+\bar v+\bar w\big)^{2k}\varphi \overset{(2.27)}{=}$$
$$= \int_\Omega \beta D_h\Big[\big(H+\bar v+\bar w\big)^{2k}\Big]D_h\bar v \overset{(2.30)}{\longrightarrow} \int_\Omega \beta\Big[\big(H+\bar v+\bar w\big)^{2k}\Big]_t \bar v_t\ ,$$

$$\int_\Omega \mathcal{R}_*\varphi \overset{(2.27)}{=} \int_\Omega D_h\mathcal{R}_* D_h\bar v \overset{(2.30)}{\longrightarrow} \int_\Omega \partial_t\mathcal{R}_*\bar v_t\,,$$

and, by the variational inequality (4.10), we obtain

$$\int_\Omega \beta\Big[(H+\bar v+\bar w)^{2k}\Big]_t \bar v_t \leq \int_\Omega -\partial_t\mathcal{R}_*\bar v_t\,. \tag{4.36}$$

By the L^∞-estimate on $\bar v$ given in (4.29), the Cauchy-Schwartz inequality and (4.9) we obtain

$$\left| \int_\Omega \partial_t \mathcal{R}_* \bar v_t \right| \le \kappa_{13} \|\bar v_t\|_{L^2} + o(1) \|\bar v_t\|_{L^2}^2 \, . \tag{4.37}$$

Since $\|\bar w\|_E = \|w(\bar v, \varepsilon)\|_E = O(\varepsilon)$ (recall (4.6)), again by (4.29)and the Cauchy-Schwartz inequality, we find

$$\begin{aligned} \int_\Omega \beta \Big[\big(H + \bar v + \bar w \big)^{2k} \Big]_t \bar v_t &= 2k \int_\Omega \beta \big(H + \bar v + \bar w \big)^{2k-1} \big(H_t + \bar v_t + \bar w_t \big) \bar v_t \\ &\ge 2k \int_\Omega \beta (\bar v + H)^{2k-1} \bar v_t^2 - o(1) \|\bar v_t\|_{L^2}^2 - \kappa_{14} \|\bar v_t\|_{L^2} \, . \end{aligned} \tag{4.38}$$

Collecting (4.36),(4.37) and (4.38), we obtain

$$\int_\Omega \beta (\bar v + H)^{2k-1} \bar v_t^2 \le \kappa_{15} \|\bar v_t\|_{L^2} + o(1) \|\bar v_t\|_{L^2}^2 \, . \tag{4.39}$$

Since $\bar v, \bar v_t \in N$ and $\bar v^{2k-1} \bar v_t^2 \in L^1(\Omega)$, it results $\int_\Omega \beta \bar v^{2k-1} \bar v_t^2 = 0$ by (2.19). Using the inequality (2.22) we obtain

$$\begin{aligned} \int_\Omega \beta (\bar v + H)^{2k-1} \bar v_t^2 &= \int_\Omega \beta \Big[(\bar v + H)^{2k-1} - \bar v^{2k-1} \Big] \bar v_t^2 \\ &\ge 4^{1-k} \int_\Omega \beta H^{2k-1} \bar v_t^2 \ge 4^{1-k} c_1 (\beta H^{2k-1}) \int_\Omega \bar v_t^2 \end{aligned} \tag{4.40}$$

where $c_1(\cdot)$ was defined in (4.11) and $\beta H^{2k-1} > 0$ by (4.4). By (4.39) and (4.40) we get

$$\|\bar v_t\|_{L^2}^2 \le \kappa_{16} \|\bar v_t\|_{L^2} + o(1) \|\bar v_t\|_{L^2}^2$$

and we finally deduce that there exists a $0 < \varepsilon_3 \le \varepsilon_2$ such that

$$\|\bar v\|_{H^1} < \kappa_{17} \qquad \forall\, |\varepsilon| \le \varepsilon_3 \, . \tag{4.41}$$

PROOF OF THEOREM 2-(i) COMPLETED. Defining $R := \kappa_{17}$ and $\varepsilon_* := \varepsilon_3$ we obtain, by (4.41), that

$$\|\bar v(\varepsilon)\|_{H^1} < R \qquad\qquad \forall\, |\varepsilon| \le \varepsilon_*$$

and $\bar v(\varepsilon)$ is an interior minimum or maximum point of Φ in $B_R := \{ \|v\|_{H^1} < R \}$. By Lemma 3.3 $\bar u = \bar v + \bar w = \bar v(\varepsilon) + w(\bar v(\varepsilon), \varepsilon)$ is a weak solution of (4.1) and

$$u := \varepsilon \Big(H + \bar v(\varepsilon^{2k}) + w(\bar v(\varepsilon^{2k}), \varepsilon^{2k}) \Big) \in E \tag{4.42}$$

is a weak solution of (1.1)-(1.2)-(1.3) satisfying $\|u\|_E \le C|\varepsilon|$. $\square$

Remark 4.4 *Let $u := u_\varepsilon = v_\varepsilon + w_\varepsilon$, $v_\varepsilon \in V$, $w_\varepsilon \in W$, be a weak solution of $\Box u_\varepsilon = \varepsilon(g(x,u_\varepsilon) + h(t,x))$ where $g \in C([0,\pi] \times \mathbb{R})$, $g(x,u) = g(x,-u) = g(\pi - x, u)$. Suppose u_ε satisfies $\|u_\varepsilon\|_{L^\infty} \leq R$, $\forall \varepsilon$ small. We claim this implies $h \in N^\perp$. Indeed, w_ε satisfies the range equation $w_\varepsilon = \varepsilon \Box^{-1} \Pi_{N^\perp}(g(x,u_\varepsilon) + h(t,x))$ and therefore $\|w_\varepsilon\|_{L^\infty} \leq C|\varepsilon|$. Moreover, by the kernel equation $\Pi_N(g(x, v_\varepsilon + w_\varepsilon) + h(t,x)) = 0$, and noting that $\Pi_N g(x, v_\varepsilon) = 0$ by* (2.18)*, we derive*

$$\Big\|\Pi_N h(t,x)\Big\|_{L^2} = \Big\|\Pi_N(g(x, v_\varepsilon + w_\varepsilon) - g(x,v_\varepsilon))\Big\|_{L^2} \leq \Big\|g(x, v_\varepsilon + w_\varepsilon) - g(x,v_\varepsilon)\Big\|_{L^2} \to 0$$

as $\|w_\varepsilon\|_{L^\infty} \to 0$ because g is uniformly continuous on $[0,\pi] \times \{|u| \leq C\}$. Therefore $\Pi_N h = 0$ and $h \in N^\perp$.

Remark 4.5 (Multiplicity) *By* (2.18)*, any forcing term $h(t,x) := -g(x, v_0(t,x))$, $v_0 \in V \setminus \{0\}$, is in $N^\perp$, if $g(x,u) = g(x,-u) = g(\pi - x, u)$. Therefore the equation $\Box u = \varepsilon(g(x,u) + h(t,x))$ possesses, beyond the ε-small solution u of Theorem 2, also the other two (not small) solutions $\pm v_0$.*

4.5 Higher regularity and Classical Solutions

We now prove Theorem 2-(ii) obtaining more regularity for the weak solution $u \in E$ of (1.1)-(1.2)-(1.3) defined in (4.42).

Since $\bar v \in V := N \cap H^1$ is a critical point of $\Phi : V \to \mathbb{R}$

$$\int_\Omega \big(\beta(H + \bar u)^{2k} + \mathcal{R}_*\big)\psi = 0 \qquad \forall\, \psi \in N \cap H^1\,, \tag{4.43}$$

which actually holds for any $\psi \in N$ since $N \cap H^1$ is dense in N with the L^2-topology[7]. Hence, taking $\psi := \varphi_t$ for any $\varphi \in N \cap H^1$ in (4.43) and integrating by parts, we find

$$\begin{aligned} 0 &= \int_\Omega \partial_t \Big(\beta(H + \bar v + \bar w)^{2k} + \mathcal{R}_*\Big)\varphi \\ &= \int_\Omega \Big[2k\beta(H+\bar u)^{2k-1}(H_t + \bar v_t + \bar w_t) + \partial_t \mathcal{R}_*\Big]\varphi \end{aligned} \tag{4.44}$$

for any $\varphi \in N \cap H^1$. Since the term into square brackets $[\ldots]$ in (4.44) is in $L^2(\Omega)$ then, again by the L^2-density of $N \cap H^1$ in N, (4.44) actually holds for any $\varphi \in N$.
Setting for brevity

$$z := z(t,x;\varepsilon) := \Big(t, x, \varepsilon^{\frac{1}{2k}}\big(H(t,x) + \bar u(t,x;\varepsilon)\big)\Big)\,,$$

we can write, from (4.7)-(4.3),

$$\mathcal{R}_*(t,x;\varepsilon) = \varepsilon^{-1}\mathcal{R}(z) \qquad \text{and} \qquad \partial_t \mathcal{R}_* = \varepsilon^{-1+\frac{1}{2k}}\partial_u \mathcal{R}(z)\bar v_t + A$$

[7] Recall that $\big[\beta(H+\bar u)^{2k} + \mathcal{R}_*\big] \in L^2(\Omega)$ since $\beta, H, \bar u, \mathcal{R}_*$ are continuous functions.

where
$$A(t,x;\varepsilon) := \varepsilon^{-1}\partial_t\mathcal{R}(z) + \varepsilon^{-1+\frac{1}{2k}}\partial_u\mathcal{R}(z)\big(H_t(t,x)+\bar{w}_t(t,x;\varepsilon)\big)\ . \tag{4.45}$$
Then (4.44) becomes
$$\int_\Omega \Big[2k\beta(H+\bar{u})^{2k-1}(H_t+\bar{v}_t+\bar{w}_t) + \varepsilon^{-1+\frac{1}{2k}}\partial_u\mathcal{R}(z)\bar{v}_t + A\Big]\varphi = 0 \qquad \forall\,\varphi\in N\ . \tag{4.46}$$

For the remainder of this subsection we shall take $\varepsilon\neq 0$, and K_i will denote suitable positive constants possibly depending[8] also on ε.

Since we are assuming that $h\in H^j\cap C^{j-1}$, $j\geq 1$, then, by (2.3), $H\in H^{j+1}\cap C^j$. Hence, to prove that $u\in H^{j+1}\cap C^j$, by (4.42), it is sufficient to show that $\bar{v},\bar{w}\in H^{j+1}\cap C^j$.

We first prove that

Lemma 4.6 $\bar{u}\in C^1(\overline{\Omega})\cap H^2(\Omega)$.

Proof: We shall divide the proof in three steps.

Step 1: $\bar{w}\in C^1(\overline{\Omega})\cap H^2(\Omega)$ and
$$\|\bar{w}\|_{C^1} + \|\bar{w}\|_{H^2} \leq K_1\ |\varepsilon|\ . \tag{4.47}$$

We have $\bar{u}=\bar{v}+\bar{w}\in C(\overline{\Omega})\cap H^1(\Omega)$, $H\in H^2(\Omega)\cap C^1(\overline{\Omega})$ and $\beta\in H^1\big((0,\pi)\big)$. Moreover, since $\mathcal{R}\in C^1(\overline{\Omega}\times\mathbb{R})$ and $z(\cdot;\varepsilon)\in C\cap H^1$, then $\mathcal{R}_*(\cdot;\varepsilon)\in C\cap H^1$ and $\|\mathcal{R}_*(\cdot;\varepsilon)\|_C + \|\mathcal{R}_*(\cdot;\varepsilon)\|_{H^1}\leq K_2$. Hence $f(t,x,\bar{u}(t,x;\varepsilon);\varepsilon)=\beta(x)\big(H(t,x)+\bar{u}(t,x;\varepsilon)\big)^{2k}$ $+\mathcal{R}_*(t,x;\varepsilon)\in C\cap H^1$ and $\|f\|_C+\|f\|_{H^1}\leq K_3$.

Therefore, since $\bar{w}$ solves the range equation $\bar{w}=\varepsilon\Box^{-1}\Pi_{N^\perp}f$, $\Pi_{N^\perp}$ satisfies (2.6)-(2.7) and $\Box^{-1}$ satisfies (2.3), we deduce that $\bar{w}\in C^1(\overline{\Omega})\cap H^2(\Omega)$ and (4.47).

Step 2: $\bar{v}_t\in L^\infty(\Omega)$.

Let define
$$\kappa_{18} := k\,4^{-k}\min_{\Omega_{1/8}}\big(\beta H^{2k-1}\big) \tag{4.48}$$
and $0<\varepsilon_4\leq\varepsilon_3$ such that, $\forall\,|\varepsilon|\leq\varepsilon_4$
$$|\varepsilon|^{-1+\frac{1}{2k}}\|\partial_u\mathcal{R}(z)\|_{L^\infty(\Omega)}\leq\frac{\kappa_{18}}{2}\,,\ \ 4k\Big|\beta(H+\bar{v}+\bar{w})^{2k-1}-\beta(H+\bar{v})^{2k-1}\Big|_{L^\infty}\leq\pi\frac{\kappa_{18}}{2} \tag{4.49}$$
(such ε_4 exists by (1.7) and since $\|w\|_E=O(\varepsilon)$). We claim that
$$\int_\Omega\Big[2k\beta(H+\bar{u})^{2k-1}+\varepsilon^{-1+\frac{1}{2k}}\partial_u\mathcal{R}(z)\Big]\,v\varphi\geq\Big(10\pi\kappa_{18}M-\kappa_{19}\|\hat{v}\|_{L^2(\mathbb{T})}\Big)\|q(\hat{v})\|_{L^1(\mathbb{T})} \tag{4.50}$$
$$\forall\,|\varepsilon|\leq\varepsilon_4\,,\quad\forall\,v=v_+-v_-\in N\,,\ \ v_\pm(t,x)=\hat{v}(t\pm x)\,,\quad\varphi:=q(v_+)-q(v_-)\in N\,,$$

[8] However, such K_i can be taken *independently* of ε if we assume the further regularity hypothesis (1.10) on $\mathcal{R}$, see remark 1.3 and remark 4.8.

where $q = q_M$ $(M > 0)$ was defined in (4.16). Noting that $\int_\Omega \beta \bar{v}^{2k-1} v\varphi = 0$ by (2.19), $v\varphi \geq 0$ and using (2.22),

$$\begin{aligned}\int_\Omega 2k\beta\big(H+\bar{v}\big)^{2k-1} v\varphi &= \int_\Omega 2k\beta\Big(\big(H+\bar{v}\big)^{2k-1} - \bar{v}^{2k-1}\Big) v\varphi \geq \int_\Omega 2k4^{1-k}\beta H^{2k-1} v\varphi \\ &\geq 2k4^{1-k}\min_{\Omega_{1/8}}\big(\beta H^{2k-1}\big)\int_{\Omega_{1/8}} v\varphi = 8\kappa_{18}\int_{\Omega_{1/8}} v\varphi\,. \end{aligned} \tag{4.51}$$

Using (2.11)-(2.12) we obtain the lower bound

$$\begin{aligned}\int_{\Omega_{1/8}} v\varphi &\geq \int_{\Omega_{1/8}} v_+q_+ + v_-q_- - \int_\Omega |v_-||q_+| + |v_+||q_-| \\ &\geq 2\pi\left(1 - 2\frac{1}{8}\right)\int_0^{2\pi} \hat{v}(s)q(\hat{v}(s))\,ds - 2\int_\Omega |\hat{v}(t-x)||q(\hat{v}(t+x))| \\ &\geq \frac{3\pi}{2}\int_0^{2\pi} \hat{v}(s)q(\hat{v}(s))\,ds - \sqrt{2\pi}\|\hat{v}\|_{L^2(\mathbb{T})}\|q(\hat{v})\|_{L^1(\mathbb{T})}\,,\end{aligned}$$

and, by (4.51), we get

$$\int_\Omega 2k\beta\big(H+\bar{v}\big)^{2k-1} v\varphi \geq 12\pi\kappa_{18}\int_0^{2\pi} \hat{v}(s)q(\hat{v}(s))\,ds - \kappa_{19}\|\hat{v}\|_{L^2(\mathbb{T})}\|q(\hat{v})\|_{L^1(\mathbb{T})}\,. \tag{4.52}$$

Since $\int_\Omega |v\varphi| = \int_\Omega v\varphi = 2\pi\int_0^{2\pi} \hat{v}(s)q(\hat{v}(s))\,ds$ and using (4.49), we get, $\forall\,|\varepsilon| \leq \varepsilon_4$,

$$\left|\int_\Omega \varepsilon^{-1+\frac{1}{2k}}\partial_u\mathcal{R}(z)v\varphi\right| \leq |\varepsilon|^{-1+\frac{1}{2k}}\|\partial_u\mathcal{R}(z)\|_{L^\infty(\Omega)}\int_\Omega v\varphi \leq \pi\kappa_{18}\int_0^{2\pi} \hat{v}(s)q(\hat{v}(s))\,ds\;.$$

Therefore, using (4.52) and (4.49), we obtain

$$\begin{aligned}&\int_\Omega \Big[2k\beta(H+\bar{u})^{2k-1} + \varepsilon^{-1+\frac{1}{2k}}\partial_u\mathcal{R}(z)\Big]\,v\varphi \\ &= \int_\Omega 2k\beta(H+\bar{v})^{2k-1}v\varphi + \varepsilon^{-1+\frac{1}{2k}}\partial_u\mathcal{R}(z)v\varphi + 2k\beta\Big[(H+\bar{v}+\bar{w})^{2k-1} - (H+\bar{v})^{2k-1}\Big]v\varphi \\ &\geq 10\pi\kappa_{18}\int_0^{2\pi} \hat{v}(s)q(\hat{v}(s))\,ds - \kappa_{19}\|\hat{v}\|_{L^2(\mathbb{T})}\|q(\hat{v})\|_{L^1(\mathbb{T})}\,, \qquad \forall\,|\varepsilon| \leq \varepsilon_4\,.\end{aligned} \tag{4.53}$$

Since $\lambda q(\lambda) \geq M|q(\lambda)|$

$$\int_0^{2\pi} \hat{v}(s)q(\hat{v}(s))\,ds \geq M\int_0^{2\pi} |q(\hat{v}(s))|\,ds = M\|q(\hat{v})\|_{L^1(\mathbb{T})}\,,$$

and, by (4.53), we finally get (4.50).

We now conclude the proof that $\bar v_t \in L^\infty(\Omega)$. Taking $v := \bar v_t$ in (4.50) and $\varphi := q(\partial_t \bar v_+) - q(\partial_t \bar v_-)$ we obtain[9]

$$\int_\Omega \Big[2k\beta (H+\bar u)^{2k-1} + \varepsilon^{-1+\frac{1}{2k}} \partial_u \mathcal{R}(z) \Big] \bar v_t \varphi \geq \big(10\pi \kappa_{18} M - \kappa_{20} \big) \| q(\hat{\bar v}') \|_{L^1(\mathbb{T})} \qquad \forall\, |\varepsilon| \leq \varepsilon_4 \,. \tag{4.54}$$

Note that, since $H, \bar w \in C^1$, by (1.7) (A is defined in (4.45))

$$\|A\|_{L^\infty} \leq K_4 \,. \tag{4.55}$$

Finally from (4.46)-(4.47)-(4.54)-(4.55) we get

$$\big(10\pi \kappa_{18} M - \kappa_{20} \big) \| q(\hat{\bar v}') \|_{L^1(\mathbb{T})} \leq K_5 \|\varphi\|_{L^1(\Omega)} \,, \qquad \forall\, |\varepsilon| \leq \varepsilon_4 \,,$$

from which, recalling $\|\varphi\|_{L^1} \leq 2\pi \|q(\hat{\bar v}')\|_{L^1(\mathbb{T})}$ (see (4.27)), we have

$$M \|q(\hat{\bar v}')\|_{L^1(\mathbb{T})} \leq K_6 \|q(\hat{\bar v}')\|_{L^1(\mathbb{T})} \,. \tag{4.56}$$

We claim that (4.56) implies $\hat{\bar v}' \in L^\infty(\mathbb{T})$. Indeed, if $\hat{\bar v}' \notin L^\infty(\mathbb{T})$, then for any $M > 0$, $\|q(\hat{\bar v}')\|_{L^1(\mathbb{T})} > 0$ and (4.56) yields $M \leq K_6$; hence $\hat{\bar v}' \in L^\infty(\mathbb{T})$. Taking $M := \|\hat{\bar v}'\|_{L^\infty(\mathbb{T})}/2$ we obtain, by (4.56), $\|\hat{\bar v}'\|_{L^\infty(\mathbb{T})} \leq 2K_6$ and $\|\bar v_t\|_{L^\infty(\Omega)} \leq 4K_6$ by (2.15).

Step 3: $\bar v_t \in H^1(\Omega)$ (and hence $\bar v \in N \cap C^1(\Omega) \cap H^2(\Omega)$).

We claim that

$$\int_\Omega \Big[2k\beta (H+\bar u)^{2k-1} + \varepsilon^{-1+\frac{1}{2k}} \big(\partial_u \mathcal{R}(z) \big) \Big] v^2 \geq \kappa_{18} \|v\|_{L^2}^2 \,, \qquad \forall\, v \in N \,, \quad |\varepsilon| \leq \varepsilon_4 \,, \tag{4.57}$$

where κ_{18} is defined in (4.48) and ε_4 is defined in (4.49). Arguing as before, using $\int_\Omega \beta \bar v^{2k-1} v^2 = 0$ (recall (2.19)), (2.22) and (4.12)-(4.11)

$$\begin{aligned} \int_\Omega 2k\beta \big(H + \bar v \big)^{2k-1} v^2 &= \int_\Omega 2k\beta \Big(\big(H + \bar v \big)^{2k-1} - \bar v^{2k-1} \Big) v^2 \\ &\geq \int_\Omega 2k 4^{1-k} \beta H^{2k-1} v^2 \\ &\geq 2k4^{1-k} c_1 (\beta H^{2k-1}) \int_\Omega v^2 = 2\kappa_{18} \|v\|_{L^2}^2 \,. \end{aligned} \tag{4.58}$$

Since, by (4.49),

$$\left| \int_\Omega \varepsilon^{-1+\frac{1}{2k}} \big(\partial_u \mathcal{R}(z) \big) v^2 \right| \leq |\varepsilon|^{-1+\frac{1}{2k}} \|\partial_u \mathcal{R}(z)\|_{L^\infty} \|v\|_{L^2}^2 \leq \frac{\kappa_{18}}{2} \|v\|_{L^2}^2 \,,$$

[9] Here $\bar v(t,x) = \hat{\bar v}(t+x) - \hat{\bar v}(t-x)$ and so $\bar v_t(t,x) = \hat{\bar v}'(t+x) - \hat{\bar v}'(t-x)$.

and

$$\begin{aligned}\Big|\int_\Omega 2k\beta[(H+\bar u)^{2k-1}-(H+\bar v)^{2k-1}]v^2\Big| &\le \Big|2k\beta[(H+\bar u)^{2k-1}-(H+\bar v)^{2k-1}]\Big|_{L^\infty}\|v\|^2_{L^2}\\ &\le \frac{\kappa_{18}}{2}\|v\|^2_{L^2}\end{aligned}$$

using (4.58) we prove (4.57).
Take $\varphi = -D_{-h}D_h\bar v_t \in N$ in (4.46). Integrating by part (recall (2.27) and (2.24)) equality (4.46), we obtain

$$\begin{aligned}0 &= \int_\Omega D_h\Big[2k\beta(H+\bar u)^{2k-1}(H_t+\bar v_t+\bar w_t)+\varepsilon^{-1+\frac{1}{2k}}\partial_u\mathcal{R}(z)\bar v_t + A\Big]D_h\bar v_t\\ &= \int_\Omega \Big[2k\beta(H+\bar u)^{2k-1}+\varepsilon^{-1+\frac{1}{2k}}\big(\partial_u\mathcal{R}(z)\big)\Big](D_h\bar v_t)^2 \qquad (4.59)\\ &\quad+\Big[2k\beta\Big(D_h\big((H+\bar u)^{2k-1}\big)\Big)\big(T_h\bar v_t\big)+2k\beta\Big(D_h\big((H+\bar u)^{2k-1}(H_t+\bar w_t)\big)\Big)\\ &\qquad+\varepsilon^{-1+\frac{1}{2k}}\Big(D_h\big(\partial_u\mathcal{R}(z)\big)\Big)\big(T_h\bar v_t\big)+D_hA\Big]D_h\bar v_t\,.\end{aligned}$$

The dominant term here is (4.59). Using (4.57) with $v := D_h\bar v_t$, we get

$$\int_\Omega\Big[2k\beta(H+\bar u)^{2k-1}+\varepsilon^{-1+\frac{1}{2k}}\big(\partial_u\mathcal{R}(z)\big)\Big](D_h\bar v_t)^2 \ge \kappa_{18}\|D_h\bar v_t\|^2_{L^2}\,,\qquad \forall\,|\varepsilon|\le\varepsilon_4\,. \quad (4.60)$$

We now estimate all the other terms. Since $\|T_h\bar v_t\|_{L^\infty(\Omega)} = \|\bar v_t\|_{L^\infty(\Omega)} \le K_7$ and

$$\Big\|D_h\big((H+\bar u)^{2k-1}\big)\Big\|_{L^2} \le \Big\|\partial_t\big((H+\bar u)^{2k-1}\big)\Big\|_{L^2} = (2k-1)\Big\|(H+\bar u)^{2k-2}(H_t+\bar u_t)\Big\|_{L^2} \le \kappa_{21}\,, \quad (4.61)$$

we obtain

$$\Big|\int_\Omega\Big[2k\beta\Big(D_h\big((H+\bar u)^{2k-1}\big)\Big)\big(T_h\bar v_t\big)\Big]D_h\bar v_t\Big| \le K_8\|D_h\bar v_t\|_{L^2}\,. \quad (4.62)$$

Since $H\in C^1\cap H^2$, $\bar v_t\in L^\infty$, $\bar w\in C^1\cap H^2$, we have

$$\begin{aligned}&\Big\|D_h\big((H+\bar u)^{2k-1}(H_t+\bar w_t)\big)\Big\|_{L^2} \le \Big\|\partial_t\big((H+\bar u)^{2k-1}(H_t+\bar w_t)\big)\Big\|_{L^2}\\ &\quad= \Big\|(2k-1)(H+\bar u)^{2k-2}(H_t+\bar u_t)(H_t+\bar w_t)+(H+\bar u)^{2k-1}(H_{tt}+\bar w_{tt})\Big\|_{L^2} \le K_9\,,\end{aligned}$$

and we deduce

$$\Big|\int_\Omega\Big[2k\beta\Big(D_h\big((H+\bar u)^{2k-1}(H_t+\bar w_t)\big)\Big)\Big]D_h\bar v_t\Big| \le K_9\|D_h\bar v_t\|_{L^2}\,. \quad (4.63)$$

From

$$\varepsilon^{-1+\frac{1}{2k}}\partial_t\big(\partial_u\mathcal{R}(z)\big) = \varepsilon^{-1+\frac{1}{2k}}\partial^2_{tu}\mathcal{R}(z)+\varepsilon^{-1+\frac{1}{k}}\partial^2_{uu}\mathcal{R}(z)(H_t+\bar u_t)$$

we derive

$$\left\| \varepsilon^{-1+\frac{1}{2k}} D_h\big(\partial_u \mathcal{R}(z)\big) \right\|_{L^2} \le \left\| \varepsilon^{-1+\frac{1}{2k}} \partial_t\big(\partial_u \mathcal{R}(z)\big) \right\|_{L^2} \le K_{10} \tag{4.64}$$

and (recall $\bar v_t \in L^\infty$)

$$\left| \int_\Omega \left[\varepsilon^{-1+\frac{1}{2k}} \Big(D_h\big(\partial_u \mathcal{R}(z)\big)\Big) \bar v_t \right] D_h \bar v_t \right| \le K_{10} \|\bar v_t\| \ \|D_h \bar v_t\|_{L^2} = K_{11} \|\bar v_t\| \, . \tag{4.65}$$

We finally estimate the term

$$\left| \int_\Omega (D_h A)(D_h \bar v_t) \right| \le \|D_h A\|_{L^2} \|D_h \bar v_t\|_{L^2} \stackrel{(2.29)}{\le} \|\partial_t A\|_{L^2} \|D_h \bar v_t\|_{L^2} \, .$$

(A is defined in (4.45)). Since

$$\begin{aligned} \partial_t A &= \varepsilon^{-1} \partial^2_{tt} \mathcal{R}(z) + \varepsilon^{-1+\frac{1}{2k}} \partial^2_{tu} \mathcal{R}(z)(2H_t + \bar v_t + 2\bar w_t) \\ &\quad + \varepsilon^{-1+\frac{1}{k}} \partial^2_{uu} \mathcal{R}(z)(H_t + \bar u_t)(H_t + \bar w_t) + \varepsilon^{-1+\frac{1}{2k}} \partial_u \mathcal{R}(z)(H_{tt} + \bar w_{tt}) \, , \end{aligned}$$

by (1.7), and using that $\bar v_t \in L^\infty(\Omega)$, $\bar w, H \in H^2(\Omega) \cap C^1(\Omega)$, then $\|\partial_t A\|_{L^2} \le K_{12}$ and

$$\left| \int_\Omega (D_h A)(D_h \bar v_t) \right| \le K_{12} \|D_h \bar v_t\|_{L^2} \, . \tag{4.66}$$

Recollecting (4.59), (4.60), (4.62), (4.63), (4.65) and (4.66) we obtain

$$\kappa_{18} \|D_h \bar v_t\|^2_{L^2} \le K_{13} \|D_h \bar v_t\|_{L^2(\Omega)}$$

and $\|D_h \bar v_t\|_{L^2(\Omega)} \le K_{14}$ for all h. Therefore, by (2.28), $\bar v_t \in N \cap H^1$ and $\|\bar v_{tt}\|_{L^2(\Omega)} \le K_{14}$.
$\square$

We now prove Theorem 2-(ii) by induction over $j \ge 1$.

Lemma 4.7 *Assuming $\bar v$, $\bar w \in C^{j-1}(\overline{\Omega}) \cap H^j(\Omega)$, then $\bar v$, $\bar w \in C^j(\overline{\Omega}) \cap H^{j+1}(\Omega)$.*

Proof: Again we divide the proof in three steps.
Step 1: $\bar w \in C^j(\overline{\Omega}) \cap H^{j+1}(\Omega)$ and

$$\|\bar w\|_{C^j} + \|\bar w\|_{H^{j+1}} \le K_1^{(j)} |\varepsilon| \, . \tag{4.67}$$

By hypotheses $H \in C^j(\overline{\Omega}) \cap H^{j+1}(\Omega)$ (since $h \in C^{j-1}(\overline{\Omega}) \cap H^j(\Omega)$), $\beta \in H^j\big((0,\pi)\big)$, $\mathcal{R} \in C^j(\overline{\Omega} \times \mathbb{R})$ and $z(\cdot;\varepsilon) \in C^{j-1}(\overline{\Omega}) \cap H^j(\Omega)$. Hence $\mathcal{R}_*(\cdot;\varepsilon) \in C^{j-1}(\overline{\Omega}) \cap H^j(\Omega)$ and $\|\mathcal{R}_*(\cdot;\varepsilon)\|_{C^{j-1}} + \|\mathcal{R}_*(\cdot;\varepsilon)\|_{H^j} \le K_2^{(j)}$. Hence $f(t,x,\bar u(t,x;\varepsilon);\varepsilon) = \beta(x)\big(H(t,x) + \bar u(t,x;\varepsilon)\big)^{2k} + \mathcal{R}_*(t,x;\varepsilon) \in C^{j-1}(\overline{\Omega}) \cap H^j(\Omega)$ and $\|f\|_{C^{j-1}} + \|f\|_{H^j} \le K_3^{(j)}$.

Since $\bar w$ solves the range equation $\bar w = \varepsilon \Box^{-1} \Pi_{N^\perp} f$, $\Pi_{N^\perp}$ satisfies (2.6)-(2.7) and $\Box^{-1}$ satisfies (2.3), we conclude that $\bar w \in C^j(\overline{\Omega}) \cap H^{j+1}(\Omega)$ and that (4.67) holds.

Step 2: $\partial_t^j \bar{v} \in L^\infty(\Omega)$.

Reasoning as for (4.44), we get

$$\begin{aligned} 0 &= \int_\Omega \partial_t^j \Big(\beta (H + \bar{v} + \bar{w})^{2k} + \mathcal{R}_* \Big) \varphi \\ &= \int_\Omega \Big[2k\beta (H + \bar{u})^{2k-1} + \varepsilon^{-1+\frac{1}{2k}} \partial_u \mathcal{R}(z) \Big] (\partial_t^j \bar{v}) \varphi + \mathcal{F}^{(j)} \varphi \end{aligned} \tag{4.68}$$

for any $\varphi \in N$. Here $\mathcal{F}^{(j)}$ depends polynomially on k, β and

$$\varepsilon^{-1+\frac{n}{2k}} \partial_t^l \partial_u^n \mathcal{R}(z) \quad l + n = j\,, \qquad \partial_t^l H\,,\ \partial_t^l \bar{w} \quad l \le j\,, \qquad \partial_t^l \bar{v} \quad l \le j-1\,,$$

but *not* on $\partial_t^j \bar{v}$.
Choosing in (4.50) $v = \partial_t^j \bar{v}$ and so $\varphi := q(\partial_t^j \bar{v}_+) - q(\partial_t^j \bar{v}_-)$, we get (recall that $\bar{v} \in H^j$), $\forall\, |\varepsilon| \le \varepsilon_4$,

$$\int_\Omega \Big[2k\beta (H + \bar{u})^{2k-1} + \varepsilon^{-1+\frac{1}{2k}} \partial_u \mathcal{R}(z) \Big] (\partial_t^j \bar{v}) \varphi \ge \Big(10\pi \kappa_{18} M - \kappa_{24} \Big) \| q(\hat{\bar{v}}^{(j)}) \|_{L^1(\mathbb{T})}\,, \tag{4.69}$$

where $\hat{\bar{v}}^{(j)}(\xi) := \frac{d^j}{d\xi^j} \hat{\bar{v}}$. On the other hand, by (4.68) we get

$$\int_\Omega \Big[2k\beta (H + \bar{u})^{2k-1} + \varepsilon^{-1+\frac{1}{2k}} \partial_u \mathcal{R}(z) \Big] (\partial_t^j \bar{v}) \varphi \le \| \mathcal{F}^{(j)} \|_{L^\infty} \| \varphi \|_{L^1} \le K_4^{(j)} \| \varphi \|_{L^1}\,. \tag{4.70}$$

Since $\|\varphi\|_{L^1} \le 2\pi \| q(\hat{\bar{v}}^{(j)}) \|_{L^1(\mathbb{T})}$ (see (4.27)), from (4.69) and (4.70) we get

$$\big(10\pi \kappa_{18} M - \kappa_{24} \big) \| q(\hat{\bar{v}}^{(j)}) \|_{L^1(\mathbb{T})} \le K_5^{(j)} \| q(\hat{\bar{v}}^{(j)}) \|_{L^1(\mathbb{T})}\,, \qquad \forall\, |\varepsilon| \le \varepsilon_4\,.$$

Then

$$M \| q(\hat{\bar{v}}^{(j)}) \|_{L^1(\mathbb{T})} \le K_6^{(j)} \| q(\hat{\bar{v}}^{(j)}) \|_{L^1(\mathbb{T})}\,, \qquad \forall\, |\varepsilon| \le \varepsilon_4\,.$$

Arguing as in (4.56) we get $\hat{\bar{v}}^{(j)} \in L^\infty(\mathbb{T})$. Finally $\partial_t^j \bar{v} \in L^\infty(\Omega)$ and $\|\partial_t^j \bar{v}\|_{L^\infty(\Omega)} \le 4K_6^{(j)}$.

Step 3: We now prove that $\partial_t^j \bar{v} \in H^1$ (and hence $\bar{v} \in N \cap C^j \cap H^{j+1}$).

Choosing $\varphi := -D_{-h} D_h \partial_t^j \bar{v}$ in (4.68), integrating by parts (recall (2.27) and (2.24))

$$\begin{aligned} 0 &= \int_\Omega \Big[2k\beta (H + \bar{u})^{2k-1} + \varepsilon^{-1+\frac{1}{2k}} \partial_u \mathcal{R}(z) \Big] \big(D_h(\partial_t^j \bar{v}) \big)^2 \\ &\quad + \Big[\Big(D_h \big(2k\beta (H + \bar{u})^{2k-1} + \varepsilon^{-1+\frac{1}{2k}} \partial_u \mathcal{R}(z) \big) \Big) \big(T_h(\partial_t^j \bar{v}) \big) + \big(D_h(\mathcal{F}^{(j)}) \big) \Big] \big(D_h(\partial_t^j \bar{v}) \big)\,. \end{aligned} \tag{4.71}$$

Using (4.57) we get

$$\int_\Omega \Big[2k\beta (H + \bar{u})^{2k-1} + \varepsilon^{-1+\frac{1}{2k}} \partial_u \mathcal{R}(z) \Big] \big(D_h(\partial_t^j \bar{v}) \big)^2 \ge \kappa_{18} \Big\| D_h(\partial_t^j \bar{v}) \Big\|_{L^2}^2\,, \qquad \forall\, |\varepsilon| \le \varepsilon_4\,. \tag{4.72}$$

¿From (4.61), (4.64) and since $\partial_t^j \bar v \in L^\infty(\Omega)$,

$$\left| \int_\Omega \Big(D_h\big(2k\beta(H+\bar u)^{2k-1} + \varepsilon^{-1+\frac{1}{2k}}\partial_u \mathcal{R}(z)\big)\Big) \big(T_h(\partial_t^j \bar v)\big)\big(D_h(\partial_t^j \bar v)\big)\right|$$
$$\le 4K_6^{(j)} \Big\| D_h\big(2k\beta(H+\bar u)^{2k-1} + \varepsilon^{-1+\frac{1}{2k}}\partial_u \mathcal{R}(z)\big)\Big\|_{L^2} \|D_h(\partial_t^j \bar v)\|_{L^2}$$
$$\le K_7^{(j)} \|D_h(\partial_t^j \bar v)\|_{L^2} \,. \tag{4.73}$$

We note that $\partial_t \mathcal{F}^{(j)}$ is a polynomial in k, β,

$$\varepsilon^{-1+\frac{n}{2k}}\partial_t^l \partial_u^n \mathcal{R}(z) \quad l+n = j+1\,, \qquad \partial_t^l H\,,\ \partial_t^l \bar w \quad l \le j+1\,, \qquad \partial_t^l \bar v \quad l \le j\,,$$

and that the terms $\partial_t^{j+1} H$, $\partial_t^{j+1} \bar w \in L^2(\Omega)$ (recall that $\bar w, H \in H^{j+1}(\Omega)$ by Step 1) appear only linearly (with no powers). Hence, using that $\partial_t^l H$, $\partial_t^l \bar w$, $\partial_t^l \bar v \in L^\infty(\Omega)$, $\forall l \le j$,

$$\begin{aligned}
\left| \int_\Omega \big(D_h(\mathcal{F}^{(j)})\big)\big(D_h(\partial_t^j \bar v)\big)\right| &\le \|D_h(\mathcal{F}^{(j)})\|_{L^2} \|D_h(\partial_t^j \bar v)\|_{L^2} \\
&\overset{(2.29)}{\le} \|\partial_t(\mathcal{F}^{(j)})\|_{L^2} \|D_h(\partial_t^j \bar v)\|_{L^2} \\
&\le K_8^{(j)} \|D_h(\partial_t^j \bar v)\|_{L^2} \,.
\end{aligned} \tag{4.74}$$

Finally, by (4.71)-(4.72)-(4.73)-(4.74) we get

$$\kappa_{18} \|D_h(\partial_t^j \bar v)\|_{L^2}^2 \le K_9^{(j)} \|D_h(\partial_t^j \bar v)\|_{L^2},$$

and therefore $\|D_h(\partial_t^j \bar v)\|_{L^2} \le K_{10}^{(j)}$. By (2.28), we conclude the proof obtaining $\partial_t^{j+1} \bar v \in L^2(\Omega)$ and $\|\partial_t^{j+1} \bar v\|_{L^2} \le K_{10}^{(j)}$. □

Remark 4.8 *If H, $\bar u \in H^i(\Omega) \cap C^{i-1}(\overline{\Omega})$ $(0 \le i \le j)$ and (1.10) holds, then R_* and $\partial_t R_*$ are bounded in $H^i(\Omega) \cap C^{i-1}(\overline{\Omega})$ by some constant κ_i independent of ε. In this case, the constants K_i of this section can be taken independently of ε, obtaining the estimates (1.11).*

4.6 Proof of Theorem 1

The following Proposition is a sort of "weak maximum principle" for the wave equation (1.1)-(1.2)-(1.3).

Proposition 4.9 *Let $h \in N^\perp$, $h \ge 0$ a.e. in Ω. Then there exists a weak solution $H \in E$ of $\Box H = h$ satisfying $H \ge 0$. In particular we can choose*

$$H(t,x) := \frac{1}{2}\int_0^\kappa \int_{t-x-\xi}^{t-x+\xi} h(\tau,\xi)\, d\tau\, d\xi - \frac{1}{2}\int_\kappa^x \int_{t-x+\xi}^{t+x-\xi} h(\tau,\xi)\, d\tau\, d\xi \tag{4.75}$$

for a suitable $\kappa \in (0,\pi)$. Moreover $h \in C^{j-1} \Rightarrow H \in C^j$ and $h \in H^j \Rightarrow H \in H^{j+1}$, for $j \ge 1$.

PROOF: We suppose that

$$\exists\, A \subseteq \Omega \quad \text{with} \quad \text{meas}(A) > 0\,, \quad h \geq \text{const} > 0 \quad \text{on} \;\; A\,, \tag{4.76}$$

since the case $h \equiv 0$ a.e. on Ω is trivial.

Step 1: *H defined in* (4.75) *belongs to* $H^1(\Omega) \cap C^{1/2}(\overline{\Omega})$ *for any* $\kappa \in (0, \pi)$ *and*

$$\begin{aligned} 2(\partial_t H)(t,x) &= \int_0^\kappa \Big(h(t-x+\xi,\xi) - h(t-x-\xi,\xi) \Big) d\xi \\ &\quad - \int_\kappa^x \Big(h(t+x-\xi,\xi) - h(t-x+\xi,\xi) \Big) d\xi \in L^2(\Omega)\,, \qquad (4.77) \\ 2(\partial_x H)(t,x) &= \int_0^\kappa \Big(- h(t-x+\xi,\xi) + h(t-x-\xi,\xi) \Big) d\xi \\ &\quad - \int_\kappa^x \Big(h(t+x-\xi,\xi) + h(t-x+\xi,\xi) \Big) d\xi \in L^2(\Omega)\,. \qquad (4.78) \end{aligned}$$

We shall prove that the first addendum in the r.h.s. of (4.75)

$$H_1(t,x) := \frac{1}{2} \int_0^\kappa \int_{t-x-\xi}^{t-x+\xi} h(\tau,\xi)\, d\tau\, d\xi$$

belongs to $C^{1/2}(\overline{\Omega}) \cap H^1(\Omega)$, the second addendum being analogous. Defining

$$T(t,x) := T(t,x;\kappa) := \Big\{ (\tau,\xi) \in \Omega \quad \Big| \quad t-x-\xi < \tau < t-x+\xi\,, \;\; 0 < \xi < \kappa \Big\}$$

we can write $H_1(t,x) := (1/2) \int_{T(t,x)} h(\tau,\xi)\, d\tau\, d\xi$.

Since $\text{meas}\big(T(t,x;\kappa)\big) = \kappa^2 \leq \pi^2$ we derive that H_1 is uniformly bounded by

$$\big|H_1(t,x)\big| \leq \frac{1}{2} \int_\Omega \mathbf{1}_{T(t,x)}(\tau,\xi) |h(\tau,\xi)|\, d\tau\, d\xi \leq \frac{\pi}{2} \|h\|_{L^2(\Omega)}$$

using Cauchy-Schwartz inequality.

For $i = 1, 2$ and $(t_i, x_i) \in \Omega$, let define $T_i := T(t_i, x_i)$. It results

$$\text{meas}(T_1 \setminus T_2) = \text{meas}(T_2 \setminus T_1) \leq \pi \big(|t_1 - t_2| + |x_1 - x_2| \big)$$

and, using again Cauchy-Schwartz inequality,

$$|H_1(t_1,x_1) - H_1(t_2,x_2)| \leq \frac{1}{2} \int_{T_1 \setminus T_2} |h| + \frac{1}{2} \int_{T_2 \setminus T_1} |h| \leq \sqrt{\pi} \big(|t_1 - t_2| + |x_1 - x_2| \big)^{1/2} \|h\|_{L^2(\Omega)}\,.$$

Therefore we have proved $H_1 \in C^{1/2}(\overline{\Omega})$.

We now prove that $H_1 \in H^1(\Omega)$ and that $\partial_t H_1 = -\partial_x H_1 = f_1$ where

$$f_1(t,x) := \frac{1}{2}\int_0^\kappa \Big(h(t-x+\xi,\xi) - h(t-x-\xi,\xi) \Big)\, d\xi \in L^2(\Omega) \ .$$

We first justify that $f_1 \in L^2(\Omega)$. Since

$$|h(t-x+\xi,\xi)|^2 + |h(t-x-\xi,\xi)|^2 \geq \frac{1}{2}|h(t-x+\xi,\xi) - h(t-x-\xi,\xi)|^2 \,,$$

by periodicity w.r.t. t we obtain that, $\forall\, x \in (0,\pi)$,

$$\|h\|^2_{L^2(\Omega)} \geq \int_0^\kappa \int_0^{2\pi} |h(t,\xi)|^2 dt\, d\xi \geq \frac{1}{4}\int_0^\kappa \int_0^{2\pi} |h(t-x+\xi,\xi) - h(t-x-\xi,\xi)|^2 dt\, d\xi \,.$$

Integrating the previous inequality in the variable x between 0 and π, applying Fubini Theorem and Cauchy-Schwartz inequality, we deduce

$$\begin{aligned}
\pi \|h\|^2_{L^2(\Omega)} &\geq \frac{1}{4}\int_0^\pi \int_0^\kappa \int_0^{2\pi} |h(t-x+\xi,\xi) - h(t-x-\xi,\xi)|^2 \, dt\, d\xi\, dx \\
&= \frac{1}{4}\int_\Omega \int_0^\kappa |h(t-x+\xi,\xi) - h(t-x-\xi,\xi)|^2 \, d\xi\, dt\, dx \\
&\geq \frac{1}{4\kappa}\int_\Omega \Big(\int_0^\kappa |h(t-x+\xi,\xi) - h(t-x-\xi,\xi)|\, d\xi \Big)^2 \, dt\, dx \,. \\
&\geq \frac{1}{\kappa}\int_\Omega f_1^2(t,x)\, dt\, dx = \frac{1}{\kappa}\|f_1\|^2_{L^2} \ .
\end{aligned}$$

Finally, we prove that $\partial_x H_1 = -f_1$, being $\partial_t H_1 = f_1$ analogous. By Fubini Theorem

$$\begin{aligned}
2\int_\Omega H_1 \varphi_x &= \int_0^{2\pi}\int_0^\kappa \int_0^\pi \int_{t-x-\xi}^{t-x+\xi} h(\tau,\xi)\varphi_x(t,x)\, d\tau\, dx\, d\xi\, dt \\
&= \int_0^{2\pi}\int_0^\kappa \Big(\int_{t+\xi-\pi}^{t+\xi} h \int_0^{t+\xi-\tau} \varphi_x\, dx\, d\tau + \int_{t-\xi}^{t+\xi-\pi} h \int_0^\pi \varphi_x\, dx\, d\tau \\
&\qquad\qquad + \int_{t-\xi-\pi}^{t-\xi} h \int_{t-\xi-\tau}^\pi \varphi_x\, dx\, d\tau \Big)\, d\xi\, dt \\
&= \int_0^{2\pi}\int_0^\kappa \Big(\int_{t+\xi-\pi}^{t+\xi} h(\tau,\xi)\varphi(t,t+\xi-\tau)\, d\tau \\
&\qquad\qquad - \int_{t-\xi-\pi}^{t-\xi} h(\tau,\xi)\varphi(t,t-\xi-\tau)\, d\tau \Big)\, d\xi\, dt \\
&= \int_0^{2\pi}\int_0^\kappa \Big(-\int_\pi^0 h(t+\xi-x,\xi)\varphi(t,x)\, dx \\
&\qquad\qquad + \int_\pi^0 h(t-\xi-x,\xi)\varphi(t,x)\, dx \Big)\, d\xi\, dt = 2\int_\Omega f_1 \varphi \,.
\end{aligned}$$

With analogue computations for the second addendum of H in (4.75) we derive (4.77) and (4.78).

Step 2: *There exists $\kappa \in (0,\pi)$ such that $H(t,x)$ verifies the Dirichlet boundary conditions $H(t,0) = H(t,\pi) = 0 \ \forall\, t \in \mathbb{T}$.*

By (4.75), the function H satisfies, for any $\kappa \in (0,\pi)$, $H(t,0) = 0 \ \forall\, t \in \mathbb{T}$. It remains to find κ imposing $H(t,\pi) = 0$. Let, for $t \in \mathbb{T}$,

$$\mathcal{T}(t) := \{ (\tau,\xi) \in \Omega \quad \text{s.t.} \quad t-\xi < \tau < t+\xi\,, \quad 0 < \xi < \pi \}\,.$$

By (4.76) it follows that

$$\exists\, t_* \in \mathbb{T} \ \text{ s.t. } \ \mathcal{T}_* := \mathcal{T}(t_*) \ \text{ verifies } \ \mathrm{meas}(A \cap \mathcal{T}_*) > 0\,, \ \mathrm{meas}(A \setminus \mathcal{T}_*) > 0\,. \tag{4.79}$$

Taking $x = \pi$ in (4.75) we obtain

$$\begin{aligned} H(t,\pi) \ &:= \ \frac{1}{2}\int_0^{\kappa}\int_{t-\pi-\xi}^{t-\pi+\xi} h(\tau,\xi)\, d\tau\, d\xi - \frac{1}{2}\int_{\kappa}^{\pi}\int_{t-\pi+\xi}^{t+\pi-\xi} h(\tau,\xi)\, d\tau\, d\xi \\ &= \ \frac{1}{2}\int_0^{\pi}\int_{t-\pi-\xi}^{t-\pi+\xi} h(\tau,\xi)\, d\tau\, d\xi - \frac{1}{2}\int_{\kappa}^{\pi}\int_{t-\pi-\xi}^{t+\pi-\xi} h(\tau,\xi)\, d\tau\, d\xi \\ &= \ \frac{1}{2}\int_{\mathcal{T}(t-\pi)} h(\tau,\xi)\, d\tau\, d\xi - \frac{1}{2}\int_{\kappa}^{\pi}\int_{0}^{2\pi} h(\tau,\xi)\, d\tau\, d\xi \\ &= \ \frac{1}{2}\int_{\mathcal{T}_*} h(\tau,\xi)\, d\tau\, d\xi - \frac{1}{2}\int_{\kappa}^{\pi}\int_{0}^{2\pi} h(\tau,\xi)\, d\tau\, d\xi \ =: \ c - \chi(\kappa)\,, \end{aligned} \tag{4.80}$$

where in the last two lines we have used the periodicity of $h(\cdot,\xi)$ and (2.4) to substitute $\mathcal{T}(t-\pi)$ with $\mathcal{T}_*$.

In order to prove that $H(t,\pi) = 0$, $\forall t \in \mathbb{T}$, we need only to solve $\chi(\kappa) = c$. By the absolute continuity of the integral (with respect to the two-dimensional measure $d\tau\, d\xi$) $\chi(\kappa)$ is a (nonincreasing) continuous function. Moreover $\chi(0) > c > 0$ by (4.79) and $\chi(\pi) = 0$. Therefore there exists $\kappa \in (0,\pi)$ solving $\chi(\kappa) = c$.

Step 3: *$H \in E$ is a weak solution of $\Box H = h$, namely*

$$\int_{\Omega} \varphi_t H_t - \varphi_x H_x + \varphi h = 0\,, \qquad \forall\, \varphi \in C^1_0(\overline{\Omega})\,. \tag{4.81}$$

By Fubini Theorem and periodicity we get

$$\begin{aligned} &\int_{\Omega}\Big(\varphi_t(t,x)\int_0^{\kappa} h(t-x+\xi,\xi)d\xi + \varphi_x(t,x)\int_0^{\kappa} h(t-x+\xi,\xi)d\xi\Big)\, dt\, dx \\ &= \int_0^{\pi}\int_0^{\kappa}\int_0^{2\pi}\Big(\varphi_t(t,x)h(t-x+\xi,\xi) + \varphi_x(t,x)h(t-x+\xi,\xi)\Big)\, dt\, d\xi\, dx \\ &= \int_0^{\pi}\int_0^{\kappa}\int_0^{2\pi}\Big(\varphi_t(t+x-\xi,x) + \varphi_x(t+x-\xi,x)\Big)h(t,\xi)\, dt\, d\xi\, dx \\ &= \int_0^{\kappa}\int_0^{2\pi} h(t,\xi)\int_0^{\pi}\frac{d}{dx}\Big(\varphi(t+x-\xi,x)\Big)\, dx\, dt\, d\xi = 0 \end{aligned} \tag{4.82}$$

by Dirichlet boundary conditions. Analogously,

$$-\int_\Omega \Big(\varphi_t(t,x)\int_0^\kappa h(t-x-\xi,\xi)d\xi + \varphi_x(t,x)\int_0^\kappa h(t-x-\xi,\xi)d\xi\Big)\,dt\,dx = 0\,. \tag{4.83}$$

Moreover, again by Fubini Theorem,

$$\begin{aligned}
&\int_\Omega \Big(-\varphi_t(t,x)\int_\kappa^x h(t+x-\xi,\xi)d\xi + \varphi_x(t,x)\int_\kappa^x h(t+x-\xi,\xi)d\xi\Big)\,dt\,dx \\
&= \int_0^\pi\int_\kappa^x\int_0^{2\pi}\Big(-\varphi_t(t,x)h(t+x-\xi,\xi)+\varphi_x(t,x)h(t+x-\xi,\xi)\Big)\,dt\,d\xi\,dx \\
&= \int_0^\pi\int_\kappa^x\int_0^{2\pi}\Big(-\varphi_t(t-x+\xi,x)+\varphi_x(t-x+\xi,x)\Big)h(t,\xi)\,dt\,d\xi\,dx \\
&= \int_0^{2\pi}\int_0^\pi h(t,\xi)\int_\xi^\pi \frac{d}{dx}\Big(\varphi(t-x+\xi,x)\Big)\,dx\,dt\,d\xi \\
&\quad -\int_0^{2\pi}\int_0^\kappa h(t,\xi)\int_0^\pi \frac{d}{dx}\Big(\varphi(t-x+\xi,x)\Big)\,dx\,dt\,d\xi = -\int_\Omega h\varphi
\end{aligned}$$

and, analogously,

$$\int_\Omega \Big(\varphi_t(t,x)\int_\kappa^x h(t-x+\xi,\xi)d\xi + \varphi_x(t,x)\int_\kappa^x h(t-x+\xi,\xi)d\xi\Big)\,dt\,dx = -\int_\Omega h\varphi\,. \tag{4.84}$$

Summing (4.82), (4.83), (4.84), (4.84) and recalling (4.77), (4.78) we get (4.81).

Step 4: *$H(t,x)\geq 0$ in Ω.*

First case: $0<x<\kappa$. By (4.75) and geometrical considerations on the domains of the integrals, we derive that, for $0<x<\kappa$, $H(t,x)=\int_\Theta h(\tau,\xi)\ d\tau d\xi$ where $\Theta:=\Theta_{t,x}$ is the trapezoidal region in Ω with a vertex in $(\tau,\xi)=(t,x)$ and delimited by the straight lines $\tau=t-x+\xi$, $\tau=t+x-\xi$, $\xi=\kappa$ and $\tau=t-x-\xi$. Since $h\geq 0$ we conclude that $H(t,x)\geq 0$.

Second case: $\kappa<x<\pi$. Since $H(t+\pi-x,\pi)=0$ we have, by (4.75),

$$\int_0^\kappa\int_{t-x-\xi}^{t-x+\xi} h(\tau,\xi)\ d\tau d\xi = \int_\kappa^\pi\int_{t-x+\xi}^{t-x-\xi+2\pi} h(\tau,\xi)\ d\tau d\xi\ .$$

Therefore, substituting in (4.75), we get, for $\kappa<x<\pi$, the expression $H(t,x)=\int_\Theta h(\tau,\xi)\ d\tau d\xi$ where, now, $\Theta:=\Theta_{t,x}$ is the trapezoidal region in Ω with a vertex in $(\tau,\xi)=(t,x)$ and delimited by the straight lines $\tau=t-x+\xi$, $\tau=t-x-\xi+2\pi$, $\xi=\kappa$ and $\tau=t+x-\xi$. Since $h\geq 0$ we conclude also in this case that $H(t,x)\geq 0$. $\square$

The following result is a sort of "strong maximum principle" for the wave equation (1.1)-(1.2)-(1.3) and directly follows by Proposition 4.9.

Proposition 4.10 *Let $h \in N^{\perp}$, $h \geq 0$ a.e in Ω and suppose $h(t,x) \geq h_0 > 0$ a.e. in Ω_α for some $0 \leq \alpha < 1/2$. Then there exists a weak solution $H \in E$ of $\Box H = h$ satisfying $H \geq 0$ and $H > 0$ in Ω_α. Moreover $h \in C^{j-1} \Rightarrow H \in C^j$ and $h \in H^j \Rightarrow H \in H^{j+1}$, for $j \geq 1$.*

PROOF: Let $\varphi \in C^\infty(\mathbb{R})$ satisfy $\varphi(x) > 0$ for $\alpha\pi < x < (1-\alpha)\pi$, $\varphi(x) = 0$ otherwise, and $-\varphi''(x) \leq h_0$ $\forall\, x \in \mathbb{R}$. Then $H_*(t,x) := \varphi(x) \in C_0^\infty(\overline{\Omega})$ is a (classical) solution of $\Box H_*(t,x) = h_*(t,x) := -\varphi''(x) \in N^\perp \cap C^\infty$.

Since $h - h_* \geq 0$ a.e. in Ω (recall $-\varphi''(x) \leq h_0$) and $h - h_* \in N^\perp$, by Proposition 4.9 there exists $H_{**} \in E$, $H_{**} \geq 0$ which is a weak solution of $\Box H_{**} = h - h_*$. Therefore $H := H_* + H_{**} \geq H_*$ $(= \varphi(x))$ is the required weak solution of $\Box H = h$ satisfying $H \geq 0$ in Ω and $H > 0$ in Ω_α. The statement on regularity follows from Proposition 4.9 and from $h_*, H_* \in C^\infty$. □

PROOF OF THEOREM 1. By Proposition 4.10 (with $\alpha = 0$) there exists a weak solution $H \in E$ of $\Box H = h$ verifying (1.8) (i.e. $H > 0$ in Ω). Therefore existence of a weak solution $u \in E$ satisfying $\|u\|_E \leq C|\varepsilon|$ follows from Theorem 2-(i) with $\beta(x) \equiv \beta$ and $\mathcal{R} \equiv 0$. The higher regularity for u and the estimate $\|u\|_{H^{j+1}(\Omega)} + \|u\|_{C^j(\overline{\Omega})} \leq C|\varepsilon|$ follow from Theorem 2-(ii) and (1.11) in remark 1.3 since assumption (1.10) is trivially verified $(\mathcal{R} \equiv 0)$. □

5 Proof of Theorem 3

In order to prove Theorem 3 we perform the Lyapunov-Schmidt reduction of section 3 and we minimize the reduced action functional Φ in $\overline{B_R} := \{\|v\|_{H^1} \leq R\}$. To conclude the existence of a solution, we have to prove that the minimum $\bar{v} \in \overline{B_R}$ is an interior minimum point in B_R for some $R > 0$.

This case is easier that the previous one since the required a-priori estimates can be deduced directly by the 0^{th}-order variational inequality (1.18) which does not vanish for $\varepsilon = 0$.

Step 1: The L^∞-estimate

Since $a(x,u)$ satisfies (1.12) or (1.13), by (2.18), $\int_\Omega a(x,\bar{v})\varphi = 0$, $\forall \varphi \in V$ (as $\bar{v} \in V$) and hence

$$\int_\Omega f(\bar{v})\varphi = \int_\Omega \widetilde{f}(\bar{v})\varphi + a(x,\bar{v})\varphi = \int_\Omega \widetilde{f}(\bar{v})\varphi\ . \tag{5.1}$$

Since $\|w(\bar{v},\varepsilon)\|_E \leq C|\varepsilon|$, by the variational inequality (1.18) and (5.1), we find

$$\int_\Omega \widetilde{f}(\bar{v})\varphi \leq o(1) C_1(\|\bar{v}\|_{L^\infty}) \|\varphi\|_{L^1} \leq o(1) C_1(R) \|\varphi\|_{L^1}\ ,$$

where $C_1(\cdot)$ is a suitable increasing function depending on f. Then there exists a decreasing function $0 < \varepsilon_1(\cdot) \leq \varepsilon_0(\cdot)$ such that

$$\int_\Omega \widetilde{f}(\bar v)\varphi \leq \|\varphi\|_{L^1} \qquad \text{for} \quad |\varepsilon| \leq \varepsilon_1(R)\,. \tag{5.2}$$

We now choose, as in subsection 4.3, the admissible variation $\varphi = q(\bar v_+) - q(\bar v_-)$ where q is defined in (4.16). By the mean value Theorem

$$\widetilde{f}(t,x,\bar v) = \widetilde{f}(t,x,0) + \widetilde{f}_u(\text{intermediate point})\bar v$$

and, by (5.2), since $\widetilde{f}_u \geq \beta > 0$ and $\bar v\varphi \geq 0$ (recall (4.18)), we obtain

$$\beta \int_\Omega \bar v \varphi \leq \kappa_1 \|\varphi\|_{L^1}\,. \tag{5.3}$$

Arguing as at the end of subsection 4.3 (see inequality (4.35), recall (4.27) and $M := \|\hat{\bar v}\|_{L^\infty(\mathbb{T})}/2$) we deduce

$$\int_\Omega \bar v\varphi \geq \kappa_2 \|\bar v\|_{L^\infty}\|\varphi\|_{L^1}$$

and, by (5.3), we deduce

$$\|\bar v\|_{L^\infty} \leq \kappa_3 \qquad \text{for} \quad |\varepsilon| \leq \varepsilon_1(R)\,.$$

Step 2: The H^1-estimate

The H^1-estimate is carried out as in subsection 4.4 taking the admissible variation $\varphi := -D_{-h}D_h\bar v$ in the variational inequality (1.18). By Lemma 2.7, denoting $\bar w := w(\bar v)$,

$$0 \geq \int_\Omega f(\bar v+\bar w)\varphi = \int_\Omega D_h f(\bar v+\bar w)D_h\bar v \xrightarrow{h\to 0} \int_\Omega \Big(f_t(\bar v+\bar w) + f_u(\bar v+\bar w)(\bar v_t+\bar w_t)\Big)\bar v_t\,. \tag{5.4}$$

Since $\|w\|_E = O(\varepsilon)$, we get, by (5.4),

$$\int_\Omega f_u(\bar v+\bar w)\bar v_t^2 \leq \kappa_4 \|\bar v_t\|_{L^2}\,. \tag{5.5}$$

Since $\widetilde{f}_u \geq \beta > 0$, $\int_\Omega a_u(x,\bar v)\bar v_t^2 = 0$ by (2.19), and $\|w\|_E = O(\varepsilon)$

$$\begin{aligned}
\int_\Omega f_u(\bar v+\bar w)\bar v_t^2 &= \int_\Omega \widetilde{f}_u(\bar v+\bar w)\bar v_t^2 + a_u(t,\bar v+\bar w)\bar v_t^2 \\
&\geq \int_\Omega \beta\bar v_t^2 + \int_\Omega \Big(a_u(t,\bar v+\bar w) - a_u(t,\bar v)\Big)\bar v_t^2 \\
&\geq \int_\Omega \beta\bar v_t^2 - o(1)\int_\Omega \bar v_t^2 \geq \frac{\beta}{2}\int_\Omega \bar v_t^2
\end{aligned} \tag{5.6}$$

for $|\varepsilon| \le \varepsilon_2(R) \le \varepsilon_1(R)$. ¿From (5.5) and (5.6) we deduce

$$\|\bar v\|_{H^1} < \kappa_5 \qquad \forall\, |\varepsilon| \le \varepsilon_2(R)\,.$$

PROOF OF THEOREM 3 COMPLETED. For $R_* := \kappa_5$ and $\varepsilon_* := \varepsilon_2(R_*)$, $\bar v$ is an interior point in B_{R_*} and $\bar u := \bar v + w(\bar v, \varepsilon)$ is a weak solution of (1.1)-(1.2)-(1.3). Regularity of the solution $\bar u$ is proved as in subsection 4.5 and Theorem 3 follows. $\square$

6 Appendix

PROOF OF LEMMA 2.2. By the periodicity of $a(t,x)$ with respect to t

$$\int_{\Omega_\alpha} a(t,x)dtdx = \int_{\widetilde\Omega_\alpha} a(t,x)dtdx$$

where $\widetilde\Omega_\alpha := \{\alpha\pi < x < \pi(1-\alpha), -x < t < -x+2\pi\}$. Under the change of variables $s_+ := t+x$, $s_- := t-x$ the domain $\widetilde\Omega_\alpha$ trasforms into the domain

$$\Big\{0 < s_+ < 2\pi,\ s_+ - 2\pi(1-\alpha) < s_- < s_+ - 2\pi\alpha\Big\}$$

and we get (2.9).

For $p, q \in L^1(\mathbb{T})$, by (2.9) we have

$$\begin{aligned}
\int_{\Omega_\alpha} p(t+x)q(t-x)\,dt\,dx &= \frac12 \int_0^{2\pi} ds_+\ p(s_+) \int_{-2\pi+s_++2\alpha\pi}^{s_+-2\alpha\pi} q(s_-)ds_- \\
&= \frac12 \int_0^{2\pi} ds_+\ p(s_+) \Big(\int_0^{2\pi} q(s)ds \ - \int_{s_+-2\alpha\pi}^{s_++2\alpha\pi} q(s_-)ds_- \Big) \\
&= \frac12 \int_0^{2\pi} p(s)ds \int_0^{2\pi} q(s)ds \\
&\quad -\frac12 \int_0^{2\pi} ds_+\ p(s_+\ ) \int_{-2\alpha\pi}^{2\alpha\pi} q(s_+ + z)dz
\end{aligned}$$

and we obtain (2.10) by Tonelli's Theorem (calling $s_+ = y$).
Formula (2.11) follows by (2.10) setting $q \equiv 1$.

We now prove (2.12). Since the change of variables $(t,x) \mapsto (t, \pi - x)$ leaves the domain Ω_α unchanged

$$\int_{\Omega_\alpha} a(t,x)dtdx = \int_{\Omega_\alpha} a(t,\pi-x)dtdx\ ,$$

and, using also the periodicity of p,

$$\begin{aligned}\int_{\Omega_\alpha} f(p(t+x))g(p(t-x))\,dt\,dx &= \int_{\Omega_\alpha} f(p(t+\pi-x))g(p(t-\pi+x))\,dt\,dx\\ &= \int_{\alpha\pi}^{\pi-\alpha\pi}\int_0^{2\pi} f(p(t+\pi-x))g(p(t+\pi+x))\,dt\,dx\\ &= \int_{\alpha\pi}^{\pi-\alpha\pi}\int_0^{2\pi} f(p(t-x))g(p(t+x))\,dt\,dx\end{aligned}$$

proving (2.12). $\square$

PROOF OF LEMMA 2.3. (2.13) follows from the equality $v^2 = v_+^2 + v_-^2 - 2v_+v_-$, (2.11) (with $p = v_+^2, v_-^2$ and $\alpha = 0$) and (2.10) (taking $p, q = \hat{v}$, $\alpha = 0$ and recalling that $\hat{v}$ has zero average).

(2.14) follows form (2.13) since $v_t(t,x) = \hat{v}'(t+x) - \hat{v}'(t-x)$ (and similarly for v_x).

Next, the first inequality of (2.15) follows from $v(t,x) = \hat{v}(t+x) - \hat{v}(t-x)$ recalling that, since $\hat{v}$ has zero average, there exist two positive measure sets in which $\hat{v} \geq 0$ and $\hat{v} \leq 0$. The second inequality of (2.15) is trivial.

We finally prove (2.16). Since $\hat{v}$ is continuous ($\hat{v} \in H^1(\mathbb{T})$) there exists ξ_M such that $\|\hat{v}\|_{L^\infty(\mathbb{T})} = |\hat{v}(\xi_M)|$. Being $\int_0^{2\pi} \hat{v} = 0$, there exists $|\xi_0 - \xi_M| \leq \pi$ such that $\hat{v}(\xi_0) = 0$. Hence

$$\|\hat{v}\|_{L^\infty(\mathbb{T})} = |\hat{v}(\xi_M)| = \left|\int_{\xi_0}^{\xi_M} \hat{v}'(s)ds\right| \leq \sqrt{\pi}\|\hat{v}'\|_{L^2(\mathbb{T})} = \frac{1}{\sqrt{2}}\|v_t\|_{L^2(\Omega)}$$

by the Cauchy-Schwartz inequality and (2.14). Finally by (2.15),

$$\|v\|_{L^\infty(\Omega)} \leq 2\|\hat{v}\|_{L^\infty(\mathbb{T})} \leq \sqrt{2}\|v_t\|_{L^2(\Omega)} = \frac{1}{\sqrt{2}}\Big(\|v_t\|_{L^2(\Omega)} + \|v_x\|_{L^2(\Omega)}\Big) \leq \|v\|_{H^1(\Omega)}\,.$$

where $\|v\|^2_{H^1(\Omega)} := \|v\|^2_{L^2(\Omega)} + \|v_x\|^2_{L^2(\Omega)} + \|v_t\|^2_{L^2(\Omega)}$. $\square$

PROOF OF LEMMA 2.4. By the change of variables $(t,x) \mapsto (t, \pi - x)$ and periodicity,

$$\begin{aligned}\int_{\Omega_\alpha} \varphi_1 \cdot \ldots \cdot \varphi_{2k+1} &= \int_{\alpha\pi}^{\pi-\alpha\pi}\int_0^{2\pi}\prod_{j=1}^{2k+1}\Big(\hat{\varphi}_j(t+x) - \hat{\varphi}_j(t-x)\Big)dtdx\\ &= \int_{\alpha\pi}^{\pi-\alpha\pi}\int_0^{2\pi}\prod_{j=1}^{2k+1}\Big(\hat{\varphi}_j(t+\pi-x) - \hat{\varphi}_j(t-\pi+x)\Big)dtdx\\ &= \int_{\alpha\pi}^{\pi-\alpha\pi}\int_0^{2\pi}\prod_{j=1}^{2k+1}\Big(\hat{\varphi}_j(t-x) - \hat{\varphi}_j(t+x)\Big)dtdx\\ &= (-1)^{2k+1}\int_{\Omega_\alpha} \varphi_1 \cdot \ldots \cdot \varphi_{2k+1}\ ,\end{aligned}$$

which implies (2.17).
With similar arguments we can prove (2.18) and (2.19). □

PROOF OF LEMMA 2.5. The inequality (2.20) follows by the convexity of $t \to t^{2k}$.

We next prove (2.21). If $b = 0$ it is trivially true. If $b \neq 0$ let us divide for b^{2k} and set $x := a/b \in \mathbb{R}$. (2.21) is equivalent to prove

$$f(x) := (x-1)^{2k} - x^{2k} - 1 + 2kx^{2k-1} + 2kx \geq 0. \tag{6.1}$$

It results $f(0) = 0$, $f'(x) = 2k[(x-1)^{2k-1} - x^{2k-1} + (2k-1)x^{2k-2} + 1]$ and so $f'(0) = 0$. Therefore to prove (6.1) it is sufficient to show that f is convex. We have $f''(x) = 2k(2k-1)g(x)$ where $g_k(x) := (x-1)^{2k-2} - x^{2k-2} + (2k-2)x^{2k-3}$, $k \geq 2$.
We now show by induction on $k \geq 2$ that $g_k(x) > 0$. It is true for $k = 2$ since $g_2(x) = (x-1)^2 - x^2 + 2x = 1 > 0$. Supposing now $g_k(x) > 0$, let us prove that $g_{k+1}(x) > 0$.

We claim that

$$g_{k+1}(x) = (x-1)^{2k} - x^{2k} + 2kx^{2k-1}$$

is a strictly convex function. Indeed

$$g'_{k+1}(x) = 2k[(x-1)^{2k-1} - x^{2k-1} + (2k-1)x^{2k-2}] \quad \text{and} \quad g''_{k+1}(x) = 2k(2k-1)g_k(x).$$

By the inductive hypothesis $g_k(x) > 0$ and therefore $g''_{k+1}(x) > 0$. Moreover, being $g_{k+1}(x) \approx \mathrm{cost} x^{2k-2}$, $\lim_{x\to\pm\infty} g_{k+1}(x) = +\infty$ and $g_{k+1}(x)$ possesses a unique point of global minimum $\bar{x}$ that is also the unique critical point. Now it is sufficient to show that $g_{k+1}(\bar{x}) > 0$.

$$g'_k(\bar{x}) = 2k[(\bar{x}-1)^{2k-1} - \bar{x}^{2k-1} + (2k-1)\bar{x}^{2k-2}] = 0$$

implies that $(\bar{x}-1)^{2k-1} = \bar{x}^{2k-1} - (2k-1)\bar{x}^{2k-2}$. Substituting this equality in the expression for $g_{k+1}(\bar{x})$, we have

$$\begin{aligned} g_{k+1}(\bar{x}) &= (\bar{x}-1)[\bar{x}^{2k-1} - (2k-1)\bar{x}^{2k-2}] - \bar{x}^{2k} + 2k\bar{x}^{2k-1} \\ &= \bar{x}^{2k} - \bar{x}^{2k-1} - (2k-1)\bar{x}^{2k-1} + (2k-1)\bar{x}^{2k-2} - \bar{x}^{2k} + 2k\bar{x}^{2k-1} \\ &= (2k-1)\bar{x}^{2k-2} > 0 \end{aligned}$$

(we use that $\bar{x} \neq 0$, in fact $g'_{k+1}(0) = -2k \neq 0$)).

Proof of (2.22). The case $k = 1$ is trivial. For $k \geq 2$, we divide by b^{2k-1} and define $x := a/b \in \mathbb{R}$. We have to prove that

$$f(x) := (x+1)^{2k-1} - x^{2k-1} \geq 4^{1-k}\,, \qquad \forall\, x \in \mathbb{R}\,.$$

Since

$$f'(x) = (2k-1)[(x+1)^{2k-2} - x^{2k-2}] = 0 \quad \Longleftrightarrow \quad (x+1)^{2k-2} = x^{2k-2} \quad \Longleftrightarrow \quad x = -\frac{1}{2}$$

and $f(x) \to \infty$ as $|x| \to \infty$, we conclude that $x = -1/2$ is the unique minimum point of $f(x)$ and therefore $f(x) \geq f(-1/2) = 4^{1-k}$. □

Proof of Lemma 2.6. ¿From the inequality (2.20) we obtain

$$\int_\Omega v^{2k} = \int_\Omega (v_+ - v_-)^{2k} \leq 2^{2k-1} \int_\Omega v_+^{2k} + v_-^{2k} = 2^{2k-1} \int_\Omega \hat{v}^{2k}(t+x) + \hat{v}^{2k}(t-x) dt dx \ ,$$

which, using (2.11), proves (2.23). □

We first prove the Lemma in the case $k \geq 2$. Using the inequality (2.21) (2.11) and (2.12), we obtain

$$\begin{aligned} \int_{\Omega_\alpha} v^{2k} &= \int_{\Omega_\alpha} (v_+ - v_-)^{2k} \geq \int_{\Omega_\alpha} v_+^{2k} + v_-^{2k} - 2k \int_{\Omega_\alpha} v_+^{2k-1} v_- + v_+ v_-^{2k-1} \\ &= 2\pi(1-2\alpha) \int_0^{2\pi} \hat{v}^{2k} - 4k \int_{\Omega_\alpha} v_+^{2k-1} v_- \ . \end{aligned} \tag{6.2}$$

By (2.10) and since $\hat{v}$ has zero average

$$\begin{aligned} \left| \int_{\Omega_\alpha} v_+^{2k-1} v_- \right| &= \frac{1}{2} \left| \int_{-2\pi\alpha}^{2\pi\alpha} \int_0^{2\pi} \hat{v}^{2k-1}(y) \hat{v}(y+z) dy dz \right| \\ &\leq \frac{1}{2} \int_{-2\pi\alpha}^{2\pi\alpha} \int_0^{2\pi} |\hat{v}(y)|^{2k-1} |\hat{v}(y+z)| dy dz \ . \end{aligned} \tag{6.3}$$

By Hölder inequality with $p := 2k/(2k-1)$ and $q := 2k$ $(1/p + 1/q = 1)$,

$$\begin{aligned} \int_0^{2\pi} |\hat{v}(y)|^{2k-1} |\hat{v}(y+z)| dy &\leq \left(\int_0^{2\pi} |\hat{v}(y)|^{2k} dy \right)^{\frac{2k-1}{2k}} \left(\int_0^{2\pi} |\hat{v}(y+z)|^{2k} dy \right)^{\frac{1}{2k}} \\ &= \int_0^{2\pi} |\hat{v}(y)|^{2k} dy \ , \end{aligned}$$

where, in the equality, we have used the periodicity of $\hat{v}$ to conclude that $\int_0^{2\pi} |\hat{v}(y+z)|^{2k} dy = \int_0^{2\pi} |\hat{v}(y)|^{2k} dy$. Inserting the last inequality in (6.3), we obtain

$$\left| \int_{\Omega_\alpha} v_+^{2k-1} v_- \right| \leq 2\pi\alpha \int_0^{2\pi} |\hat{v}(y)|^{2k} dy \tag{6.4}$$

Inserting (6.4) in (6.2) the Proposition follows in the case $k \geq 2$. The case $k = 1$ is similar:

$$\int_{\Omega_\alpha} v^2 = \int_{\Omega_\alpha} v_+^2 + v_-^2 - 2 v_+ v_- = 2\pi(1-2\alpha) \int_0^{2\pi} \hat{v}^2 - 2 \int_{\Omega_\alpha} v_+ v_-$$

and we conclude by (6.4). □

Proof of Lemma 2.7. Formula (2.24) follows from

$$\begin{aligned} D_h(fg)(t,x) &= \frac{f(t+h,x)g(t+h,x)-f(t,x)g(t,x)}{h} \\ &= \frac{f(t+h,x)-f(t,x)}{h}g(t,x)+f(t+h,x)\frac{g(t+h,x)-g(t,x)}{h} \,. \end{aligned}$$

We prove (2.25) by induction. It is obvious for $m=1$. We suppose (2.25) holds for m and prove it for $m+1$: by (2.24) we have

$$\begin{aligned} D_h(f^m\cdot f) &= (D_hf^m)f+T_hf^mD_hf \\ &= (D_hf)\left[\sum_{j=0}^{m-1}f^{m-j}T_hf^j+T_hf^m\right] \\ &= (D_hf)\sum_{j=0}^{m}f^{(m+1)-j-1}T_hf^j \,. \end{aligned}$$

Formula (2.26) follows by (2.24) performing the change of variables $s=t+h$,

$$\begin{aligned} \int_\Omega D_h(fg)(t,x) &= \int_\Omega (D_hf)g+\int_\Omega f(t+h,x)\frac{g(t+h,x)-g(t,x)}{h}dtdx \\ &= \int_\Omega (D_hf)g+\int_\Omega f(s,x)\frac{g(s,x)-g(s-h,x)}{h}dsdx \,. \end{aligned}$$

We now prove formula (2.27) for integration by parts. Due to the periodicity of f and g with respect to t

$$\begin{aligned} \int_\Omega f(D_{-h}g) &= -\frac{1}{h}\int_\Omega f(t,x)\Big[g(t-h,x)-g(t,x)\Big]dtdx \\ &= -\frac{1}{h}\int_\Omega \Big[f(t+h,x)-f(t,x)\Big]g(t,x)dtdx=-\int_\Omega (D_hf)g \,. \end{aligned}$$

The proof of (2.28) is standard. Let $\varphi\in C^1(\Omega)$. By (2.27)

$$\int_\Omega (D_hf)\varphi=-\int_\Omega f(D_{-h}\varphi) \,. \tag{6.5}$$

Now the sequence $f(D_{-h}\varphi)$ converges to $f\varphi_t$ a.e. and, since

$$|f(t,x)\,(D_{-h}\varphi)(t,x)|\le\|\varphi\|_{C^1(\Omega)}|f(t,x)|\in L^1(\Omega) \,,$$

we can use the Lebesgue Theorem to obtain

$$\int_\Omega f(D_{-h}\varphi)\overset{h\to0}{\longrightarrow}\int_\Omega f\varphi_t. \tag{6.6}$$

Since, by hypothesis, $D_h f$ is bounded in $L^2(\Omega)$, $D_h f \stackrel{L^2}{\rightharpoonup} g$, up to a subsequence. Passing to the limit in (6.5) for $h \to 0$ we find $\int_\Omega g\varphi = -\int_\Omega f\varphi_t$. Therefore f has a weak derivative $f_t = g$ and by the weakly lower semicontinuity of the norm

$$\|f_t\|_{L^2} \leq \liminf \|D_h f\|_{L^2} \leq C \ .$$

In order to prove (2.29) assume temporarily f is smooth. ¿From the fundamental Theorem of calculus

$$(D_h f)(t,x) = \frac{f(t+h,x) - f(t,x)}{h} = \int_0^1 f_t(t+hs,x)ds \ .$$

By Cauchy-Schwartz inequality, Fubini Theorem and periodicity we obtain

$$\begin{aligned}
\int_\Omega |D_h f(t,x)|^2 dtdx &= \int_0^\pi \int_0^{2\pi} \Big| \int_0^1 f_t(t+hs,x)ds \Big|^2 dtdx \\
&\leq \int_0^\pi \int_0^{2\pi} \int_0^1 |f_t(t+hs,x)|^2 dsdtdx \\
&= \int_0^\pi \int_0^1 \int_0^{2\pi} |f_t(t+hs,x)|^2 dtdsdx \\
&= \int_0^\pi \int_0^1 \int_0^{2\pi} |f_t(t,x)|^2 dtdsdx \\
&= \int_0^1 \int_0^\pi \int_0^{2\pi} |f_t(t,x)|^2 dtdxds = \|f_t\|^2_{L^2(\Omega)} \ .
\end{aligned}$$

Inequality (2.29) is valid, for any f having a weak derivative $f_t \in L^2(\Omega)$, by approximation.

In order to prove (2.30) we first show the weak L^2-convergence. Let $\varphi \in C^1(\Omega)$. By (2.27), applying as before the Lebesgue Theorem, and since f has a weak derivative f_t

$$\int_\Omega (D_h f)\varphi = -\int_\Omega f(D_{-h}\varphi) \stackrel{h\to 0}{\longrightarrow} -\int_\Omega f\varphi_t = \int_\Omega f_t \varphi \ , \qquad \forall\, \varphi \in C^1(\Omega) \ . \tag{6.7}$$

Since, by (2.29), $D_h f$ is bounded in L^2, and (6.7) holds in the dense subset $C^1(\Omega) \subset L^2(\Omega)$, we conclude the weak L^2-convergence $D_h f \stackrel{L^2}{\rightharpoonup} f_t$.

By the weakly lower semicontinuity of the norm and (2.29)

$$\|f_t\|_{L^2(\Omega)} \leq \liminf \|D_h f\|_{L^2(\Omega)} \leq \|f_t\|_{L^2(\Omega)} \quad \Longrightarrow \quad \lim \|D_h f\|_{L^2(\Omega)} = \|f_t\|_{L^2(\Omega)} \ . \tag{6.8}$$

Since $L^2(\Omega)$ is a Hilbert space, weak convergence $D_h f \stackrel{L^2}{\rightharpoonup} f_t$ and (6.8) imply the strong convergence $D_h f \stackrel{L^2}{\to} f_t$. $\square$

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