A topological version of Levinson's theorem

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April 2005

Abstract

In the framework of scattering theory, we show how the scattering matrix can be related to the projection on the bound states by an index map of K-theory. Pairings with appropriate cyclic cocyles lead naturally to a topological version of Levinson's theorem.

1 Introduction

Let us consider the self-adjoint operators $H_0 := -\Delta$ and $H := H_0 + V$ in the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^n)$, where $|V(x)| \leq c\,(1+|x|)^{-\beta}$ with $\beta > 1$. It is well known that for such short range potentials V, the wave operators

$$\Omega_\pm := s - \lim_{t\to\pm\infty} e^{itH}\, e^{-itH_0} \tag{1}$$

exist and have same range. The complement of this range is generated by the eigenvectors of H, we let P denote the projection on this subspace. The scattering matrix S for this system is defined by the product $\Omega_+^*\Omega_-$, where Ω_+^* is the adjoint of Ω_+.

Levinson's theorem establishes a relation between an expression in terms of the unitary operator S and an expression depending on the projection P. There exist many presentations of this theorem, but we recall only the one of [14] in the case $n = 3$. We refer to [5], [9] and [15] for other versions of a similar result.

Let $\mathcal{U} : \mathcal{H} \to L^2\big(\mathbb{R}_+; L^2(\mathbb{S}^{n-1})\big)$ be the unitary transformation that diagonalizes H_0, *i.e.* that satisfies $[\mathcal{U}H_0 f](\lambda,\omega) = \lambda[\mathcal{U}f](\lambda,\omega)$, with f in the domain of H_0, $\lambda \in \mathbb{R}_+$ and $\omega \in \mathbb{S}^{n-1}$. Since the operator S commutes with H_0, there exists a family $\{S(\lambda)\}_{\lambda\in\mathbb{R}_+}$ of unitary operators in $L^2(\mathbb{S}^{n-1})$ satisfying $\mathcal{U}S\,\mathcal{U}^* = \{S(\lambda)\}$ almost everywhere in λ [3, Chap. 5.7]. Under suitable hypotheses on V [14] Levinson's theorem takes the form

$$\int_0^\infty \mathrm{d}\lambda\, \big\{\mathrm{tr}[iS(\lambda)^* \tfrac{\mathrm{d}S}{\mathrm{d}\lambda}(\lambda)] - \tfrac{\nu}{\sqrt{\lambda}}\big\} = 2\pi\,\mathrm{Tr}[P], \tag{2}$$

where tr is the trace on $L^2(\mathbb{S}^{n-1})$, Tr the trace on $\mathcal{H}$ and $\nu = (4\pi)^{-1}\int_{\mathbb{R}^3} \mathrm{d}x\, V(x)$. Clearly the r.h.s. of this equality is invariant under variations of V that do not change the number of bound states of H. But it is not at all clear how this stability comes about in the l.h.s.

In this note we propose a modification of the l.h.s. of (2) in order to restore the topological nature of this equality. The idea is very natural from the point of view of non-commutative topology: we rewrite the l.h.s. of (2) as the result of a pairing between K-theory and cyclic cohomology. Beyond formula (2), we show that the unitary S is related to the projection P at the level of K-theory by the index map, *cf.* Theorem 2.2. Let us point out that the wave operators play a key role in this work. Sufficient conditions on Ω_- imply that H has only a finite set of bound states, but also give informations on the behaviour of $S(\cdot)$ at the origin.

2 The algebraic framework

In this section we show how the scattering matrix S can be related to the projection P on the bound states via a boundary map of K-theory. Consider the short exact sequence

$$0 \to C_0\big(\mathbb{R};\mathscr{K}\big)\rtimes_\tau\mathbb{R} \to C_0\big(\mathbb{R}\cup\{+\infty\};\mathscr{K}\big)\rtimes_\tau\mathbb{R} \overset{ev_\infty}{\to} \mathscr{K}\rtimes\mathbb{R} \to 0, \tag{3}$$

where $\mathscr{K}$ is the algebra of compact operators in some Hilbert space. The sequence (3) is the Wiener-Hopf extension of the crossed product $\mathscr{K}\rtimes\mathbb{R}$ with trivial $\mathbb{R}$-action on $\mathscr{K}$; τ is the action on $C_0\big(\mathbb{R}\cup\{+\infty\}\big)$ by translation, leaving the point $\{+\infty\}$ invariant, and the surjection ev_∞ is induced by evaluation at $\{+\infty\}$. Our goal is to identify P as an element in the ideal and S as an element of the unitisation of the quotient, and to verify that the boundary map $\mathrm{ind} : K_1(\mathscr{K}\rtimes\mathbb{R}) \to K_0\big(C_0(\mathbb{R};\mathscr{K})\rtimes_\tau\mathbb{R}\big)$ maps the K_1-class of S to (minus) the K_0-class of P. To do so we represent the above short exact sequence in the physical Hilbert space $\mathcal{H}$.

Following the developments of [10] we first consider the case $\mathscr{K} = \mathbb{C}$ and let A, B be (unbounded) self-adjoint operators in $\mathcal{H}$ both with purely absolutely continuous spectrum equal to $\mathbb{R}$ and commutator given formally by $[iA, B] = -1$. We can then represent $C_0\big(\mathbb{R}\cup\{+\infty\};\mathscr{K}\big)\rtimes_\tau\mathbb{R}$ faithfully as the norm closure $\mathscr{C}'$ in $\mathcal{B}(\mathcal{H})$ of the set of finite sums of the form $\varphi_1(A)\,\eta_1(B) + \ldots + \varphi_m(A)\,\eta_m(B)$ where $\varphi_i \in C_0\big(\mathbb{R}\cup\{+\infty\}\big)$ and $\eta_i \in C_0(\mathbb{R})$. We denote by $\mathscr{J}'$ the ideal obtained by choosing functions φ_i that vanish at $\{+\infty\}$. Furthermore, we can represent $\mathscr{K}\rtimes\mathbb{R}$ faithfully in $\mathcal{B}(\mathcal{H})$ by elements of the form $\eta(B)$ with $\eta \in C_0(\mathbb{R})$. This algebra is denoted by $\mathscr{E}'$.

In [10] position and momentum operators were chosen for A and B but we take $A := -\frac{i}{2}(Q\cdot\nabla + \nabla\cdot Q)$ and $B := \frac{1}{2}\ln H_0$. We refer to [11] for a thorough description of A in various representations. Let us notice that a typical element of $\mathscr{C}'$ is of the form $\varphi(A)\,\eta(H_0)$ with $\varphi \in C_0\big(\mathbb{R}\cup\{+\infty\}\big)$ and $\eta \in C_0(\mathbb{R}_+)$, the algebra of continuous functions on $\mathbb{R}_+$ that vanish at the origin and at infinity. We shall now consider $\mathscr{K} = \mathcal{K}\big(L^2(\mathbb{S}^{n-1})\big)$ from the decomposition $\mathcal{H} \cong L^2\big(\mathbb{R}_+; L^2(\mathbb{S}^{n-1})\big)$ in spherical coordinates. Since A and H_0 are rotation invariant the presence of a larger $\mathscr{K}$ does not interfere with the above argument. Thus we set $\mathscr{C} := \mathscr{C}'\otimes\mathscr{K}$, $\mathscr{J} := \mathscr{C}'\otimes\mathscr{K}$ and $\mathscr{E} := \mathscr{E}'\otimes\mathscr{K}$. These algebras are all represented in the same Hilbert space $\mathcal{H}$, although $\mathscr{E}$ is a quotient of $\mathscr{C}$. The surjection ev_∞ becomes the map $\mathcal{P}_\infty$, where $\mathcal{P}_\infty[T] := T_\infty$, with T_∞ uniquely defined by the conditions $\|\chi(A \geq t)\,(T - T_\infty)\| \to 0$ and

$\|\chi(A \geq t)\,(T^* - T^*_\infty)\| \to 0$ as $t \to +\infty$, χ denoting the characteristic function. We easily observe that $\mathcal{P}_\infty[\varphi(A)\,\eta(H_0)] = \varphi(+\infty)\,\eta(H_0)$ for any $\varphi \in C_0\big(\mathbb{R} \cup \{+\infty\}\big)$ and $\eta \in C_0\big(\mathbb{R}_+;\mathscr{K}\big)$, where $\varphi(+\infty)$ is simply the value of the function φ at the point $\{+\infty\}$. Let us summarise our findings:

Lemma 2.1. *All three algebras of* (3) *are represented faithfully in $\mathcal{H}$ by $\mathscr{J}$, $\mathscr{C}$ and $\mathscr{E}$. In $\mathcal{B}(\mathcal{H})$ the surjection ev_∞ becomes $\mathcal{P}_\infty$.*

Note that $\mathscr{J}$ is equal to the set of compact operators in $\mathcal{H}$. For suitable potentials V, the operator $S-1$ belongs to $\mathscr{E}$ [11, 12] and P is a compact operator. The key ingredient below is the use of Ω_- to make the link between the K_1-class $[S]_1$ of S and the K_0-class $[P]_0$ of P.

Theorem 2.2. *Assume that $\Omega_- - 1$ belongs to $\mathscr{C}$. Then $S-1$ is an element of $\mathscr{E}$, P belongs to $\mathscr{J}$ and one has at the level of K-theory:*

$$\mathrm{ind}[S]_1 = -\,[P]_0. \tag{4}$$

Proof. Let $T \in \mathscr{C}$. Then $T_\infty = \mathcal{P}_\infty(T) \in \mathscr{E}$ satisfies $\|\chi(A \geq t)(T - T_\infty)\| \to 0$ as $t \to +\infty$. Equivalently, $\|\chi(A \geq 0)\,[U(t)TU(t)^* - T_\infty]\| \to 0$ as $t \to +\infty$, since T_∞ commutes with $U(t) := e^{\frac{i}{2}t\ln H_0}$ for all $t \in \mathbb{R}$. It is then easily observed that $s-\lim_{t\to+\infty} U(t)\,T\,U(t)^* = T_\infty$. Now, if T is replaced by $\Omega_- - 1$, the operator T_∞ has to be equal to $S-1$, since $s-\lim_{t\to+\infty} U(t)\,\Omega_-\,U(t)^*$ is equal to S. Indeed, this result directly follows from the intertwining relation of Ω_- and the invariance principle [1, Thm. 7.1.4].

We thus have shown that $\Omega_- - 1$ is a preimage of $S - 1$ in $\mathscr{C}$. It is well known that $\Omega_-\Omega_-^* = 1 - P$ and $\Omega_-^*\Omega_- = 1$. In particular Ω_- is a partial isometry so that $\mathrm{ind}[S]_1 = [\Omega_-\Omega_-^*]_0 - [\Omega_-^*\Omega_-]_0 = -[P]_0$, see *e.g.* [17, Prop. 9.2.2]. ◻

Remark 2.3. It seems interesting that the condition $\Omega_- - 1 \in \mathscr{C}$ implies the finiteness of the set of eigenvalues of H. Another consequence of this hypothesis is that $S(0) = 1$, a result which is also not obvious. See [12, Sec. 5] for a detailed analysis of the behaviour of $S(\cdot)$ near the origin.

It is important to express the above condition on Ω_- in a more traditional way, *i.e.* in terms of scattering conditions. The following lemma is based on an alternative description of the C^*-algebra $\mathscr{C}$. Its easy proof can be obtained by mimicking some developments given in Section 3.5 of [10]. We also use the convention of that reference, that is: if a symbol like $T^{(*)}$ appears in a relation, it means that this relation has to hold for T and for its adjoint T^*.

Lemma 2.4. *The operator Ω_- belongs to $\mathscr{C}$ if and only if the following conditions are satisfied:*

(i) $\|\chi(H_0 \leq \varepsilon)\,(\Omega_- - 1)^{(*)}\| \to 0$ *as* $\varepsilon \to 0$, *and* $\|\chi(H_0 \geq \varepsilon)\,(\Omega_- - 1)^{(*)}\| \to 0$ *as* $\varepsilon \to +\infty$,

(ii) $\|\chi(A \leq t)\,(\Omega_- - 1)^{(*)}\| \to 0$ *as* $t \to -\infty$, *and* $\|\chi(A \geq t)\,(\Omega_- - S)^{(*)}\| \to 0$ *as* $t \to +\infty$.

Equivalently, the condition (ii) *can be rewritten as*

(ii') $\|\chi(A \leq 0)\;U(t)\;(\Omega_- - 1)^{(*)}\;U(t)^*\| \to 0$ *as* $t \to -\infty$, *and* $\|\chi(A \geq 0)\;U(t)\;(\Omega_- - S)^{(*)}\,U(t)^*\| \to 0$ *as* $t \to +\infty$.

3 A new version of Levinson's theorem

In the next statement, it is required that the map $\mathbb{R}_+ \ni \lambda \mapsto S(\lambda) \in \mathcal{B}\big(L^2(\mathbb{S}^{n-1})\big)$ is differentiable. We refer for example to [11, Thm. 3.6] for sufficient conditions on V for that purpose. Trace class conditions on $S(\lambda) - 1$ for all $\lambda \in \mathbb{R}_+$ are common requirements [8]. Unfortunately, similar conditions on $S'(\lambda)$ were much less studied in the literature.

Theorem 3.1. *Let $\Omega_- - 1$ belong to $\mathscr{C}$. Assume furthermore that the map $\mathbb{R}_+ \ni \lambda \mapsto S(\lambda) \in \mathcal{B}\big(L^2(\mathbb{S}^{n-1})\big)$ is differentiable, and that $\lambda \mapsto \mathrm{tr}[S'(\lambda)]$ belongs to $L^1\big(\mathbb{R}_+, \mathrm{d}\lambda\big)$. Then the following equality holds:*

$$\int_0^\infty \mathrm{d}\lambda\, \mathrm{tr}\big[i(S(\lambda)-1)^*\, S'(\lambda)\big] = 2\pi\, \mathrm{Tr}[P]. \tag{5}$$

Proof. The boundary maps in K-theory of the exact sequence (3) are the inverses of the Connes-Thom isomorphism (which here specialises to the Bott-isomorphism as the action in the quotient is trivial) and have a dual in cyclic cohomology [6], or rather on higher traces [7, 13], which gives rise to an equality between pairings which we first recall: Tr is a 0-trace on the ideal $C_0\big(\mathbb{R};\mathscr{K}\big) \rtimes_\tau \mathbb{R} \cong \mathcal{K}\big(L^2(\mathbb{R})\big) \otimes \mathcal{K}\big(L^2(\mathbb{S}^{n-1})\big)$ which we factor $\mathrm{Tr} = \mathrm{Tr}' \otimes \mathrm{tr}$. Then $\hat{\mathrm{tr}} : \mathscr{K} \rtimes \mathbb{R} \to \mathbb{C}$, $\hat{\mathrm{tr}}[a] = \mathrm{tr}[a(0)]$ is a trace on the crossed product and $(a, b) \mapsto \hat{\mathrm{tr}}[a\delta(b)]$ a 1-trace where $[\delta(b)](t) = itb(t)$. With these ingredients

$$\hat{\mathrm{tr}}[i(u-1)^*\delta(u)] = -2\pi \mathrm{Tr}[p] \quad \text{if} \quad \mathrm{ind}[u]_1 = [p]_0, \tag{6}$$

provided u is a representative of its K_1-class $[u]_1$ on which the 1-trace can be evaluated. This is for instance the case if $\delta(u)$ is $\hat{\mathrm{tr}}$-traceclass. To apply this to our situation, in which u is the unitary represented by the scattering matrix and p is represented by the projection onto the bound states, we express δ and $\hat{tr}$ on $\mathcal{U}\mathscr{E}\mathcal{U}^*$ where $\mathcal{U}$ is the unitary from Section 1 diagonalising H_0. Then δ becomes $\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}$ and $\hat{\mathrm{tr}}$ becomes $\int_{\mathbb{R}_+} \frac{\mathrm{d}\lambda}{\lambda}\, \mathrm{tr}$. Our hypothesis implies the neccessary trace class property so that the l.h.s. of (6) corresponds to $\int_0^\infty \mathrm{d}\lambda\, \mathrm{tr}\big[i(S(\lambda)-1)^* S'(\lambda)\big]$ and the r.h.s. to $2\pi\,\mathrm{Tr}[P]$. □

Remark 3.2. Expressions very similar to (5) already appeared in [5] and [9]. However, it seems that they did not attract the attention of the respective authors and that a formulation closer to (2) was preferred. One reason is that the operator $\{S(\lambda)^* S'(\lambda)\}_{\lambda\in\mathbb{R}_+}$ has a physical meaning: it represents the *time delay* of the system under consideration. We refer to [2] for more explanations and results on this operator.

Remark 3.3. At present our approach does not allow to say anything about a *half-bound state*. We refer to [12], [15] or [16] for explanations on that concept and to [15] or [16] for corrections of Levinson's theorem in the presence of such a *0-energy resonance*.

4 Further prospects

We outline several improvements or extensions that ought to be carried out or seem natural in view of this note. We hope to express some of these in a further publication.

- Our main hypothesis of Theorem 3.1, that $\Omega_- - 1$ belongs to the C^*-algebra $\mathscr{C}$, is crucial and we have provided estimates in Lemma 2.4 which would guarantee it. Such estimates are rather difficult to obtain and we were not able to locate similar conditions in the literature. They clearly need to be addressed.

- Similar results should hold for a more general operator H_0 with absolutely continuous spectrum. In that case, the role of A would be played by an operator conjugate to H_0. We refer to [1, Prop. 7.2.14] for the construction of such an operator in a general framework.

- More general short range potentials or trace class perturbations can also be treated in a very similar way. By our initial hypothesis on V we have purposely eliminated positive eigenvalues of H, but it would be interesting to have a better understanding of their role with respect to Theorems 2.2 and 3.1.

- In principle, Theorem 2.2 is stronger than Levinson's theorem and one could therefore expect new topological relations from pairings with other cyclic cocycles. In the present setting these do not yet show up as the ranks of the K-groups are too small. But in more complicated scattering processes this could well be the case.

- In the literature one finds also the so-called *higher-order Levinson's Theorems* [4]. In the case $n = 3$ and under suitable hypotheses they take the form [4, eq. 3.28]

$$\int_0^\infty \mathrm{d}\lambda\, \lambda^N \big\{ \mathrm{tr}\big[iS(\lambda)^* S'(\lambda) \big] - C_N(\lambda) \big\} = 2\pi \sum_j e_j^N,$$

  where N is any natural number, C_N are correction terms, and $\{e_j\}$ is the set of eigenvalues of H with multiplicities counted. The correction terms can be explicitly computed in terms of H_0 and V [4] and we expect that they can be absorbed in a similar manner into the S-matrix as above.

Acknowledgements

Serge Richard thanks the Swiss National Science Foundation for its financial support.

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