SOLVABILITY $p-$LAPLACE EQUATION WITH DILATIONS AND COMPRESSIONS

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ABSTRACT. We consider $p-$Laplace, $p > 1$ boundary value problem with certain functional operator situated in major terms and prove weak existence theorem.

1. INTRODUCTION

The $p-$Laplacian appears in the study of flow through porus media ($p = 3/2$), nonlinear elasticity ($p \geq 2$) and glaciology ($p \in (1, 4/3]$). We refer to [6] for more background material.

For $p-$Laplace equation with right hand side homogeneous in u, existence and nonexistence results were obtained by many authors, see for example [4], [9], [6], [7]. Variational methods were employed in [10] and others when trying to find positive solutions.

Functional differential operators placed in the Laplacian including dilation/compression operators were studied in [3], [5].

We consider elliptic problem with nonlocal linear operators situated inside the Laplacian:

$$\Delta_p Au = f(x) \in H^{-1,p^*}(M), \quad u\mid_{\partial M}= 0, \quad \frac{1}{p} + \frac{1}{p^*} = 1, \qquad (1.1)$$

here A is a certain bounded linear operator. If operator A is a small perturbation of the identity then, as it is not hard to show, the problem remains coercive, as for monotonicity, this property is much delicate to check, and possibly it is destroyed even under the small perturbation. It should be noted that in the linear case ($p = 2$) the monotonicity property is steady under the small perturbation.

On the other hand, using standard variational technique we have to consider weak convergent sequences but the operator A has not necessarily to be continuous with respect to weak convergence.

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Our version of variational method allows to prove the existence theorem avoiding of using the monotonicity or weak convergence.

Linear case of equation (1.1) with nonlocal operator of the argument's dilation/compression

$$Au = \sum_{i=-k}^{k} a_i u(q^i x), \quad q > 1, \quad a_j \in \mathbb{R}$$

was studied in [12], [13]. The case of coefficients a_k dependent on x was considered in [14] in the linear setup.

Boundary value problems for elliptic functional-differential equations have been studied in the articles [2], [11] and others.

Boundary value problems for elliptic equations with shifts in the space variables were considered in [8], [17].

A theory of boundary value problems for elliptic differential-difference equation in a bounded domain was constructed in [15].

2. Main theorem

Let M be a bounded domain in $\mathbb{R}^m$ with smooth boundary ∂M and $x = (x_1, \ldots, x_m)$ be coordinates in $\mathbb{R}^m$. Denote by ∂_i the partial derivative in the variable x_i.

For any $v \in L^r(M), \quad r \geq 1$ and $w \in L^{r^*}(M), \quad 1/r + 1/r^* = 1$ we put

$$(v, w) = \int_M v(x) w(x)\, dx.$$

Denote by $\mathcal{W}$ a space of bounded linear operators $G : H_0^{1,p}(M) \to H^{1,p}(M), \quad p > 1$ with the following properties.

For any operator $G \in \mathcal{W}$ there exists a bounded operator $G^+ : L^p(M) \to L^p(M)$ such that:

$$\partial_i G = G^+ \partial_i, \quad \|G^+ w\|_{L^p(M)} \geq c_0 \|w\|_{L^p(M)}. \tag{2.1}$$

Positive constant c_0 depends only on G. Let

$$G^{+*} : L^{p^*}(M) \to L^{p^*}(M), \quad \frac{1}{p} + \frac{1}{p^*} = 1$$

be conjugated operator, then there is a bounded operator $G^{+*+} : H^{-1,p^*}(M) \to H^{-1,p^*}(M)$ satisfying the equation:

$$\partial_i G^{+*} = G^{+*+} \partial_i. \tag{2.2}$$

It is clear, the space $\mathcal{W}$ is an associative algebra with unit.

Denote by $\mathcal{L}(H_0^{1,p}(M))$ the space of bounded linear operators of $H_0^{1,p}(M)$ to itself, and if $T \in \mathcal{L}(H_0^{1,p}(M))$ then

$$\|Tv\|_{H_0^{1,p}(M)} \geq c_5 \|v\|_{H_0^{1,p}(M)}, \tag{2.3}$$

here c_5 is another positive constant dependent only on T.

Introduce an operator $A : H_0^{1,p}(M) \to H^{1,p}(M)$ by the formula:

$$A = GT, \quad G \in \mathcal{W}, \quad T \in \mathcal{L}(H_0^{1,p}(M)). \tag{2.4}$$

Another operator to regard is $B = T^* G^{+*+} : H^{-1,p^*}(M) \to H^{-1,p^*}(M)$.

Consider the p-Laplace operator:

$$\Delta_p v = \sum_{i=1}^{m} \partial_i (|\partial_i v|^{p-2} \partial_i v).$$

The main object of our study is the following elliptic problem:

$$\Delta_p Au = f(x) \in L^{p^*}(M), \quad u \mid_{\partial M} = 0. \tag{2.5}$$

Theorem 1. *If $Bf \in L^{p^*}(M)$ and $\ker B = 0$ then problem (2.5) has a weak solution $u \in H_0^{1,p}(M)$ i.e. for any $h \in H_0^{1,p}(M)$ one has*

$$-\sum_{i=1}^{m} (|\partial_i Au|^{p-2} \partial_i Au, \partial_i h) = (f, h).$$

3. Application: Dilation and Compression Operators

Assume the domain M to be star-shaped with respect to the origin.

Define an operator $\sigma : L^r(M) \to L^r(\mathbb{R}^m), \quad r \geq 1$ as follows:

$$\sigma v = \begin{cases} v(x) & \text{if} \quad x \in M, \\ 0 & \text{if} \quad x \in \mathbb{R}^m \backslash \overline{M}. \end{cases}$$

Construct an operator: $R_a v = (\sigma v)(ax), \quad a > 0$. This operator dilates/compresses the graph of the function $v(x)$ in a times, here a is a constant.

One can show that the operator R_a is bounded as an operator of the space $H_0^{1,r}(M), \quad r \geq 1$ to $H^{1,r}(M)$ and as an operator of $L^r(M)$ to itself. Changing the variable in the integral one obtains that

$$\|R_a\|_{L^p(M) \to L^p(M)} \leq a^{-\frac{m}{p}}. \tag{3.1}$$

The operator R_a commutes with the partial derivatives in the following fashion:

$$\partial_i R_a = R_a^+ \partial_i, \quad R_a^+ = a R_a.$$

The operator R_a^* can also be written in the explicit form: letting $R_a^* = a^{-m} R_{a^{-1}}$ and by the change of variable $x \mapsto a^{-1} x$ in the integral we make sure that $(R_a v, w) = (v, R_a^* w)$. So, one has: $R_a^{+*} = a^{1-m} R_{a^{-1}}$ and $R_a^{+*+} = a^{-m} R_{a^{-1}} = R_a^*$.

To define operator R_a^{+*+} on $H^{-1,p^*}(M)$ recall a theorem.

Theorem 2 ([1])**.** *Any element $g \in H^{-1,p^*}(M)$ presents as follows*

$$g = g_0 + \sum_{i=1}^{m} \partial_i g_i, \quad g_j \in L^{p^*}(M), \quad j = 0, \ldots, m. \tag{3.2}$$

Moreover,

$$\|g\|^{p^*}_{H^{-1,p^*}(M)}=\min\sum_{i=0}^m\|g_i\|^{p^*}_{L^{p^*}(M)}.$$

The minimum being taken over, and attained on the set of all functions $g_i \quad i=0,\dots,m$ *for which (3.2) holds.*

Now we put:

$$(R_a^{+*+}g,h)=(g_0,R_ah)-\sum_{i=1}^m(g_i,\partial_iR_ah),\quad h\in H^{1,p}_0(M).$$

With the help of Theorem 2 one finds:

$$\|R_a^{+*+}\|_{H^{-1,p^*}(M)\to H^{-1,p^*}(M)}\le\max\{a^{-\frac{m}{p}},a^{1-\frac{m}{p}}\}. \tag{3.3}$$

Introduce an operator

$$G=G_1\dots G_n,\quad G_i=\mathrm{id}_{H^{1,p}_0(M)}+\lambda_iR_{a_i},\quad \lambda_i\in\mathbb{C},\quad i=1,\dots,n.$$

We claim that under certain assumptions on constants λ_i the operators G_i belong to the space $\mathcal{W}$ and thus the operator G also belongs to $\mathcal{W}$. Moreover, if the operator B is taken with such G and

$$T=\mathrm{id}_{H^{1,p}_0(M)},$$

then the conditions of Theorem 1 hold and thus, problem (2.5) being considered with the operator $A=G$, has the weak solution.

We take constants λ_i such that if a function u is real-valued then the function Gu is also real-valued.

Evidently, if the operators $G_i^+,G_i^{+*+},\quad i=1,\dots,n$ are invertible, then all the specified conditions are fulfilled.

Consider operators $G_i^+=\mathrm{id}_{L^p(M)}+\lambda_ia_iR_{a_i}$. By formula (3.1), for invertibility of these operators it is sufficient to have

$$|\lambda_i|a_i\|R_{a_i}\|_{L^p(M)}\le|\lambda_i|a_i^{1-\frac{m}{p}}<1,\quad i=1,\dots,m.$$

By the same arguments and formula (3.3) for invertibility of operators G_i^{+*+} we put

$$|\lambda_i|\max\{a_i^{-\frac{m}{p}},a_i^{1-\frac{m}{p}}\}<1,\quad i=1,\dots,m.$$

As a result of these observations formulate the following proposition.

Proposition 1. *If*

$$|\lambda_i|\max\{a_i^{-\frac{m}{p}},a_i^{1-\frac{m}{p}}\}<1,\quad i=1,\dots,m. \tag{3.4}$$

and the operator G takes any real-valued function to the real-valued one, then problem (2.5) being considered with the operator $A=GT$, has the weak solution.

4. Proof of Theorem 1

Consider a linear function
$$J : H_0^{1,p}(M) \to \mathbb{R}, \quad J(v) = (Bf, v).$$
We are going to minimize this function on the level set of a function
$$F : H_0^{1,p}(M) \to \mathbb{R}, \quad F(v) = \int_M \sum_{i=1}^m |\partial_i Av|^p \, dx.$$
Such a variational scheme is not quite standard: problems with homogeneous right hand side are usually treated in the opposite way: it is the function $F(v)$ whose conditional extremum is looked for.

Lemma 1. *There are positive constants c_2, c_4 such that for any $v \in H_0^{1,p}(M)$ one has*
$$c_4 \|v\|^p_{H_0^{1,p}(M)} \geq F(v) \geq c_2 \|v\|^p_{H_0^{1,p}(M)}.$$

Proof. The estimate from above immediately follows from the boundedness of the operator A:
$$F(v) \leq \|Av\|^p_{H^{1,p}(M)} \leq c_4 \|v\|^p_{H_0^{1,p}(M)}.$$
Using formulas (2.1), (2.3) we derive
$$\sum_{j=1}^m \int_M |\partial_j GTu|^p \, dx = \sum_{j=1}^m \int_M |G^+ \partial_j Tu|^p \, dx \geq c_0^p \|Tu\|^p_{H_0^{1,p}(M)}$$
$$\geq c_0^p c_5^p \|u\|^p_{H_0^{1,p}(M)}.$$
□

The following Corollary is the main of importance.

Corollary 1. *A function*
$$\nu(v) = (F(v))^{\frac{1}{p}}$$
is an equivalent norm of $H_0^{1,p}(M)$ and thus, the set
$$S = \{v \in H_0^{1,p}(M) \mid F(v) = 1\}$$
is a unit sphere of $H_0^{1,p}(M)$ with respect to the norm ν.

Lemma 2. *The function $J\,|_S$ attains its minimum, say at $\hat{v}$.*

Proof. By the conditions of the Theorem, the function J can be extended continuously to the space $L^p(M)$, we do not introduce another notation for this extension.

Theorem 3 ([16])**.** *Let V, W be Banach spaces, and let the space V be reflexive. If a linear operator $Q : V \to W$ is compact, then the image of the closed unit ball $B \subset V$ under Q is compact.*

Applying this theorem to the embedding $H_0^{1,p}(M)\subset L^p(M)$ we see that the ball

$$B=\{v\in H_0^{1,p}(M)\mid \nu(v)\le 1\}$$

is a compact subset of $L^p(M)$. Thus the function J attains its minimum at B, denote this minimum by $\hat v$. Since J is a linear function we have: $\hat v\in S=\partial B$. $\square$

Consider weak derivatives:

$$J'h=\frac{d}{ds}\Big|_{s=0}J(\hat v+sh)=(Bf,h)=J(h),\quad h\in H_0^{1,p}(M), \tag{4.1}$$

$$F'(\hat v)h=\frac{d}{ds}\Big|_{s=0}F(\hat v+sh)=p\sum_{i=1}^m(|\partial_iA\hat v|^{p-2}\partial_iA\hat v,\partial_iAh). \tag{4.2}$$

There is standard relation between these derivatives. This relation is described as follows.

Lemma 3. *The following inclusion holds:*

$$\ker F'(\hat v)\subseteq \ker J'. \tag{4.3}$$

Proof. Let

$$h\in\ker F'(\hat v). \tag{4.4}$$

Define a function of two real arguments by the formula:

$$\varphi(y,t)=F(y\hat v+th).$$

We want to show that for some positive t_0 there exists such a function $y(t)\in C^1(-t_0,t_0),\quad y(0)=1$ that satisfies an equation

$$\varphi(y(t),t)=1. \tag{4.5}$$

If it would be true, then the set $\{y(t)\hat v+th\mid |t|<t_0\}$ is a curve on the manifold S. This curve passes across the point $\hat v$ and has h as a tangent vector at this point.

Observing that $\varphi(1,0)=F(\hat v)=1$ and

$$\begin{aligned}\varphi_y(y,t)\mid_{(y,t)=(1,0)}&=F'(y\hat v+th)\hat v\mid_{(y,t)=(1,0)}\\&=F'(\hat v)\hat v=pF(\hat v)=p\ne 0,\end{aligned} \tag{4.6}$$

we see that equation (4.5) satisfies to the conditions of the implicit function theorem, thus we obtain desired function $y(t)$.

Note that the derivative of the function $y(t)$ vanishes at zero:

$$y_t(0)=0. \tag{4.7}$$

Indeed, since $F(y(t)\hat v+th)=1$ for $|t|<t_0$, we have

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}F(y(t)\hat v+th)&=F'(y(0)\hat v)(y_t(0)\hat v+h)\\&=y_t(0)F'(\hat v)\hat v+F'(\hat v)h=0.\end{aligned} \tag{4.8}$$

Combining formula (4.8) with (4.4) and (4.6) we obtain (4.7).

Since $y(t)\hat{v}+th\in S$ for $|t|<t_0$, and by Lemma 2 a function $\rho(t)=J(y(t)\hat{v}+th)$ attains its local minimum at $t=0$ so we have:

$$\rho_t(0)=J'(y_t(0)\hat{v}+h)=0.$$

This formula and formula (4.7) imply: $J'h=0$. □

According to the Lagrange multipliers theorem, formula (4.3) implies that there exists such a constant λ that $\lambda F'(\hat{v})=J'$. In other words for any $h\in H_0^{1,p}(M)$ formulas (4.1), (4.2) give:

$$p\lambda\sum_{i=1}^{m}(|\partial_i A\hat{v}|^{p-2}\partial_i A\hat{v},\partial_i Ah)=(Bf,h). \tag{4.9}$$

Let us transform the left side of (4.9). Put $w_i=|\partial_i A\hat{v}|^{p-2}\partial_i A\hat{v}$, and write

$$(w_i,\partial_i Ah)=(w_i,\partial_i GTh). \tag{4.10}$$

Using formulas (2.1), (2.2) we calculate:

$$\begin{gathered}(w_i,\partial_i GTh)=(w_i,G^{+}\partial_i Th)=(G^{+*}w_i,\partial_i Th)\\ =-(\partial_i G^{+*}w_i,Th)=-(G^{+*+}\partial_i w,Th)=-(T^{*}G^{+*+}\partial_i w_i,h).\end{gathered}$$

By this formula and (4.10) it follows that

$$(w_i,\partial_i Ah)=-(B\partial_i w_i,h). \tag{4.11}$$

Substituting (4.11) to (4.9) one has:

$$-p\lambda\sum_{i=1}^{m}B\partial_i(|\partial_i A\hat{v}|^{p-2}\partial_i A\hat{v})=Bf. \tag{4.12}$$

By assumption of the Theorem $\ker B=0$, thus equation (4.12) is equivalent to the following one:

$$-p\lambda\sum_{i=1}^{m}\partial_i(|\partial_i A\hat{v}|^{p-2}\partial_i A\hat{v})=f. \tag{4.13}$$

Let a constant τ be a root of the equation $-p\lambda|\tau|^{p-2}\tau=1$. Then substituting in (4.13) the expression $\hat{v}=\tau u$, we see that the function u solves (2.5).

The Theorem is proved.

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References

[1] R. A. Adams, Sobolev Spaces. Academic Press, NY, 1975.

[2] A. B. Antonevich, The index and normal solvability of general elliptic boundary value problems with a finite group of shifts on the boundary. Differential Equations, 8 (1972).

[3] K. Cooke and J. Wiener Distributional and analytic solutons of functional-differential equations, J. Math. Anal. Appl. 98 (1984), 11-129.

[4] M. Guedda and L. Veron, Local and global properties of solutions of quasilinear elliptic equations. J. Differential Equations 76 (1988) 159-189.

[5] A. Iserles and Y. Liu, On functional-differential equations with proportional delays, J. Math. Anal. Appl. 207 (1997), 73-95.

[6] J. L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites Non-linéaires. Dunod, Paris, 1969.

[7] A. Szulkin, Ljusternik-Schnirelmann theory on C^1 manifolds. Ann. Inst. H. Poincaré. Anal. Non Linéaire 5 (1988) 119-139.

[8] G.G. Onanov A. L. Skubachevskiĭ, Differential equations with diviating arguments in stationary problems in the mechanics of deforming media., Soviet Appl. Mech. 15 (1979), 391-397. MR 80e:73055.

[9] M. Otani, Existence and nonexistence of nontrivial solutions of some nonlinear degenerate elliptic equations. J. Funct. Anal. 76 (1988) 140-159.

[10] M. Otani and T. Teshima, On the first eigenvalue of some quasilinear elliptic equations. Proc. Japan Acad. Ser. A Math. Sci. 64 (1988) 8-10

[11] V. S. Rabinovich, The solvability of differential-difference equations in $\mathbb{R}^n$ and in a half-space, Soviet Math. Dokl. 19 (1978) 1498-1502 (1979). MR 80h:35125.

[12] L.E. Rossovskiĭ, The coercivity of functional-differential equations. Math. Notes, 59(1996), 75-82, MR 97b:35184.

[13] L.E. Rossovskiĭ, Boundary Value Problems for Elliptic Functional-Differential Equations with Dilatations and Compressions of the Arguments. Proceedings of Moscow Math. Soc. vol. 62 (2001).

[14] L.E. Rossovskiĭ, On the boundary value problems for the elliptic functional-differential equation with conntractions. Functional Differential Equations, (2001), 8, pp. 395-406.

[15] A. L. Skubachevskiĭ, Elliptic functional differential equations and applications, Operator Theory, Advances and Applications, Birkhäuser, Basel-Boston-Berlin, 1996.

[16] M. E. Taylor, Partial Differential Equations, Vol. 1, Springer, New York, 1996.

[17] M A. Vorontsov N. G. Iroshnikov and R. L. Abernathy, Diffractive patterns in a nonlinear optical two-dimensional fidback system with field rotation, Chaos, Solitons and Fractals, 4 (1994)

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