On a Ruelle–Perron–Frobenius type theorem

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Abstract

We consider the problem of uniqueness of a stationary measure for Markov semigroups on a Polish space. Assuming that the transition function is "uniformly Feller" for a family of functions $\mathcal{R}$ and uniformly irreducible, we show that any two measures coincide on $\mathcal{R}$. In particular, if $\mathcal{R}$ is a determining family, then the above conditions ensure the uniqueness of stationary measure. The result obtained has applications in the ergodic theory of randomly forced PDE's.

1 Statement of the result

Let X be a Polish space, let $\mathcal{B}_X$ be the Borel σ-algebra on X, let $C_b(X)$ be the space of bounded continuous functions $f\colon X \to \mathbb{R}$, and let $\mathcal{P}(X)$ be the set of probability Borel measures on X. For $f \in C_b(X)$ and $\mu \in \mathcal{P}(X)$, we set

$$(f, \mu) = \int_X f(u)\,\mu(du) = \int_X f(u)\,d\mu. \tag{1}$$

For any $\mu \in \mathcal{P}(X)$, let $L^1(X, \mu)$ be the space of measurable functions $f\colon X \to \mathbb{R}$ such that

$$\|f\|_\mu := \int_X |f(u)|\,d\mu < \infty.$$

Let us consider a Feller transition function $P(t, u, \Gamma)$, $u \in X$, $\Gamma \in \mathcal{B}_X$, defined for $t \in \mathcal{T}$, where either $\mathcal{T} = \{t \in \mathbb{R} : t \geq 0\}$ or $\mathcal{T} = \{t \in \mathbb{Z} : t \geq 0\}$. We denote by

$$P_t : C_b(X) \to C_b(X), \quad P_t^* : \mathcal{P}(X) \to \mathcal{P}(X), \quad t \in \mathcal{T},$$

the Markov semi-groups associated with $P(t, u, \Gamma)$ (see [2, Section VI.11] for definitions). Recall that a measure $\mu \in \mathcal{P}(X)$ is said to be *stationary* for P_t^* if $P_t^*\mu = \mu$ for all $t \in \mathcal{T}$

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For any function $f(u)$, we denote by f^+ and f^- its positive and negative parts, respectively:

$$f^+ = \frac{1}{2}(|f| + f), \quad f^- = \frac{1}{2}(|f| - f).$$

Let $\mathcal{R} \subset C_b(X)$ be a family of functions. We say that two measures $\mu_1, \mu_2 \in \mathcal{P}(X)$ coincide on $\mathcal{R}$ if $(f, \mu_1) = (f, \mu_2)$ for any $f \in \mathcal{R}$. The family $\mathcal{R}$ is said to be *determining for* $\mathcal{P}(X)$ if any two measures $\mu_1, \mu_2 \in \mathcal{P}(X)$ that coincide on $\mathcal{R}$ are equal.

To formulate the main result, we introduce the following condition.

(H) *For any $f \in \mathcal{R}$ and $c \in \mathbb{R}$ the function $f - c$ belongs to $\mathcal{R}$, and the following property holds: if a function $f \in \mathcal{R}$ is such that*

$$\sup_{u \in X} f_t^+(u) \geq \alpha \quad \text{for all} \quad t \in \mathcal{T}, \tag{2}$$

$$\sup_{u \in X} f_t^-(u) \geq \alpha \quad \text{for all} \quad t \in \mathcal{T}, \tag{3}$$

where $f_t^+ = \big(P_t f\big)^+$, $f_t^- = \big(P_t f\big)^-$, and $\alpha = \alpha(f) > 0$ is a constant not depending on t, then there is a constant $A_f > 1$ and an increasing sequence t_k such that

$$\sup_{u \in X} \big(P_{d_k} f_{t_k}^+\big)(u) \leq A_f \inf_{u \in X} \big(P_{d_k} f_{t_k}^+\big)(u), \tag{4}$$

$$\sup_{u \in X} \big(P_{d_k} f_{t_k}^-\big)(u) \leq A_f \inf_{u \in X} \big(P_{d_k} f_{t_k}^-\big)(u), \tag{5}$$

where we set $d_k = t_{k+1} - t_k$.

Sufficient conditions guaranteeing the validity of (H) are given in Section 3; see conditions (H_1) and (H_2). The following theorem is the main result of this paper.

Theorem 1. *Assume that condition* (H) *holds for a family $\mathcal{R}$. Then the assertions below take place.*

(i) *Let $\mu \in \mathcal{P}(X)$ be a stationary measure for P_t^*. Then, for any $f \in \mathcal{R}$,*

$$P_t f \to (f, \mu) \quad \text{as} \quad t \to \infty \quad \text{in} \quad L^1(X, \mu). \tag{6}$$

(ii) *Any two stationary measures for P_t^* coincide on $\mathcal{R}$. In particular, if $\mathcal{R}$ is a determining family for $\mathcal{P}(X)$, then P_t^* has at most one stationary measure.*

In the case when X is the set $\{0,1\}^{\mathbb{N}}$ endowed with the Tikhonov topology, Theorem 1 is close to the classical Ruelle result used by him in [4] to prove uniqueness of a 1D Gibbs measure. A weaker version of Theorem 1 was established in our paper [3] (see Theorem 4.1). The result obtained in this paper enables one to establish the uniqueness of a stationary measure for some infinite-dimensional RDS.

2 Proof of Theorem 1

Step 1. We begin with the proof of assertion (i). Let $\mu \in \mathcal{P}(X)$ be a stationary measure and let $f \in \mathcal{R}$. Without loss of generality, we can assume that

$$(f, \mu) = \int_X f(u)\,\mu(du) = 0. \tag{7}$$

The general case can be reduced to the former by the change $f \mapsto f - (f, \mu)$. Thus, we must prove that

$$\|P_t f\|_\mu = \int_X \big|P_t f\big|\,d\mu \to 0 \quad \text{as} \quad k \to \infty. \tag{8}$$

Since $P_s^* \mu = \mu$ for any $s \geq 0$, we have

$$\begin{aligned}
\|P_{t+s} f\|_\mu &= \int_X \big|P_s(P_t f)\big|\,d\mu \leq \int_X P_s\big|(P_t f)\big|\,d\mu \\
&= \int_X \big|(P_t f)\big|\,d(P_s^* \mu) = \int_X \big|(P_t f)\big|\,d\mu = \|P_t f\|_\mu.
\end{aligned}$$

This means that $\|P_t f\|_\mu$ is a non-increasing function of t. Hence, the convergence (8) will be established if we show that for any $\varepsilon > 0$ there is $t = t_\varepsilon > 0$ such that

$$\|P_{t_\varepsilon} f\|_\mu \leq \varepsilon. \tag{9}$$

Step 2. We first assume that there is a sequence $\{s_k\} \subset \mathbb{R}$ for which

$$\sup_{u \in X} f_{s_k}^+(u) \to 0 \quad \text{as} \quad k \to \infty.$$

In this case,

$$\int_X (P_{s_k} f)^+(u)\,d\mu = \int_X f_{s_k}^+(u)\,d\mu \to 0 \quad \text{as} \quad k \to \infty. \tag{10}$$

Moreover, it follows from (7) that

$$(f, \mu) = (P_t^* \mu, f) = (f_t, \mu) = 0, \quad t \in \mathcal{T},$$

where $f_t = P_t f$, and therefore

$$(f_t^+, \mu) = (f_t^-, \mu) \quad \text{for any} \quad t \in \mathcal{T}. \tag{11}$$

Combining this with (10), we derive

$$(f_{s_k}^+, \mu) = (f_{s_k}^-, \mu) \to 0 \quad \text{as} \quad k \to \infty,$$

whence we conclude that (9) holds. A similar argument shows that if

$$\sup_{u \in X} f_{s_k}^-(u) \to 0 \quad \text{as} \quad k \to \infty$$

for a sequence $\{s_k\} \subset \mathbb{R}$, then (9) is fulfilled.

Step 3. Thus, we can assume that (2) and (3) hold with a constant $\alpha > 0$. By condition (H), there is an increasing sequence $\{t_k\}$ such that (4) and (5) are satisfied. We claim that

$$\|P_{t_k} f\|_\mu \leq a_f^{k-1} \|f\|_\mu \quad \text{for all } k \geq 1, \tag{12}$$

where $a_f = 1 - A_f^{-1} < 1$.

The proof is by induction on k. Inequality (12) is obvious for $k = 1$. Assuming that (12) is established for $k \leq r$, we now prove it for $k = r + 1$. Note that, by (4) and (5), we have [1]

$$\int_X f_{t_r}^{\pm} d\mu = \int_X P_{d_r} f_{t_r}^{\pm} d\mu \leq \sup_{u \in X} \bigl(P_{d_r} f_{t_r}^{\pm}\bigr)(u) \leq A_f \inf_{u \in X} \bigl(P_{d_r} f_{t_r}^{\pm}\bigr)(u),$$

whence

$$P_{d_r} f_{t_r}^{\pm}(u) - A_f^{-1} \|f_{t_r}^{\pm}\|_\mu \geq 0 \quad \text{for any} \quad u \in X.$$

It follows that

$$\int_X \bigl|P_{d_r} f_{t_r}^{\pm} - A_f^{-1} \|f_{t_r}^{\pm}\|_\mu\bigr| \, d\mu = \int_X \bigl(P_{d_r} f_{t_r}^{\pm} - A_f^{-1} \|f_{t_r}^{\pm}\|_\mu\bigr) \, d\mu = a_f \|f_{t_r}^{\pm}\|_\mu.$$

We now estimate the expression $\|P_{t_{r+1}} f\|_\mu = \|P_{d_r} f_{t_r}\|_\mu$. In view of (11), we have

$$\|f_t^+\|_\mu = \|f_t^-\|_\mu \quad \text{for any} \quad t \in \mathcal{T}, \tag{13}$$

and therefore

$$\begin{aligned}
\int_X \bigl|P_{d_r} f_{t_r}\bigr| d\mu &= \int_X \bigl|P_{d_r}(f_{t_r}^+ - f_{t_r}^-)\bigr| d\mu \\
&\leq \int_X \bigl|P_{d_r} f_{t_r}^+ - A_f^{-1} \|f_{t_r}^+\|_\mu\bigr| \, d\mu + \int_X \bigl|P_{d_r} f_{t_r}^- - A_f^{-1} \|f_{t_r}^-\|_\mu\bigr| \, d\mu \\
&\leq a_f \bigl(\|f_{t_r}^+\|_\mu + \|f_{t_r}^-\|_\mu\bigr) = a_f \|f_{t_r}\|_\mu.
\end{aligned}$$

Using the induction hypothesis, we derive

$$\int_X \bigl|P_{t_{r+1}} f\bigr| d\mu \leq a_f \|P_{t_r} f\|_\mu \leq a_f^r \|f\|_\mu,$$

which completes the proof of (12). Inequality (9) is an obvious consequence of (12).

Step 4. Let us turn to assertion (ii). We first show that if $\mu_1, \mu_2 \in \mathcal{P}(X)$ are two stationary measures that do not coincide on $\mathcal{R}$, then they are singular.

[1] Here and henceforth a formula involving the symbol $\pm$ is a brief writing for the two formulas corresponding to the upper and lower signs.

To this end, we apply a well-known argument (for instance, see [1, Proposition 3.2.5]). Let $f \in \mathcal{R}$ be a function such that

$$(f, \mu_1) \neq (f, \mu_2). \tag{14}$$

By assertion (i),

$$P_t f \to (f, \mu_i) \quad \text{as} \quad k \to \infty \quad \text{in} \quad L^1(X, \mu_i), \quad i = 1, 2.$$

Therefore, there is a sequence $s_k \to +\infty$ such that

$$P_{s_k} f \to (f, \mu_i) \quad \text{as} \quad k \to \infty \quad \mu_i\text{-almost everywhere}, \quad i = 1, 2. \tag{15}$$

Denote by C_i, $i = 1, 2$, the set of points $u \in X$ for which (15) takes place. We have $\mu_1(C_1) = \mu_2(C_2) = 1$ and, in view of (14), $C_1 \cap C_2 = \varnothing$. This means that μ_1 and μ_2 are singular.

We now assume that $\mu_1, \mu_2 \in \mathcal{P}(X)$ are two stationary measures for P_t^* that do not coincide on $\mathcal{R}$. Consider the measure $\mu = (\mu_1 + \mu_2)/2$. It is clear that $\mu \in \mathcal{P}(X)$ is a stationary measure and that μ and μ_1 do not coincide on $\mathcal{R}$. As was shown above, the measures μ and μ_1 must be singular. On the other hand, the definition of μ implies that they are not singular. The contradiction obtained completes the proof of Theorem 1.

3 Sufficient conditions for application of Theorem 1

Let $\mathcal{R} \subset C_b(X)$ be a family of functions such that $f - c \in \mathcal{R}$ for any $f \in \mathcal{R}$ and $c \in \mathbb{R}$. We consider a transition function $P(t, u, \Gamma)$, $t \in \mathcal{T}$, $u \in X$, $\Gamma \in \mathcal{B}_X$, satisfying the following conditions.

($\mathbf{H_1}$) *For any $f \in \mathcal{R}$ there is an increasing sequence $s_m \to +\infty$ such that the family $\{P_{s_m} f, m \geq 1\}$ is uniformly equicontinuous.*

($\mathbf{H_2}$) *For every $r > 0$ there are $\varepsilon > 0$ and $l \geq 1$ such that*

$$P(l, u, B_X(v, r)) \geq \varepsilon \quad \text{for any} \quad u, v \in X, \tag{16}$$

where $B_X(v, r)$ denotes a ball in X of radius r centred at v.

Theorem 2. *Let conditions* (H_1) *and* (H_2) *be satisfied. Then* (H) *is fulfilled for $\mathcal{R}$, and therefore assertions* (i) *and* (ii) *of Theorem 1 hold. Moreover, the support of μ coincides with X, and for any $f \in \mathcal{R}$ we have*

$$P_t f \to (f, \mu) \quad \text{as} \quad t \to \infty \quad \text{in} \quad C_b(X). \tag{17}$$

Proof. Let $f \in \mathcal{R}$ be an arbitrary function satisfying inequalities (2) and (3), where $f_t^{\pm} = (P_t f)^{\pm}$ and α is a positive constant. We must construct a sequence t_k for which (4) and (5) hold.

In view of condition (H_1), there is $r > 0$ and for any $m \geq 1$ there is $u_{s_m} \in X$ such that

$$\inf_{v \in B_X(u_{s_m}, r)} f_{s_m}^+(v) \geq \sup_{v \in X} f_{s_m}^+(v) - \frac{\alpha}{2} \geq \frac{\alpha}{2}. \tag{18}$$

Let $\varepsilon > 0$ and $l \geq 1$ be the constants entering condition (H_2). Since (16) holds uniformly with respect to $u \in X$, the Chapman–Kolmogorov relation implies that

$$P(l', u, B_X(v, r)) \geq \varepsilon \quad \text{for any } u, v \in X,\ l' \geq l. \tag{19}$$

We now construct a subsequence $\{t_k\} \subset \{s_m\}$ possessing the required property. We set $t_1 = s_1$ and assume that $t_2, \dots, t_k$ are already constructed. Let t_{k+1} be the smallest element of $\{s_m\}$ greater than $t_k + l$. By definition, there is an integer $m \geq 1$ such that $s_m = t_k$. In view of (18) and (19), we have

$$\begin{aligned}\big(P_{d_k} f_{t_k}^+\big)(u) &= \int_X P(d_k, u, dv) f_{t_k}^+(v) \geq \int_{B_X(u_{t_k}, r)} P(d_k, u, dv) f_{t_k}^+(v) \\ &\geq P\big(l, u, B_X(u_{t_k}, r)\big) \inf_{v \in B_X(u_{t_k}, r)} f_{t_k}^+(v) \geq \frac{\alpha\varepsilon}{2},\end{aligned} \tag{20}$$

where $d_k = t_{k+1} - t_k$. On the other hand,

$$\big(P_{d_k} f_{t_k}^+\big)(u) \leq \sup_{u \in X} f_{t_k}^+(u) \leq \sup_{u \in X} \big|f_{t_k}(u)\big| \leq \sup_{u \in X} \big|f(u)\big|. \tag{21}$$

Combining (20) and (21), we arrive at (4) with

$$A_f = 2(\alpha\varepsilon)^{-1} \sup_{u \in X} \big|f(u)\big|.$$

Inequality (5) can be proved in a similar way.

We now assume that μ is a stationary measure for P_t^* and show that its support coincides with X. To this end, it suffices to check that

$$\mu\big(B_X(v, r)\big) > 0 \quad \text{for any } v \in X \text{ and } r > 0. \tag{22}$$

In view of the invariance of μ and inequality (16), we have

$$\mu\big(B_X(v, r)\big) = \int_X P\big(l, u, B_X(v, r)\big)\, \mu(du) \geq \varepsilon,$$

which implies (22).

It remains to prove (17). To this end, we first show that X is compact. Indeed, assuming the contrary, we can find a constant $r > 0$ and a sequence $\{v_i\}$

such that $d(v_i, v_j) > 2r$ for $i \neq j$. In view of (16), there is an integer $l \geq 1$ and a constant $\varepsilon > 0$ such that

$$P(l, u, B_X(v_i, r)) \geq \varepsilon \quad \text{for any } i \geq 1,\ u \in X.$$

Noting now that the balls $B_X(v_i, r)$ are pairwise distinct, we arrive at a contradiction.

We can now establish (17). Let $\mu \in \mathcal{P}(X)$ be a stationary measure. We fix any function $f \in \mathcal{R}$ and assume, without loss of generality, that $(f, \mu) = 0$. By condition (H_1) and the Arzelà–Ascoli theorem, there exists a subsequence $\{r_m\} \subset \{s_m\}$ such that $P_{r_m} f(u)$ converges uniformly to a function $g(u)$. In view of (6), the function g must vanish on the support of μ. Since $\operatorname{supp} \mu = X$, we have $g \equiv 0$. In remains to note that the function $t \mapsto \sup_X |P_t f(u)|$ is non-increasing, and therefore we obtain (17). The proof of Theorem 2 is complete. $\square$

References

[1] G. Da Prato, J. Zabczyk, Ergodicity for Infinite-Dimensional Systems, London Mathematical Society Lecture Note Series, vol. 229, Cambridge University Press, Cambridge, 1996.

[2] W. Feller, An Introduction to Probability Theory and Its Applications, Vol. II, John Wiley & Sons, New York–London–Sydney, 1971.

[3] S. Kuksin, A. Shirikyan, Stochastic dissipative PDE's and Gibbs measures, Comm. Math. Phys. 213 (2000), 291–330.

[4] D. Ruelle, Statistical mechanics of a one-dimensional lattice gas, Comm. Math. Phys. 9 (1968), p. 267–278.