A New Constant Arising in Non-linear Maps

Milton M. Chowdhury
16 Caledonian Avenue, FY3 8RB, England,U.K.
The University of Manchester Institute of Science and Technology

Abstract

We present a new constant which arises in non-linear maps. We can show (unlike other constants in mathematical physics) that this new constant is irrational.

Key Words

Non-linear map, period-doublings

1. Theorem 1 - Result On Full Logistic Map

Define $u(x)$ by

$$u(x) = 1 \; if \, x < 0$$
$$u(x) = 0 \; if \; x \geq 0$$

The logistic map [2] which is $x_{n+1} = \lambda x_n(1 - x_n),\ \lambda \in R$ has been investigated we give results for $\lambda = 4$ this is the the full logistic map. The full logistic map has the non-recursive representation $x_n = \frac{(1-\cos(2^n \arccos(1-2x_0)))}{2}$ [3]. It is known that $u(\cos 2^{n-1})$, $n \geq 1$ is the nth bit of $\frac{1}{\pi} BitXor \frac{1}{2\pi}$ see [4] and appendix. We now give theorem 1 that with the full logistic map and $x_0 = \frac{1-\cos 1}{2}$ then

$$\sum_{n=0}^{\infty} \frac{1}{2^n} u(\frac{1}{2} - x_n) = \frac{1}{2\pi} BitXor \frac{1}{\pi}$$

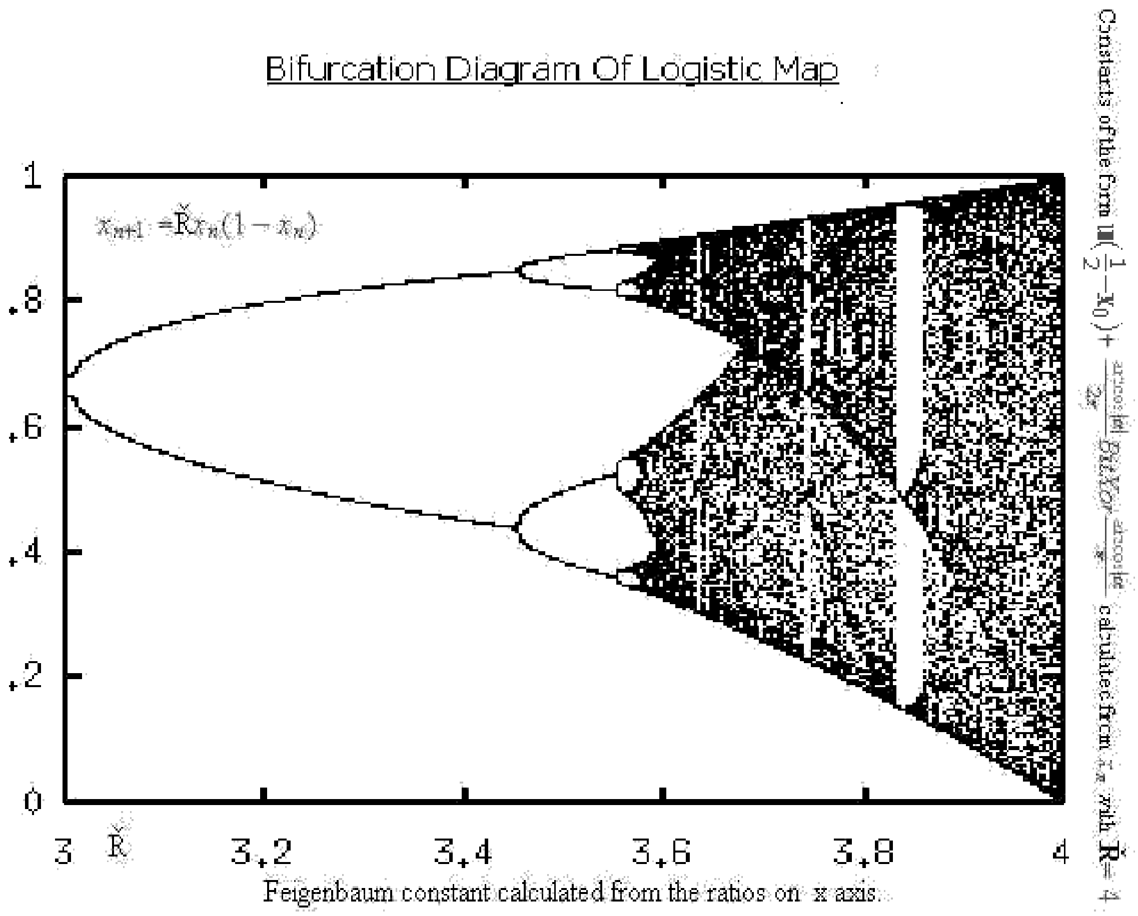
and more generally it can be shown with $x_0 = \frac{1-\alpha}{2}$ and $|\alpha| \leq 1$ then

$$\sum_{n=0}^{\infty} \frac{1}{2^n} u(\frac{1}{2} - x_n) = \frac{u(\frac{1}{2} - x_0)}{2} + \frac{\arccos|\alpha|}{2\pi} BitXor \frac{\arccos|\alpha|}{\pi} \; if \; 0 < |\alpha| \leq 1$$
$$= u(0) + \frac{1}{2} = \frac{1}{4} if \, \alpha = 0$$

.

The birfucation diagram below shows the relation of Feigenbaum constant [1] and constants of the form $\frac{u(\frac{1}{2}-x_0)}{2} + \frac{\arccos|\alpha|}{2\pi} BitXor \frac{\arccos|\alpha|}{\pi}$ this gives a solution of to the problem we consider of relating known constants to Feigenbaum constant. We can show constants of the form

$\frac{u(\frac{1}{2}-x_0)}{2} + \frac{\arccos|\alpha|}{2\pi} BitXor \frac{\arccos|\alpha|}{\pi}$ are irrational we sketch the proof, it is known $\cos(p\pi) = q$ when p & q are rational $p = 0, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm\frac{2}{3}$ and 1 and $q = 1, \frac{1}{2}, 0, -\frac{1}{2}$ and -1 respectively are the only possible values see [6]. We immeadiatly see that $\frac{\arccos p}{\pi} = q$ is irrational for nearly all p. It can be shown by using properties of binary expansions $aBitXor\frac{a}{2}$ is rational if a is rational.



More generally we define a condition for a systems of iteratons $\gamma_{n,1}, \gamma_{n,2}, ..., \gamma_{n,m}$ to be possibly chaotic if at bifurcation parameters $\lambda_1, \lambda_2, ..., \lambda_m$

$$\sum_{n=0}^{\infty} \frac{\Psi(u(g_1(\gamma_{n,1})), u(g_2(\gamma_{n,2})), ..., u(g_m(\gamma_{n,m})), \lambda_1, \lambda_2, ..., \lambda_n)}{(b+1)^n} = C$$

if C is a different (recognisable) (ir)rational for nearly all different $\gamma_{0,1}, \gamma_{0,2}, ..., \gamma_{0,m}$
for real valued functions $g_1, g_2, ..., g_m$ and an integer valued function Ψ this definition can be based on the work in [4], $b \in N^+$.
The result given above can recast as, if $x_{n+1} = 4x_n(1-x_n)$, and $x_n = \frac{(1-\cos(2^n \arccos(1-2x_0)))}{2}$ then $x_0 = \frac{1}{2} - \frac{1}{2}\beta_0$ and $x_n = \frac{1}{2} - \frac{1}{2}\beta_n$ with $\beta_{n+1} = 2\beta_n^2 - 1$, $-1 \leq \beta_0 \leq 1$, $n \geq 0$, with $\beta_0 = \delta$, $-1 \leq \delta \leq 1$ we have

$$\sum_{n=0}^{\infty} \frac{u(\beta_n)}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{u(1-2x_n)}{2^{n+1}} = \theta$$

$$\theta = \frac{\arccos(\delta)}{2\pi} BitXor \frac{\arccos(\delta)}{\pi} \text{ if } 0 < \delta \leq 1$$

$$\theta = \frac{1}{2} + \frac{\arccos(|\delta|)}{2\pi} BitXor \frac{\arccos(|\delta|)}{\pi} \text{ if } -1 \leq \delta < 0$$

$$\theta = \frac{1}{4} = u(0) + \frac{1}{4} \text{ if } \delta = 0$$

The set of points generated by the iteration $z_{n+1} = z_n^2$ with $z_n \in C$ and $|z_0| = 1$ generate points of a Julia set. This set is closely related to the Mandelbrot set [7]. Theorem 2, if z_0=cos α+isin α then $u(\mathrm{Im}(z_n))$, $u(\mathrm{Re}(z_n))$ correspond to bits of $\frac{\alpha}{\pi}$ and $\frac{\alpha}{2\pi} BitXor \frac{\alpha}{\pi}$ this can be shown.

2. Generalisations

The series $\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u(\sin 2^n) = \frac{1}{2\pi}$ and $\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u(\tan 2^n) = \frac{1}{\pi}$ are given in [8], [9]. Replacing tan by cos gives $\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u(\cos 2^n) = \frac{1}{\pi} BitXor \frac{1}{2\pi}$ see [4]. Now define $a(n,x_k)$ below recursively, $k \in N$. x is understood to be x_k for some k. $a(n,x_k) = a_n$ with initial value x_k we use similiar definitions for $b(n,x_k) = b_n$ etc.

$$\begin{aligned} a(n,x_k) &= \sin(2^n \arcsin(a_0)) \\ &= a_0 \ = x_k \text{ if } n = 0 \ 0 < x_k < 1 \\ &= 2a_0\sqrt{1-a_0^2} \text{ if } n = 1 \\ &= 2a_{n-1}(1-2a_{n-2}) \text{ if } n \geq 2 \end{aligned}$$

this recursive definition and the similar ones that follow are derived using the double angle formulae for tan, sine and cos etc. Then

$$\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u(a(n,x_k)) = \frac{\arcsin(x_k)}{2\pi}$$

see [8]. Define b_n by

$$\begin{aligned} b(n,x_k) &= \cos(2^n \arccos(b_0)) \\ &= x_k = b_0 \ 0 < x_k < 1 \text{ if } n = 0 \\ &= 2b_{n-1}^2 - 1 \text{ if } n \geq 1 \end{aligned}$$

then it can be shown (for example a proof based on theorem 1) that

$$\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u(b(n,x_k)) = \frac{\arccos(x_k)}{\pi} BitXor \frac{\arccos(x_k)}{2\pi}$$

which is a result. Define the Plouffe recursion [9] with c_n by

$$
\begin{aligned}
c(n,x_k) &= \tan(2^n \arctan(c_0)) \\
&= c_0 = x_k \text{ if } n = 0 \\
&= \frac{2c_{n-1}^2}{1-c_{n-1}^2} \text{ if } n \geq 1,\ |c_k| \neq 1 \\
&= -\infty \text{ if } n \geq 1, |c_k| = 1
\end{aligned}
$$

we consider $0 < x_k < 1$

$$
\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u(c(n,x_k)) = \frac{\arctan(x_k)}{\pi}
$$

see [8]. Let $d(n,x), e(n,x), f(n,x)$ be the analogous recursions for sec, csc and cot respectively obtained by using the double angle formula so for example for $d(n,x)$ we have

$$
\begin{aligned}
d(n,x_k) &= x_k = d_0 = 0 < x_k < 1 \text{ if } n = 0 \\
&= \frac{1}{-1 + \frac{2}{d_{n-1}^2}} \text{ if } n \geq 1
\end{aligned}
$$

then it can be shown that

$$
\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u(d(n,x_k)) = \frac{\text{arcsec}(x_k)}{\pi} BitXor \frac{\text{arcsec}(x_k)}{2\pi}
$$

Now define

$$
\sum_{\forall n} BitXor f(v_n) = f(v_1) BitXor f(v_2) ... f(v_{n-1}) BitXor (v_n)
$$

and

$$
\sum_{\forall n} \overset{\wedge}{BitXor}\ f(v_n) = f(v_1) BitXor \frac{f(v_1)}{2} ... f(v_n) BitXor \frac{f(v_n)}{2}
$$

$$
\sum_{\forall n} \overset{\wedge\wedge}{BitXor}\ f(v_n) = f(v_1) BitXor \frac{f(v_1)}{2} BitAnd ... BitAnd f(v_n) BitXor \frac{f(v_n)}{2}
$$

$$
\sum_{\forall n} \overset{\wedge\wedge\wedge}{BitXor}\ f(v_n) = f(v_1) BitXor \frac{f(v_1)}{2} BitOr ... BitOr f(v_n) BitXor \frac{f(v_n)}{2}
$$

Lemma-If $v_1, v_2, ..., v_n \in [-1,1]$ then $u(v_1 v_2 ... v_n) = u(v_1)\ Xor\ u(v_2) ... Xor\ u(v_n)$

by the above lemma we have Theorem 3

$$
\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u(\prod_{\forall A} a(n,x_A) \prod_{\forall B} b(n,x_B) \prod_{\forall C} c(n,x_C) \prod_{\forall D} (d(n,x_D) \prod_{\forall E} e(n,x_E) \prod_{\forall F} f(n,x_F)) =
$$

$(\sum_{\forall A} BitXor \frac{1}{\pi}\arcsin(x_A)) BitXor (\sum_{\forall B} \overset{\wedge}{BitXor} \frac{1}{\pi}\arccos(x_B)) BitXor (\sum_{\forall C} BitXor$
$\frac{1}{\pi}\arctan(x_C)) BitXor (\sum_{\forall D} \overset{\wedge}{BitXor} \frac{1}{\pi}\operatorname{arcsec}(x_D)) BitXor (\sum_{\forall E} BitXor \frac{1}{\pi}\operatorname{arccsc}(x_E))$
$BitXor (\sum_{\forall F} BitXor \frac{1}{\pi}\operatorname{arccot}(x_F))$

Now consider a variant of theorem 3 the recursions involving cos only we have

$$\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u(\prod_{\forall B} b(n, x_B)) = \sum_{\forall B} \overset{\wedge}{BitXor} \frac{1}{\pi}\arccos(x_B)$$

Consider the system of logistic equations $T_{n+1,k} = 4T_{n,k}(1 - T_{n,k}), n \geq 0$.
We will demonstrate that for a system of logistic equation the system gives
$\frac{\alpha}{\pi}$ to an given number binary digits from iterations of the system. We have
$T_{n,k} = \frac{1}{2}(1 - \cos(2^n \arccos(1 - 2T_{0,k}))$
$T_{n,k} = \frac{1}{2} - \frac{1}{2}b(n, x_k)$ for $n \geq 0$
let $1 \leq k \leq z$ (so we have a system of z logistic equations).
Let $x_k = \cos(\frac{\alpha}{2^k}) \in [-1, 1],\ \alpha \in [0, 2\pi]$ or let
$x_k = \cos(m_k\alpha) \in [-1, 1],\ \alpha \in [0, 2\pi],$ with $m_k = 1$ or $m_k = \frac{1}{2^v}, v = 1, 2, 3, \ldots$
Let $g = \sum_{n=0}^{z} u(x_n)$, By properties of $BitXor$,
$\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u(\prod_{k=1}^{z}(1 - 2T_{n,k})) = \frac{\alpha}{\pi} BitXor \frac{\alpha}{2^z\pi} = \sum_{\forall B} \overset{\wedge}{BitXor} \frac{\alpha}{2^B\pi}$

we have theorem 4 (we omit the proof)

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u(\prod_{k=1}^{z}(1 - 2T_{n,k})) &= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u(\prod_{k=1}^{z} b(n, x_k)) \\
&= \frac{\alpha}{\pi} BitXor \frac{\alpha}{2^z\pi} \text{ if } \forall k,\ 0 < x_k \leq 1 \\
&= \frac{\alpha}{\pi} BitXor \frac{\alpha}{2^z\pi} \text{ if } \forall k, x_k \neq 0, -1 \leq x_k < 1 \ \&\ g \text{ even} \\
&= \frac{1}{2} + \frac{\alpha}{\pi} BitXor \frac{\alpha}{2^z\pi} \text{ if } \forall k, x \neq 0, -1 \leq x_k < 1 \ \&\ g \text{ odd} \\
&= \frac{1}{4} \text{if } \forall k,\ x_k = 0 \ \&\ z \text{ odd} \\
&= 0 \text{ if } \forall k,\ x_k = 0 \ \&\ z \text{ even} \\
&= \frac{1 - (-1)^z}{8} + u(-|V|)(\frac{\alpha}{\pi} BitXor \frac{\alpha}{2^z\pi} + \\
&\quad u(V)\frac{1 - (-1)^g}{4} - \frac{1 - (-1)^z}{8}) \\
V &= \arg\min_{\forall k} x_k
\end{aligned}$$

we can now give Theorem 5 it can be shown that

$$\lim_{z\to\infty}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}u(\prod_{k=1}^{z}(1-2T_{n,k})) = \lim_{z\to\infty}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}u(\prod_{k=1}^{z}b(n,x_k))$$
$$= \frac{\alpha}{\pi} \text{ if } \forall k,\ 0 < x_k \le 1$$
$$= \frac{\alpha}{\pi} \text{ if } \forall k, x_k \ne 0, -1 \le x_k < 1 \ \& \ g \text{ even}$$
$$= \frac{1}{2} + \frac{\alpha}{\pi} \text{ if } \forall k, x \ne 0, -1 \le x_k < 1 \ \& \ g \text{ odd}$$
$$= \frac{1}{4} \text{if } \forall k,\ x_k = 0 \ \& \ z \text{ odd}$$
$$= 0 \text{ if } \forall k,\ x_k = 0 \ \& \ z \text{ even}$$
$$= \frac{1-(-1)^z}{8} + u(-|V|)(\frac{\alpha}{\pi} + u(V)\frac{1-(-1)^g}{4} - \frac{1-(-1)^z}{8})$$
$$V = \arg\min_{\forall k} x_k$$

We now give two variants of theorem 4 (and we could generalise these theorems to based on theorem 3 to obtain two variants of theorem 3.)

Lemmas, If $v_1, v_2, ..., v_n \in [-1,1]$ then $u(v_1)u(v_2)...u(v_n) = u(v_1)\ And...And\ u(v_n)$

If $v_1, v_2, ..., v_n \in [-1,1]$ then $u(-(u(v_1)+u(v_2)...u(v_n))) = u(v_1)\ Or...Or\ u(v_n)$

hence by properties of *BitOr* and *BitAnd* we have theorem 6

$$\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\prod_{k=1}^{z}u(1-2T_{n,k}) = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\prod_{k=1}^{z}u(b(n,x_k)) = \sum_{n=1}^{n=z+1}\overset{\wedge\wedge}{BitXor}\ \frac{\alpha}{2^{n-1}\pi}$$

$$\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}u(\sum_{k=1}^{z}-u(1-2T_{n,k})) = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}u(\sum_{k=1}^{k=z}-u(b(n,x_k))) = \sum_{n=1}^{n=z+1}\overset{\wedge\wedge\wedge}{BitXor}\ \frac{\alpha}{2^{n-1}\pi}$$
$$= \sum_{n=1}^{n=z+1}\overset{\wedge}{BitXor}\ \frac{\alpha}{2^{n-1}\pi}$$
$$= \frac{1-(-1)^z}{8} + u(-|V|)(\sum_{n=1}^{n=z+1}\overset{\wedge}{BitXor}\ \frac{\alpha}{2^{n-1}\pi} + u(V)\frac{1-(-1)^g}{4} - \frac{1-(-1)^z}{8})$$
$$V = \arg\min_{\forall k} x_k$$

now define

$T_{n,k} = \frac{1}{2} - \frac{1}{2}b''(n,x_k)$ for $n \ge 0$ with $b''(n,x_k) = (-1)^k b(n,x_k)$ we give theorem 7, $z \ge 1, k \ge 0$,

$x_k = \cos((\frac{1}{2}(1+(-1)^k)+1)(\frac{\alpha}{2\pi}\left(-\frac{1}{2}\right)^k - \frac{1}{3}\left(-\frac{1}{2}\right)^k + \frac{1}{3}))$

$$\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\prod_{k=1}^{z}u(1-2T_{n,k}) = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\prod_{k=1}^{z}u(b''(n,x_k)) = \sum_{n=1}^{n=z+1}\overset{\wedge\wedge}{BitXor}\ \frac{\alpha}{2^{n-1}\pi}$$

$$\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}u(-\sum_{k=1}^{z}u(1-2T_{n,k})) = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}u(-\sum_{k=1}^{k=z}u(b''(n,x_k))) = \sum_{n=1}^{n=z+1}\overset{\wedge\wedge\wedge}{BitXor}\ \frac{\alpha}{2^{n-1}\pi}$$

$$= (\frac{1}{2} \pm \frac{-1}{2}) \pm (\frac{1-(-1)^z}{8} + u(-|V|)(\frac{\alpha}{\pi}BitXor(\frac{\alpha}{2\pi}\left(-\frac{1}{2}\right)^z - \frac{1}{3}\left(-\frac{1}{2}\right)^z + \frac{1}{3}) + u(V)\frac{1-(-1)^g}{4} - \frac{1-(-1)^z}{8}))$$

$$V = \arg\min_{\forall k} x_k$$

we can now give theorem 8

$$\lim_{z\to\infty}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\prod_{k=1}^{z}u(1-2T_{n,k}) = \lim_{z\to\infty}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\prod_{k=1}^{z}u(b''(n,x_k)) = \lim_{z\to\infty}\sum_{n=1}^{n=z+1}\overset{\wedge\wedge}{BitXor}\ \frac{\alpha}{2^{n-1}\pi} =$$

$$\lim_{z\to\infty}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}u(-\sum_{k=1}^{z}u(1-2T_{n,k})) = \lim_{z\to\infty}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}u(-\sum_{k=1}^{k=z}u(b''(n,x_k))) = \lim_{z\to\infty}\sum_{n=1}^{n=z+1}\overset{\wedge\wedge\wedge}{BitXor}\ \frac{\alpha}{2^{n-1}\pi}$$

$$= (\frac{1}{2} \pm \frac{-1}{2}) \pm \frac{1-(-1)^z}{8} + u(-|V|)(\frac{\alpha}{\pi}BitXor\frac{1}{3} + u(V)\frac{1-(-1)^g}{4} - \frac{1-(-1)^z}{8})$$

$$V = \arg\min_{\forall k} x_k$$

3. A Method for S.o.i.c.

If we can show that if the R.H.S. of theorem 3 is (ir)rational etc. depending on the initial conditions then this gives a method to detect chaos - sensitivity of initial conditions (s.o.i.c.) we can show detection of s.o.i.c. for some sequence. We give the method below after first proving giving theorems A and B below.

Theorem A

There are two representations of $\beta = \frac{F}{2}BitXorF$
with $F \in [0,1]$ with the two values of $F = \alpha_1, \alpha_2$
$\alpha_1 \in [0,1]$ and $\alpha_2 = 1 - \alpha_1$.

Proof

See appendix.

Theorem B

$\frac{\arccos|\alpha|}{2\pi} BitXor \frac{\arccos|\alpha|}{\pi}$ is irrational when $\frac{\arccos|\alpha|}{2\pi}$ is irrational.

Proof

This is straight foward by the proof titled 'If $\beta = \frac{\alpha}{2} BitXor \alpha$ is....' (see appendix pages 13-16) that $\frac{\arccos|\alpha|}{2\pi} BitXor \frac{\arccos|\alpha|}{\pi}$ is irrational when $\frac{\arccos|\alpha|}{\pi}$ is irrational. $\frac{\arccos|\alpha|}{\pi}$ is strictly monotonic decreasing on [0,1] this will allow us to use Theorem A. We know the values for which $\frac{\arccos|\alpha|}{\pi}$ is rational for rational α this shows that $\frac{\arccos|\alpha|}{2\pi} BitXor \frac{\arccos|\alpha|}{\pi}$ is rational for the rational values.

Proof of s.o.i.c. for the logistic equation

We are measuring the behavior of the iterates by looking at the signs of the iterates.

Define $f(\alpha) = -\frac{u(\frac{1}{2} - x_0)}{2} + \sum_{n=0}^{\infty} \frac{1}{2^n} u(\frac{1}{2} - x_n) = \frac{\arccos|\alpha|}{2\pi} BitXor \frac{\arccos|\alpha|}{\pi}$

with $x_0 = \frac{1-\alpha}{2}$ and $|\alpha| \leq 1$ for iterates of the logistic equation.

We consider $0 < \alpha \leq 1$ hence $x_0 > \frac{1}{2}$. We have to show that for small $e \in (k, l)$ that $f(\alpha \pm e) \neq f(\alpha)$ equality occurs by using theorem A when

$\frac{\arccos(\alpha)}{2\pi} BitXor \frac{\arccos(\alpha)}{\pi} = (1 - \frac{\arccos(\alpha)}{2\pi}) BitXor (1 - \frac{\arccos(\alpha)}{\pi})$

hence we should not choose $e = \cos(\pi - \arccos(\alpha)) = \theta$.

Hence we set arbitrary small k, l for α in $f(\alpha)$ such that $k < \alpha < l$ and $\theta \notin (k, l)$ we see that $k - l < e < l - \alpha$.

By theorem B we see that for all such e above that $f(\alpha \pm e)$ is a different (ir)rational and because $\frac{\arccos(\alpha)}{\pi}$ is strictly monotonic decreasing on $[0, 1]$ this shows we have detected s.o.i.c. for $x_0 > \frac{1}{2}$.

A very similar proof shows we have detected s.o.i.c. for $x_0 < \frac{1}{2}$.

For $x_0 = \frac{1}{2}$, we know that $f(0)$ is rational, we know all the values of rational x_0 such that $f(x_0)$ is rational. Now we set arbitrary k, l such that for any $e \in (k, l)$ then $f(e)$ isirrational. Then because $f(e)$ is irrational this gives a different type of behaviour for the iterates this shows we have detected s.o.i.c. the same method of proof works for any rational values of x_0. It can be shown by using the the proof titled'If $\beta = \frac{\alpha}{2} BitXor \alpha$ is....' twice that $\frac{\alpha}{\pi} BitXor \frac{\alpha}{4\pi}$ is irrational.

Now consider theorem 4

with $z = 2 \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} u(\prod_{k=1}^{2}(1-2T_{n,k})) = \frac{\alpha}{\pi} BitXor \frac{\alpha}{4\pi}$

we can extend the above proof to show s.o.i.c. for the sequence $\{\prod_{k=1}^{2}(1-2T_{n,k})\}_{n=1}^{n=\infty}$. The logistic equation

$y_{n+1} = Py_n(1-\frac{y_n}{k})$ can be derived from

$\frac{dy}{dt} = ry(1-\frac{y}{K})$ with $\frac{dy}{dt} = (y_{n+1}-y_n)/h$ hence

the above resutls in section 1&2 can be used to estimate

the solutions of a system of one or more logistic equations.

.

References

[1] Feigenbaum, M. J., The Universal Metric Properties of Nonlinear Transformations, *J. Stat. Phys.*, J. Stat. Phys. 21, 1979, 669-706

[2] J. Glieck, *Chaos*: *Making a New Science*, Viking Penguin, 1987.

[3] H.G. Schuster, *Deterministic Chaos*, VCH Weinhein, 1989.

[4] M. M. Chowdhury, New Algorithm to Calculate Products Involving $\frac{1}{\pi}$, Unpublished note, 2002

[5] Mathworld, http://mathworld.wolfram.com/FeigenbaumConstant.html

[6] Niven, Zuckerman and Montgomery, *An Introduction To The Theory of Numbers*, 5th Edition,Wiley,1991

[7] H. Jurgens, *Chaos and Fractals*. *New frontiers of science*, Springer Verlag, 1992.

[8] J. Ardnt and C. Haenel, *π Unleashed*, Springer, 2000.

[9] J. M. Borwien and R. Girgensohn, Addition Theorem and Binary Expansions, *Cand. J. Math*. 47, 1995, 262-273

Appendix

Proof of Closed Form of The Series $\sum_{n=0}^{\infty}(2^{-n-1})$u[cos($2^n$)] = 0.4756260707...

Define xor for two binary digits(bits) j and k

j	k	j or k
0	0	0
0	1	1
1	0	1
1	1	0

BitXor does the Xor operation on the nth bit of the fractional part of the numbers to give the nth bit in the result. Xor is a standard operation on two bits see [10]. The closed form of the series is $\frac{1}{2\pi}$ BitXor $\frac{1}{\pi}$ (hence binary digit in the binary expansion of $\frac{1}{\pi}BitXor\frac{1}{2\pi}$ is 1 if and only if the corresponding binary digits in the expansions of $\frac{1}{\pi}$and $\frac{1}{2\pi}$ are different).Verifying this closed form for the first 10 bits of $\sum_{n=0}^{\infty}(2^{-n-1})$u[cos($2^n$)] , $\frac{1}{2\pi}$ = 0.0010100010_2 , , $\frac{1}{\pi}$ = 0.0101000101_2 this gives 0.0111100111_2 when Xor on each of the corresponding bits of $\frac{1}{\pi}$and $\frac{1}{2\pi}$, and making the integer part zero.

Proof

The nth digit of a x∈ R in base b can be obtained by computing the fractional part of (xb^n) because it is the leading digit of this fractional part .
For cos[2πx], let x ∈ R, x>0 , let FractionalPart[x] = F[x].

F[x]∈ (,)	(0,.25)	(0.25,0.5)	(0.5,0.75)	(0.75,1)
u[cos[2πx]]	0	1	1	0

Hence the nth digit of $\frac{1}{\pi}$ in base 4 gives the values, u[Cos[2*(4^n)*($\frac{1}{\pi}$)*π]] = u[Cos[2*(4^n)]] ,(n=0 corresponds to the 1st digit after the base 4 point, the first digit after the base 4 point of x is defined as Floor[4*F[x]]), (n≥0) are 0 if the nth digit in base 4 is 0 or 3, is 1 if the nth digit is 1 or 2. The nth digit of $\frac{1}{2\pi}$ in base 4 gives the values, u[Cos[2*(4^n)*($\frac{1}{2\pi}$)*π]] = u[Cos[(4^n)]] ,(n=0 corresponds to the 1st digit after the base 4 point), (n≥0) are 0 if the nth digit in base 4 is 0 or 3, is 1 if the nth digit is 1 or 2.
The truncated binary expansion of $\frac{1}{2\pi}$ is 0.00101000101111110011_2 with the binary expansion of $\frac{1}{\pi}$ starting at the second bit in the expansion. Using the fact that two binary digits can be represented exactly as a digit in base 4, and u[Cos[4^n]] = u[Cos[2^{2n}]] is 1 if the corresponding base 2 bits of the base 4 digits are 01 or 10 and 0 if the digits are 00 or 11 in the expansion of $\frac{1}{2\pi}$.
u[Cos[2*4^n]] = u[Cos[2^{2n+1}]] is 1 if the corresponding base 2 bits of the base 4 digits are 01 or 10 and 0 if the digits are 00 or 11 in the expansion of $\frac{1}{\pi}$.
We can see that u[Cos[2^{2n}]] can be calculated by Xor on consecutive digits on the binary expansion of $\frac{1}{2\pi}$, generally Xor on the mth and (m+1)th digits in the expansion of $\frac{1}{2\pi}$, with m=0 corresponding to the first bit after the binary point. If m is even m≥0, this gives the value u[Cos[2^{2n}]] for n≥0, If m is odd m≥1, this gives the value u[Cos[2^{2n+1}]] for n≥0,
This gives the identity with n≥0

$$u[Cos(2^n)] \;\; = u[Sin(2^n)] \; Xor \; u[Sin(2^{n+1})].$$

The above is equivalent to Xor on the nth bit in the expansion of $\frac{1}{\pi}$ and $\frac{1}{2\pi}$ to calculate u[Cos(2^n)] for n≥0, so $\frac{1}{2\pi}$ BitXor $\frac{1}{\pi}$ must be the closed form of the sum of the series.

.

Theorem A

There are two representations of $\beta = \frac{F}{2} BitXorF$
with $F \in [0,1]$ with the two values of $F = \alpha_1, \alpha_2$
$\alpha_1 \in [0,1]$ and $\alpha_2 = 1 - \alpha_1$.

Proof

Consider the two expansions below

$\alpha_1 = a_1 \quad a_2 \quad a_3 ... \quad ...a_n...$
$\beta = \quad b_1 \quad b_2 ... \quad ...b_{n-1}$

$\alpha_2 = a_1` \quad a_2` \quad a_3`... \quad ...a_n`...$
$\beta = \quad b_1 \quad b_2 ... \quad ...b_{n-1}$

We have to show that for fixed $\beta \in [0,1]$ there are two values of F we use the values of β. Given α_1 β is uniquely determined. It is sufficient to consider β irrational and it is sufficient to show there finite α_d for any real β. Now we consider all possible values of α_2. Suppose the first digit that α_2 is different from α_1 is at a_2 (so $a_2` = 1 - a_2$) we see from the definition of Xor that b_1 calcualted from α_2 is different from the b_1 calculated from $\alpha_1$hence α_2 cannot be different at a_2. We see in general that if a_p $p \geq 2$, is the first digit where α_2 differs from α_1 that b_{p-1} calcualted from α_2 is different from the b_{p-1} calculated from α_1 hence α_2 can differ from α_1 at $a_1`$
($a_1` = 1 - a_1$). Consider the table of possible digits below

a_1	a_2	b_1
0	0	0
0	1	1
1	0	1
1	1	0

hence $a_2` = 1 - a_2$. It is straight foward to show by induction that $a_p` = 1 - a_p,\ p \geq 1$ hence there are only two values of F with $\alpha_2 = 1 - \alpha_1$.

continued in four separate files