# On the Semigroup Decomposition of the Time Evolution of Quantum Mechanical Resonances

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**Abstract**

A way of utilizing Lax-Phillips type semigroups for the description of time evolution of resonances for scattering problems involving Hamiltonians with a semibounded spectrum was recently introduced by Y. Strauss. In the proposed framework the evolution is decomposed into a background term and an exponentially decaying resonance term evolving according to a semigroup law given by a Lax-Phillips type semigroup; this is called the semigroup decomposition. However, the proposed framework assumes that the $S$-matrix in the energy representation is the boundary value on the positive real axis of a bounded analytic function in the upper half-plane. This condition puts strong restrictions on possible applications of this formalism. In this paper it is shown that there is a simple way of weakening the assumptions on the $S$-matrix analyticity while still obtaining the semigroup decomposition of the evolution of a resonance.

## 1 Introduction

There has been a recent effort to adapt the formalism of the scattering theory developed by P.D. Lax and R.S. Phillips[1] into the framework of quantum mechanics. An initial effort in this direction[2,3,4] was followed by the introduction of a more general formalism by Y. Strauss, E. Eisenberg and L.P. Horwitz[5] which was subsequently applied to certain Lee-Friedrichs type models in relativistic quantum field theory[6,7] and, more recently, to the analysis of the Stark effect[8]. In a parallel work H. Baumgartel[9] has used a modification of the Lax-Phillips scattering theory in order to deal with quantum mechanical resonances. In particular, he has shown the relevancy of this modified structure for the construction of appropriate Gamow vectors for resonances of certain scattering problems.

It is readily observed that the class of problems which can be analyzed within the framework introduced in reference [5] is limited by the very fact that it essentially maintains the original structure of the Lax-Phillips theory.

In this formalism the generator of evolution is required to have an unbounded spectrum from below as well as from above, thus a large class of quantum mechanical scattering problems is excluded from its range of applicability. A way of overcoming this difficulty, when dealing with scattering problems for which the generator of evolution has a semibounded absolutely continuous spectrum, was recently proposed in reference [10].

The basic setting analyzed in reference [10] is a scattering problem involving a "free" unperturbed Hamiltonian $\mathbf{H}_0$ and a perturbed Hamiltonian $\mathbf{H}$ defined on a Hilbert space $\mathcal{H}$ where we assume that the absolutely continuous spectrum of both $\mathbf{H}_0$ and $\mathbf{H}$ satisfies $\operatorname{ess\,Supp}\sigma_{ac}(\mathbf{H}_0) = \operatorname{ess\,Supp}\sigma_{ac}(\mathbf{H}) = \mathbb{R}^+$ and that the Møller wave operators $\Omega^{\pm}(\mathbf{H}_0, \mathbf{H})$ exist and are complete. In order to obtain the desired result, described below, it is assumed further that the $S$-matrix in the energy representation, denoted by $\tilde{S}$, is a boundary value on $\mathbb{R}^+$ of an $\mathcal{H}^\infty(\Pi)$ function where, denoting by $\Pi$ the upper half of the complex plane, $\mathcal{H}^\infty(\Pi)$ is the class of functions which are bounded analytic in $\Pi$.

Suppose that under the assumptions mentioned above $\tilde{S}$, as an analytic function in $\Pi$, has a simple zero in the upper half-plane at a point $\overline{\mu}$ with $\operatorname{Im}\mu < 0$ and $\operatorname{Re}\mu > 0$. It is then easy to show that there exists an analytic continuation of $\tilde{S}$ across the positive real axis and that this analytic continuation has a simple pole below the real axis at the point $z = \mu$ which is considered to be associated with a scattering resonance (this is usually referred to as a second sheet pole of the $S$-matrix). Denote by $\mathbf{U}(t)$ the unitary evolution generated by the full scattering Hamiltonian $\mathbf{H}$ and by $\mathcal{H}_{ac}$ the subspace of $\mathcal{H}$ corresponding to the a.c. spectrum of $\mathbf{H}$. It is shown in reference [10] that the pole of $\tilde{S}$ at $z = \mu$ (or rather the zero at $\overline{\mu}$) induces a decomposition of any matrix element $(g, \mathbf{U}(t)f)_{\mathcal{H}_{ac}}$, for $t \geq 0$ and $f$ and $g$ belonging to a certain dense set in $\mathcal{H}_{ac}$, into a term evolving according to a semigroup law and a background term. In a sense to be made precise in the next section the semigroup term is of Lax-Phillips type and the eigenvalue of the generator of the semigroup is exactly $\mu$ i.e., the location of the pole of the $S$-matrix $\tilde{S}$. One may say that the semigroup part of the evolution is driven by the pole of the $S$-matrix. The identification of the eigenvalue of the generator of the semigroup with the location of the $S$-matrix pole is made through a mechanism originating from the Sz.-Nagy-Foias theory of contractions on Hilbert space[11]. The decomposition of the matrix element $(g, \mathbf{U}(t)f)_{\mathcal{H}_{ac}}$ in the form

$$(g, \mathbf{U}(t)f)_{\mathcal{H}_{ac}} = R^{sg}(g, f; t) + \alpha(g, f)e^{-i\mu t}, \quad t \geq 0 \tag{1}$$

induced via the Sz.-Nagy-Foias mechanism by a pole of the scattering matrix $\tilde{S}$ at $z = \mu$ will be called the *semigroup decomposition* of the time evolution of a resonance. The second term in Eq. (1) is the semigroup contribution and the first term is the background term.

One important drawback of the framework developed in reference [10] is the strong assumption made on the analyticity properties of the $S$-matrix. While the assumption that $\tilde{S}$ is the boundary value of an $\mathcal{H}^\infty(\Pi)$ function allows for the application of the proposed framework in certain situations including, for example, certain Friedrichs type models or compactly supported perturbations of the Laplacian, it excludes large classes of quantum mechanical scattering problems for which the scattering matrix does not have the necessary analyticity properties. The main focus of the present paper is on an attempt to overcome this obstacle.

the rest of the paper is organized as follows: A short summary of the framework for the description of the time evolution of resoances developed in reference [10] is given in Section 2 below. As mentioned above this framework assumes certain strong analyticity properties for the $S$-matrix in the upper half-plane. The weakening of these strong assumptions is dealt with in Section 3. Section 4 contains a few comments which are included in order to elucidate some important points in the formalism introduced in the previous two sections. Final remarks are made in Section 5.

## 2 The semigroup decomposition for resonance evolution

This section provides a short summary of the formalism introduced in reference [10]. As mentioned in Section 1 above, the result, Eq. (1), is obtained under the assumption that $\tilde{S}$, the $S$-matrix in the energy representation, is the boundary value on $\mathbb{R}^+$ of a function in $\mathcal{H}^\infty(\Pi)$. In the next section it is shown that it is possible to obtain the same results with a weaker assumption on the analyticity of the $S$-matrix.

We start the discussion in this section with the definition of a Lax-Phillips type semigroup. Consider a Hilbert space $\mathcal{H}$ and an evolution group of unitary operators $\{\mathbf{U}(t)\}_{t\in\mathbb{R}}$ on $\mathcal{H}$. The starting point for the Lax-Phillips scattering theory is the assumption that there exist in $\mathcal{H}$ two distinguished subspaces $\mathcal{D}_-$ and $\mathcal{D}_+$ with the properties

$$\begin{aligned} \mathcal{D}_- &\perp \mathcal{D}_+ \\ \mathbf{U}(t)\mathcal{D}_- &\subset \mathcal{D}_-, \quad t \leq 0 \\ \mathbf{U}(t)\mathcal{D}_+ &\subset \mathcal{D}_+, \quad t \geq 0 \\ \cap_t \mathbf{U}(t)\mathcal{D}_\pm &= \{0\} \\ \overline{\cup_t \mathbf{U}(t)\mathcal{D}_\pm} &= \mathcal{H}\,. \end{aligned} \tag{2}$$

The subspaces $\mathcal{D}_-$ and $\mathcal{D}_+$ are called respectively the *incoming subspace* and *outgoing subspace* for the evolution $\mathbf{U}(t)$. The main object investigated in the Lax-Phillips theory is the Lax-Phillips semigroup which is defined to be the family $\{\mathbf{Z}(t)\}_{t\geq 0}$ of operators on $\mathcal{H}$ given by

$$\mathbf{Z}(t) = \mathbf{P}_+ \mathbf{U}(t) \mathbf{P}_-, \quad t \geq 0\,. \tag{3}$$

Here $\mathbf{P}_-$ is the orthogonal projection of $\mathcal{H}$ onto the orthogonal complement of $\mathcal{D}_-$ and $\mathbf{P}_+$ is the orthogonal projection of $\mathcal{H}$ onto the orthogonal complement of $\mathcal{D}_+$. The family $\{\mathbf{Z}(t)\}_{t\geq 0}$ forms a strongly continuous contractive semigroup on $\mathcal{K} = \mathcal{H} \ominus (\mathcal{D}_- \oplus \mathcal{D}_+)$ with $s - \lim_{t\to\infty} \mathbf{Z}(t)x = 0$ for every $x \in \mathcal{K}$.

Under the assumptions in Eq. (2) Lax and Phillips prove the existence of two *translation representations* for $\mathcal{H}$. In the *incoming translation representation* $\mathcal{H}$ is mapped onto the Hilbert space $L^2_{\mathcal{N}}(\mathbb{R})$ of functions taking their values in a Hilbert space $\mathcal{N}$ (called auxiliary space), $\mathcal{D}_-$ is mapped onto $L^2_{\mathcal{N}}(\mathbb{R}^-)$ and the evolution $\mathbf{U}(t)$ is represented as translation to the right by $t$ units. Analogously, in the *outgoing translation representation* $\mathcal{H}$ is mapped onto $L^2_{\mathcal{N}}(\mathbb{R})$, $\mathcal{D}_+$ is mapped onto $L^2_{\mathcal{N}}(\mathbb{R}^+)$ and the evolution $\mathbf{U}(t)$ is again represented by translation to the right. The mapping $S^{LP}$ of the incoming translation representation onto the outgoing translation representation is the *Lax-Phillips S-matrix*. One usually does not work with the translation representations but rather with their Fourier transforms called respectively the *incoming spectral representation* and *outgoing spectral representation*. According to the Paley-Wiener theorem[12] in the incoming spectral representation $\mathcal{D}_-$ is represented by $\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})$ where $\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})$ is the space of boundary values on $\mathbb{R}$ of functions in $\mathcal{H}^2_{\mathcal{N}}(\Pi)$, the space of (vector valued) Hardy class functions on the upper half-plane. By the same theorem $\mathcal{D}_+$ is represented in the outgoing spectral representation by the function space $\mathcal{H}^-_{\mathcal{N}}(\mathbb{R})$ containing boundary values of functions in $\mathcal{H}^2_{\mathcal{N}}(\overline{\Pi})$ where $\overline{\Pi}$ is the lower half-plane. The Lax-Phillips $S$-matrix in the spectral representation, i.e. the Fourier transform of the Lax-Phillips $S$-matrix, will be denoted by $\mathcal{S}^{LP}$. The operator $\mathcal{S}^{LP} : L^2_{\mathcal{N}}(\mathbb{R}) \mapsto L^2_{\mathcal{N}}(\mathbb{R})$ is realized as a multiplicative, operator valued function $\Theta(\cdot)$, such that $\Theta(\sigma)$ maps $\mathcal{N}$ onto $\mathcal{N}$ for each $\sigma \in R$. The operator valued function $\Theta(\cdot)$ is characterized by its action on $\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})$ as being an *inner function*[13,14,15] (the notation in this paper does not distinguish between an operator defined on the space $\mathcal{H}^2_{\mathcal{N}}(\Pi)$ and the corresponding operator defined on the space of boundary value functions $\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})$; since the two spaces are isomorphic the same notation is being used for both).

As mentioned above the main object of interest in the Lax-Phillips theory is the Lax-Phillips semigroup. In the outgoing spectral representation an element $\mathbf{Z}(t) : \mathcal{K} \mapsto \mathcal{K}$ of the Lax-Phillips semigroup is represented by $\hat{Z}(t) : \hat{\mathcal{K}} \mapsto \hat{\mathcal{K}}$ defined by

$$\hat{Z}(t) = T_{u(t)}|\hat{\mathcal{K}}, \quad t \geq 0 \tag{4}$$

where

$$\hat{\mathcal{K}} = L^2_{\mathcal{N}}(\mathbb{R}) \ominus (\mathcal{H}^-_{\mathcal{N}}(\mathbb{R}) \oplus \mathcal{S}^{LP}\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})) = \\ = L^2_{\mathcal{N}}(\mathbb{R}) \ominus (\mathcal{H}^-_{\mathcal{N}}(\mathbb{R}) \oplus \Theta(\cdot)\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})) \tag{5}$$

and where $T_{u(t)} : \mathcal{H}^+_{\mathcal{N}}(\mathbb{R}) \mapsto \mathcal{H}^+_{\mathcal{N}}(\mathbb{R})$ is a *Toeplitz operator*[13,16,17] with symbol $u(t)$ define by

$$[u(t)f](\sigma) = e^{-i\sigma t} f(\sigma), \quad f \in L^2_{\mathcal{N}}(\mathbb{R}),\ \sigma \in \mathbb{R}\,. \tag{6}$$

The structure of the semigroup $\hat{Z}(t)$, representing $\mathbf{Z}(t)$ in the outgoing spectral representation, can be undrestood in the context of the construction of *functional models* for continuous contractive semigroups on Hilbert space, a part of the Sz.-Nagy-Foias theory of contraction operators on Hilbert space[11]. We call an operator a *model operator* for a class of operators if every operator in that class is similar to a multiple of a part of it (a part of an operator is defined to be the restriction of the operator to one of its invariant subspaces). By a functional model we mean that the model operator is defined on sutible function spaces. In fact, Eq. (4) provides a functional model for a Lax-Phillips type semigroup. From this point of view the semigroup is a fundamental object, the Lax-Phillips Hilbert space and incoming and outgoing subspaces are obtained in the process of a *unitary dilation* of the semigroup and there always exits a similarity, in fact a unitary, transformation of $\mathbf{Z}(t)$ into its functional model representation in terms of $\hat{Z}(t)$.

We notice that Eq. (5) can be written in the form

$$\hat{\mathcal{K}} = \mathcal{H}^+_{\mathcal{N}}(\mathbb{R}) \ominus \Theta(\cdot)\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})\,. \tag{7}$$

In fact, Eq. (4) together with Eq. (7) are considered to be the canonical functional model for a Lax-Phillips type semigroup. Here we consider an *isometric dilation* of the semigroup and we end up with a functional model defined on the Hardy space $\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})$ [10,11]. We are interested in this canonical functional model for the Lax-Phillips semigroup and accordingly, for $\Theta(\cdot)$ an inner function, we call a semigroup of the form

$$\hat{Z}(t) = T_{u(t)} | (\mathcal{H}^+_{\mathcal{N}}(\mathbb{R}) \ominus \Theta(\cdot)\mathcal{H}^+_{\mathcal{N}}(\mathbb{R})), \quad t \geq 0 \tag{8}$$

a semigroup of *Lax-Phillips type* (in the mathematical literature a functional model for $\mathbf{Z}(t)$ in the form of Eq. (8) follows from the observation that the contractive semigroup $\{\mathbf{Z}^*(t)\}_{t\geq 0}$ belongs to the class $C_{\cdot 0}$).

A semigroup of the form given in Eq. (8) is one ingredient entering into the formalism developed in reference [10]. Another important ingredient is the notion of Hilbert space nesting introduced into the study of quantum mechanical resonances by A. Grossman[18]. A nesting map of a Hilbert space $\mathcal{H}_1$ into a Hilbert space $\mathcal{H}_0$ is a linear mapping $\theta : \mathcal{H}_1 \mapsto \mathcal{H}_0$ such that:

1. The domain of $\theta$ is $\mathcal{H}_1$ and $\theta$ is continuous on $\mathcal{H}_1$.

2. The range $\theta\mathcal{H}_1 \subset \mathcal{H}_0$ is dense in $\mathcal{H}_0$.

3. $\theta$ is injective.

A map with the properties 1.-3. is also known as a *quasi-affine map* (for interesting properties of such maps see for example reference [11]). The adjoint of a nesting map $\theta$, defined by

$$(f, \theta g)_{\mathcal{H}_0} = (\theta^* f, g)_{\mathcal{H}_1} \,,$$

is a nesting of $\mathcal{H}_0$ into $\mathcal{H}_1$. A slightly more extended version of the following theorem was proved in reference [10]:

**Theorem 1 (outgoing\incoming contractive nesting)** *Let* $\mathbf{H}_0$ *and* $\mathbf{H}$ *be selfadjoint operators on a Hilbert space* $\mathcal{H}$. *Let* $\{\mathbf{U}(t)\}_{t\in\mathbb{R}}$ *be the unitary evolution group on* $\mathcal{H}$ *generated by* $\mathbf{H}$ *(i.e.,* $\mathbf{U}(t) = \exp(-i\mathbf{H}t)$*). Denote by* $\mathcal{H}_{ac}^0$ *and* $\mathcal{H}_{ac}$, *respectively, the absolutely continuous subspaces of* $\mathbf{H}_0$ *and* $\mathbf{H}$. *Assume that the absolutely continuous spectrum of* $\mathbf{H}_0$ *and* $\mathbf{H}$ *has multiplicity one and that* $\operatorname{ess\,Supp}\sigma_{ac}(\mathbf{H}_0) = \operatorname{ess\,Supp}\sigma_{ac}(\mathbf{H}) = \mathbb{R}^+$. *Assume furthermore that the Møller wave operators* $\mathbf{\Omega}^{\pm}(\mathbf{H}_0, \mathbf{H}) : \mathcal{H}_{ac}^0 \mapsto \mathcal{H}_{ac}$ *exist and are complete. Then there are mappings* $\hat{\mathbf{\Omega}}_{\pm} : \mathcal{H}_{ac} \mapsto \mathcal{H}^+(\mathbb{R})$ *such that:*

($\alpha$) $(\mathcal{H}_{ac}, \mathcal{H}^+(\mathbb{R}), \hat{\mathbf{\Omega}}_{\pm})$ *are contractive Hilbert space nestings of* $\mathcal{H}_{ac}$ *into* $\mathcal{H}^+(\mathbb{R})$.

($\beta$) *For every* $t \geq 0$ *and every* $f \in \mathcal{H}_{ac}$ *we have*

$$\hat{\mathbf{\Omega}}_{\pm}\mathbf{U}(t)f = T_{u(t)}\hat{\mathbf{\Omega}}_{\pm}f \tag{9}$$

*where* $T_{u(t)}$ *is the Toeplitz operator with symbol* $u(t)$. □

The nesting $(\mathcal{H}_{ac}, \mathcal{H}^+(\mathbb{R}), \hat{\mathbf{\Omega}}_-)$ is called below the *incoming contractive nesting* of $\mathcal{H}_{ac}$ and we denote $f_{in} = \hat{\mathbf{\Omega}}_- f$. Similarly $(\mathcal{H}_{ac}, \mathcal{H}^+(\mathbb{R}), \hat{\mathbf{\Omega}}_+)$ is called the *outgoing contractive nesting* and we denote $f_{out} = \hat{\mathbf{\Omega}}_+ f$. The natural definition of the *nested S-matrix* is then $S_{nest} \equiv \hat{\mathbf{\Omega}}_+\hat{\mathbf{\Omega}}_-^{-1}$ and we have

$$f_{out} = \hat{\mathbf{\Omega}}_+\hat{\mathbf{\Omega}}_-^{-1} f_{in} = S_{nest} f_{in} \,.$$

Let $U : \mathcal{H}_{ac}^0 \mapsto L^2(\mathbb{R}^+)$ be the unitary transformation of $\mathcal{H}_{ac}^0$ into the spectral representation for $\mathbf{H}_0$ (the energy representation). If $\mathbf{S} = (\mathbf{\Omega}^-)^*\mathbf{\Omega}^+$ is the scattring operator associated with $\mathbf{H}_0$ and $\mathbf{H}$ then $\tilde{S} : L^2(\mathbb{R}^+) \mapsto L^2(\mathbb{R}^+)$ defined by

$$\tilde{S} \equiv U\mathbf{S}U^*$$

is the energy representation of the $S$-matrix. Define a map $\theta : \mathcal{H}^+(\mathbb{R}) \mapsto L^2(\mathbb{R}^+)$ by taking, for each function $f \in \mathcal{H}^2(\Pi)$ the restriction on $\mathbb{R}^+$ of the boundary value of $f$ on $\mathbb{R}$. Then, by a theorem of C. Van Winter[19], $\theta$ is a nesting of $\mathcal{H}^+(\mathbb{R})$ into $L^2(\mathbb{R}^+)$. The map $\theta^* : L^2(\mathbb{R}^+) \mapsto \mathcal{H}^+(\mathbb{R})$ is well defined and is a nesting of $L^2(\mathbb{R}^+)$ into $\mathcal{H}^+(\mathbb{R})$. It is shown in reference [10] that

$$S_{nest} = \theta^* \tilde{\mathbf{S}} (\theta^*)^{-1} \tag{10}$$

(in fact Eq. (10) is taken in reference [10] to be the definition of $S_{nest}$).

In Section 3 below we will need an explicit form for the map $\theta^*$. The following Lemma provides the needed expression[10]:

**Lemma 1** *Define the inclusion map $I : L^2(\mathbb{R}^+) \mapsto L^2(\mathbb{R})$ by*

$$(If)(\sigma) = \begin{cases} f(\sigma) & \sigma \geq 0 \\ 0 & \sigma < 0 \end{cases} \tag{11}$$

*Let $P_+$ be the orthogonal projection of $L^2(\mathbb{R})$ onto $\mathcal{H}^+(\mathbb{R})$. Then for any $f \in L^2(\mathbb{R}^+)$ we have*

$$\theta^* f = P_+ I f\,. \tag{12}$$

□

We are now able to state the semigroup decomposition result following from the $\mathcal{H}^\infty(\Pi)$ assumption on the $S$-matrix mentioned in Section 1. Assume therefore that $\tilde{S}$, the $S$-matrix in the energy representation, is the boundary value on $\mathbb{R}^+$ of an $\mathcal{H}^\infty(\Pi)$ function which will be denoted by $\mathcal{S}$. If $\mathcal{S}$ has only a simple zero in the upper half-plane then, according to the canonical factorization theorems for $\mathcal{H}^p$ functions[14,15], we can write $\mathcal{S}$ in the form

$$\mathcal{S}(z) = \mathcal{B}_\mu(z)\mathcal{G}(z) \tag{13}$$

where $\mathcal{B}_\mu$ is a simple *Blaschke* factor of the form (for the definition of Blaschke products see, for example, references [14,15])

$$\mathcal{B}_\mu(z) = \frac{z-\overline{\mu}}{z-\mu} \tag{14}$$

and $\mathcal{G} \in \mathcal{H}^\infty(\Pi)$ has no zeros in $\Pi$. Under the above assumptions we have the following result on the semigroup decomposition for the $\mathcal{H}^\infty$ case[10]:

**Proposition 1 ($\mathcal{H}^\infty$ case)** *Assume that the $S$-matrix $\tilde{S} : L^2(\mathbb{R}^+) \mapsto L^2(\mathbb{R}^+)$ is the boundary value on $\mathbb{R}^+$ of some function $\mathcal{S} \in \mathcal{H}^\infty(\Pi)$. Suppose, furthermore that $\mathcal{S}$ has a single, simple zero at the point $z = \overline{\mu}$, $\operatorname{Im}\mu < 0$ in $\Pi$. For any $f \in \mathcal{H}_{ac}$ let $f_{in} = \hat{\mathbf{\Omega}}_- f$ and $f_{out} = \hat{\mathbf{\Omega}}_+ f$. We have*

$$f_{out} = S_{nest} f_{in} = \mathcal{B}_\mu \theta^* \tilde{\mathcal{G}} (\theta^*)^{-1} f_{in} - i2\operatorname{Im}\mu [P_- \mathcal{G} f_{in}^-](\mu) x_\mu \tag{15}$$

*where $P_-$ is the orthogonal projection of $L^2(\mathbb{R})$ on $\mathcal{H}^-(\mathbb{R})$, $f_{in}^- \in \mathcal{H}^-(\mathbb{R})$ is such that $P_{\mathbb{R}^-}(f_{in} + f_{in}^-) = 0$, $\tilde{\mathcal{G}}$ is the boundary value on $\mathbb{R}^+$ of a function $\mathcal{G} \in \mathcal{H}^\infty(\Pi)$ and $\mathcal{G}$ has no zeros in $\Pi$. The vector $x_\mu \in \mathcal{H}^+(\mathbb{R})$ is given by $x_\mu(\sigma) = (\sigma - \mu)^{-1}$ ( $[P_- \mathcal{G} f_{in}^-](\mu)$ is the value of the function $[P_- \mathcal{G} f_{in}^-] \in \mathcal{H}^-(\mathbb{R})$ at $\mu \in \overline{\Pi}$ ).* □

Define

$$\hat{\mathcal{K}}_\mu \equiv \mathcal{H}^+(\mathbb{R}) \ominus \mathcal{B}_\mu \mathcal{H}^+(\mathbb{R})\,. \tag{16}$$

then $\hat{\mathcal{K}}_\mu$ is a one dimensional subspace of $\mathcal{H}^+(\mathbb{R})$ and $x_\mu \in \hat{\mathcal{K}}_\mu$. Since $\mathcal{B}_\mu$ is an inner function for $\mathcal{H}^+(\mathbb{R})$ we can define the Lax-Phillips type semigroup $\{\hat{Z}(t)\}_{t\geq 0}$ by

$$\hat{Z}(t) \equiv T_{u(t)}|\hat{\mathcal{K}}_\mu, \quad t \geq 0\,. \tag{17}$$

Then $x_\mu$ is an eigenvector of the generator of $\{\hat{Z}(t)\}_{t\geq 0}$ and

$$\hat{Z}(t)x_\mu = e^{-i\mu t}x_\mu, \quad t \geq 0\,.$$

Thus, for any $f \in \mathcal{H}_{ac}$ we find that

$$\begin{aligned}\hat{\mathbf{\Omega}}_+\mathbf{U}(t)f &= T_{u(t)}f_{out} = \\ &= T_{u(t)}\mathcal{B}_\mu\theta^*\tilde{\mathcal{G}}(\theta^*)^{-1}f_{in} - i(2\mathrm{Im}\,\mu[P_-\mathcal{G}f_{in}^-](\mu))\hat{Z}(t)x_\mu = \\ &= T_{u(t)}\mathcal{B}_\mu\theta^*\tilde{\mathcal{G}}(\theta^*)^{-1}f_{in} - i(2\mathrm{Im}\,\mu[P_-\mathcal{G}f_{in}^-](\mu))e^{-i\mu t}x_\mu, \quad t \geq 0\,.\end{aligned} \tag{18}$$

Define

$$\Lambda_{\hat{\mathbf{\Omega}}_+} = \hat{\mathbf{\Omega}}_+^*\hat{\mathbf{\Omega}}_+\mathcal{H}_{ac}\,. \tag{19}$$

The linear space $\Lambda_{\hat{\mathbf{\Omega}}_+} \subset \mathcal{H}_{ac}$ is dense in $\mathcal{H}_{ac}$. By the injective property of both $\hat{\mathbf{\Omega}}_+$ and $\hat{\mathbf{\Omega}}_+^*$, for any element $g \in \Lambda_{\hat{\mathbf{\Omega}}_+}$ we can find a unique $h_g \in \mathcal{H}_{ac}$ such that $g = \hat{\mathbf{\Omega}}_+^*\hat{\mathbf{\Omega}}_+h_g$. For any $g \in \Lambda_{\hat{\mathbf{\Omega}}_+}$, $f \in \mathcal{H}_{ac}$ and for $t \geq 0$ we have

$$\begin{aligned}(g, \mathbf{U}(t)f)_{\mathcal{H}_{ac}} &= (\hat{\mathbf{\Omega}}_+^*\hat{\mathbf{\Omega}}_+h_g, \mathbf{U}(t)f)_{\mathcal{H}_{ac}} = (\hat{\mathbf{\Omega}}_+h_g, \hat{\mathbf{\Omega}}_+\mathbf{U}(t)f)_{\mathcal{H}^+(\mathbb{R})} = \\ &= (h_{g,out}, T_{u(t)}f_{out})_{\mathcal{H}^+(\mathbb{R})} = \\ = (h_{g,out}, u(t)&\mathcal{B}_\mu\theta^*\tilde{\mathcal{G}}(\theta^*)^{-1}f_{in})_{\mathcal{H}^+(\mathbb{R})} \\ &- i(2\mathrm{Im}\,\mu[P_-\mathcal{G}f_{in}^-](\mu))e^{-i\mu t}(h_{g,out}, x_\mu)_{\mathcal{H}^+(\mathbb{R})}\,.\end{aligned} \tag{20}$$

Eq. (20) is of the form given in Eq. (1) and its r.h.s. provides (for $t \geq 0$) the semigroup decomposition of the matrix element $(g, \mathbf{U}(t)f)_{\mathcal{H}_{ac}}$. The zero of the $S$-matrix at $z = \overline{\mu}$ in $\Pi$ is related, via the Sz.-Nagy-Foias mechanism described in reference [10], to the Lax-Phillips type semigroup structure leading to the exponential decay of the second term on the r.h.s. of Eq. (20).

As discussed in detail in reference [10], Eq. (20) is a direct result of the assumption on the analyticity properties of the $S$-matrix. In particular, this result is a consequence of the fact that the $\mathcal{H}^\infty$ assumption imply the canonical factorization in Eq. (13). As mentioned in Section 1 above, the assumption that the $S$-matrix $\tilde{S}$ is a boundary value of a bounded analytic function in the upper half-plane is stronger than what one would consider as desirable. Large classes of models for quantum mechanical scattering phenomena do not posses the assumed analyticity properties and hence cannot be analyzed within the framework developed in reference [10]. A way of resolving this difficulty is suggested in the next section.

## 3 Modified assumptions on $S$-matrix analyticity

In this section it is shown that weaker assumptions on the analyticity of the $S$-matrix lead to a result very similar to the semigroup decomposition of Eq. (20). Thus, we assume that the $S$-matrix is analytic in a certain region above the real axis, that it can be analytically continued across $\mathbb{R}^+$ and that the resulting function is meromorphic in an open region $\Sigma$ containing $\mathbb{R}^+$ with a single, simple pole at $z = \mu$, $\operatorname{Im}\mu < 0$ inside $\Sigma$ as depicted in Fig. 1. We have the following theorem:

**Theorem 2** *Under the assumptions of Theorem 1, let* $\mathbf{S}$ *be the scattering operator associated with* $\mathbf{H}_0$ *and* $\mathbf{H}$ *and let* $\tilde{\mathcal{S}} : L^2(\mathbb{R}^+) \mapsto L^2(\mathbb{R}^+)$ *be the S-matrix in the energy representation (i.e.* $\tilde{\mathcal{S}} \equiv U\mathbf{S}U^*$ *as above). Assume that* $\tilde{\mathcal{S}}$ *has an extension to a meromorphic function* $\mathcal{S}$ *defined in the region* $\Sigma$ *with a single, simple, pole at* $z = \mu$, $\operatorname{Im}\mu < 0$, *and no other pole in* $\overline{\Sigma}$, *the closure of* $\Sigma$, *as in Fig. 1.*

*For any* $f \in \mathcal{H}_{ac}$ *use the nesting maps* $\hat{\mathbf{\Omega}}_\pm$ *to define* $f_{out} = \hat{\mathbf{\Omega}}_+ f$ *and* $f_{in} = \hat{\mathbf{\Omega}}_- f$. *Then there exists a unique element* $\psi_\mu \in \mathcal{H}_{ac}$, *such that*

$$f_{out} = S_{nest} f_{in} = \mathcal{B}_\mu \theta^* \tilde{\mathcal{G}}(\theta^*)^{-1} f_{in} + i2\operatorname{Im}\mu\,(\psi_\mu, f)_{\mathcal{H}_{ac}} x_\mu\,. \tag{21}$$

*where* $\theta^*$ *is the map given in Lemma 1,* $x_\mu \in \mathcal{H}^+(\mathbb{R})$ *is given by* $x_\mu(\sigma) = (\sigma-\mu)^{-1}$, $\mathcal{B}_\mu$ *is the Blaschke factor defined in Eq. (14) and the complex valued function* $\tilde{\mathcal{G}}$ *is defined on* $\mathbb{R}^+$ *and is the restriction to* $\mathbb{R}^+$ *of a function* $\mathcal{G}$ *holomorphic in* $\Sigma$ *and having no zeros on* $\overline{\Sigma}$. □

**Proof:**

Let $\tilde{\mathcal{S}}$ be the $S$-matrix in the energy representation and assume that it has a meromorphic extension $\mathcal{S}$ in $\Sigma$ with a simple pole at $\mu$. Then $\mathcal{S}$ has in $\Sigma$ the representation

$$\mathcal{S}(z) = (z-\mu)^{-1}\mathcal{G}'(z) \tag{22}$$

where $\mathcal{G}'$ is analytic in $\Sigma$ and has no zero at $z = \mu$. Now, if $\mathcal{S}$ is expressed by Eq. (22) below the real axis then, by the unitarity of $\tilde{\mathcal{S}}$, the restriction of $\mathcal{S}$ to $\mathbb{R}^+$, we find that in the region $\Sigma \cap \Pi$ we can write $\mathcal{S}$ in the form

$$\mathcal{S}(z) = (\overline{\mathcal{S}}(\overline{z}))^{-1} = (z-\overline{\mu})(\overline{\mathcal{G}'}(\overline{z}))^{-1}, \quad \operatorname{Im} z > 0$$

Since $\mathcal{G}'$ has no zero at $z = \mu$ then $(\overline{\mathcal{G}'}(\overline{z}))^{-1}$ does not have a pole at $z = \overline{\mu}$ and we conclude that $\mathcal{S}$ has a representation in $\Sigma$ expressed by

$$\mathcal{S}(z) = \frac{z-\overline{\mu}}{z-\mu}\,\mathcal{G}(z) = \mathcal{B}_\mu(z)\mathcal{G}(z), \quad z \in \Sigma \tag{23}$$

where $\mathcal{G}(z)$ has no zeros or poles in $\Sigma$. We see that $\mathcal{S}$ has in $\Sigma$ a representation similar to Eq. (13) with the difference being in the fact that this representation is limited to the region $\Sigma$.

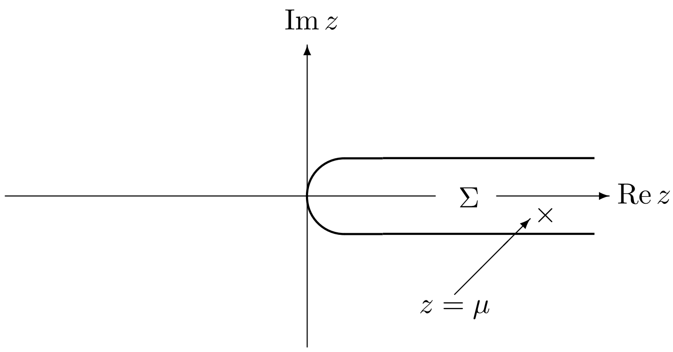


Figure 1: $S$-matrix analyticity properties. We assume that $\tilde{S}$ has a meromorphic extension to the region $\Sigma$ with a simple pole at $z = \mu$.

The $S$-matrix $\tilde{S}$ is given by the restriction of $\mathcal{S}$ in Eq. (23) to $\mathbb{R}^+$ i.e., $\tilde{S}(\lambda) = \mathcal{B}_\mu(\lambda)\tilde{\mathcal{G}}(\lambda)$, $\lambda \in \mathbb{R}^+$. Plugging this form of the $S$-matrix into the expression of the nested $S$-matrix $S_{nest}$ in Eq. (10), we cannot use the methods of reference [10] to obtain the desired results since we no longer assume that $\mathcal{S}$ is an $\mathcal{H}^\infty(\Pi)$ function. However, we can avoid the need for this assumption by writing

$$\begin{aligned} f_{out} = S_{nest} f_{in} = \theta^* \tilde{S}(\theta^*)^{-1} f_{in} = \theta^* \mathcal{B}_\mu \tilde{\mathcal{G}}(\theta^*)^{-1} f_{in} = \\ = P_+ I \mathcal{B}_\mu \tilde{\mathcal{G}}(\theta^*)^{-1} f_{in} = P_+ \mathcal{B}_\mu (P_+ + P_-) I \tilde{\mathcal{G}}(\theta^*)^{-1} f_{in} = \\ = \mathcal{B}_\mu \theta^* \tilde{\mathcal{G}}(\theta^*)^{-1} f_{in} + P_+ \mathcal{B}_\mu P_- \overline{\theta}^* \tilde{\mathcal{G}}(\theta^*)^{-1} f_{in} \end{aligned} \tag{24}$$

where $\overline{\theta}^* = P_- I$. We see that the first term on the r.h.s. of Eq. (24) is identical in form to the first term on the r.h.s. of Eq. (15). In the second term on the r.h.s of Eq. (24) the operator $P_+ \mathcal{B}_\mu P_- : \mathcal{H}^-(\mathbb{R}) \mapsto \mathcal{H}^+(\mathbb{R})$ is a *Hankel operator* with a one dimensional range. In fact, for any $g \in \mathcal{H}^+(\mathbb{R})$ and $f \in \mathcal{H}^-(\mathbb{R})$ we have

$$(\mathcal{B}_\mu g, P_+ \mathcal{B}_\mu P_- f)_{\mathcal{H}^+(\mathbb{R})} = (\mathcal{B}_\mu g, \mathcal{B}_\mu f)_{L^2(\mathbb{R})} = 0 \,.$$

Hence $Ran\,(P_+ \mathcal{B}_\mu P_-) = \hat{\mathcal{K}}_\mu$, where $\hat{\mathcal{K}}_\mu$ is given in Eq. (16). Define the subspace $\hat{\mathcal{K}}_{\overline{\mu}} \subset \mathcal{H}^-(\mathbb{R})$ by

$$\hat{\mathcal{K}}_{\overline{\mu}} \equiv \mathcal{H}^-(\mathbb{R}) \ominus \mathcal{B}_{\overline{\mu}} \mathcal{H}^-(\mathbb{R})$$

with $\mathcal{B}_{\overline{\mu}}(z) = (z - \mu)/(z - \overline{\mu})$ ($\mathcal{B}_{\overline{\mu}}$ is an inner function for $\mathcal{H}^-(\mathbb{R})$) and denote by $P_{\hat{\mathcal{K}}_{\overline{\mu}}}$ the orthogonal projection on this subspace. Then we also have $P_+ \mathcal{B}_\mu P_- = P_+ \mathcal{B}_\mu P_- P_{\hat{\mathcal{K}}_{\overline{\mu}}}$ since

$$P_+ \mathcal{B}_\mu P_- (\mathcal{B}_{\overline{\mu}} \mathcal{H}^-(\mathbb{R})) = 0 \,.$$

We conclude that

$$P_+\mathcal{B}_\mu P_- = P_{\hat{\mathcal{K}}_\mu} P_+\mathcal{B}_\mu P_- P_{\hat{\mathcal{K}}_{\overline{\mu}}} \,. \tag{25}$$

Using Eq. (25) in Eq. (24) we obtain

$$f_{out} = \mathcal{B}_\mu \theta^* \tilde{\mathcal{G}}(\theta^*)^{-1} f_{in} + P_{\hat{\mathcal{K}}_\mu} P_+\mathcal{B}_\mu P_- P_{\hat{\mathcal{K}}_{\overline{\mu}}} \overline{\theta}^* \tilde{\mathcal{G}}(\theta^*)^{-1} f_{in} \,. \tag{26}$$

According to Eq. (16) and Eq. (26) we expect that the second term on the r.h.s. of Eq. (26) is proportional to the vector $x_\mu \in \mathcal{H}^+(\mathbb{R})$. Indeed this is verified by explicit calculation. For the projection operators $P_\pm$ we have the standard expressions

$$[P_\pm f](\sigma') = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{1}{\sigma' \pm i\epsilon - \sigma} f(\sigma)\, d\sigma, \quad f \in L^2(\mathbb{R})$$

Hence, for $f \in \mathcal{H}^-(\mathbb{R})$ we have

$$\begin{aligned}[P_+\mathcal{B}_\mu P_- f](\sigma) = [P_+\mathcal{B}_\mu f](\sigma) &= \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{1}{\sigma + i\epsilon - \sigma'} \frac{\sigma' - \overline{\mu}}{\sigma' - \mu} f(\sigma') = \\ &= \frac{1}{\sigma - \mu}\, i2\mathrm{Im}\,\mu f(\mu)\,. \end{aligned} \tag{27}$$

If the $S$-matrix $\tilde{\mathcal{S}}$ has a meromorphic extension $\mathcal{S}$ in $\Sigma$ with a simple pole at $z = \mu$ then the holomorphic factor of $\mathcal{S}$ in $\Sigma$ can be found from Eq. (23) and is given by $\mathcal{G} = \mathcal{B}_{\overline{\mu}}\mathcal{S}$. Hence we have

$$\begin{aligned} &[\overline{\theta}^* \mathcal{G}(\theta^*)^{-1} f_{in}](\mu) = [P_- I \mathcal{B}_{\overline{\mu}} \tilde{\mathcal{S}}(\theta^*)^{-1} f_{in}](\mu) = \\ &\qquad = \frac{1}{2\pi i}\int_{\mathbb{R}^+} \frac{1}{\lambda - \mu} (\mathcal{B}_{\overline{\mu}} \tilde{\mathcal{S}}(\theta^*)^{-1} f_{in})(\lambda) = \\ &= \frac{1}{2\pi i}\int_{\mathbb{R}^+} \frac{1}{\lambda - \overline{\mu}} (\tilde{\mathcal{S}}(\theta^*)^{-1} f_{in})(\lambda) = [\theta^* \tilde{\mathcal{S}}(\theta^*)^{-1} f_{in}](\overline{\mu}) = f_{out}(\overline{\mu})\,. \end{aligned} \tag{28}$$

Furthermore, it was shown in reference [10] that the nesting maps $\hat{\mathbf{\Omega}}_\pm$ are given by $\hat{\mathbf{\Omega}}_\pm = \theta^* U(\mathbf{\Omega}^\mp)^*$ (where, as above, $U : \mathcal{H}^0_{ac} \mapsto L^2(\mathbb{R}^+)$ is the mapping onto the $\mathbf{H}_0$ spectral representation) so that, for every $f \in \mathcal{H}_{ac}$, we have

$$f_{out}(\overline{\mu}) = \frac{1}{2\pi i}\int_{\mathbb{R}^+} \frac{1}{\lambda - \overline{\mu}} (U(\mathbf{\Omega}^-)^* f)(\lambda) = (\psi_\mu, f)_{\mathcal{H}_{ac}} \tag{29}$$

where we define $\psi_\mu = \mathbf{\Omega}^- U^* \tilde{\psi}_\mu$ with $\tilde{\psi}_\mu \in L^2(\mathbb{R}^+)$, $\tilde{\psi}_\mu(\lambda) = (2\pi i)^{-1}(\lambda - \mu)^{-1}$. Combining Eq. (26), Eq. (27), Eq. (28) and Eq. (29) we obtain the result Eq. (21). ■

In order to see how Theorem 2 is used we go back to Eq. (20). From this equation, together with Eq. (26) and Eq. (21) we find, for any $f \in \mathcal{H}_{ac}$ and any $g \in \Lambda_{\hat{\mathbf{\Omega}}_+}$,

$$
\begin{aligned}
(g, \mathbf{U}(t) f)_{\mathcal{H}_{ac}} =& (h_{g,out}, T_{u(t)} f_{out})_{\mathcal{H}^+(\mathbb{R})} = \\
&= (h_{g,out}, u(t)\mathcal{B}_\mu \theta^* \tilde{\mathcal{G}}(\theta^*)^{-1} f_{in})_{\mathcal{H}^+(\mathbb{R})} \\
&\qquad + (h_{g,out}, \hat{Z}(t) P_{\hat{\mathcal{K}}_\mu} P_+ \mathcal{B}_\mu P_- P_{\hat{\mathcal{K}}_{\overline{\mu}}} \overline{\theta}^* \tilde{\mathcal{G}}(\theta^*)^{-1} f_{in})_{\mathcal{H}^+(\mathbb{R})} = \\
&= (h_{g,out}, u(t)\mathcal{B}_\mu \theta^* \tilde{\mathcal{G}}(\theta^*)^{-1} f_{in})_{\mathcal{H}^+(\mathbb{R})} \\
&\qquad + e^{-i\mu t} (h_{g,out}, x_\mu)_{\mathcal{H}^+(\mathbb{R})} (\psi_\mu, f)_{\mathcal{H}_{ac}} .
\end{aligned}
\tag{30}
$$

Eq. (30) has the general form of Eq. (1) and provides the semigroup decomposition in the case that the $S$-matrix $\tilde{S}$ has a meromorphic extension to the region $\Sigma$ in Fig. 1. It can be easily shown that Eq. (30) reduces to Eq. (20) if $\tilde{S}$ is in fact the boundary value on $\mathbb{R}^+$ of a function belonging to $\mathcal{H}^\infty(\Pi)$.

We observe that if in Eq. (30) the state $f \in \mathcal{H}_{ac}$ is chosen to be orthogonal to $\psi_\mu$ then the exponentially decaying term on the r.h.s. of Eq. (30), originating from the Lax-Phillips type semigroup evolution associated with the pole of the $S$-matrix, does not appear. This enables us to make a direct correspondence between the state $\psi_\mu$ and the resonance contribution to the time evolution. Since, as is seen in Section 4 below, no state in $\mathcal{H}_{ac}$ can be mapped into an exact resonance state in the Hardy space $\mathcal{H}^+(\mathbb{R})$ and there always exists some nonzero background contribution, the vector $\psi_\mu \in \mathcal{H}_{ac}$ will be called an *approximate resonance state*. Using Dirac's notation, let us denote by $\{|E^-\rangle\}_{E\in\mathbb{R}^+}$ the set of outgoing scattering states, i.e., outgoing solutions of the Lipmann-Schwinger equation. It is then easy to see that for $f \in \mathcal{H}_{ac}$ we have

$$
(U(\mathbf{\Omega}^-)^* f)(E) = \langle E^- | f \rangle, \qquad E \in \mathbb{R}^+
$$

and the definition of the state $\psi_\mu$ (see Eq. (29)) implies that it is given by the simple expression

$$
|\psi_\mu\rangle = \frac{1}{2\pi i} \int_{\mathbb{R}^+} dE \, \frac{1}{E-\mu} |E^-\rangle \,. \tag{31}
$$

## 4 Comments

This section contains some comments on the framework described in Section 2 and further extended in Section 3. The discussion below is presented in a slightly more general form than is strictly necessary in order to relate it to the formalism of Sections 2 and 3. The more general form of the statements made below places the remarks at the end of the present section, on the

Bohm-Gadella rigged Hardy space formalism for the resonance problem[20], into their natural context.

Let $\mathcal{S}$ denote the *Schwartz class* of rapidly decreasing functions in $C^\infty(\mathbb{R})$. Let $\mathcal{S}'$ denote the space of *tempered distributions* on $\mathcal{S}$. We shall need the following definition[21]:

**Definition 1 (The space $\mathcal{H}^p(\mathbb{C}\backslash\mathbb{R})$)** *For any fixed $p \in (0,\infty)$ let $\mathcal{H}^p(\mathbb{C}\backslash\mathbb{R})$ denote the space of analytic functions on $\mathbb{C}\backslash\mathbb{R}$ for which*

$$\|f\| \equiv \sup_{y\neq 0}\left\{\int_{\mathbb{R}} |f(x+iy)|^p dx\right\}^{1/p} < \infty$$

It can be shown[21] that every function $F \in \mathcal{H}^p(\mathbb{C}\backslash\mathbb{R})$ is associated with a unique tempered distribution $\ell_F \in \mathcal{S}'$ defined by

$$\ell_F(\psi) = \lim_{y\to 0^+}\int \{F(x+iy) - F(x-iy)\}\psi(x)\,dx, \quad \psi \in \mathcal{S} \tag{32}$$

Denote the set of all distributions arising in this way by $H^p(\mathbb{R})$. Then, conversely, for any $p \in (0,\infty)$ and for any distribution $\ell \in H^p(\mathbb{R})$, one can find the unique function $F_\ell \in \mathcal{H}^p(\mathbb{C}\backslash\mathbb{R})$ that defines the distribution $\ell$ through Eq. (32) via the formula[21]

$$F_\ell(z) = \frac{1}{2\pi i}\,\ell(\frac{1}{\cdot - z})\,. \tag{33}$$

Eq. (33) can be thought of as a generalization of the Cauchy integral formula for the recovery of an $\mathcal{H}^p$ function from its boundary value on $\mathbb{R}$.

Even though Eq. (32) and Eq. (33) are valid for any $p \in (0,\infty)$, for $p \in (1,\infty)$ we have the further identification of the space $H^p(\mathbb{R})$ with the space $L^p(\mathbb{R})$ in the sense that any function $f \in L^p(\mathbb{R})$ defines a tempered distribution on $\mathcal{S}$ by

$$\ell_f(\psi) = \int_{\mathbb{R}} f(x)\psi(x)dx, \quad \psi \in \mathcal{S} \tag{34}$$

and that for any $f \in L^p(\mathbb{R})$ there exists a unique $F_f \in \mathcal{H}^p(\mathbb{C}\backslash\mathbb{R})$ such that $\ell_{F_f} = \ell_f$ i.e., Eq. (32) and Eq. (34) define the same tempered distribution on $\mathcal{S}$.

Finally, we will also need the following result[21]:

**Proposition 2** *A distribution $\ell \in H^p(\mathbb{R})$ has support which omits an open interval $I \in \mathbb{R}$ if and only if the corresponding function $F_\ell \in \mathcal{H}^p(\mathbb{C}\backslash\mathbb{R})$ given by Eq. (33) has an analytic continuation across the interval $I$.*

Consider now the map $\theta^* : L^2(\mathbb{R}^+) \mapsto \mathcal{H}^+(\mathbb{R})$ and its inverse $(\theta^*)^{-1} : \theta^* L^2(\mathbb{R}^+) \mapsto L^2(\mathbb{R}^+)$. An explicit expression for $\theta^*$ is given in Eq. (12). Breaking the action of $\theta^*$ into two steps we first have, for any $f \in L^2(\mathbb{R}^+)$

$$If = P_+ If + P_- If = \theta^* f + \overline{\theta}^* f = f_+ + f_-, \quad f_+ \in \mathcal{H}^+(\mathbb{R}),\ f_- \in \mathcal{H}^-(\mathbb{R}) \tag{35}$$

where $f_+ = P_+ If$, $f_- = P_- If$. In the second step we take the $\mathcal{H}^+(\mathbb{R})$ piece i.e.,

$$\theta^* f = P_+ If = f_+ \, .$$

On the other hand the discussion preceeding Proposition 2 implies that if we apply the inclusion map $I : L^2(\mathbb{R}^+) \mapsto L^2(\mathbb{R})$ then, for any element $f \in L^2(\mathbb{R}^+)$, the element $If \in L^2(\mathbb{R})$ is associated with a unique function $F_f \in \mathcal{H}^2(\mathbb{C}\backslash\mathbb{R})$ such that

$$(If)(\sigma) = \lim_{\epsilon \to 0^+} \{F_f(\sigma + i\epsilon) - F_f(\sigma - i\epsilon)\} \tag{36}$$

In fact, the function $F_f \in \mathcal{H}^2(\mathbb{C}\backslash\mathbb{R})$ is easily found from Eq. (35). We first use the isomorphism of $\mathcal{H}^+(\mathbb{R})$ and $\mathcal{H}^2(\Pi)$ to extend the map $\theta^*$ to a mapping $\theta^*_\pi : L^2(\mathbb{R}^+) \mapsto \mathcal{H}^2(\Pi)$. Subsequently we simply define

$$F_f(z) = \begin{cases} f_+(z) = (\theta^*_\pi f)(z), & \operatorname{Im} z > 0 \\ -f_-(z) = -(\overline{\theta}^*_\pi f)(z), & \operatorname{Im} z < 0 \end{cases} \tag{37}$$

Moreover, Proposition 2 shows that $F_f$ defined in Eq. (37) is in fact analytic on $\mathbb{C}\backslash\mathbb{R}^+$. Denoting the subspace of $\mathcal{H}^2(\mathbb{C}\backslash\mathbb{R})$ of functions analytic on $\mathbb{R}^-$ by $\mathcal{H}^2(\mathbb{C}\backslash\mathbb{R}^+)$ we conclude that there exists a surjective map $A : L^2(\mathbb{R}^+) \mapsto \mathcal{H}^2(\mathbb{C}\backslash\mathbb{R}^+)$ with $Af = F_f$ for $f \in L^2(\mathbb{R}^+)$, $F_f \in \mathcal{H}^2(\mathbb{C}\backslash\mathbb{R}^+)$. Note that

$$(Af)(z) = F_f(z) = (\theta^*_\pi f)(z), \quad \operatorname{Im} z > 0 \, . \tag{38}$$

In addition, we note that Eq. (36)-(38) provide us with a procedure for the construction of the map $(\theta^*)^{-1}$. Given $f_+ \in \theta^* L^2(\mathbb{R}^+) \subset \mathcal{H}^+(\mathbb{R})$ we use the Cauchy integral formula to obtain the function $f_+ \in \mathcal{H}^2(\Pi)$. We know from Eq. (38) and Eq. (37) that there is a unique function $f \in L^2(\mathbb{R}^+)$ such that $f_+ \in \mathcal{H}^2(\Pi)$ is the restriction to $\Pi$ of a function $F_f = Af \in \mathcal{H}^2(\mathbb{C}\backslash\mathbb{R}^+)$. Hence we can analytically continue $f_+$ across $\mathbb{R}^-$ into the lower half-plane and obtain the full function $F_f$. The reconstruction of the corresponding function $f \in L^2(\mathbb{R}^+)$ is then obtained by using Eq. (36). The process of analytically continuing $f_+$ across $\mathbb{R}^-$ is done in reference [10] essentially by using the Van Winter theorem and explicit integral expressions for the map $(\theta^*)^{-1}$ are obtained.

Next we turn to a discussion of the resonance states. Let

$$R_s \equiv \{x_\mu \mid x_\mu \in \mathcal{H}^+(\mathbb{R}),\ x_\mu(\sigma) = (\sigma - \mu)^{-1}, \sigma \in \mathbb{R},\ \operatorname{Im} \mu < 0\} \tag{39}$$

Obviously, it is not possible to analytically continue any element $x_\mu \in R_s$ across $\mathbb{R}^-$ in order to obtain a function in $\mathcal{H}^2(\mathbb{C}\backslash\mathbb{R}^+)$. By Eq. (38) we obtain:

**Lemma 2** *Define the set $R_s \subset \mathcal{H}^+(\mathbb{R})$ according to Eq. (39). Then $R_s \subset \mathcal{H}^+(\mathbb{R})\backslash\hat{\mathbf{\Omega}}_\pm \mathcal{H}_{ac}$.*

Now, for any given $S$-matrix $\tilde{\mathcal{S}}$ the eigenvector of the Lax-Phillips type semigroup in the second term on the r.h.s. of Eq. (30) is proportional to some $x_\mu \in R_s$. Identifying the Hardy space state $x_\mu \in \mathcal{H}^+(\mathbb{R})$ as a pure, exponentialy decaying, resonance state, Lemma 2 provides a formal verification for the impossibility of the association of any unique state in the original Hilbert space $\mathcal{H}_{ac}$ with a pure resonance. For this reason the time evolution (e.g. the survival amplitude) of any state in $\mathcal{H}_{ac}$ always contains some background contribution and is never purely exponentially decaying.

The final remark in this section is concerned with the Bohm-Gadella rigged Hilbert space formalism for the problem of resonances[20]. The main tool in this formalism is a Gelfand triplet $\Delta_+ \subset \mathcal{H}^+(\mathbb{R}) \subset \Delta_+^*$ constructed by a rigging of the Hardy space $\mathcal{H}^+(\mathbb{R})$. The smaller sector $\Delta_+$ of the Gelfand triple is taken to be $\Delta_+ \equiv \mathcal{H}^+(\mathbb{R}) \cap \mathcal{S}$ where $\mathcal{S}$ again denotes the Schwartz space. The larger sector $\Delta_+^*$ contains all the continuous linear functionals on $\Delta_+$. One then uses a pullback procedure in order to obtain a rigged Hilbert space $\Phi \subset L^2(\mathbb{R}^+) \subset \Phi^*$ centered around the Hilbert space $L^2(\mathbb{R}^+)$. The procedure of pull back uses the map $\theta$. We first define $\Phi \equiv \theta\Delta_+$ and then the pull back procedure is used in order to define the set $\Phi^*$ of functionals on $\Phi$. Denoting the evaluation of the functional $F$ on a test function $f$ in the rigged Hilbert spaces $\Phi \subset L^2(\mathbb{R}^+) \subset \Phi^*$ or $\Delta_+ \subset \mathcal{H}^+(\mathbb{R}) \subset \Delta_+^*$ by $\langle f, F\rangle_{L^2}$ and $\langle f, F\rangle_{\mathcal{H}^+}$ respectively, the pull back of a functional $F \in \Delta_+^*$ is defined to be

$$\langle f, \widehat{(\theta^{-1})^*} F\rangle_{L^2} \equiv \langle \theta^{-1} f, F\rangle_{\mathcal{H}^+}, \quad f \in \Phi,\ F \in \Delta_+^* \tag{40}$$

The map $\widehat{(\theta^{-1})^*} : \Delta_+^* \mapsto \Phi^*$ on the l.h.s. of Eq. (40) is an extension to $\Delta_+^*$ of the map $(\theta^{-1})^*$. But on its domain of definition in $\mathcal{H}^+(\mathbb{R})$ we have $(\theta^{-1})^* = (\theta^*)^{-1}$. Hence Eq. (40) can serve just as well to define an extension of the map $(\theta^*)^{-1}$ i.e., we have

$$\langle f, \widehat{(\theta^*)^{-1}} F\rangle_{L^2} \equiv \langle \theta^{-1} f, F\rangle_{\mathcal{H}^+}, \quad f \in \Phi,\ F \in \Delta_+^* \tag{41}$$

For any function $F \in \theta^* L^2(\mathbb{R}^+) \subset \mathcal{H}^+(\mathbb{R}) \subset \Delta_+^*$ we have $(f, \widehat{(\theta^*)^{-1}} F\rangle_{L^2} = (f, (\theta^*)^{-1} F)_{L^2(\mathbb{R}^+)}$ and $\widehat{(\theta^*)^{-1}} F$ is then identified with a function in $L^2(\mathbb{R}^+)$ by the procedure for the construction of $(\theta^*)^{-1}$ described above. Here $F \in \theta^* L^2(\mathbb{R}^+)$ is a function in $\mathcal{H}^2(\mathbb{C}\backslash\mathbb{R}^+)$. However, the map $\widehat{(\theta^*)^{-1}}$, defined on $\Delta_+^*$, is certainly well defined for the whole Hardy space $\mathcal{H}^+(\mathbb{R})$ and, in particular, it is well defined in the distributional sense for any resonance state $x_\mu \in R_s \subset \mathcal{H}^+(\mathbb{R})$. Indeed, a resonance in the Bohm-Gadella theory has the form $\widehat{(\theta^*)^{-1}} x_\mu$ for some $x_\mu \in R_s$. Moreover, since the elements of $R_s$ cannot be analytically continued into functions in $\mathcal{H}^2(\mathbb{C}\backslash\mathbb{R}^+)$, they belong to $\Delta_+^* \backslash \theta^* L^2(\mathbb{R}^+)$.

## 5 Conclusions

A way of utilizing Lax-Phillips type semigroups for the description of the time evolution of quantum mechanical resonances was suggested in reference [10]. The present paper addresses the main difficulty with the framework introduced in reference [10] and described in Section 2 above i.e., the assumption that the $S$-matrix is the boundary value on $\mathbb{R}^+$ of a function in $\mathcal{H}^\infty(\Pi)$. Such a requirment is not satisfied by large classes of quantum mechanical scattering problems. It is shown in Section 3 above that this condition can be weakened to the assumption that the given $S$-matrix is analytic in a region $\Sigma$ as in Fig. 1. In addition, it is shown in Section 4 that, if we regard a resonance as a quantum object and we look for a Hilbert space state describing it, our expectation that no such state can be found in the Hilbert space $\mathcal{H}$ for the scattering problem is valid. In fact a resonance, identified as an eigenvector of the Lax-Phillips type semigroup responsible for the exponential decay of the second term on the r.h.s. of the semigroup decomposition in Eq. (30), exists as an element of the Hilbert space $\mathcal{H}^+(\mathbb{R})$, but cannot be associated with any element in $\mathcal{H}_{ac}$ in the sense that it is outside of the range of the nesting map $\hat{\mathbf{\Omega}}_+$. This implies that the background term, i.e., the first term on the r.h.s. of Eq. (30), exists for any choice of the vectors $g, f \in \mathcal{H}_{ac}$ and is never zero. However, the formalism developed above does provide a clear identification of a well defined approximate resonance state $\psi_\mu \in \mathcal{H}_{ac}$ associated with the resonance contribution to the time evolution (i.e. the semigroup term).

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