Schrödinger operators with oscillating potentials *

Itaru Sasaki
Department of Mathematics,
Hokkaido University,
Sapporo 060-0810, Japan
e-mail: i-sasaki@math.sci.hokudai.ac.jp

December 10, 2004

Abstract

Schrödinger operators H with oscillating potentials such as $\cos x^2$ are considered. Such potentials are not relatively compact with respect to the free Hamiltonian. But we show that they do not change the essential spectrum. Moreover we derive upper bounds for negative eigenvalue sums of H.

Key words: Schrödinger operator; oscillating potentials; eigenvalue sum;

* *Mathematics Subject Classification.* 35J10, 35P15, 81Q10.

1 Introduction

In this paper, we consider Schrödinger operators H with oscillating potentials such as $\cos|x|^2$. To our knowledge, the spectral analysis of such Schrödinger operators H has no antecedent.

First we show that a class of oscillating potentials V does not change the essential spectrum of the free Hamiltonian(i.e. $\sigma_{\mathrm{ess}}(-\triangle + V) = [0,\infty)$). This means that the negative part of the $-\triangle + V$ is compact operator. We remark that the potentials we consider are not compact with respect to the free Hamiltonian.

It is well known that the moment of the eigenvalues of the Schrödinger operator $-\triangle_d + V$(on $L^2(\mathbb{R}^d)$) has the following estimate:

$$\sum_{j=0}^{\infty} |e_j|^\gamma \le L_{\gamma,d} \int_{\mathbb{R}^d} |V(x)|_-^{\gamma+d/2} \mathrm{d}x, \quad (d = 1,2,3,\cdots), \tag{1}$$

where $|V(x)|_- := -\min\{0, V(x)\}$, $e_0 \le e_1 \le e_2 \le \cdots$ are negative eigenvalues of $-\triangle + V$ and $L_{\gamma,d}$ is a universal constant([4, Theorem 12.4],[5]). For the potential $V(x) = \cos(|x^2|)$, the left hand side of (1) can be defined by compactness of the negative part of H, $|V(x)|_-^{\gamma+d/2}$ is not *integrable*$(d = 1,2,\ldots)$:

$$\int_{\mathbb{R}^d} |V(x)|_-^{\gamma+d/2} \mathrm{d}x = \infty, \quad V(x) = \cos|x|^2,$$

but we show that $\sum_{j=0}^{\infty} |e_j|^\gamma$ is *finite* in the following cases:

$$\begin{cases} \gamma \ge \frac{1}{2}, & \text{for } d = 1, \\ \gamma > 0, & \text{for } d = 2,3,\ldots \end{cases} \tag{2}$$

Moreover in a general case we give new criteria for $\sum_{j=0}^{\infty} |e_j|^\gamma < \infty$ and derive upper bounds for negative eigenvalue sums of H.

In analysis of the Schrödinger operator with an oscillating potential, the positive part of the potential is essential. Because, for a low energy state u, the expectation value $|\langle u, Vu\rangle|$ becomes small by the oscillation of the potential. But $|\langle u, Vu\rangle|$ does not become small if the positive part of V is cut off.

2 Essential Spectrum

We consider the Schrödinger operator on $L^2(\mathbb{R}^d)$:

$$H := H_0 + V, \quad H_0 = -\triangle_d, \tag{3}$$

where $\triangle_d$ is the d-dimensional Laplacian and $V \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ is a real-valued function. Let S_d be the d-dimensional unit sphere, and let Θ be the stantard measure on S_d. We write $x \in \mathbb{R}^d$ as $x = r\theta$, $r = |x|, \theta \in S_d$. We denote the Laplace-Beltrami operator on S_d by Λ_d.

Throughout this section, we assume that the potential V has the following properties:

[V.1] $V : \mathbb{R}^d \mapsto \mathbb{R}$ is bounded Borel measurable, and
for $d = 1$,

$$\lim_{R\to\infty} \sup_{x\in[R,\infty)} \left| \int_R^x V(y)\mathrm{d}y \right| = 0, \quad \lim_{R\to-\infty} \sup_{x\in(-\infty,R]} \left| \int_x^R V(y)\mathrm{d}y \right| = 0; \tag{4}$$

for $d \geq 2$

$$\lim_{R\to\infty} \sup_{r\in[R,\infty)} \sup_{\theta\in S_d} \left| \int_R^r V(r\theta)\mathrm{d}r \right| = 0. \tag{5}$$

Example 2.1. The following functions V_1 and V_2 satisfy condition [V.1]:

$$\begin{aligned} &V_1(r) := a\sin(br^\ell),\ V_2(r) := a\cos(br^\ell) \quad a, b \in \mathbb{R}\setminus\{0\}, \\ &r = |x|,\ d \in \mathbb{N},\ \ell \geq 2. \end{aligned} \tag{6}$$

Under condition [V.1], H is self-adjoint with $D(H) = D(H_0)$ and bounded below. For a self-adjoint operator A, we denote by A_+, A_- the positive and negative part of A respectively:

$$A_+ = \int_{[0,\infty)} \lambda \mathrm{d}E_A(\lambda), \quad A_- = \int_{(-\infty,0)} \lambda \mathrm{d}E_A(\lambda), \tag{7}$$

where $E_A(\cdot)$ is the spectral measure associated with A. When A is bounded from below, we set

$$\Sigma(A) := \inf \sigma_{\mathrm{ess}}(A). \tag{8}$$

Theorem 2.2. *Assume that V satisfies condition [V.1]. Then*

$$\sigma_{\mathrm{ess}}(H) = [0, \infty). \tag{9}$$

In particular H_- is compact.

Remark. The potentials V_1 and V_2 with $\ell \geq 2$ in Example 2.1 are not H_0^n-compact($n = 1, 2, \ldots$), and $|V_1|$ and $|V_2|$ are not H_0-form compact. Indeed, if $\cos bx^\ell(H_0^n + 1)^{-1}$ is compact, then $\sin bx^\ell \cdot \cos bx^\ell = (\sin 2bx^\ell)/2$ is H_0^n-compact. Hence $\sin bx^\ell(H_0^n + 1)^{-1}$ is compact. Therefore $[(\sin bx^\ell)^2 + (\cos bx^\ell)^2](H_0^n + 1)^{-1} = (H_0 + 1)^{-1}$ is compact, but $(H_0^n + 1)^{-1}$ is not compact which is a contradiction. Therefore V_2 is not H_0^n-compact. Similarly we can show that V_1 is not H_0^n-compact. Therefore Theorem 2.2 is nontrivial.

Remark. Let $V = V_1$(or V_2). If $d = 1$ and

$$\lim_{L\to\infty}\int_{\mathbb{R}} V_1(x)e^{-|x|/L}\mathrm{d}x < 0, \quad \Big(\text{or } \lim_{L\to\infty}\int_{\mathbb{R}} V_2(x)e^{-|x|/L}\mathrm{d}x < 0\Big), \tag{10}$$

then $H_- \neq 0$. Indeed, for $\psi_L(x) := \exp(-|x|/2L) \in L^2(\mathbb{R})$, we have

$$\lim_{L\to\infty}\langle\psi_L, H\psi_L\rangle < 0. \tag{11}$$

In particular, in the case $l = 2$, $H_- \neq 0$ for all $a < 0$, $b > 0$. If $d \geq 2$, there exist a constants $\alpha > 0$ and $\beta < 0$ such that for all $|a| > \alpha$ and $|b| > \beta$, $H_- \neq 0$ (see [1, Lemma 4.3]).

Proof of Theorem 2.2. For $R \geq 0$, we denote by χ_R the characteristic function of $\{x \in \mathbb{R}^d | |x| \leq R\}$. Then $\chi_R V$ is H_0-compact([8, p.117, Example 6]). For all $u \in C_0^\infty(\mathbb{R}^d)$, we have

$$\langle u, Vu\rangle = \langle u, \chi_R Vu\rangle + \int_{S_d}\mathrm{d}\Theta(\theta)\int_{[R,\infty)} r^{d-1}\mathrm{d}rV(r\theta)|u(r\theta)|^2. \tag{12}$$

Let

$$W(R, r; \theta) := \int_{[R,r]} V(s\theta)\mathrm{d}s. \tag{13}$$

Then, for almost every $\theta \in S_d$,

$$\int_{[R,\infty)} r^{d-1}\mathrm{d}rV(r\theta)|u(r\theta)|^2 = -\int_{[R,\infty)} W(R, r; \theta)\frac{\mathrm{d}}{\mathrm{d}r}\Big(|u(r\theta)|^2 r^{d-1}\Big)\,\mathrm{d}r. \tag{14}$$

Therefore

$$\begin{aligned}|(\text{l.h.s}(14))| \leq \Big(1 + \frac{d-1}{R}\Big)\sup_{r\geq R}|W(R, r; \theta)|\int_{[0,\infty)}|u(r\theta)|^2 r^{d-1}\mathrm{d}r \\ + \sup_{r\geq R}|W(R, r; \theta)|\int_{[0,\infty)}\left|\frac{\mathrm{d}u(r\theta)}{\mathrm{d}r}\right|^2 r^{d-1}\mathrm{d}r,\end{aligned} \tag{15}$$

By the definition of Λ_d we have

$$\langle u, H_0 u\rangle = \int_{S_d}\mathrm{d}\Theta(\theta)\int_0^\infty\left[\left|\frac{\mathrm{d}u(r\theta)}{\mathrm{d}r}\right|^2 - \frac{u(r\theta)^*}{r^2}(\Lambda_d u)(r\theta)\right] r^{d-1}\mathrm{d}r,$$

and

$$-\int_{S_d}\mathrm{d}\Theta(\theta)\int_0^\infty u(r\theta)^*(\Lambda_d u)(r\theta)r^{d-1}\mathrm{d}r \geq 0. \tag{16}$$

Therefore, for all $u \in D(H_0)$ and $R > 0$, we have

$$|\langle u, Vu\rangle| \le |\langle u, \chi_R V u\rangle| + a(R)\|u\|^2 + b(R)\langle u, H_0 u\rangle, \tag{17}$$

where

$$a(R) := \left(1 + \frac{d-1}{R}\right) \sup_{\substack{r\ge R\\ \theta\in S_d}} |W(R,r;\theta)|,$$

$$b(R) := \sup_{\substack{r\ge R\\ \theta\in S_d}} |W(R,r;\theta)|.$$

By condition [V.1],

$$\lim_{R\to\infty} a(R) = \lim_{R\to\infty} b(R) = 0. \tag{18}$$

Hence, the following operator inequality on $D(H_0)$ holds:

$$H \ge (1 - b(R))H_0 - |\chi_R V| - a(R), \quad (R > 0). \tag{19}$$

By the min-max principle,

$$\Sigma(H) \ge -a(R), \tag{20}$$

for all R with $1 \ge b(R)$. Taking $R \to \infty$, we have $\Sigma(H) \ge 0$. Therefore $\sigma_{\rm ess}(H) \subset [0,\infty)$. This means that H_- is compact.

Next we show that $\sigma_{\rm ess}(H) \supset [0,\infty)$. Let $u \in C_0^\infty(\mathbb{R}^d)$ be a normalized vector and set

$$u_L(x) := u(x/L)/\sqrt{L^d}, \quad x \in \mathbb{R}^d. \tag{21}$$

It is easy to see that

$$\|u_L\| = 1, \quad u_L \overset{\rm w}{\to} 0\,(L\to\infty), \quad \langle u_L, H_0 u_L\rangle \to 0\,(L\to\infty). \tag{22}$$

Using (17), one can show that

$$\lim_{L\to\infty} \langle u_L, V u_L\rangle = 0. \tag{23}$$

Hence

$$\|H_+^{1/2} u_L\|^2 = \langle u_L, H u_L\rangle - \langle u_L, H_- u_L\rangle \to 0, \quad (L\to\infty), \tag{24}$$

where we have used the fact that H_- is compact. Therefore $0 \in \sigma_{\rm ess}(H_+^{1/2})$. This means that $0 \in \sigma_{\rm ess}(H)$. Therefore there exists a sequence $\{v_n\}_{n=0}^\infty \subset C_0^\infty(\mathbb{R}^d)$ such that

$$\|v_n\| = 1, \quad v_n \overset{\rm w}{\to} 0(n\to\infty), \quad \|Hv_n\| \to 0(n\to\infty). \tag{25}$$

It is easy to see that $\langle v_n, H_0 v_n\rangle$ is uniformly founded. By this fact, a suitable subsequence $\{H_0^{1/2} v_{n_j}\}_{j=0}^{\infty}$ has a weak limit. Since $v_n \overset{\mathrm{w}}{\to} 0$, we obtain $H_0^{1/2} v_{n_j} \overset{\mathrm{w}}{\to} 0 (j \to \infty)$. Thus, by using [4, Theorem 8.6], $\chi_R v_{n_j}$ converges in norm. By (17), we have

$$(1 - b(R))\langle v_{n_j}, H_0 v_{n_j}\rangle \le |\langle v_{n_j}, H v_{n_j}\rangle| + |\langle v_{n_j}, \chi_R V v_{n_j}\rangle| + a(R), \quad (R > 0).$$

Therefore, $H_0^{1/2} v_{n_j} \overset{\mathrm{s}}{\to} 0\,(j \to \infty)$. For each $k \in \mathbb{R}^d$, we set

$$w_j(x) = e^{ik\cdot x} v_{n_j}(x), \quad j = 0, 1, 2, \dots. \tag{26}$$

Then, $\{w_j\}_{j=1}^{\infty}$ satisfy following:

$$\{w_j\}_{j=1}^{\infty} \subset C_0^{\infty}(\mathbb{R}^d), \quad \|w_j\| = 1, \quad w_j \overset{\mathrm{w}}{\to} 0(j \to \infty). \tag{27}$$

It is not so hard to see that

$$\|(H - k^2) w_j\| = \|H v_{n_j}\| + 2|k| \|H_0^{1/2} v_{n_j}\| \to 0 \ (j \to \infty). \tag{28}$$

Since $k \in \mathbb{R}^d$ is arbitrary, we obtain $\sigma_{\mathrm{ess}}(H) \supset [0, \infty)$. □

3 Bounds for Eigenvalue Sums

We assume the following:

[V.2] In the case $d = 1$, there exist constants $\mathsf{R}_2 < \mathsf{R}_1$ such that

$$\lim_{x\to\infty} \int_{\mathsf{R}_1}^{x} V(y)\mathrm{d}y \in [0, \infty), \quad \lim_{x\to-\infty} \int_{x}^{\mathsf{R}_2} V(y)\mathrm{d}y \in [0, \infty). \tag{29}$$

In the case $d \ge 2$, there exists a constant $\mathsf{R} \ge 0$ such that for almost every $\theta \in S_d$,

$$\lim_{r\to\infty} \int_{\mathsf{R}}^{r} V(r\theta)\mathrm{d}r \in [0, \infty). \tag{30}$$

Example 3.1. The functions V_1 and V_2 in Example 2.1 satisfy [V.2].

Proof. It is enough to show [V.2] in the case $d \ge 2$. If $d \ge 2$, $\ell = 2$, and $a, b > 0$, by Fresnel's formula, we have

$$\lim_{r\to\infty} \int_0^r a \sin(bs^2)\mathrm{d}s = \lim_{r\to\infty} \int_0^r a\cos(bs^2)\mathrm{d}s = \sqrt{\frac{\pi a^2}{8b}} > 0. \tag{31}$$

Therefore [V.2] holds with $\mathsf{R} = 0$. In the case $a < 0$, $b > 0$, it is easy to see that

$$-\int_{\sqrt{\pi/b}}^{\infty} \sin br^2 \mathrm{d}r > 0, \tag{32}$$

$$-\int_{\sqrt{\pi/2b}}^{\infty} \cos br^2 \mathrm{d}r > 0. \tag{33}$$

Therefore [V.2] holds with $\mathsf{R} = \sqrt{\pi/b}$ or $\mathsf{R} = \sqrt{\pi/2b}$. In the case $\ell > 2$, it is not so hard to see that

$$\int_{0}^{\infty} \sin r^\ell \mathrm{d}r \geq 0, \qquad \int_{(2\pi)^{1/\ell}}^{\infty} \sin r^\ell \mathrm{d}r \leq 0 \tag{34}$$

$$\int_{(\pi/2)^{1/\ell}}^{\infty} \cos r^\ell \mathrm{d}r \leq 0, \qquad \int_{(3\pi/2)^{1/\ell}}^{\infty} \cos r^\ell \mathrm{d}r \geq 0. \tag{35}$$

This means that [V.2] holds with $\mathsf{R} = 0, (2\pi)^{1/\ell}, (\pi/2)^{1/\ell}, (3\pi/2)^{1/\ell}$. □

For $d \geq 2$, V, and R satistying [V.2], we define

$$\bar{W}(\theta) := \lim_{r\to\infty} W(\mathsf{R}, r; \theta), \tag{36}$$

$$\widetilde{V}(r\theta) := |\bar{W}(\theta) - W(\mathsf{R}, r; \theta)|(1 - \chi_{\mathsf{R}}). \tag{37}$$

For a self-adjoint operator T, we set

$$E_n(T) := \sup_{\phi_1,\dots,\phi_{n-1}} \inf_{\substack{\psi\in D(T);\|\psi\|=1 \\ \psi\in[\phi_1,\dots,\phi_{n-1}]^\perp}} \langle \psi, T\psi\rangle, \tag{38}$$

where $[\phi_1, \dots, \phi_{n-1}]^\perp$ is a shorthand for $\{\psi | \langle \psi, \phi_i\rangle = 0, i = 1, \dots, n-1\}$. By the min-max principle([8, Theorem XIII.1]), $E_n(T)$ is nth eigenvalues below the bottom of the essential spectrum of T or the bottom of the essential spectrum.

Our main theorem is:

Theorem 3.2. *Let $d \geq 2$. Suppose that V satisfies condition [V.1] and [V.2]. Assume that*

$$\int_{\mathbb{R}^d} \left|\widetilde{V}/r\right|^{\gamma+d/2} \mathrm{d}x + \int_{\mathbb{R}^d} \left|\widetilde{V}\right|^{2\gamma+d} \mathrm{d}x < \infty, \tag{39}$$

where $\gamma > 0$ for $d = 2$ and $\gamma \geq 0$ for $d \geq 3$. Then,

$$\sum_{n\geq 0} |E_n(H)|^\gamma \tag{40}$$

$$\leq L_{\gamma,d} \inf_{0<\epsilon<1} (1-\epsilon)^{-d/2} \int_{\mathbb{R}^d} \left[|\chi_{\mathsf{R}} V_-|^{\gamma+d/2} + \left| \frac{d-1}{r}\widetilde{V} + \frac{\widetilde{V}^2}{\epsilon} \right|^{\gamma+d/2} \right] \mathrm{d}x.$$

where $L_{\gamma,d}$ is a universal constant(given in [2], [3], [4, Theorem 12.4], and references therein).

In the case $d = 1$, we define

$$\widetilde{V}(x) := \begin{cases} \left| \lim_{r\to\infty} \int_x^r V(y)\mathrm{d}y \right|, & x \geq \mathsf{R}_1, \\ 0, & \mathsf{R}_1 < x < \mathsf{R}_2, \\ \left| \lim_{r\to-\infty} \int_r^x V(y)\mathrm{d}y \right|, & x \leq \mathsf{R}_2. \end{cases} \tag{41}$$

Theorem 3.3. *Let $d = 1$. Assume [V.1] and [V.2]. For a $\gamma \geq 1/2$, we assume $\widetilde{V} \in L^{2\gamma+1}(\mathbb{R})$. Then*

$$\sum_{n=0}^{\infty} |E_n(H)|^\gamma \leq L_{\gamma,1} \int_{\mathbb{R}} \left[|V_-(x)|^{\gamma+1/2} \chi_{[\mathsf{R}_1,\mathsf{R}_2]}(x) + |\widetilde{V}|^{2\gamma+1}(x) \right] \mathrm{d}x, \tag{42}$$

where $L_{\gamma,1}$ is a universal constant(given in [4, Theorem 12.4]).

Example 3.4. In the case $d = 1$, potentials V_1 and V_2 in Example 2.1 satisfy the condition

$$\widetilde{V} \in L^{2\gamma+1}(\mathbb{R}), \quad \gamma \geq \frac{1}{2}, \tag{43}$$

for all $\ell \geq 2$. In the case $d \geq 2$ and $\ell = 2$, V_1 and V_2 satisfy the condition (39) for $\gamma > 0$. In the case $d \geq 2$ and $\ell > 2$, V_1 and V_2 satisfy (39) for all $\gamma \geq 0$.

Proof. We give proof only in the case where $V = V_1$ and $a = b = 1$. If $d \geq 2$, we have

$$|\widetilde{V}(r\theta)| = (1 - \chi_{\mathsf{R}}(r)) \left| \int_r^\infty \cos s^\ell \mathrm{d}s \right| = (1 - \chi_{\mathsf{R}}(r)) \left| \int_r^\infty \frac{1}{\ell s^{\ell-1}} \frac{\mathrm{d}(\sin s^\ell)}{\mathrm{d}s} \mathrm{d}s \right|.$$

By integration by parts, we obtain

$$|\widetilde{V}(r\theta)| \leq (1 - \chi_{\mathsf{R}}(r)) \frac{2}{\ell r^{\ell-1}}. \tag{44}$$

Therefore

$$\int_{\mathbb{R}^d} |\widetilde{V}/r|^{\gamma+d/2} \mathrm{d}x \leq \left(\frac{2}{\ell}\right)^{\gamma+d/2} \Theta(S_d) \int_{\mathsf{R}}^\infty \left(\frac{1}{r}\right)^{\ell(\gamma+d/2)-d+1} \mathrm{d}r,$$

$$\int_{\mathbb{R}-d} |\widetilde{V}|^{2\gamma+d} \mathrm{d}x \leq \left(\frac{2}{\ell}\right)^{2\gamma+d} \Theta(S_d) \int_{\mathsf{R}}^\infty \left(\frac{1}{r}\right)^{(\ell-1)(2\gamma+d)-d+1} \mathrm{d}r.$$

Since $\mathsf{R} = (3\pi/2)^{1/\ell}$, we obtain the desired result. □

Proof of Theorem 3.2. For almost every $\theta \in S_d$ and for all $u \in C_0^\infty(\mathbb{R}^d)$, we have

$$
\begin{aligned}
&-\int_{\mathsf{R}}^{\infty} W(\mathsf{R}, r; \theta) \frac{\mathrm{d}}{\mathrm{d}r} (r^{d-1} |u(r\theta)|^2) \mathrm{d}r \\
&= \int_{\mathsf{R}}^{\infty} (\bar{W}(\theta) - W(\mathsf{R}, r; \theta)) \frac{\mathrm{d}}{\mathrm{d}r} (r^{d-1} |u(r\theta)|^2) \mathrm{d}r + \bar{W}(\theta) \mathsf{R}^{d-1} |u(\mathsf{R}\theta)|^2 \\
&\geq \int_{\mathsf{R}}^{\infty} (\bar{W}(\theta) - W(\mathsf{R}, r; \theta)) \frac{\mathrm{d}}{\mathrm{d}r} (r^{d-1} |u(r\theta)|^2) \mathrm{d}r. \\
&\geq -\int_{\mathsf{R}}^{\infty} \widetilde{V}(r\theta) \left[(d-1) r^{d-2} |u(r\theta)|^2 + 2 r^{d-1} |\mathrm{d}u(r\theta)/\mathrm{d}r| |u(r\theta)| \right] \mathrm{d}r,
\end{aligned}
$$

where we have used condition [V.2]. By using equation (12) and (14), for any $\epsilon > 0$ we obtain

$$
\langle u, Vu \rangle \geq \langle u, \chi_{\mathsf{R}} V u \rangle - \epsilon \langle u, H_0 u \rangle - \left\langle u, \left[\frac{d-1}{r} \widetilde{V} + \frac{\widetilde{V}^2}{\epsilon} \right] u \right\rangle. \tag{45}
$$

Therefore, for all $u \in D(H_0)$, we have

$$
\langle u, Hu \rangle \geq (1-\epsilon) \langle u, H_0 u \rangle - \left\langle u, \left[\chi_{\mathsf{R}} |V_-| + \frac{d-1}{r} \widetilde{V} + \frac{\widetilde{V}^2}{\epsilon} \right] u \right\rangle. \tag{46}
$$

Thus we can apply [4, Theorem 12.4] to obtain

$$
\begin{aligned}
&\sum_{n \geq 0} |E_n(H)|^\gamma \\
&\leq (1-\epsilon)^\gamma \sum_{n \geq 0} \left| E_n \left(H_0 - \frac{1}{1-\epsilon} \left[\chi_{\mathsf{R}} |V_-| + \frac{d-1}{r} \widetilde{V} + \frac{\widetilde{V}^2}{\epsilon} \right] \right) \right|^\gamma \\
&\leq L_{\gamma,d} (1-\epsilon)^{-d/2} \int_{\mathbb{R}^d} \left[|\chi_{\mathsf{R}} V_-|^{\gamma+d/2} + \left| \frac{d-1}{r} \widetilde{V} + \frac{\widetilde{V}^2}{\epsilon} \right|^{\gamma+d/2} \right] \mathrm{d}x,
\end{aligned}
$$

for any $0 < \epsilon < 1$. □

Proof of Theorem 3.3. Similar to the proof of Theorem 3.2 □

Acknowledgements

The author is grateful to Professor A. Arai of Hokkaido university for discussions and helpful comments.

References

[1] A. Arai, Kunimitsu Hayashi, and Itaru Sasaki, Spectral Properties of a Dirac Operator in the Chiral Quark Soliton Model, preprint, 2004.

[2] D. Hundertmark, A. Laptev, and T. Weidl, New bounds on the Lieb-Thirring constants, *Invent. math.* **140** (2000), 693-704.

[3] A. Laptev, and T. Weidl, Sharp Lieb-Thirring inequalities in high dimensions, *Acta Math.* **184** (2000), 87-111.

[4] E. H. Lieb and M. Loss, Analysis, Graduate Studies in Mathematics, American Mathematical Society, 2nd edition, 2001.

[5] E. H. Lieb and W. Thirring, Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities, Studies in Math. Phys, Essays in Honor of Valentine Bargmann, Princeton, (1976), 269-303.

[6] M. Reed and B. Simon, Methods of Modern Mathematical Physics Vol. I, Academic Press, New York, 1972.

[7] M. Reed and B. Simon, Methods of Modern Mathematical Physics Vol. II, Academic Press, New York, 1975.

[8] M. Reed and B. Simon, Methods of Modern Mathematical Physics Vol. IV, Academic Press, New York, 1978.