On the Recovery Of a 2-D Function From the Modulus Of Its Fourier Transform

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Abstract

A uniqueness theorem is proven for the problem of the recovery of a complex valued compactly supported 2-D function from the modulus of its Fourier transform. An application to the phase problem in optics is discussed.

1 Introduction

Let $\Omega \subset R^2$ be a bounded domain and $f(\xi,\eta) \in C^2\left(\overline{\Omega}\right)$ be a complex valued function. Consider its Fourier transform

$$F(x,y) = \iint\limits_{\Omega} f(\xi,\eta)\, e^{ix\xi} e^{iy\eta} d\xi d\eta, (x,y) \in R^2. \tag{1.1}$$

Let

$$G(x,y) = |F(x,y)|^2, (x,y) \in R^2. \tag{1.2}$$

We are interested in the question of the uniqueness of the following

Problem. *Given the function $G(x,y)$, determine the function $f(\xi,\eta)$.*

The right hand side of (1.1) can often be interpreted as an optical signal whose amplitude and phase are $|F(x,y)|$ and $\arg(F(x,y))$ respectively, see, e.g., [2]. This problem is also called the phase problem in optics (PPO) meaning that only the amplitude of such an optical signal is measured. The latter reflects the fact that it is often impossible to measure the phase in optics, except of the case when the so-called "reference" signal is present (e.g., the case of holography [18]), see, e.g., [3]-[9], [11]-[16], [19], and [20].

We assume that $\Omega = (0,1) \times (0,1)$ is a square and the function $f(\xi,\eta)$ has the form

$$f(\xi,\eta) = \exp[i\varphi(\xi,\eta)], \tag{1.3}$$

where the real valued function $\varphi \in C^4(\overline{\Omega})$. Let Γ be the boundary of the square Ω and δ be a small positive number. Denote

$$\Omega_\delta(\Gamma) = \{(\xi,\eta) \in \Omega : dist[(\xi,\eta),\Gamma] < \delta\},$$

where $dist[(\xi,\eta),\Gamma]$ is the Hausdorf distance between the point (ξ,η) and Γ. Hence, the subdomain $\Omega_\delta(\Gamma) \subset \Omega$ is a small neighborhood of the boundary Γ. The following uniqueness theorem is the main result of this paper

Theorem 1. *Assume that two functions* $f_1(\xi,\eta) = \exp[i\varphi_1(\xi,\eta)]$ *and* $f_2(\xi,\eta) = \exp[i\varphi_2(\xi,\eta)]$ *of the form (1.3) are solutions of the equation (1.2) with real valued functions* $\varphi_1, \varphi_2 \in C^4(\overline{\Omega})$ *satisfying conditions (1.4) and (1.5), where*

$$\varphi(1-\xi,\eta) = \varphi(\xi,\eta), \forall (\xi,\eta) \in \Omega, \tag{1.4}$$

$$\varphi(\xi,1-\eta) = \varphi(\xi,\eta), \forall (\xi,\eta) \in \Omega. \tag{1.5}$$

Also, assume that either

$$\varphi_{j\xi}(0,0) > 0 \;\; \textit{and} \; \varphi_{j\eta}(0,0) > 0 \;\; \textit{for} \; j = 1,2 \tag{1.6a}$$

or

$$\varphi_{j\xi}(0,0) < 0 \;\; \textit{and} \; \varphi_{j\eta}(0,0) < 0 \;\; \textit{for} \; j = 1,2. \tag{1.6b}$$

In addition, let $\varphi_1(0,0) = \varphi_2(0,0) = 0$ *and both functions* $\varphi_1(\xi,\eta)$ *and* $\varphi_2(\xi,\eta)$ *are analytic in a small neighborhood* $\Omega_\delta(\Gamma)$ *of the boundary* Γ *of the domain* Ω *as functions of two real variables* ξ,η. *Then* $\varphi_1(\xi,\eta) = \varphi_2(\xi,\eta)$ *in* $\overline{\Omega}$.

Remarks. **a.** We need conditions (1.4) and (1.5) for proofs of lemmata 2 and 7. We need conditions (1.6a,b) for the proof of Lemma 2, and, in a weaker form for the proof of Lemma 7. The condition of the analyticity of functions $\varphi_1(\xi,\eta)$ and $\varphi_2(\xi,\eta)$ in $\Omega_\delta(\Gamma)$ can be replaced with the assumption that $\varphi_1(\xi,\eta) = \varphi_2(\xi,\eta)$ in $\Omega_\delta(\Gamma)$. Such an assumption is often acceptable in the field of inverse problems. It should be pointed out that Lemma 2 does not guarantee the uniqueness in the entire domain Ω. Now, if one would assume *a priori* that $\varphi_1(\xi,\eta) = \varphi_2(\xi,\eta)$ in $\Omega_\delta(\Gamma)$ (thus focusing one the "search" of the function $\varphi(\xi,\eta)$ in the "major part" $\Omega \diagdown \Omega_\delta(\Gamma)$ of the domain Ω), then it would be sufficient for the proof of Lemma 7 to replace (1.6a,b) with $\varphi_\xi(0,0) \neq 0$, see (3.36) and (3.37).

b. To explain the assumption $\varphi_1(0,0) = \varphi_2(0,0) = 0$, we note that if the function $f(\xi,\eta)$ is a solution of the equation (1.2), then functions $f(\xi,\eta)e^{ic}$ and $\overline{f}(-\xi,-\eta)e^{ic}$ with an arbitrary real constant c are also solutions of this equation. Throughout the paper $\overline{f}$ denotes the complex conjugation. Hence, Theorem 1 can be reformulated by taking into account functions $\varphi(\xi,\eta) + c$ and $-\varphi(\xi,\eta) + c$, along with the function $\varphi(\xi,\eta)$.

Throughout the paper we assume that conditions of Theorem 1 are satisfied. Everywhere below $j=1,2$. Denote

$$F_j(x,y)=\iint\limits_{\Omega} f_j\left(\xi,\eta\right)e^{-ix\xi}e^{-iy\eta}d\xi d\eta,\ \ (x,y)\in\mathbb{R}^2, \tag{1.7}$$

$$G_j(x,y)=\left|F_j(x,y)\right|^2, \tag{1.8}$$

where $f_j\left(\xi,\eta\right)=\exp\left(i\varphi_j\left(\xi,\eta\right)\right)$. So, we need to prove that the equality

$$G_1(x,y)=G_2(x,y) \text{ in } \mathbb{R}^2 \tag{1.9}$$

implies that $f_1\left(\xi,\eta\right)=f_2\left(\xi,\eta\right)$ in Ω.

The form (1.3) is chosen for two reasons. First of all, if both functions $\left|f\left(\xi,\eta\right)\right|$ and $\arg\left[f\left(\xi,\eta\right)\right]$ would be unknown simultaneously, then (1.2) would be one equation with two unknown functions. It is unlikely that a uniqueness result might be proven for such an equation without some stringent additional assumptions. Another indication of this is an example of the non-uniqueness in section 3. Second, the representation (1.3) is quite acceptable in optics, see, e.g., [6], [12] and [13]. Derivations in these references are similar and, briefly, are as follows. Suppose that the plane $\{x_3=0\}$ in the space $\mathbb{R}^3=\{(x_1,x_2,x_3)\}$ is an opaque sheet from which an aperture Ω is cut off. Suppose that a phase screen S is placed in the aperture Ω. The "phase screen" means a thin lens which changes only the phase of the optical signal transmitted through it, but it does not change its amplitude. For each point $(x_1,x_2,0)\in\Omega$ consider the intersection of the straight line $L\left(x_1,x_2\right)$ orthogonal to the plane $\{x_3=0\}$ with S, i.e., consider $L\left(x_1,x_2\right)\cap S$. Let $\psi\left(x_1,x_2\right)$ and $n(x_1,x_2)$ be respectively the thickness and the refraction index of this intersection. Suppose, a plane wave $u_0=\exp\left(ikx_3\right)$ propagates in the half-space $\{x_3<0\}$. Consider the positive half-space $\{x_3>0\}$. Then the function $F(x,y)$ of the form (1.1), (1.3) with $\varphi\left(x_1,x_2\right):=k\left[n(x_1,x_2)-1\right]\psi\left(x_1,x_2\right)$ is approximately proportional to the wave field in the so-called Fraunhofer zone [2], i.e., with $kx_3>>1$. Hence, our problem can be viewed as an *inverse* problem of the determination of the function $\varphi\left(x_1,x_2\right)$ characterizing the phase screen from the amplitude of the scattered field measured far away from that screen. In addition, see, e.g., the paper [7], where the function $\varphi\left(x_1,x_2\right)$ is called "the aberration function (phase errors)" and its reconstruction seems to be the subject of the main interest of [7].

The function $F(x,y)$ can be continued in the complex plane $\mathbb{C}$ as an entire analytic function with respect to any of two variables x or y, while another one is kept real. It follows from the Paley-Wiener theorem [10] that the resulting function $F(z,y)$ will be an entire analytic function of the first order of the variable $z\in\mathbb{C}$. The major difference between 1-D and 2-D cases is that, unlike the 1-D case zeros of an analytic function of two or more complex variables are not necessarily isolated. For this reason, we consider below the analytic continuation of $F(x,y)$ with respect to x only and keep $y\in\mathbb{R}$. Thus, we consider the function $F(z,y), z\in\mathbb{C}, y\in\mathbb{R}$. The example of the non-uniqueness in section 3 indicates that our main effort should be focused on the proof that complex zeros of the function $F(z,y)$ can be determined uniquely for each $y \in (a, b)$, where $(a, b) \subset \mathbb{R}$ is a certain interval. This is achieved in two stages. First, we prove that "asymptotic" zeros can be determined uniquely (Lemma 7). Next, it is shown that the rest of zeros can also be uniquely determined (sections 4 and 5).

The first uniqueness theorem for the PPO was proven by Calderon and Pepinsky [5]; also see the paper of Wolf [20] for a similar result. These publications were concerned with the case of a real valued centro-symmetrical function f, which is different from our case of the complex valued centro-symmetrical function f satisfying conditions (1.3)-(1.6). The latter causes a substantial difference in proofs of corresponding uniqueness results.

Many publications discuss a variety of aspects of the PPO, see, e.g., above cited ones and references cited there; a recently published introduction to the PPO can be found in [8]. We also refer to the paper [17], which is concerned with the inverse problem of shape reconstruction from the modulus of the far field data; the mathematical statement of this problem is different from the above. A uniqueness result for the discrete case was proven in [3], where the function f is a linear combination of $\delta-$ functions. For the "continuous" 2-D case, uniqueness theorems for the problem (1.1)-(1.3) were proven in [12] and [13] assuming that $\varphi \in C^{\infty}\left(\overline{\Omega}\right)$. The goal of this publication is to replace the C^{∞} with the C^4 via exploring some new ideas. The *main* new idea is presented in section 4. It an opinion of the author that the proof of Lemma 8 of this section is the most difficult element of this paper.

The rest of the paper is devoted to the proof of Theorem 1. In section 2 we prove that the function $\varphi(x, y)$ can be reconstructed uniquely near the boundary Γ of the domain Ω. In section 3 five lemmata are proven. In section 4 one more lemma is proven. We finalize the proof of Theorem 1 in section 5.

2 Uniqueness In $\Omega_{\delta}(\Gamma)$

Results, similar with lemmata 1 and 2 of this section were proven in [13] (see lemmata 2.5-2.7 and the proof of Theorem 3.1 in this reference). However, since the reference [13] is not easily available, it makes sense to present full proofs of lemmata 1 and 2 here. In addition, these proofs are both significantly simplified and clarified compared with those of [13]. To prove that $\varphi_1 = \varphi_2$ in $\Omega_{\delta}(\Gamma)$, we need to analyze some integral equations. For any number $\varepsilon \in (0, 1)$ denote $P_{\varepsilon} = \{0 < x, y < \varepsilon\}$, a subdomain of the square Ω.

Lemma 1. *Let the number $\varepsilon \in (0,1)$. Let complex valued functions $q, K_s\,(s = 1, 2, 3)$ be such that*

$$q(x, y) \in C^2\left(\overline{P}_{\varepsilon}\right), K_1(x, y, \xi), K_2(x, y, \eta) \in C^2\left(\overline{P}_{\varepsilon} \times [0, \varepsilon]\right),$$

and $K_3(x, y, \xi, \eta) \in C^2\left(\overline{P}_{\varepsilon} \times \overline{P}_{\varepsilon}\right)$. Also, let

$$K_1(0, 0, 0) = K_2(0, 0, 0) = 1 \tag{2.1}$$

and

$$q(0, 0) = q_x(0, 0) = q_y(0, 0) = 0. \tag{2.2}$$

Suppose that the complex valued function $u(x,y) \in C^2\left(\overline{P}_\varepsilon\right)$ *satisfies the integral equation*

$$[\alpha x + \beta y + q(x,y)]\, u(x,y) =$$

$$\alpha \int_0^x K_1(x,y,\xi)\, u(\xi,y)\, d\xi + \beta \int_0^y K_2(x,y,\eta)\, u(x,\eta)\, d\eta \tag{2.3}$$

$$+ \int_0^x \int_0^y K_3(x,y,\xi,\eta)\, u(\xi,\eta)\, d\eta d\xi, \text{ in } P_\varepsilon,$$

and

$$u(0,0) = 0, \tag{2.4}$$

where α *and* β *are two real numbers such that*

$$\alpha\beta > 0. \tag{2.5}$$

Then there exists such a number $\varepsilon_0 \in (0,\varepsilon]$ *depending only on numbers* α, β *and functions* $q, K_s\,(s = 1,2,3)$ *that* $u(x,y) = 0$ *for* $(x,y) \in \overline{P}_{\varepsilon_0}$.

Note that because of the presence of the factor $[\alpha x + \beta y + q(x,y)]$ in the left hand side of the equation (2.3), this is not a standard Volterra equation. Indeed, we cannot simply divide both sides of (2.3) by $[\alpha x + \beta y + q(x,y)]$, since $[\alpha x + \beta y + q(x,y)]\,|_{x=y=0} = 0$.

Proof of Lemma 1. We first prove that there exists a number $\widetilde{\varepsilon} \in (0,\varepsilon]$ such that

$$u(x,0) = 0 \text{ and } u(0,y) = 0, \text{ for } x,y \in (0,\widetilde{\varepsilon}). \tag{2.6}$$

Set in (2.3) $y = 0$. Denote $v(x) = u(x,0), q_0(x) = q(x,0)$ and $K_0(x,\xi) = K_1(x,0,\xi) - 1$. Then

$$[\alpha x + q_0(x)]\, v(x) = \alpha \int_0^x [1 + K_0(x,\xi)]\, v(\xi)\, d\xi, x \in (0,\varepsilon).$$

Differentiating this equation with respect to x, we obtain for $x \in (0,\varepsilon)$

$$[\alpha x + q_0(x)]\, v'(x) + q_0'(x) v(x) = \alpha K_0(x,x)\, v(x) + \alpha \int_0^x K_{0x}(x,\xi)\, v(\xi)\, d\xi. \tag{2.7}$$

By (2.1) and (2.2) we have for $x \in (0,\varepsilon)$

$$K_0(x,x) = \int_0^x \frac{d\,[K_0(\xi,\xi)]}{d\xi} d\xi, \quad q_0(x) = \int_0^x q_0''(\xi)\,(x-\xi)\, d\xi, \quad q_0'(x) = \int_0^x q_0''(\xi) d\xi.$$

In particular, this means that $q_0(x) = o(x)$ as $x \to 0$. Hence, since by (2.5) $\alpha \neq 0$, then there exists a number $\varepsilon_1 \in (0, \varepsilon]$ depending on the number α and the function $q_0(x)$ such that functions

$$\frac{x}{\alpha x + q_0(x)}, \quad \widetilde{K}(x) = \frac{K_0(x, x)}{\alpha x + q_0(x)} \quad \text{and} \quad \widetilde{q}(x) = \frac{q_0'(x)}{\alpha x + q_0(x)}$$

are bounded in $[0, \varepsilon_1]$.

Divide (2.7) by the function $\alpha x + q_0(x)$ and integrate the resulting equality then. We obtain for $x \in (0, \varepsilon_1)$

$$v(x) = \int_0^x \left[\widetilde{K} - \widetilde{q}\right](\xi)\, v(\xi)\, d\xi + \alpha \int_0^x \frac{d\tau}{\alpha\tau + q_0(\tau)} \int_0^\tau K_{0x}(\tau, \xi)\, v(\xi)\, d\xi. \tag{2.8}$$

Let $|\alpha K_{0x}(\tau, \xi)| \leq M$ for $(\tau, \xi) \in P_{\varepsilon_1} \times P_{\varepsilon_1}$, where M is a positive number. Then

$$\left|\alpha \int_0^x \frac{d\tau}{\alpha\tau + q_0(\tau)} \int_0^\tau K_{0x}(\tau, \xi)\, v(\xi)\, d\xi\right| \leq M \int_0^x \frac{d\tau}{|\alpha\tau + q_0(\tau)|} \int_0^\tau |v(\xi)|\, d\xi \tag{2.9}$$

$$\leq M_1 V(x) \cdot x, \quad x \in (0, \varepsilon_1),$$

where

$$V(x) = \max_{0 \leq \xi \leq x} |v(\xi)| \tag{2.10}$$

and the positive number M_1 depends only on M, α and $\|q_0\|_{C[0,\varepsilon_1]}$. Hence, (2.8) and (2.9) imply that the following estimate takes place with another positive constant M_2 depending only on M, α and norms $\left\|\widetilde{K} - \widetilde{q}\right\|_{C[0,\varepsilon_1]}$, $\|q_0\|_{C[0,\varepsilon_1]}$

$$|v(x)| \leq M_2 V(x) \cdot x, \quad x \in (0, \varepsilon_1). \tag{2.11}$$

Let $t \in (0, \varepsilon_1)$ be an arbitrary number. By (2.11),

$$\max_{0 \leq x \leq t} |v(x)| \leq \max_{0 \leq x \leq t} \left[M_2 V(x) \cdot x\right].$$

Since the function $V(x)x$ is monotonically increasing, then the latter inequality leads to

$$V(t) \leq M_2 V(t) \cdot t, \quad t \in (0, \varepsilon_1). \tag{2.12}$$

Choose a number $\widetilde{\varepsilon} \in (0, \varepsilon_1)$ such that $M_2 \widetilde{\varepsilon} < 1/2$. Then (2.12) leads to

$$V(t) \leq \frac{V(t)}{2}, \quad t \in (0, \widetilde{\varepsilon}).$$

Hence $V(x) = 0$ for $x \in (0, \widetilde{\varepsilon})$. This and (2.10) imply that $u(x, 0) := v(x) = 0$ in $(0, \widetilde{\varepsilon})$, which is the first equality (2.6). The second equality (2.6) can be proven similarly.

Denote

$$\widehat{K}_1(x,y,\xi) = K_1(x,y,\xi) - 1, \ \widehat{K}_2(x,y,\eta) = K_2(x,y,\eta) - 1. \tag{2.13}$$

Let in (2.3) $(x,y) \in P_{\widetilde{\varepsilon}}$, where the number $\widetilde{\varepsilon} \in (0,\varepsilon)$ is the same as in (2.6). Apply the operator $\partial_y \partial_x$ to both sides of (2.3). Using (2.1), we obtain

$$\begin{aligned}
(\alpha x + \beta y + q(x,y))\, u_{xy} + q_x u_y + q_y u_x &= \left[\alpha \widehat{K}_1(x,y,x)\, u_y + \beta \widehat{K}_2(x,y,y)\, u_x\right] \\
&+ \left[\alpha \widehat{K}_{1y}(x,y,x) + \beta \widehat{K}_{2x}(x,y,y) + K_3(x,y,x,y)\right] \cdot u \\
&+ \left[\alpha \int_0^x \widehat{K}_{1x}(x,y,\xi)\, u_y(\xi,y)\, d\xi + \beta \int_0^y \widehat{K}_{2y}(x,y,\eta)\, u_x(x,\eta)\, d\eta\right] \\
&+ \alpha \int_0^x \widehat{K}_{1xy}(x,y,\xi)\, u(\xi,y)\, d\xi + \beta \int_0^y \widehat{K}_{2xy}(x,y,\eta)\, u(x,\eta)\, d\eta \\
&+ \int_0^x K_{3x}(x,y,\xi,y)\, u(\xi,y)\, d\xi + \int_0^y K_{3y}(x,y,x,\eta)\, u(x,\eta)\, d\eta \\
&+ \int_0^x \int_0^y K_{3xy}(x,y,\xi,\eta)\, u(\xi,\eta)\, d\eta d\xi, (x,y) \in P_{\widetilde{\varepsilon}}.
\end{aligned} \tag{2.14}$$

The Taylor's formula and (2.2) imply that the function $q_1(x,y) = q(x,y)/(x^2+y^2)$ is bounded in $\overline{P}_{\widetilde{\varepsilon}}$. Hence, (2.5) implies that there exists such a number $\varepsilon_2 \in (0,\widetilde{\varepsilon}]$ that functions

$$\frac{x}{\alpha x + \beta y + q(x,y)} \quad \text{and} \quad \frac{y}{\alpha x + \beta y + q(x,y)} \tag{2.15}$$

are bounded in $\overline{P}_{\varepsilon_2}$. To see this, it is sufficient to introduce polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ with $\theta \in [0,\pi/2]$. Further, the Taylor's formula, (2.1) and (2.13) imply that functions

$$\frac{\alpha \widehat{K}_1(x,y,x)}{\alpha x + \beta y + q(x,y)} \quad \text{and} \quad \frac{\beta \widehat{K}_2(x,y,y)}{\alpha x + \beta y + q(x,y)} \tag{2.16}$$

are also bounded in $\overline{P}_{\varepsilon_2}$. In addition, by (2.4)

$$u_y(0,y) = u_x(x,0) = 0 \quad \text{and} \quad u(x,y) = \int_0^y u_y(x,\eta)\, d\eta = \int_0^x u_x(\xi,y)\, d\xi. \tag{2.17}$$

For $t \in (0,\varepsilon_2)$ denote

$$w(t) = \max_{0 \le x,y \le t} \left[|u_x(x,y)| + |u_y(x,y)|\right]. \tag{2.18}$$

Using (2.17), substitute

$$u(x,y)=\int\limits_0^y u_y\left(x,\eta\right)d\eta \tag{2.19}$$

in the right hand side of (2.14). Next, divide both sides of (2.14) by the function
$[\alpha x+\beta y+q(x,y)]$ and apply the operator

$$\int\limits_0^x \left(...\right)d\xi \tag{2.20}$$

to both sides of the resulting equality. Note that all kernels of integral operators in (2.14) are bounded. Also, since functions (2.15) are bounded, then

$$\int\limits_0^x \frac{d\xi}{\left|\alpha\xi+\beta y+q(\xi,y)\right|}\int\limits_0^y \left|u_x\left(\xi,\eta\right)\right|d\eta \leq Qw(t)\cdot t,\ \text{for}\ (x,y)\in\overline{P}_t,\quad t\in(0,\varepsilon_2) \tag{2.21}$$

and

$$\int\limits_0^x \frac{d\xi}{\left|\alpha\xi+\beta y+q(\xi,y)\right|}\int\limits_0^\xi \left|u_y\left(\tau,y\right)\right|d\tau \leq Qw(t)\cdot t,\ \text{for}\ (x,y)\in\overline{P}_t,\quad t\in(0,\varepsilon_2)\,. \tag{2.22}$$

Here and below in this proof Q denotes different positive constants independent on the parameter $t\in(0,\varepsilon_2)$ and functions u and w. Thus, using the fact that functions (2.15) and (2.16) are bounded and using also estimates (2.21) and (2.22), we conclude that the application of the operator (2.20) to the equality, which is obtained from (2.14) after the substitution (2.19) and division by the function $[\alpha x+\beta y+q(x,y)]$ leads to the following estimate

$$|u_y(x,y)|\leq Qw(t)\cdot t,\ \text{for}\ (x,y)\in\overline{P}_t,\quad t\in(0,\varepsilon_2)\,. \tag{2.23}$$

On the other hand, substituting in (2.14)

$$u(x,y)=\int\limits_0^x u_x\left(\xi,y\right)d\xi,$$

dividing it then by the function $[\alpha x+\beta y+q(x,y)]$ and applying the operator

$$\int\limits_0^y (...)d\eta,$$

we similarly obtain

$$|u_x(x,y)|\leq Qw(t)\cdot t,\ \text{for}\ (x,y)\in\overline{P}_t,\quad t\in(0,\varepsilon_2)\,. \tag{2.24}$$

Summing up (2.23) and (2.24), we obtain

$$|u_x(x,y)| + |u_y(x,y)| \leq Qw(t)\cdot t, \quad \text{for } (x,y)\in \overline{P}_t, \quad t\in(0,\varepsilon_2).$$

By (2.18), this is equivalent with

$$|u_x(x,y)| + |u_y(x,y)| \leq Qt\cdot \max_{0\leq x,y\leq t}\left[|u_x(x,y)| + |u_y(x,y)|\right],$$

for $(x,y)\in \overline{P}_t,\ t\in(0,\varepsilon_2)$. Hence,

$$\max_{0\leq x,y\leq t}\left[|u_x(x,y)| + |u_y(x,y)|\right] \leq Qt\cdot \max_{0\leq x,y\leq t}\left[|u_x(x,y)| + |u_y(x,y)|\right], \quad t\in(0,\varepsilon_2).$$

The latter inequality and (2.18) lead to

$$w(t)\leq Qw(t)\cdot t, \text{ for } t\in(0,\varepsilon_2).$$

Choose the number $\varepsilon_0\in(0,\varepsilon_2)$ such that $Q\varepsilon_0<1/2$. Then the latter inequality implies that

$$w(t)\leq \frac{w(t)}{2}, \text{ for } t\in(0,\varepsilon_0).$$

Hence, $w(t)=0$ for $t\in(0,\varepsilon_0)$. This, (2.6) and (2.18) imply that $u(x,y)=0$ for $(x,y)\in\overline{P}_{\varepsilon_0}$. □

Lemma 2. $\varphi_1=\varphi_2$ in $\Omega_\delta(\Gamma)$.

Proof. For the sake of definiteness, we assume in this proof that the condition (1.6a) is fulfilled. The proof in the case (1.6b) is similar. Consider the function $h(x,y)$,

$$h(x,y)=2\sin\left[\left(\frac{\varphi_1-\varphi_2}{2}\right)(x,y)\right]$$

Since $(\varphi_1-\varphi_2)(0,0)=0$, then

$$h(0,0)=0. \tag{2.25}$$

Since both functions φ_1 and φ_2 are analytic in $\Omega_\delta(\Gamma)$, it is sufficient to prove that $h(x,y)=0$ for $(x,y)\in P_\sigma$ for a number $\sigma\in(0,1)$.

Let $\widetilde{G}(x,y)$ be the inverse Fourier transform of the function $G(x,y)$ defined in (1.2). Then

$$\widetilde{G}(x,y)=\iint_{\mathbb{R}^2} f(x+\xi,y+\eta)\overline{f}(\xi,\eta)\,\chi(x+\xi,y+\eta)\,\chi(\xi,\eta)\,d\xi d\eta, \tag{2.26}$$

where $\chi(\xi,\eta)$ is the characteristic function of the square Ω. Assuming that $(x,y)\in\Omega$, we can rewrite the equality (2.26) in the form

$$\widetilde{G}(x,y)=\int_0^{1-x}\int_0^{1-y} f(x+\xi,y+\eta)\overline{f}(\xi,\eta)\,d\xi d\eta. \tag{2.27}$$

Suppose that $\varphi_1(\xi,\eta) \neq \varphi_2(\xi,\eta)$ in $\Omega_\delta(\Gamma)$. Recall that $f_1(\xi,\eta) = \exp[i\varphi_1(\xi,\eta)]$ and $f_2(\xi,\eta) = \exp[i\varphi_2(\xi,\eta)]$. Denote $g(\xi,\eta) = f_1(\xi,\eta) - f_2(\xi,\eta)$. Then (2.27) leads to

$$\int_0^{1-x}\int_0^{1-y} f_1(x+\xi,y+\eta)\overline{f}_1(\xi,\eta)\,d\xi d\eta - \int_0^{1-x}\int_0^{1-y} f_2(x+\xi,y+\eta)\overline{f}_2(\xi,\eta)\,d\xi d\eta = 0. \tag{2.28}$$

Since

$$f_1(x+\xi,y+\eta)\cdot\overline{f}_1(\xi,\eta) - f_2(x+\xi,y+\eta)\cdot\overline{f}_2(\xi,\eta)$$
$$= f_1(x+\xi,y+\eta)\cdot\overline{g}(\xi,\eta) + g(x+\xi,y+\eta)\cdot\overline{f}_2(\xi,\eta),$$

then (2.28) implies that

$$\int_0^{1-x}\int_0^{1-y} f_1(x+\xi,y+\eta)\overline{g}(\xi,\eta)\,d\eta d\xi + \int_0^{1-x}\int_0^{1-y} g(x+\xi,y+\eta)\overline{f}_2(\xi,\eta)\,d\eta d\xi = 0. \tag{2.29}$$

Consider the second integral in (2.29). Changing variables, we obtain

$$\int_0^{1-x}\int_0^{1-y} g(x+\xi,y+\eta)\overline{f}_2(\xi,\eta)\,d\eta d\xi = \int_x^{1}\int_y^{1} g(\xi,\eta)\overline{f}_2(\xi-x,\eta-y)\,d\eta d\xi. \tag{2.30}$$

Note that by (1.4) and (1.5) $g(\xi,\eta) = g(1-\xi,1-\eta)$. Substituting this in the integral in the right hand side of (2.30) and changing variables $(\xi',\eta') = (1-\xi,1-\eta)$, we obtain

$$\int_x^{1}\int_y^{1} g(\xi,\eta)\overline{f}_2(\xi-x,\eta-y)\,d\eta d\xi = \tag{2.31}$$

$$\int_0^{1-x}\int_0^{1-y} g(\xi',\eta')\,\overline{f}_2(1-(\xi'+x),1-(\eta'+y))\,d\eta' d\xi'.$$

Since by (1.4) and (1.5) $\overline{f}_2(1-(\xi'+x),1-(\eta'+y)) = \overline{f}_2(\xi'+x,\eta'+y)$, then (2.29)-(2.31) lead to

$$\int_0^{1-x}\int_0^{1-y} f_1(x+\xi,y+\eta)\overline{g}(\xi,\eta)\,d\eta d\xi \tag{2.32}$$

$$+\int_0^{1-x}\int_0^{1-y} \overline{f}_2(x+\xi,y+\eta)\,g(\xi,\eta)d\eta d\xi = 0, \text{ for } (x,y)\in\Omega.$$

It is convenient to make another change of variables $(x,y) \Leftrightarrow (x',y') = (1-x, 1-y)$ and still keep the same notations for these new ones (for brevity). Since by (1.4) and (1.5) $f_j(1-x+\xi, 1-y+\eta) = f_j(x-\xi, y-\eta), j = 1,2$, then (2.32) becomes

$$\int_0^x \int_0^y \left[f_1(x-\xi, y-\eta)\overline{g}(\xi,\eta) + \overline{f}_2(x-\xi, y-\eta)\, g(\xi,\eta) \right] d\eta d\xi = 0, \text{ for } (x,y) \in \Omega.$$

Apply the operator $\partial_y \partial_x$ to this equality. Note that $f_1(0,0) = f_2(0,0) = 1$. We obtain for $(x,y) \in \Omega$

$$\begin{aligned} g(x,y) + \overline{g}(x,y) = &- \int_0^x \left[f_{1x}(x-\xi, 0)\, \overline{g}(\xi,y) + \overline{f}_{2x}(x-\xi,0)\, g(\xi,y) \right] d\xi \\ &- \int_0^y \left[f_{1y}(0, y-\eta)\overline{g}(x,\eta) + \overline{f}_{2y}(0,y-\eta) g(x,\eta) \right] d\eta \\ &- \int_0^x \int_0^y \left[f_{1xy}(x-\xi, y-\eta)\, \overline{g}(\xi,\eta) + \overline{f}_{2xy}(x-\xi, y-\eta)\, g(\xi,\eta) \right] d\eta d\xi. \end{aligned} \tag{2.33}$$

We have

$$\begin{aligned} g &= e^{i\varphi_1} - e^{i\varphi_2} = (\cos\varphi_1 - \cos\varphi_2) + i(\cos\varphi_1 - \cos\varphi_2) \\ &= 2\sin\left(\frac{\varphi_2 - \varphi_1}{2}\right)\left[\sin\left(\frac{\varphi_1+\varphi_2}{2}\right) - i\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)\right]. \end{aligned}$$

Hence,

$$g(x,y) = h(x,y)\left[\sin\left(\frac{\varphi_1+\varphi_2}{2}\right) - i\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)\right]. \tag{2.34}$$

Hence,

$$g(x,y) + \overline{g}(x,y) = 2h(x,y)\sin\left[\left(\frac{\varphi_1+\varphi_2}{2}\right)(x,y)\right]. \tag{2.35}$$

Denote

$$r_1(x,y) = 2\sin\left[\left(\frac{\varphi_1+\varphi_2}{2}\right)(x,y)\right].$$

Since $\varphi_1(0,0) = \varphi_2(0,0) = 0$, then the Taylor's formula implies that the function $r_1(x,y)$ can be represented in the form

$$r_1(x,y) = \alpha x + \beta y + q(x,y), \tag{2.36}$$

where

$$\alpha = \varphi_{1x}(0,0) + \varphi_{2x}(0,0), \quad \beta = \varphi_{1y}(0,0) + \varphi_{2y}(0,0) \tag{2.37}$$

and the function $q(x,y) \in C^2\left(\overline{\Omega}\right)$ satisfies conditions (2.2). By (1.6a) and (2.37)

$$\alpha, \beta > 0. \tag{2.38}$$

Hence (2.35) and (2.36) imply that the function $g(x,y)+\overline{g}(x,y)$ can be represented as

$$g(x,y)+\overline{g}(x,y) = \left[\alpha x + \beta y + q(x,y)\right] h(x,y). \tag{2.39}$$

Consider first two integrals in the right hand side of (2.33). Using the Taylor's formula, (2.34) and (2.37), we obtain

$$\begin{gathered}
-\left[f_{1x}\left(x-\xi,0\right)\overline{g}(\xi,y)+\overline{f}_{2x}\left(x-\xi,0\right)g(\xi,y)\right] \\
= -i\varphi_{1x}(0,0)\left[1+r_2(x-\xi)\right]\times \\
\left[\sin\left(\frac{\varphi_1+\varphi_2}{2}\right)(\xi,y)+i\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)(\xi,y)\right]\cdot h(\xi,y)+ \\
i\varphi_{2x}(0,0)\left[1+r_3(x-\xi)\right]\times \\
\left[\sin\left(\frac{\varphi_1+\varphi_2}{2}\right)(\xi,y)-i\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)(\xi,y)\right]\cdot h(\xi,y) \\
= \left[\varphi_{1x}(0,0)+\varphi_{2x}(0,0)\right]\cdot\left[1+G_1(x,y,\xi)\right]\cdot h(\xi,y) = \alpha\left[1+G_1(x,y,\xi)\right]\cdot h(\xi,y),
\end{gathered} \tag{2.40}$$

where functions $r_2(x), r_3(x) \in C^2\left[0,1\right]$, the function $G_1(x,y,\xi) \in C^2\left(\overline{\Omega}\times[0,1]\right)$ and $r_2(0) = r_3(0) = G_1(0,0,0) = 0$. Similarly

$$-\left[f_{1y}\left(0,y-\eta\right)\overline{g}(x,\eta)+\overline{f}_{2y}\left(0,y-\eta\right)g(x,\eta)\right] = \beta\left[1+G_2(x,y,\eta)\right]\cdot h(x,\eta), \tag{2.41}$$

where the function $G_2(x,y,\eta) \in C^2\left(\overline{\Omega}\times[0,1]\right)$ and $G_2(0,0,0)=0$.

Denote

$$K_1\left(x,y,\xi\right) := \left[1+G_1(x,y,\xi)\right], K_2\left(x,y,\eta\right) := 1+G_2(x,y,\eta),$$

$$K_3\left(x,y,\xi,\eta\right) := f_{1xy}\left(x-\xi,y-\eta\right)\cdot\left[\sin\left(\frac{\varphi_1+\varphi_2}{2}\right)(\xi,\eta)+i\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)(\xi,\eta)\right]$$

$$+\overline{f}_{2xy}\left(x-\xi,y-\eta\right)\cdot\left[\sin\left(\frac{\varphi_1+\varphi_2}{2}\right)(\xi,\eta)-i\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)(\xi,\eta)\right].$$

Thus,

$$K_1, K_2 \in C^2\left(\overline{\Omega}\times[0,1]\right), K_3 \in C^2\left(\overline{\Omega}\times\overline{\Omega}\right).$$

Further, (2.34) and (2.37)-(2.41) imply that the function $h(x,y)$ satisfies the following integral equation

$$\begin{gathered}
\left[\alpha x+\beta y+q(x,y)\right]h(x,y) = \\
\alpha\int\limits_0^x K_1\left(x,y,\xi\right)u\left(\xi,y\right)d\xi+\beta\int\limits_0^y K_2\left(x,y,\eta\right)u\left(x,\eta\right)d\eta
\end{gathered} \tag{2.42}$$

$$+\int_0^x\int_0^y K_3\left(x,y,\xi,\eta\right)u\left(\xi,\eta\right)d\eta d\xi,\text{ in }\Omega,$$

where numbers α,β and functions q,K_1,K_2 and K_3 satisfy conditions of Lemma 1. Thus, by Lemma 1 there exists a number $\sigma\in(0,1)$ such that $h(x,y)=0$ for $(x,y)\in P_\sigma$. $\square$

Note, however that it does not follow from the proof of Lemma 2 that the function $[\alpha x+\beta y+q(x,y)]^{-1}$ has no singularities at points $(x,y)\in\Omega$, which are located far from the boundary Γ. Hence, it is unclear what does the equation (2.42) imply for these points. Thus, we should proceed with the proof of Theorem 1.

3 Lemmata

By (1.2)

$$G(x,y)=F(x,y)\cdot\overline{F}(x,y),\quad\forall\,(x,y)\in\mathbb{R}^2.\tag{3.1}$$

Hence, the analytic continuation $G(z,y)$ of the function $G(x,y)$ is

$$G(z,y)=\left[\iint_\Omega f(\xi,\eta)e^{-iz\xi}e^{-iy\eta}d\xi d\eta\right]\cdot\left[\iint_\Omega \overline{f}(\xi,\eta)e^{iz\xi}e^{iy\eta}d\xi d\eta\right].\tag{3.2}$$

Denote $\widehat{F}(z,y)=\overline{F}(\overline{z},y)$. Then one can rewrite (3.2) as

$$G(z,y)=F(z,y)\widehat{F}(z,y).\tag{3.3}$$

Hence, $G(z,y)$ and $\widehat{F}(z,y)$ are entire analytic functions of the first order of the variable $z\in\mathbb{C}$ for every $y\in\mathbb{R}$. Since functions $F_1(z,y)$ and $F_2(z,y)$ are analytic with respect to $y\in\mathbb{R}$ as functions of the real variable, it is sufficient to prove that $F_1(z,y)=F_2(z,y)$ for $z\in C$ and for every $y\in(a,b)$ for an interval $(a,b)\subset\mathbb{R}$. And this is what is done in this paper below.

Consider an example of the non-uniqueness, which is sometimes called the "complex zero-flipping" in the physics literature, see, e.g., [8]. The function $F(z,y)$ can be represented in the form [1]

$$F(z,y)=z^{k(y)}e^{g(z,y)}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n(y)}\right)\exp\left(\frac{z}{a_n(y)}\right),\quad\forall y\in\mathbb{R},\tag{3.4}$$

where $k(y)\geq 0$ is an integer, $g(z,y)$ is a linear function of z and $\{a_n(y)\}_{n=1}^{\infty}$ is the set of zeros of the function $F(z,y)$. Each zero is counted as many times as its multiplicity is. The integer $k(y)$, the function $g(z,y)$ and zeros $\{a_n(y)\}_{n=1}^{\infty}$ depend on y as on a parameter. Specific types of such dependencies (e.g., analytic, continuous, etc.) do not affect the rest of the proof of Theorem 1. Thus, for brevity we will not indicate dependencies of these on

y in some (but not all) formulas below. Suppose, for example that $\operatorname{Im} a_1 \neq 0$. Consider the function $F_\star(x,y)$,

$$F_\star(x,y) = \frac{x - \overline{a}_1}{x - a_1} \cdot F(x,y).$$

Note that

$$\left| \frac{x - \overline{b}}{x - b} \right| = 1, \quad \forall x \in \mathbb{R}, \forall b \in \mathbb{C}. \tag{3.5}$$

Hence, by (3.5) $|F_\star(x,y)| = |F(x,y)|$ for all $(x,y) \in \mathbb{R}^2$. In addition, it can be easily shown that the inverse Fourier transform $f_\star(\xi,\eta)$ of $F_\star(x,y)$ has its support in Ω. Thus, the most difficult aspect of the PPO is to determine complex zeros in (3.4).

Lemma 3. *For an $y \in (-\infty,\infty)$ let $\{a_n\}_{n=1}^\infty \subset \mathbb{C}$ be the set of all zeros of the function $F(z,y)$ as indicated in (3.4). Then $\{\overline{a}_n\}_{n=1}^\infty$ is the set of all zeros of the function $\widehat{F}(z,y)$. Thus, $F(a,y) = 0 \Leftrightarrow \widehat{F}(\overline{a},y) = 0$. The multiplicity of each zero $z = a$ of the function $F(z,y)$ equals the multiplicity of the zero $z = \overline{a}$ of the function $\widehat{F}(z,y)$. The set of zeros of the function $G(z,y)$ is $\{a_n\}_{n=1}^\infty \cup \{\overline{a}_n\}_{n=1}^\infty$.*

Proof. Let $F(a,y) = 0$. This means that

$$\iint\limits_\Omega f(\xi,\eta) e^{-iz\xi} e^{-iy\eta} d\xi d\eta \mid_{z=a} = 0.$$

Consider the complex conjugate of both sides,

$$0 = \overline{\iint\limits_S f(\xi,\eta) e^{-iz\xi} e^{-iy\eta} d\xi d\eta} \mid_{z=a} = \iint\limits_S \overline{f}(\xi,\eta) e^{i\overline{z}\xi} e^{iy\eta} d\xi d\eta \mid_{z=a} = \overline{F}(\overline{a},y) = \widehat{F}(\overline{a},y).$$

Further, let $z = a$ be a zero of the multiplicity s. Then differentiating last two formulas k $(1 \leq k \leq s)$ times with respect to z, we obtain the statement of this lemma about the multiplicity. $\square$

Lemma 4. *For each $y \in \mathbb{R}$ real zeros of functions $F_1(z,y)$ and $F_2(z,y)$ coincide.*

Proof. By (1.9) $G_1(z,y) = G_2(z,y), \forall z \in \mathbb{C}, \forall y \in \mathbb{R}$. Hence, for any fixed $y \in \mathbb{R}$ all zeros (real and complex) of functions $G_1(z,y)$ and $G_2(z,y)$ coincide. By (3.3) and Lemma 3 the multiplicity of each real zero $x = a$ of the function $G_j(x,y)$ is twice the multiplicity of the zero $x = a$ of the function $F_j(x,y)$. $\square$

First, consider the problem of the determination of the number $k(y)$ and the function $g(z,y)$ in (3.4). For a positive integer m denote $I_m = (2m\pi + \pi/2, 2m\pi + 3\pi/2)$.

Lemma 5. *Suppose that there exists a positive number N_0 such that for every integer $m > N_0$ sets of zeros of functions $F_1(z,y)$ and $F_2(z,y)$ coincide for every $y \in I_m$. Then there exists a number $N > N_0$ such that for every integer $m > N$ and for every number $y \in I_m$ corresponding numbers $k_1(y)$ and $k_2(y)$ and functions $g_1(z,y)$ and $g_2(z,y)$ in products (3.4) for functions F_1 and F_2 coincide.*

Proof. Denote

$$p_j(\xi,y)=\int_0^1 e^{-iy\eta}f_j\left(\xi,\eta\right)d\eta. \tag{3.11}$$

By Lemma 2 $p_1(\xi,y)=p_2(\xi,y)$ for $\xi\in[0,\delta)\cup(1-\delta,1]$. Hence, we can denote

$$p\left(\xi,y\right):=p_1(\xi,y)=p_2(\xi,y) \text{ for } \xi\in[0,\delta)\cup(1-\delta,1]. \tag{3.12}$$

By (1.4)

$$p(1,y)=p(0,y) \tag{3.13}$$

and

$$p_\xi\left(1,y\right)=-p_\xi\left(0,y\right). \tag{3.14}$$

Using (1.7), (3.11) and (3.13) we obtain

$$F_j(z,y)=-\frac{1}{iz}\left[\left(e^{-iz}-1\right)p(1,y)-\int_0^1 e^{-iz\xi}p_{j\xi}(\xi,y)d\xi\right]. \tag{3.15}$$

Hence,

$$F_j(z,y)=-\frac{e^{-iz}}{iz}\left[p(1,y)+o(1)\right], \text{ for } \operatorname{Im}z\to\infty \tag{3.16a}$$

and

$$F_j(z,y)=\frac{1}{iz}\left[p(1,y)+o(1)\right], \quad \text{for } \operatorname{Im}z\to-\infty. \tag{3.16b}$$

Setting in (3.11) $\xi:=1$, integrating by parts and recalling that $f(1,1)=f(0,0)=1$, we obtain

$$p(1,y)=-\frac{1}{iy}\left[\left(e^{-iy}-1\right)-\int_0^1 e^{-iy\eta}f_\xi\left(1,\eta\right)d\eta.\right] \tag{3.17}$$

Because of the choice of intervals I_m, we have

$$\left|e^{-iy}-1\right|\geq\sqrt{2},\ \forall y\in I_m,\quad \forall m=1,2,... \tag{3.18}$$

The Riemann-Lebesgue lemma implies that one can choose a positive integer $N>N_0$ so large that

$$\left|\int_0^1 e^{-iy\eta}f_\xi\left(1,\eta\right)d\eta\right|\leq 0.1,\quad \forall y\in\{y>N\}. \tag{3.19}$$

Hence, (3.17)-(3.19) imply that

$$|p(1,y)|\geq\frac{1}{y},\quad \forall m>N,\quad \forall y\in I_m. \tag{3.20}$$

Choose an arbitrary integer $m > N$. Then zeros of functions $F_1(z, y)$ and $F_2(z, y)$ coincide for all $y \in I_m$. Also, (3.16a,b) and (3.20) imply that

$$F_j(z, y) = -\frac{e^{-iz}}{iz} p(1, y)(1 + o(1)), \quad \text{for} \quad \operatorname{Im} z \to \infty, \forall y \in I_m, \tag{3.21a}$$

$$F_j(z, y) = \frac{1}{iz} p(1, y)(1 + o(1)), \quad \text{for} \quad \operatorname{Im} z \to -\infty, \forall y \in I_m, \tag{3.21b}$$

and

$$p(1, y) \neq 0, \ \ \forall y \in I_m. \tag{3.22}$$

Let

$$F_j(z, y) = z^{k_j(y)} e^{g_j(z,y)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n(y)}\right) \exp\left(\frac{z}{a_n(y)}\right).$$

Then for all $y \in I_m$

$$\log\left[\frac{F_1(z, y)}{F_2(z, y)}\right] = (k_1(y) - k_2(y)) \log z + g_1(z, y) - g_2(z, y). \tag{3.23}$$

On the other hand, by (3.21a) and (3.22)

$$\log\left[\frac{F_1(z, y)}{F_2(z, y)}\right] = o(1), \quad \text{for} \quad \operatorname{Im} z \to \infty, \forall y \in I_m.. \tag{3.24}$$

Since $g_1(z, y)$ and $g_2(z, y)$ are linear functions of the variable z (for every $y \in \mathbb{R}$), then comparison of (3.23) and (3.24) shows that $k_1(y) - k_2(y) = 0$ and $g_1(z, y) - g_2(z, y) = 0$ for all $y \in I_m$. □

Lemma 6. *There exists a positive number $N = N(F_1, F_2)$ and a positive number $T = T(N)$ such that for every integer $m > N$ and for every $y \in I_m$ all zeros of functions $F_1(z, y)$ and $F_2(z, y)$ are located in the strip $\{|\operatorname{Im} z| < T\}$.*

Proof. Choose a number $N = N(F_1, F_2)$ such that (3.19) and (3.20) hold. Let $m > N$ be an integer. By (3.21a,b) and (3.22) there exists a positive number $T = T(N)$ independent on m (as long as $m > N$) such that

$$|F_j(z, y)| \geq \frac{\exp(\operatorname{Im} z)}{2|z|} |p(1, y)| \neq 0 \text{ for } \operatorname{Im} z \geq T, y \in I_m$$

and

$$|F_j(z, y)| \geq \frac{1}{2|z|} |p(1, y)| \neq 0 \text{ for } \ \operatorname{Im} z \leq -T, y \in I_m.$$

□

Lemma 7. *There exists a positive number $N = N(F_1, F_2)$ such that there exists a positive number $M = M(N)$ such that for all integers $m > N$ and for every $y \in I_m$ zeros of functions $F_1(z, y)$ and $F_2(z, y)$ coincide in $\{|z| > M\}$.*

Proof. We use notations of the proof of Lemma 5. Choose a positive number $N_1 = N_1(F_1, F_2)$ such that (3.19) and (3.20) are fulfilled. Let $T = T(N_1)$ be the number of Lemma 6. First, we prove that for every integer $m > N_1$ and for every $y \in I_m$ both functions $F_1(z, y)$ and $F_2(z, y)$ have infinitely many zeros. Fix an $y_0 \in I_m$. Let, for example the function $F_1^0(z) = F_1(z, y_0)$ has only a finite number $s \geq 0$ zeros in $\mathbb{C}$. Then (3.4) implies that

$$F_1^0(z) = P_s(z)e^{\gamma z}, \tag{3.25}$$

where γ is a complex number and $P_s(z)$ is a polynomial of the degree s. However, by (3.15) and the Riemann-Lebesgue lemma

$$h(x) = -\frac{1}{ix}\left[\left(e^{-ix} - 1\right)p(1, y_0) + o\left(\frac{1}{x}\right)\right] \text{ for } x \to \infty, x \in \mathbb{R}. \tag{3.26}$$

Since by (3.20) $p(1, y_0) \neq 0$, then (3.26) contradicts with (3.25).

The latter and Lemma 6 imply that for every integer $m > N_1$, for every $y \in I_m$ and for each positive number K there exists a zero $z_j(K) \in \{|z| > K\} \cap \{|\operatorname{Im} z| < T\}$ of the function $F_j(z, y)$. Integrating by parts in (3.15) and using (3.13) and (3.14), we obtain for every integer $m > N_1$ and for all $y \in I_m$

$$-iz \cdot F_j(z, y) = \left(e^{-iz} - 1\right)p(1, y) + \frac{1}{iz}\left(e^{-iz} + 1\right)p_\xi(1, y) \tag{3.27}$$

$$-\frac{1}{z^2}\left(e^{-iz} - 1\right)p_{\xi\xi}(1, y) + \frac{1}{z^2}\int_0^1 e^{-iz\xi}\partial_\xi^3 p_j(\xi, y)d\xi.$$

Lemma 6 tells one that in order to find the asymptotic behavior of zeros of functions $F_j(z, y)$, one should investigate the behavior of these functions at $|\operatorname{Re} z| \to \infty$ with $|\operatorname{Im} z| < T$. Integrating by parts in (3.11) and using (1.5), we obtain

$$\partial_\xi^k p_j(\xi, y) = -\frac{1}{iy}\left[\left(e^{-iy} - 1\right)\partial_\xi^k f_j(\xi, 1) + \int_0^1 e^{-iy\eta}\partial_\xi^k \partial_\eta f_j\left(\xi, \eta\right)d\eta\right], k = 0, 1, 2, 3. \tag{3.28}$$

It follows from (3.12), (3.20) and (3.28) that one can the choose a number $M_1 = M_1(N_1)$ so large that

$$\left|p(1, y) + \frac{1}{iz}p_\xi(1, y)\right| \geq \frac{1}{2y}, \text{ for } m > N_1, y \in I_m, |z| > M_1. \tag{3.29}$$

Also, (3.20) implies that

$$\frac{1}{p(1, y) + p_\xi(1, y)/iz} = \frac{1}{p(1, y)}\left(1 - \frac{p_\xi(1, y)}{izp(1, y)} + \frac{\widetilde{p}(z, y)}{z^2}\right), \tag{3.30}$$

for $m > N_1, y \in I_m, |z| > M_1$, where $|\widetilde{p}(z, y)| \leq C$. Here and below in this proof C denotes different positive numbers which are independent on $z \in \{|\dot{z}| > 1\} \cap \{|\operatorname{Im} z| < T\}, N_1, N_2, M_k$

$(k=1,...,5)$, complex numbers z_j (which are chosen below) and the parameter $y\in I_m$, as long as the integer $m>N_1$. Although, in principle at least each function $F_j, j=1,2$ "has its own" constant C_j, but we always choose $C=\max(C_1,C_2)$.

By (3.27) and (3.28) we have

$$-iz\cdot F_j(z,y)=$$
$$e^{-iz}\left(p(1,y)+\frac{1}{iz}p_\xi(1,y)\right)-\left(p(1,y)-\frac{1}{iz}p_\xi(1,y)+B_j\left(z,y\right)\right), \tag{3.31}$$
$$\forall\ z\in\{|z|>M_4\}\cap\{|\operatorname{Im}z|<T\}\,, m>N_1 \text{ and } y\in I_m,$$

where the function B_j can be estimated as

$$|B_j\left(z,y\right)|\leq\frac{C}{y\left|z\right|^2}. \tag{3.32}$$

Dividing (3.31) by the function $[p(1,y)+p_\xi(1,y)/iz]$, using (3.29), (3.30) and (3.32), we obtain for $m>N_1, \forall y\in I_m, |\operatorname{Im}z|<T$

$$-iz\cdot F_j(z,y)\cdot\left(p(1,y)+\frac{1}{iz}p_\xi(1,y)\right)^{-1}=e^{-iz}-1+\frac{2p_\xi(1,y)}{izp(1,y)}+\widetilde{B}_j\left(z,y\right), \tag{3.33}$$
$$\forall\ z\in\{|z|>M_4\}\cap\{|\operatorname{Im}z|<T\}\,, m>N_1 \text{ and } y\in I_m,$$

where the function $\widetilde{B}_j\left(z,y\right)$ satisfies the estimate

$$\left|\widetilde{B}_j\left(z,y\right)\right|\leq\frac{C}{\left|z\right|^2}, \tag{3.34}$$
$$\forall\ z\in\{|z|>M_4\}\cap\{|\operatorname{Im}z|<T\}\,, m>N_1 \text{ and } y\in I_m.$$

Choose an integer $m_0>N_1$ and fix a number $y_0\in I_{m_0}$. Let $z_j\in\{|z|>M_1\}\cap\{|\operatorname{Im}z|<T\}$ be a zero of the function $F_j\left(z,y_0\right)$. Then (3.33) implies that

$$\exp(-iz_j)=1-\frac{2p_\xi(1,y_0)}{iz_jp(1,y_0)}-\widetilde{B}_j\left(z_j,y_0\right).$$

Since $\exp(-iz_j)=\exp(-iz_j+2i\pi s)$ for any integer s, then there exists an integer $n\left(z_j\right)$ such that

$$-iz_j+2i\pi n\left(z_j\right)=\log\left(1-\frac{2p_\xi(1,y_0)}{izp(1,y_0)}-\widetilde{B}_j\left(z,y_0\right)\right). \tag{3.35}$$

By (1.6a,b)

$$\varphi_\xi(0,0)\neq 0 \tag{3.36}$$

Hence, (1.4), (1.5) and (3.36) imply that

$$\varphi_\xi(0,0)=-\varphi_\xi(1,1)\neq 0. \tag{3.37}$$

Thus, (1.3), (3.17), (3.18), (3.28), (3.37) and the Riemann-Lebesgue lemma imply that one can choose the number $N_2 = N_2(F_1, F_2) \geq N_1$ and

$$\frac{2p_\xi(1, y_0)}{ip(1, y_0)} = 2\varphi_\xi(1,1)\left(1 + g\left(y_0\right)\right) \neq 0, \tag{3.38}$$

where $g(y)$ is a complex valued function such that

$$|g(y)| < \frac{1}{8}, \forall m > N_2, \forall y \in I_m. \tag{3.39}$$

Without loss of generality we assume from now on in this proof that the integer m_0, which was chosen after (3.34) is so large that $m_0 > N_2$ and the fixed number $y_0 \in I_{m_0}$. By (3.34), (3.38) and (3.39) one can choose the number $M_2 = M_2(N_2) \geq M_1$ so large that

$$\log\left(1 - \frac{2p_\xi(1, y_0)}{izp(1, y_0)} - \widetilde{B}_j\left(z, y_0\right)\right) = -\frac{2\varphi_\xi(1,1)}{z}\left(1 + \widetilde{g}_j\left(z, y_0\right)\right), \tag{3.40}$$

where the function $\widetilde{g}_j(z, y)$ is such that

$$|\widetilde{g}_j(z, y)| < \frac{1}{4}, \tag{3.41}$$

$$\forall\ z \in \{|z| > M_2\} \cap \{|\operatorname{Im} z| < T\}, \forall m > N_2, \forall y \in I_m.$$

Substituting the right hand side of (3.40) in the right hand side of (3.35) and setting $y := y_0$, we obtain

$$z_j = 2\pi n\left(z_j\right) - i\frac{2\varphi_\xi(1,1)}{z_j}\left(1 + \widetilde{g}_j\left(z_j, y_0\right)\right). \tag{3.42}$$

Since we are concerned in this lemma with the asymptotic behavior of zeros of functions $F_j(z, y)$, then we can assume now that the zero $z_j \in \{|z| > M_2\} \cap \{|\operatorname{Im} z| < T\}$. Choose the number $M_3 = M_3(N_2) \geq M_2$ so large that $C/M_3 < M_3/8$. Similarly with the above we now assume that $z_j \in \{|z| > M_3\} \cap \{|\operatorname{Im} z| < T\}$. Hence, (3.41) and (3.42) lead to

$$|\operatorname{Im}(z_j)| \leq \left|\frac{2\varphi_\xi(1,1)}{z_j}\left(1 + \widetilde{g}_j\left(z_j, y_0\right)\right)\right| < \frac{C}{M_3} < \frac{M_3}{8}. \tag{3.43}$$

On the other hand, since $|z_j| > M_3$, then (3.42) and (3.43) imply that $M_3 < |z_j| < |2\pi n\left(z_j\right)| + M_3/8$. Hence,

$$|2\pi n\left(z_j\right)| > \frac{7}{8}M_3. \tag{3.44}$$

Combining (3.44) with (3.42) and (3.43), we obtain

$$z_j = 2\pi n\left(z_j\right)\left[1 + \lambda_j\left(z_j, y_0\right)\right], \tag{3.45}$$

where the function $\lambda_j(z,y)$ satisfies the following estimate

$$|\lambda_j(z,y)| < \frac{1}{7}, \tag{3.46}$$

$$\forall\ z \in \{|z| > M_3\} \cap \{|\operatorname{Im} z| < T\}, \forall m > N_2, \forall y \in I_m.$$

It follows from (3.45) and (3.46) that

$$\operatorname{Re}(z_j) \neq 0 \tag{3.47a}$$

and

$$sgn\left[\operatorname{Re}(z_j)\right] = sgn\left(n(z_j)\right), \tag{3.47b}$$

where $sgn(x) = 1$ if $x > 0$ and $sgn(x) = -1$ if $x < 0$ for $x \in \mathbb{R}$.

Note that for each above zero z_j of the function F_j there exists only one integer $n(z_j)$. Indeed, if there exists a second one $n'(z_j)$, then (3.42) implies that

$$z_j = 2\pi n'(z_j) - i\frac{2\varphi_\xi(1,1)}{z_j}\left(1 + \widetilde{g}_j(z_j, y_0)\right).$$

Subtracting this formula from (3.42), we obtain $2\pi\left[n'(z_j) - n(z_j)\right] = 0$.

Consider now $sgn(\operatorname{Im}(z_j))$. It follows from (3.41)-(3.46) that one can choose a number $M_4 = M_4(N_2) \geq M_3$ so large that for any zero $z_j \in \{|z| > M_4\} \cap \{|\operatorname{Im} z| < T\}$ of the function $F_j(z, y_0)$ the following equality is true

$$\operatorname{Im}(z_j) = -\frac{\varphi_\xi(1,1)}{\pi n(z_j)}\left(1 + \mu_j(z_j, y_0)\right), \tag{3.48}$$

along with (3.47), where the function $\mu_j(z,y)$ satisfies the estimate

$$\left|\mu_j(z,y)\right| \leq \frac{1}{2}, \tag{3.49}$$

$$\forall\ z \in \{|z| > M_4\} \cap \{|\operatorname{Im} z| < T\}, \forall m > N_2, \forall y \in I_m.$$

Since $\varphi_\xi(1,1) \neq 0$, then (3.47a,b)-(3.49) lead to

$$\operatorname{Im}(z_j) \neq 0 \tag{3.50}$$

and

$$sgn(\operatorname{Im}(z_j)) = -sgn\left[\varphi_\xi(1,1)\right] \cdot sgn\left[\operatorname{Re}(z_j)\right] = -sgn\left[\varphi_\xi(1,1)\right] \cdot sgn\left(n(z_j)\right). \tag{3.51}$$

We show now that the multiplicity of the zero z_j is one, as long as
$z_j \in \{|z| > M_5\} \cap \{|\operatorname{Im} z| < T\}$, where the number $M_5 \geq M_4$ is chosen below. Since the formula (3.31) was derived from formulas (3.27) and (3.28), then (3.31) can be rewritten in the form

$$-iz \cdot F_j(z,y) =$$

$$e^{-iz}\left(p(1,y)+\frac{1}{iz}p_\xi(1,y)\right)-\left(p(1,y)-\frac{1}{iz}p_\xi(1,y)\right) \tag{3.52}$$

$$-\frac{1}{z^2}\left(e^{-iz}-1\right)p_{\xi\xi}(1,y)+\frac{1}{z^2}\int\limits_0^1 e^{-iz\xi}\partial_\xi^3 p_j(\xi,y)d\xi,$$

$$\forall\ z\in\{|z|>M_4\}\cap\{|\operatorname{Im} z|<T\},\forall m>N_2,\forall y\in I_m.$$

Differentiating both sides of the formula (3.52) with respect to z, setting then $m:=m_0, y:=y_0, z:=z_j\in\{|z|>M_4\}\cap\{|\operatorname{Im} z|<T\}$ and assuming that $F_j(z_j,y_0)=\partial_z F_j(z_j,y_0)=0$, we obtain

$$\exp(-iz_j)\left(p(1,y_0)+\frac{1}{iz_j}p_\xi(1,y_0)\right)=\frac{H_j(z_j,y_0)}{y_0 z_j^2}, \tag{3.53}$$

where the function $H_j(z,y)$ satisfies the estimate

$$|H_j(z,y)|\le C, \tag{3.54}$$

$$\forall\ z\in\{|z|>M_4\}\cap\{|\operatorname{Im} z|<T\},\forall m>N_2,\forall y\in I_m.$$

Hence, dividing (3.53) by $(p(1,y_0)+p_\xi(1,y_0)/iz_j)$ and using (3.29) and (3.54), we obtain

$$\exp(-iz_j)=\frac{\widetilde{H}_j(z_j,y_0)}{z_j^2}, \tag{3.55}$$

where the function $\widetilde{H}_j(z,y)$ satisfies the following estimate

$$\left|\widetilde{H}_j(z,y)\right|\le C. \tag{3.56}$$

$$\forall\ z\in\{|z|>M_4\}\cap\{|\operatorname{Im} z|<T\},\forall m>N_2,\forall y\in I_m.$$

Since $|\exp(-iz_j)|\ge\exp(-|\operatorname{Im}(z_j)|)$ and $|z_j|>M_4$, then replacing in (3.43) M_3 with M_4, we obtain $|\exp(-iz_j)|\ge\exp(-C/M_4)$. Hence, (3.55) and (3.56) imply that

$$\frac{C}{M_4^2}\ge\exp\left(-\frac{C}{M_4}\right). \tag{3.57}$$

Choose a number $M_5=M_5(N_2)\ge M_4$ so large that

$$\frac{C}{M_5^2}<\frac{1}{2}\exp\left(-\frac{C}{M_5}\right). \tag{3.58}$$

Again, we can assume (similarly with the above) that $z_j\in\{|z|>M_5\}\cap\{|\operatorname{Im} z|<T\}$. On the other hand, it follows from (3.57) that if the multiplicity of the zero z_j is greater than 1, then one should have

$$\frac{C}{M_5^2}\ge\exp\left(-\frac{C}{M_5}\right). \tag{3.59}$$

Inequalities (3.58) and (3.59) contradict with each other. This contradiction proves that the multiplicity of the zero z_j is 1, as long as $z_j \in \{|z| > M_5\} \cap \{|\operatorname{Im} z| < T\}$.

Set now $N = N(F_1, F_2) := N_2$ and $M = M(N) := M_5$. For the sake of definiteness, let $j = 1$. Consider the zero $z_1 \in \{|z| > M\} \cap \{|\operatorname{Im} z| < T\}$ of the function $F_1(z, y_0)$, i.e., $F_1(z_1, y_0) = 0$. We are going to prove now that $F_2(z_1, y_0) = 0$, which would be sufficient for the validity of Lemma 7. By (1.9) and (3.3) $F_1(z_1, y_0) \cdot \widehat{F}_1(z_1, y_0) = F_2(z_1, y_0) \cdot \widehat{F}_2(z_1, y_0) = 0$. Suppose that $F_2(z_1, y_0) \neq 0$. Then $\widehat{F}_2(z_1, y_0) = 0$. Hence, Lemma 3 implies that $F_2(\overline{z}_1, y_0) = 0$. Since $\operatorname{Re} z_1 = \operatorname{Re} \overline{z}_1$, then (3.47a) implies that $\operatorname{Re} z_1 = \operatorname{Re} \overline{z}_1 \neq 0$. Formulas (3.50) and (3.51) are valid for any zero $z_j \in \{|z| > M\} \cap \{|\operatorname{Im} z| < T\}$ of the function F_j for $j = 1, 2$. Denote $z_2 := \overline{z}_1$. Then $F_2(z_2, y_0) = 0$. Hence, using (3.50) and (3.51), we obtain

$$\operatorname{Im}(\overline{z}_1) \neq 0 \ \text{ and } sgn(\operatorname{Im}(\overline{z}_1)) = -sgn\left[\varphi_\xi(1,1)\right] \cdot sgn\left(\operatorname{Re} z_1\right).$$

But since $F_1(z_1, y_0) = 0$, then formulas (3.50) and (3.51) are also true for z_1, i.e.,

$$\operatorname{Im}(z_1) \neq 0 \ \text{ and } \ sign(\operatorname{Im}(z_1)) = -sgn\left[\varphi_\xi(1,1)\right] \cdot sgn\left(\operatorname{Re} z_1\right).$$

Thus, we have obtained that $\operatorname{Im}(z_1) \cdot \operatorname{Im}(\overline{z}_1) \neq 0$ and $sign(\operatorname{Im}(z_1)) = sgn(\operatorname{Im}(\overline{z}_1))$, which is impossible, since $\operatorname{Im}(z_1) = -\operatorname{Im}(\overline{z}_1)$. This proves that $F_2(z_1, y_0) = 0$. □

4 Zeros In $\{|z| < M\}$

Both in this and next sections numbers $N = N(F_1, F_2)$ and $M = M(N)$ are those, which were chosen in Lemma 7. Let $m > N$ be an integer. Fix an arbitrary number $y_1 \in I_m$. So, in both sections 4 and 5 we assume that $y = y_1$ and do not indicate the dependence on the parameter y (for brevity), keeping in mind, however that this dependence exists. Recall that we assume the existence of two functions $f_1(\xi, \eta)$ and $f_2(\xi, \eta)$, which correspond to the same function G in (1.2). Hence, (1.9) and (3.3) imply that

$$F_1(z) \cdot \widehat{F}_1(z) = F_2(z) \cdot \widehat{F}_2(z), \forall z \in \mathbb{C}. \tag{4.1}$$

Let $\Phi(z), z \in \mathbb{C}$ be an entire analytic function. Denote $Z(\Phi)$ the set of all zeros of this function. Also, denote

$$Z(M, \Phi) = Z(\Phi) \cap \{|z| < M\},$$

$$Z_0(\Phi) = Z(\Phi) \cap \{\operatorname{Im} z = 0\},$$

$$Z_+(M, \Phi) = Z(M, \Phi) \cap \{\operatorname{Im} z > 0\}$$

and

$$Z_-(M, \Phi) = Z(M, \Phi) \cap \{\operatorname{Im} z < 0\}.$$

Using Lemma 4, we obtain

$$Z_0(F_1) = Z_0(F_2). \tag{4.2}$$

By Lemma 7

$$Z(F_1)\diagdown Z(M,F_1) = Z(F_2)\diagdown Z(M,F_2). \tag{4.3}$$

Hence, Lemma 5 and (4.2) imply that in order to establish Theorem 1, it is sufficient to prove that

$$Z_+(M,F_1) = Z_+(M,F_2) \tag{4.4}$$

and

$$Z_-(M,F_1) = Z_-(M,F_2) \tag{4.5}$$

Let

$$\{a_k\}_{k=1}^{n_1} = Z_+(M,F_1) \text{ and } \{b_k\}_{k=1}^{n_2} = Z_+(M,F_2). \tag{4.6}$$

In both cases each zero is counted as many times as its multiplicity is. Consider functions $B_1(z)$ and $B_2(z)$,

$$B_1(z) = F_1(z) \cdot \prod_{k=1}^{n_1} \frac{z-\overline{a}_k}{z-a_k}, \tag{4.7}$$

$$B_2(z) = F_2(z) \cdot \prod_{k=1}^{n_2} \frac{z-\overline{b}_k}{z-b_k}. \tag{4.8}$$

The main result of this section is

Lemma 8. $Z(B_1) = Z(B_2)$.

Proof. By (4.2) and (4.6)-(4.8)

$$Z_0(B_1) = Z_0(B_2) \tag{4.9}$$

and

$$Z_+(M,B_1) = Z_+(M,B_2) = \varnothing. \tag{4.10}$$

Also, it follows from (4.3) and (4.6)-(4.8) that

$$Z(B_1)\diagdown Z(M,B_1) = Z(B_2)\diagdown Z(M,B_2). \tag{4.11}$$

Thus, (4.9)-(4.11) imply that all what we need to prove in this lemma is that

$$Z_-(M,B_1) = Z_-(M,B_2). \tag{4.12}$$

Let

$$\{\overline{c}_k\}_{k=1}^{s} = Z_-(M,B_1). \tag{4.13}$$

In (4.13) we count each zero $\overline{c}$ of the function B_1 as many times as its multiplicity is. The *main idea* of this proof is to show that a combination of Lemma 3 and (4.1) with (4.7) and (4.8) leads to

$$\{\overline{c}_k\}_{k=1}^{s} \subseteq Z_-(M,B_2). \tag{4.14}$$

To establish (4.14), we should consider several possible cases for zeros $\{\overline{c}_k\}_{k=1}^{s}$. Consider the zero $\overline{c}_1 \in Z_-(M,B_1)$. Either $F_1(c_1) = 0$ or $F_1(c_1) \neq 0$. We consider both these cases.

Case 1. Assume first that

$$F_1(c_1) = 0 \tag{4.15}$$

By (4.1) and (4.15) at least one of the two equalities (4.16), (4.17) takes place,

$$F_2(c_1) = 0, \tag{4.16}$$

$$\widehat{F}_2(c_1) = 0. \tag{4.17}$$

Assuming that (4.15) is true, consider cases (4.16) and (4.17) separately. We denote them C_{11} and C_{12} respectively.

Case $\mathbf{C}_{11}$. Suppose that (4.16) is true. Since by (4.13) $\operatorname{Im} c_1 > 0$, then $c_1 \in Z_+(M, F_2)$. Let, for example $c_1 = b_1$. Then $\overline{c}_1 = \overline{b}_1$. Therefore, $\overline{c}_1$ is present in the nominator of the first term of the product in (4.8), which implies that $B_2(\overline{c}_1) = 0$. Hence,

$$\overline{c}_1 \in Z_-(M, B_2). \tag{4.18}$$

Case $\mathbf{C}_{12}$. Assume that (4.16) is invalid, i.e.,

$$F_2(c_1) \neq 0. \tag{4.19}$$

Then (4.17) holds. Because of (4.19), the number c_1 is not present in denominators of the product in (4.8). On the other hand, (4.1), (4.15), (4.19) and Lemma 3 imply that $F_2(\overline{c}_1) = 0$. Hence, by (4.8) $B_2(\overline{c}_1) = 0$, which implies (4.18).

Thus, the assumption (4.15) led us to (4.18) in both possible cases C_{11} and C_{12}. Consider now the Case 2, which is opposite to the Case 1.

Case 2. Suppose that

$$F_1(c_1) \neq 0. \tag{4.20}$$

By (4.13)

$$B_1(\overline{c}_1) = 0. \tag{4.21}$$

Since $\operatorname{Im} c_1 > 0$, then (4.20) implies that $c_1 \notin Z_+(M, F_1)$, which means that $\overline{c}_1$ is not present in nominators of the product in (4.7). Hence, (4.7) and (4.21) lead to $F_1(\overline{c}_1) = 0$. Hence, by (4.1) at least one of the two equalities (4.22), (4.23) takes place

$$F_2(\overline{c}_1) = 0, \tag{4.22}$$

$$\widehat{F}_2(\overline{c}_1) = 0. \tag{4.23}$$

Assuming that (4.20) is true, consider cases (4.22) and (4.23) separately. We denote these two cases C_{21} and C_{22} respectively.

Case $\mathbf{C}_{21}$. Suppose that (4.22) is true. Since $\operatorname{Im} \overline{c}_1 < 0$, then $\overline{c}_1$ is not present in denominators of the product in (4.8). Hence, $B_2(\overline{c}_1) = 0$. This means that (4.18) is true.

Case $\mathbf{C}_{22}$. Assume now that (4.22) is invalid. Hence, (4.23) holds. Hence, Lemma 3 implies that (4.16) holds (note that we cannot now refer to the above case C_{11}, because being "inside of Case 2", we do not assume that (4.15) is valid). Since $\operatorname{Im} c_1 > 0$, then

(4.16) means that $c_1 \in Z_+(M, F_2)$. Let, for example $c_1 = b_1$. Hence, $\overline{c}_1 = \overline{b}_1$. Therefore, $\overline{c}_1$ is present in the nominator of the first term of the product in (4.8), which implies that $B_2(\overline{c}_1) = 0$. This means that (4.18) is true.

Thus, the conclusion from Cases 1 and 2 is that (4.18) holds.

We show now that

$$\overline{c}_2 \in Z_-(M, B_2). \tag{4.24}$$

Hence, using (4.13) and (4.18), we obtain that

$$B_{11}(z) = \frac{B_1(z)}{z - \overline{c}_1} \quad \text{and} \quad B_{21}(z) = \frac{B_2(z)}{z - \overline{c}_1}. \tag{4.25}$$

are entire analytic functions. Also, by (4.13) and (4.25)

$$\{\overline{c}_k\}_{k=2}^{s} = Z_-(M, B_{11}). \tag{4.26}$$

It follows from (4.25) that in order to prove (4.24), it is sufficient to prove that

$$\overline{c}_2 \in Z_-(M, B_{21}). \tag{4.27}$$

We again consider two possible cases.

Case 3. Suppose that (4.15) is true. Because of (4.1) and (4.15), at least one of equalities (4.16) or (4.17) holds. We again consider cases (4.16) and (4.17) separately and denote them C_{31} and C_{32} respectively. Since (4.15) holds, we can assume that $c_1 = a_1$.

Case C_{31}. Suppose that (4.16) holds. Let, for example $c_1 = a_1 = b_1$. Introduce functions $F_{11}(z)$ and $F_{21}(z)$ by

$$F_{11}(z) = \frac{F_1(z)}{z - a_1} \text{ and } F_{21}(z) = \frac{F_2(z)}{z - a_1}. \tag{4.28}$$

Since $F_1(a_1) = F_2(a_1) = 0$, then $F_{11}(z)$ and $F_{21}(z)$ are entire analytic functions. Hence, $Z_+(M, F_{11}) = \{a_k\}_{k=2}^{n_1}$ and $Z_+(M, F_{21}) = \{b_k\}_{k=2}^{n_2}$. It follows from (4.7), (4.8), (4.15), (4.25), and (4.28) that formulas for functions $B_{11}(z)$ and $B_{21}(z)$ can be written as

$$B_{11}(z) = F_{11}(z) \prod_{k=2}^{n_1} \frac{z - \overline{a}_k}{z - a_k} \text{ and } B_{21}(z) = F_{21}(z) \prod_{k=2}^{n_2} \frac{z - \overline{b}_k}{z - b_k}. \tag{4.29}$$

Note that by (4.28)

$$\widehat{F}_{11}(z) = \overline{F}_{11}(\overline{z}) = \frac{\widehat{F}_1(z)}{z - \overline{a}_1} \text{ and } \widehat{F}_{21}(z) = \overline{F}_{21}(\overline{z}) = \frac{\widehat{F}_2(z)}{z - \overline{a}_1}.$$

Hence,

$$F_{j1}(z) \cdot \widehat{F}_{j1}(z) = \frac{F_j(z) \cdot \widehat{F}_j(z)}{(z - a_1)(z - \overline{a}_1)}.$$

Hence, (4.1) leads to

$$F_{11}(z)\cdot\widehat{F}_{11}(z)\equiv F_{21}(z)\cdot\widehat{F}_{21}(z). \tag{4.30}$$

Relations (4.15), (4.25), (4.26) and (4.28)-(4.30) enable us to repeat arguments of the above Case 1 replacing $\overline{c}_1$ with $\overline{c}_2$, $B_1(z)$ with $B_{11}(z), B_2(z)$ with $B_{21}(z), F_1(z)$ with $F_{11}(z)$, and $F_2(z)$ with $F_{21}(z)$. Thus, we obtain (4.27), which, in turn leads to (4.24).

Case $\mathbf{C_{32}}$. Assume now that (4.16) is invalid. Then (4.17) holds. Since we are still "within Case 3", then $c_1=a_1$. Hence, (4.17) and Lemma 3 imply that $F_2(\overline{a}_1)=0$. Introduce entire analytic functions $F_{12}(z)$ and $F_{22}(z)$ as

$$F_{12}(z)=\frac{F_1(z)}{z-a_1},\ F_{22}(z)=\frac{F_2(z)}{z-\overline{a}_1}. \tag{4.31}$$

Then

$$F_{j2}(z)\cdot\widehat{F}_{j2}(z)=\frac{F_j(z)\cdot\widehat{F}_j(z)}{(z-a_1)(z-\overline{a}_1)}.$$

Hence, (4.1) implies that

$$F_{12}(z)\cdot\widehat{F}_{12}(z)\equiv F_{22}(z)\cdot\widehat{F}_{22}(z). \tag{4.32}$$

Hence, using (4.7), (4.8), (4.15), (4.25), and (4.31), we conclude that formulas for functions $B_{11}(z)$ and $B_{21}(z)$ can be written as

$$B_{11}(z)=F_{12}(z)\prod_{k=2}^{n_1}\frac{z-\overline{a}_k}{z-a_k}\quad\text{and } B_{21}(z)=F_{22}(z)\prod_{k=1}^{n_2}\frac{z-\overline{b}_k}{z-b_k}. \tag{4.33}$$

Therefore, relations (4.15), (4.25), (4.26), and (4.31)-(4.33) enable us to repeat arguments of the above Case 1 replacing $\overline{c}_1$ with $\overline{c}_2$, $B_1(z)$ with $B_{11}(z)$,$B_2(z)$ with $B_{21}(z), F_1(z)$ with $F_{12}(z)$, and $F_2(z)$ with $F_{22}(z)$. This leads to (4.27), which, in turn implies (4.24).

Thus, both cases C_{31} and C_{32} led us to (4.24). This proves that if $F_1(c_1)=0$, then $\overline{c}_2\in Z_-(M,B_2)$.The alternative (to the Case 3) Case 4 with $F_1(c_1)\neq 0$ is considered similarly. The only difference is that instead of Case 1 we should refer to Case 2 for the repetition of the arguments. Thus, we have established that both zeros $\overline{c}_1,\overline{c}_2\in Z_-(M,B_2)$. To prove that $\overline{c}_3\in Z_-(M,B_2)$, we need to consider entire analytic functions

$$B_{12}(z)=\frac{B_{11}(z)}{z-\overline{c}_2}\quad\text{and}\quad B_{12}(z)=\frac{B_{21}(z)}{z-\overline{c}_2}$$

and repeat the above. Therefore, repeating this process, we obtain (4.14). Hence, (4.13) and (4.14) lead to $Z_-(M,B_1)\subseteq Z_-(M,B_2)$. Similarly, $Z_-(M,B_1)\subseteq Z_-(M,B_2)$. Thus, (4.12) is true. $\square$

Consider now zeros of functions $F_1(z)$ and $F_2(z)$ in $\{\operatorname{Im} z<0\}\cap\{|z|<M\}$. Let $\{a_k'\}_{k=1}^{n_3}=Z_-(M,F_1)$ and $\{b_k'\}_{k=1}^{n_4}=Z_-(M,F_2)$. Similarly with (4.7) and (4.8) we introduce functions $B_1^-(z)$ and $B_2^-(z)$ by

$$B_1^-(z)=F_1(z)\cdot\prod_{k=1}^{n_3}\frac{z-\overline{a}_k'}{z-a_k'}\quad\text{and}\quad B_2^-(z)=F_2(z)\cdot\prod_{k=1}^{n_4}\frac{z-\overline{b}_k'}{z-b_k'}.$$

Lemma 9. $Z\left(B_1^-\right) = Z\left(B_2^-\right)$.

We omit the proof of this lemma, because it is quite similar with the proof of Lemma 8.

5 Proof of Theorem 1

We recall that in this section numbers $N = N(F_1, F_2)$ and $M = M(N)$ are those, which were chosen in Lemma 7, $m > N$ is an integer, an arbitrary number $y_1 \in I_m$ is fixed, and we set $y := y_1$. So, for brevity we do not indicate in this section the dependence on the parameter y. It was established in the beginning of section 4 that in order to prove Theorem 1, it is sufficient to proof (4.4) and (4.5).

By (1.8), (1.9), (3.5), (4.7) and (4.8) $|B_1(x)|^2 = |B_2(x)|^2$, $\forall x \in \mathbb{R}$. Hence, lemmata 5 and 8 imply that the function $g(z)$ and the integer k in analogs of infinite products (3.4) for functions B_1 and B_2 are the same for both these functions. Hence, Lemma 8 and (3.4) imply that $B_1(z) = B_2(z), \forall z \in \mathbb{C}$. Hence, (4.7) and (4.8) lead to

$$F_1(z) + \left(\prod_{k=1}^{n_1} \frac{z - \overline{a}_k}{z - a_k} - 1\right) F_1(z) = F_2(z) + \left(\prod_{k=1}^{n_2} \frac{z - \overline{b}_k}{z - b_k} - 1\right) F_2(z), \forall z \in \mathbb{C}. \tag{5.1}$$

Set in (5.1) $z := x \in (-\infty, \infty)$ and apply the inverse Fourier transform with respect to x,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (...)\, e^{ix\xi} dx.$$

We can write each of functions

$$Q_1(x) = \left[\prod_{k=1}^{n_1} \frac{x - \overline{a}_k}{x - a_k} - 1\right] \text{ and } Q_2(x) = \left[\prod_{k=1}^{n_2} \frac{x - \overline{b}_k}{x - b_k} - 1\right]$$

as a sum of partial fractions, i.e., as a sum of

$$\frac{D_{sk}}{(x - d_k)^s},\ \ \operatorname{Im} d_k > 0,\ j = 1, ..., t, \quad t \leq \max(n_1, n_2)$$

with certain constants D_{sk}, where $d_k \in Z_+(M, F_1) \cup Z_+(M, F_2)$. The theory of residuals [1] implies that

$$\int_{-\infty}^{\infty} \frac{1}{(x - d_k)^s} \cdot e^{ix\xi} dx = H(\xi)\, P_{s-1}(\xi)\, e^{id_k\xi}, \tag{5.2}$$

where $H(\xi) = 1$ for $\xi > 0$ and $H(\xi) = 0$ for $\xi < 0$ is the Heaviside function and $P_{s-1}(\xi)$ is a polynomial of the degree $s - 1$.

Let $V_1(\xi)$ and $V_2(\xi)$ be the inverse Fourier transforms of functions $Q_1(x)$ and $Q_2(x)$ respectively. By (5.2) $V_1(\xi) = V_2(\xi) = 0$ for $\xi < 0$. Thus, (5.1) and (5.2) imply that

$$\widehat{p}_1(\xi) + \int\limits_0^{\xi} \widehat{p}_1(\xi - \theta) V_1(\theta) d\theta = \widehat{p}_2(\xi) + \int\limits_0^{\xi} \widehat{p}_2(\xi - \theta) V_2(\theta) d\theta, \tag{5.3}$$

where functions $\widehat{p}_1(\xi)$ and $\widehat{p}_2(\xi)$ are defined as $\widehat{p}_1(\xi) := p_1(\xi, y_1), \widehat{p}_2(\xi) := p_2(\xi, y_1)$ and functions $p_1(\xi, y)$ and $p_2(\xi, y)$ were defined in (3.11). By (3.12) $\widehat{p}_1(\xi) = \widehat{p}_2(\xi) := \widehat{p}(\xi)$ for $\xi \in (0, \delta)$. Denote $W(\xi) = V_1(\xi) - V_2(\xi)$. Hence, (5.3) leads to

$$\int\limits_0^{\xi} \widehat{p}(\xi - \theta) W(\theta) d\theta = 0, \quad \xi \in (0, \delta). \tag{5.4}$$

Differentiate (5.4) with respect to ξ and note that by (3.13) $\widehat{p}(1) = \widehat{p}(0)$ and by (3.22) $\widehat{p}(0) \neq 0$. We obtain the following Volterra integral equation with respect to the function $W(\xi)$

$$W(\xi) + \frac{1}{\widehat{p}(0)} \int\limits_0^{\xi} \widehat{p}(\xi - \theta) W(\theta) d\theta = 0, \quad \xi \in (0, \delta). \tag{5.5}$$

Hence, $W(\xi) = 0$ for $\xi \in (0, \delta)$. Since the function $W(\xi)$ is a linear combination of functions $P_{s-1}(\xi) e^{id_k \xi}$ for $\xi > 0$, then $W(\xi)$ is analytic with respect to $\xi \in (0, \infty)$ and can be continued in $\mathbb{C}$ as an entire analytic function $W(z)$. Therefore, $W(z) = 0$ for all $z \in \mathbb{C}$. Hence, $Z_+(M, F_1) = Z_+(M, F_2)$, which proves (4.4). We omit the proof of (4.5), since it can be carried out quite similarly via the use of Lemma 9 instead of Lemma 8. □

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