PEANO TYPE THEOREM FOR ABSTRACT PARABOLIC EQUATION

OLEG ZUBELEVICH

DEPARTMENT OF DIFFERENTIAL EQUATIONS AND MATHEMATICAL PHYSICS
PEOPLES FRIENDSHIP UNIVERSITY OF RUSSIA
ORDZHONIKIDZE ST., 3, 117198, MOSCOW, RUSSIA
E-MAIL: OZUBEL@YANDEX.RU

Abstract. Parabolic equation with non-Lipschitz nonlinearity is considered. Peano-type existence theorem is proved. New class of parabolic equations that have analytic solutions is obtained.

1. Introduction

This paper is devoted to quasi-linear parabolic equations with a non-Lipschitz nonlinearity. In the classical setup a quasi-linear initial value parabolic problem has the form

$$u_t = f(t, u, \nabla^k u) + Au, \quad u\mid_{t=0} = \hat{u}. \tag{1.1}$$

Here A is a linear elliptic operator of order n and the term $\nabla^k u$ symbolizes the derivatives of u up to order k. Besides this, equation (1.1) must be provided with the boundary conditions.

If the function $\hat{u}$ belongs to a suitable space, the mapping f is Lipschitz in a certain sense and $k < n$ then problem (1.1) has a unique local-in-time solution. This simple observation easily follows from the contracting mapping principle.

There is a vast literature devoted to local and global existence theorems for different quasi-linear parabolic equations. But all the known results have at least two common points. First, they use Lipschitz-type conditions for the mapping f (or for its approximations) and second, they are based on the assumption that $k < n$.

At first glance both of these points are seemed to be independent from each other. Actually if we suppose that the Lipschitz-type conditions are fulfilled then to get existence theorem we have to assume that $k < n$.

2000 *Mathematics Subject Classification.* 35K90, 35R05, 35R10.

Key words and phrases. Peano theorem, abstract Cauchy problem, nonlocal problems, functional-differential equations, integro-differential equations, quasilinear parabolic equations.

Partially supported by grants RFBR 02-01-00400.

If we reject the Lipschitz hypothesis then we obtain a large class of equations that are also free from the second assumption but have an existence theorem.

This effect takes place not only for parabolic equations. If we consider the Cauchy-Kowalewski problem in the non-Lipschitz setup [10] then there are equations such that the order of derivatives in the right side is greater than in the left one but the solution exists.

Such a type problems do not belong to the classical partial differential equations but to the functional-differential equations and the differential equations with nonlocal terms.

The main mathematical tool we use is a locally convex space version of the Schauder fixed point theorem and theory of scales of Banach spaces. Another approaches to the abstract parabolic problems in the Lipschitz setup contain in [1], [3].

2. Main theorem

Consider a scale of Banach spaces $\{E_s, \|\cdot\|_s\}_{s>0}$ such that all the embeddings $E_{s+\delta}\subseteq E_s,\quad \delta>0$ are completely continuous and

$$\|\cdot\|_s\le\|\cdot\|_{s+\delta}. \tag{2.1}$$

The parameter s may not be necessarily run through all the positive real numbers. We do not use the spaces E_s with big s and one can assume for example that $s\in(0,1)$. It is just for simplicity's sake that we consider $s>0$.

Let $B_s(r)$ be an open ball of the space E_s with radius r and center at the origin.

Introduce constants $C,T,R>0,\quad \alpha\ge 0$.

Let $A:E_{s+\delta}\to E_s$ be a bounded linear operator. Assume that this operator generates a strongly continuous semigroup $e^{At}:E_s\to E_s,\quad t\ge 0$ such that for any $u\in E_{s+\delta}$ and for any $v\in E_s$ the following formulas are valid

$$\lim_{h\to 0+}\Big\|\frac{1}{h}\Big(e^{Ah}-\mathrm{id}_{E_{s+\delta}}\Big)u-Au\Big\|_s=0,\quad \|e^{At}v\|_s\le C\|v\|_s. \tag{2.2}$$

Definition 1. *The semigroup e^{At} is said to be parabolic if there exists a constant $\gamma>1$ such that for any $\delta,t>0,\quad \delta^\gamma<t$ we have*

$$\|e^{At}u\|_{s+\delta}\le C\|u\|_s. \tag{2.3}$$

Suppose a function $f:(0,T]\times\overline{B}_{s+\delta}(R)\to E_s$ to be continuous and such that if $(s+\delta)^\gamma<t\le T$ and $u\in\overline{B}_{s+\delta}(R)$ then the following inequality holds

$$\|f(t,u)\|_s\le\frac{C}{\delta^\alpha}. \tag{2.4}$$

A case when

$$\|f(t,u)\|_s\le\frac{C}{t^\beta\delta^\alpha},\quad \beta>0$$

is rather usual but since $\delta^\gamma < t$ this case reduces to (2.4): $C/(t^\beta \delta^\alpha) \le C/\delta^{\beta\gamma+\alpha}$.

Consider a problem

$$u_t = f(t,u) + Au, \tag{2.5}$$

$$u\,|_{t=0} = 0. \tag{2.6}$$

The sense of initial condition (2.6) will be clear in the sequel.

Now we give a definition.

Definition 2. *We shall say that problem (2.5) is parabolic if the semigroup e^{At} is parabolic and*

$$\chi = \frac{\alpha}{\gamma} < 1.$$

Let a space $E^1(T)$, $\quad T > 0$ be given by the formula

$$E^1(T) = \bigcap_{0<s^\gamma<\tau<T} C^1((\tau,T),E_s). \tag{2.7}$$

This space consists of all functions u that map any number $t \in (0,T)$ to the element $u(t) \in \bigcap_{0<s^\gamma<t} E_s$ and the restriction $u\,|_{(\tau,T)}$ belongs to the space $C^1((\tau,T),E_s)$ for all $s \in (0,\tau^{1/\gamma})$.

Theorem 1. *Suppose that problem (2.5) is parabolic. Then there exists a constant $T_* > 0$ such that this problem has a solution $u(t) \in E^1(T_*)$, and for any constant $c \in (0,1)$ one has*

$$\|u(t)\|_{ct^{1/\gamma}} \to 0 \quad \text{as} \quad t \searrow 0. \tag{2.8}$$

The constant T_* depends only on C, α, γ.

The proof of theorem 1 contains in sections 4, 5.

In the next section to illustrate the effect discussed in the Introduction, theorem 1 is applied to a nonlocal parabolic problem. To compare our result with the known one we also consider the Navier-Stokes equation.

If A is the classical Laplace operator and the parabolic equation is considered in a suitable domain then $\gamma = 2$ and the inequality from formula (2.7) takes the form $0 < s^2 < \tau$.

The parameter s symbolizes a spatial variable, so that this inequality specifies the parabolic domain in the plane (τ, s). This endows the term "parabolic equation" with the new sense.

The main difficulty one encounters while using theorem 1 is to obtain estimate (2.3). The general scheme to prove this inequality is as follows. First one must estimate the heat kernel say with the help of the Gaussian upper bounds (see for example [4] and references therein) then use the same argument as in the proof of lemma 1 (see below).

Let us remark that if $E_s = \mathbb{R}^m$, $\quad \|\cdot\|_s = |\cdot|$, $\quad s > 0$ and $A = 0$ then theorem 1 generalizes classical Peano's theorem to the case when the right side of the equation satisfies (2.4) with $s = \delta = (t/3)^{1/\gamma}$.

3. Applications

In the sequel we denote all the inessential positive constants by the same letter c.

Let $\mathbb{T}^m=\mathbb{R}^m/(2\pi\mathbb{Z})^m$ be the $m-$dimensional torus. All the technique developed below can be transferred almost literally to the case of the problem with zero boundary conditions on the $m-$dimensional cube.

By $x=(x_1,\ldots,x_m)$ denote an element of $\mathbb{R}^m$.

Let $\mathbb{T}^m_s=\{z=x+iy\in\mathbb{C}^m\mid x\in\mathbb{T}^m,\quad |y_j|<s,\quad j=1,\ldots,m\}$ be the complex neighborhood of the torus $\mathbb{T}^m$.

Define a set $E_s,\quad s>0$ as follows $E_s=C(\overline{\mathbb{T}}^m_s)\bigcap\mathcal{O}(\mathbb{T}^m_s)$. Here $\mathcal{O}(\mathbb{T}^m_s)$ stands for the set of analytic functions in $\mathbb{T}^m_s$.

The set E_s is a Banach space with respect to the norm $\|u\|_s=\max_{z\in\overline{\mathbb{T}}^m_s}|u(z)|$. By the Montel theorem the embeddings $E_{s+\delta}\subset E_s,\quad \delta>0$ are completely continuous. By definition put $E_0=C(\overline{\mathbb{T}}^m)$.

Let Δ stands for the standard Laplace operator

$$\Delta=\sum_{j=1}^m\partial_j^2,\quad \partial_j=\frac{\partial}{\partial x_j}.$$

Lemma 1. *There exists a positive constant c such that for any $u\in E_s,\quad s\geq 0$ the following inequality holds*

$$\|e^{t\Delta}u\|_{s+\delta}\leq c\exp\Big(\frac{\delta^2}{4t}\Big)\|u\|_s,\quad t,\delta>0.$$

The constant c depends only on m.

Proof. The assertion of the lemma easily follows from the well-known formula:

$$(e^{t\Delta}u)(x)=\frac{1}{(4\pi t)^{m/2}}\int_{\mathbb{R}}e^{-(\xi_1-x_1)^2/(4t)}\,d\xi_1\ldots\int_{\mathbb{R}}e^{-(\xi_m-x_m)^2/(4t)}\,d\xi_m u(\xi).$$

In all these integrals one must shift the contour of integration to the complex plane and then the desired inequality follows from the standard estimates. □

By lemma 1 the semigroup $e^{t\Delta}$ is parabolic with $\gamma=2$.

Lemma 2. *Take a constant $\rho\in(0,1/2]$. For any $\varepsilon\in(0,2\rho)$ there is a positive constant $c=c(\varepsilon)$ such that if $u\in E_{s+\delta}$ then*

$$\|\Delta^{-\rho}\partial_j u\|_s\leq\frac{c}{\delta^{1-2\rho+\varepsilon}}\|u\|_{s+\delta},\quad s\geq 0,\quad \delta>0, \tag{3.1}$$

$$\|\Delta^{\rho}u\|_s\leq\frac{c}{\delta^{2\rho+\varepsilon}}\|u\|_{s+\delta}. \tag{3.2}$$

Proof. Let us prove formula (3.1). Using the standard facts on Sobolev's spaces we have

$$\|\Delta^{-\rho}\partial_j u\|_s\leq c\|\Delta^{-\rho}\partial_j u\|_{H^{\varepsilon,p}(\mathbb{T}^m_s)}\leq c\|u\|_{H^{\varepsilon+1-2\rho,p}(\mathbb{T}^m_s)},\quad \varepsilon p>2m.$$

Then the desired result follows from the interpolation formula and the Cauchy inequality:

$$\|u\|_{H^{\varepsilon+1-2\rho,p}(\mathbb{T}^m_s)} \le c\|u\|^{\varepsilon+1-2\rho}_{H^{1,p}(\mathbb{T}^m_s)}\|u\|^{2\rho-\varepsilon}_{L^p(\mathbb{T}^m_s)}, \quad \|u\|_{H^{1,p}(\mathbb{T}^m_s)} \le \frac{c}{\delta}\|u\|_{s+\delta}.$$

Formula (3.2) is derived in the same way. □

Proposition 1 ([9]). *For any constants* $a \ge r \ge 0$ *one has*

$$\|e^{t\Delta}u\|_{H^a(\mathbb{T}^m)} \le \frac{c}{t^{(a-r)/2}}\|u\|_{H^r(\mathbb{T}^m)}.$$

If $a > m/2$ *then* $\|u\|_0 \le c\|u\|_{H^a(\mathbb{T}^m)}$.

The first of the following two examples illustrates the effect described in the Introduction, the second one is to compare our result with the known one.

3.1. **Integro-differential parabolic equation.** Let us focus our attention on a one dimensional ($m = 1$) system.

Consider a problem

$$u_t = \|\Delta^n u\|^\lambda_{L^2(\mathbb{T})} + \Delta u, \quad u\,|_{t=0} = \hat{u}(x) = \sum_{|k|\ge 2} \frac{e^{ikx}}{|k|^{1/2}\log|k|} \in L^2(\mathbb{T}). \quad (3.3)$$

Here λ is a positive parameter, $n \in \mathbb{N}$.

Parabolic equations with right side depending on L^p norms of the unknown function arise in the theory of incompressible viscous fluid [7].

Let us show that if $n\lambda < 1$ (non-Lipschitz case) then problem (3.3) has a solution in the sense of theorem 1.

After the change of variable $u = e^{t\Delta}\hat{u} + v$ our problem takes the form

$$v_t = f(t,v) + \Delta v, \quad v\,|_{t=0} = 0, \quad f(t,v) = \|\Delta^n e^{t\Delta}\hat{u} + \Delta^n v\|^\lambda_{L^2(\mathbb{T})}. \quad (3.4)$$

So that one has $|f(t,v)| \le c(\|e^{t\Delta}\hat{u}\|^\lambda_{H^{2n}(\mathbb{T})} + \|\Delta^n v\|^\lambda_{L^2(\mathbb{T})})$. Then using proposition 1 we obtain $\|e^{t\Delta}\hat{u}\|_{H^{2n}(\mathbb{T})} \le ct^{-n}\|\hat{u}\|_{L^2(\mathbb{T})}$. The Cauchy inequality gives

$$\|\Delta^n v\|_{L^2(\mathbb{T})} \le c\|\Delta^n v\|_s \le c\delta^{-2n}\|v\|_{s+\delta}, \quad \delta > 0.$$

Combining these inequalities with each other and taking into account that $(s+\delta)^2 < t$ we have

$$|f(t,v)| \le c\delta^{-2n\lambda}(\|\hat{u}\|^\lambda_{L^2(\mathbb{T})} + \|v\|^\lambda_{s+\delta}).$$

Thus $\chi = n\lambda$ and if $n\lambda < 1$ then by theorem 1 the problem has at least one analytic solution.

Consider the case $\lambda = 1$ (Lipschitz case) and let for simplicity $n = 1$.

Denote by u_k the Fourier coefficients of a function u: $u(x) = \sum_{k\in\mathbb{Z}} u_k e^{ikx}$. Notice that the norm of $L^2(\mathbb{T})$ can be presented as follows

$$\|u\|^2_{L^2(\mathbb{T})} = c\sum_{k\in\mathbb{Z}} |u_k|^2.$$

Then separating the variables in problem (3.3) we obtain

$$u_0 = c\int_0^t \Big(\sum_{|k|\ge 2}\frac{|k|^3 e^{-2\xi|k|^2}}{(\log|k|)^2}\Big)^{\frac12}\,d\xi, \tag{3.5}$$
$$u_k = 0,\quad \text{if}\quad |k|=1,$$
$$u_k = \frac{e^{-t|k|^2}}{|k|^{1/2}\log|k|},\quad \text{if}\quad |k|\ge 2.$$

It is not difficult to show that

$$\Big(\sum_{|k|\ge 2}\frac{|k|^3 e^{-2\xi|k|^2}}{(\log|k|)^2}\Big)^{\frac12} \ge -\frac{c}{\xi\log\xi},\quad \xi\in(0,1).$$

So that the integral in formula (3.5) does not exist and thus there are no solutions in this case.

3.2. **3-D Navier-Stokes equation.** In this section we use the Einstein summation convention.

Consider the Navier-Stokes equation in the divergence free setup. After Leray's projection the Navier-Stokes equation takes the well-known form

$$\begin{aligned}(u^k)_t &= A^k_l\partial_j(u^j u^l)+\Delta u^k,\quad A^k_l=(\Delta^{-1}\partial_k\partial_l-\delta_{kl}),\\ u^k\mid_{t=0} &= \hat u^k\in H^r(\mathbb{T}^3),\end{aligned} \tag{3.6}$$

where $\delta_{kl}=1$ for $k=l$ and 0 otherwise; $k,l,j=1,2,3$.

From [6], [5] it follows that if $r=1/2$ then problem (3.6) has a solution $u^i(t,x)$ which is regular in the spatial variables for all $t\in(0,T_*)$. Here T_* is a small positive constant.

Let us show that by theorem 1 the analytic solution exists for all $r>1/2$. This indicates that in terms of paper [1] theorem 1 allows us to carry out only the subcritical case. This is no surprise since theorem 1 is too general.

Assume a parameter $\rho\in(0,1/2)$ to be close 1/2 and let us change the variable in (3.6): $u^k=e^{t\Delta}\hat u^k+\Delta^\rho v^k$. Then the problem have the form

$$v^k_t = f^k(t,v)+\Delta v^k,\quad v^k\mid_{t=0}=0,$$

here

$$f^k(t,v)=A^k_l\partial_j\Delta^{-\rho}(e^{t\Delta}\hat u^j e^{t\Delta}\hat u^l+e^{t\Delta}\hat u^j\Delta^\rho v^l+\Delta^\rho v^j e^{t\Delta}\hat u^l+\Delta^\rho v^j\Delta^\rho v^l).$$

Estimate the function f term by term. Using lemma 2 we have

$$\|A^k_l\partial_j\Delta^{-\rho}(\Delta^\rho v^j\Delta^\rho v^l)\|_s \le \frac{c}{\delta^{\varepsilon+1-2\rho}}\sum_{j,l=1}^3\|\Delta^\rho v^j\Delta^\rho v^l\|_{s+\delta/2}$$
$$\le \frac{c}{\delta^{\varepsilon+1-2\rho}}\sum_{j,l=1}^3\|\Delta^\rho v^j\|_{s+\delta/2}\|\Delta^\rho v^l\|_{s+\delta/2}\le\frac{c}{\delta^{\varepsilon+1+2\rho}}\sum_{j,l=1}^3\|v^j\|_{s+\delta}\|v^l\|_{s+\delta}.$$

Now one must choose the parameters $\varepsilon > 0, \quad \rho \in (0, 1/2)$ such that

$$\frac{\varepsilon + 1 + 2\rho}{2} < 1. \tag{3.7}$$

Let us estimate another term of the function f by using lemmas1, 2 and proposition 1 ($(s+\delta)^2 < t$):

$$\begin{aligned}
\|A_l^k \partial_j \Delta^{-\rho}(e^{t\Delta}\hat{u}^j e^{t\Delta}\hat{u}^l)\|_s &\le \frac{c}{\delta^{1+\varepsilon-2\rho}} \sum_{j,l=1}^{3} \|e^{t\Delta}\hat{u}^j\|_{s+\delta}\|e^{t\Delta}\hat{u}^l\|_{s+\delta} \\
&\le \frac{c}{\delta^{1+\varepsilon-2\rho}} \sum_{j,l=1}^{3} \|e^{t\Delta/2}\hat{u}^j\|_0\|e^{t\Delta/2}\hat{u}^l\|_0 \\
&\le \frac{c}{\delta^{1+\varepsilon-2\rho}} \sum_{j,l=1}^{3} \|e^{t\Delta/2}\hat{u}^j\|_{H^a(\mathbb{T}^3)}\|e^{t\Delta/2}\hat{u}^l\|_{H^a(\mathbb{T}^3)} \\
&\le \frac{c}{\delta^{1+\varepsilon-2\rho}t^{a-r}} \sum_{j,l=1}^{3} \|\hat{u}^j\|_{H^r(\mathbb{T}^3)}\|\hat{u}^l\|_{H^r(\mathbb{T}^3)},
\end{aligned}$$

here $a > 3/2$. We need to have

$$\frac{1 + \varepsilon - 2\rho}{2} + a - r < 1. \tag{3.8}$$

In the same manner we obtain

$$\|A_l^k \partial_j \Delta^{-\rho}(e^{t\Delta}\hat{u}^j \Delta^{\rho} v^l)\|_s \le \frac{c}{\delta^{\varepsilon+1}t^{(a-r)/2}} \sum_{j,l=1}^{3} \|\hat{u}^j\|_{H^r(\mathbb{T}^3)}\|v^l\|_{s+\delta}.$$

Thus there must be

$$\varepsilon + 1 + a - r < 2. \tag{3.9}$$

It is not difficult to show that for any $r > 1/2$ there exist the small parameter $\varepsilon > 0$, the parameter a close to $3/2$ from above and the parameter ρ close to $1/2$ from below such that inequalities (3.7), (3.8), (3.9) are fulfilled.

4. Preliminaries in functional analysis

In this section we collect several facts from functional analysis. These facts will be useful in the section 5 when we prove theorem 1.

Consider the spaces

$$C([\tau, T], E_{\mu\tau^{1/\gamma}}), \quad 0 < \mu < 1, \quad 0 < \tau < T$$

with standard norms. Now we construct the projective limit of these spaces. Define a space $E(T)$ as follows

$$E(T) = \bigcap_{0<\mu<1} \bigcap_{0<\tau<T} C([\tau, T], E_{\mu\tau^{1/\gamma}}).$$

There is another equivalent definition of the space $E(T)$:

$$E(T)=\bigcap_{0<s^\gamma<\tau<T} C([\tau,T],E_s).$$

Being endowed with a collection of seminorms

$$\|u\|_{\tau,\mu}=\max_{\tau\le\xi\le T}\|u(\xi)\|_{\mu\tau^{1/\gamma}},\quad u\in E(T) \tag{4.1}$$

the space $E(T)$ becomes a locally convex topological space.

These seminorms obviously satisfy the following inequalities

$$\|u\|_{\tau,\mu}\le\|u\|_{\tau,\mu+\delta},\quad \delta>0, \tag{4.2}$$

$$\|u\|_{\tau,r\mu}\le\|u\|_{r^\gamma\tau,\mu},\quad 0<r\le 1. \tag{4.3}$$

Indeed, formula (4.2) follows from (2.1) directly. Formula (4.3) is a result of the estimate

$$\|u\|_{\tau,r\mu}=\max_{\tau\le\xi\le T}\|u(\xi)\|_{\mu(r^\gamma\tau)^{1/\gamma}}\le\max_{r^\gamma\tau\le\xi\le T}\|u(\xi)\|_{\mu(r^\gamma\tau)^{1/\gamma}}=\|u\|_{r^\gamma\tau,\mu}.$$

Formulas (4.2), (4.3) imply that the space $E(T)$ is first countable: the topology of this space can be defined by the seminorms (4.1) only with $\mu,\tau\in\mathbb{Q}$.

Recall the Arzela-Ascoli theorem [8]:

Theorem 2. *Let $H\subset C([0,T],X)$ be a set in the space of continuous functions with values in a Banach space X. Assume that the set H is closed, bounded, uniformly continuous and for every $t\in[0,T]$ the set $\{u(t)\in X\}$ is a compact set in the space X. Then the set H is a compact set in the space $C([0,T],X)$.*

Now we shall establish an analogue of this result.

Proposition 2. *Suppose that a set $K\subset E(T)$ is closed. Then K is a compact set if the following two conditions are fulfilled.*

The set K is bounded.

For any $\varepsilon>0$ and for any $\tau\in(0,T)$, $\mu\in(0,1)$ there exists a constant $\delta>0$ such that if $t',t''\in[\tau,T]$, $|t'-t''|<\delta$ then

$$\sup_{u\in K}\|u(t')-u(t'')\|_{\mu\tau^{1/\gamma}}<\varepsilon.$$

(This means that K is a uniformly continuous set.)

First prove a lemma.

Lemma 3. *Let $\{v_j\}\subseteq K$ be a sequence. Then for any $\tau\in(0,T)$ the sequence $\{v_j\}$ contains a subsequence that is convergent in all the norms $\|\cdot\|_{\tau,\mu}$, $\mu\in(0,1)$.*

Proof. Indeed, take an increasing sequence $\mu_k\to1$, $\mu_1>0$ and fix any value of $\tau\in(0,T)$. Since the sequence $\{v_j\}$ is bounded and uniformly continuous in $C([\tau,T],E_{\mu_2\tau^{1/\gamma}})$ then by theorem 2 it contains a subsequence $\{v^1_j\}$ that is convergent in $C([\tau,T],E_{\mu_1\tau^{1/\gamma}})$.

Further since the sequence $\{v_j^1\}$ is bounded and uniformly continuous in $C([\tau,T],E_{\mu_3\tau^{1/\gamma}})$ one can pick a subsequence $\{v_j^2\}\subseteq\{v_j^1\}$ such that the sequence $\{v_j^2\}$ is convergent in $C([\tau,T],E_{\mu_2\tau^{1/\gamma}})$ etc.

By inequality (4.2) the diagonal sequence $\{v_j^j\}$ converges in all the norms $\|\cdot\|_{\tau,\mu},\quad \mu\in(0,1)$ with this fixed τ. □

Proof of proposition 2. A set $P=\mathbb{Q}\bigcap(0,T)$ is countable. So we can number its elements as follows $P=\{\tau_i\}_{i\in\mathbb{N}}$.

We must show that any sequence $\{u_j\}\subseteq K$ contains a convergent subsequence $\{u_{j_k}\}$.

By lemma 3 there is a subsequence $\{u_j^1\}\subseteq\{u_j\}$ that is convergent in all the norms $\|\cdot\|_{\tau_1,\mu}\quad \mu\in(0,1)$. By the same argument there is a subsequence $\{u_j^2\}\subseteq\{u_j^1\}$ that is convergent in all the norms $\|\cdot\|_{\tau_2,\mu}\quad \mu\in(0,1)$ etc.

The diagonal sequence $\{u_j^j\}$ is convergent in all the norms $\|\cdot\|_{\tau_k,\mu},\quad k\in\mathbb{N},\quad \mu\in(0,1)$.

By inequality (4.3) the sequence $\{u_j^j\}$ is convergent in all the norms $\|\cdot\|_{\tau,\mu},\quad \tau\in(0,T),\quad \mu\in(0,1)$.

Proposition 2 is proved.

Lemma 4. *Let X,Y be Banach spaces. Suppose that $A_a:X\to Y,\quad a'>a>0$ is a collection of bounded linear operators such that for each $x\in X$ we have*

$$\sup_{a'>a>0}\|A_ax\|_Y<\infty,\quad \|A_ax\|_Y\to 0\quad\text{as}\quad a\to 0.$$

Then for any compact set $B\subset X$ it follows that

$$\sup_{x\in B}\|A_ax\|_Y\to 0\quad\text{as}\quad a\to 0.$$

This result is a direct consequence of the Banach-Steinhaus theorem [8].

Let us recall a generalized version of the Schauder fixed point theorem.

Theorem 3 ([2]). *Let W be a closed convex subset of the locally convex space E. Then a compact continuous mapping $f:W\to W$ has a fixed point $\hat u$ i.e. $f(\hat u)=\hat u$.*

5. Proof of Theorem 1

By definition put

$$W(T_*)=\{u\in E(T_*)\mid \|u\|_{\tau,\nu}\le R,\quad 0<\tau<T_*,\quad 0<\nu<1\}.$$

The constant $T_*>0$ will be defined.

We want to find a fixed point of a mapping

$$F(u)=\int_0^t e^{A(t-\xi)}f(\xi,u(\xi))\,d\xi.$$

Then we show that this fixed point is the desired solution to problem (2.5).

Lemma 5. *If the constat T_* is small enough then the mapping F takes the set $W(T_*)$ to itself.*

Proof. Let constants t, s be taken as follows $0 < s < t^{1/\gamma}, \quad t \le T_*$. Suppose $u \in W(T_*)$ then estimate a function $v(t) = F(u)$:

$$\|v(t)\|_s \le \int_0^t \|e^{A(t-\xi)} f(\xi, u(\xi))\|_s \, d\xi = X + Y, \tag{5.1}$$

here we use the notation

$$X = \int_0^{t-s^\gamma} \|e^{A(t-\xi)} f(\xi, u(\xi))\|_s \, d\xi, \quad Y = \int_{t-s^\gamma}^t \|e^{A(t-\xi)} f(\xi, u(\xi))\|_s \, d\xi.$$

To estimate X take constants ε and μ such that

$$0 < \varepsilon < \frac{s}{t^{1/\gamma}} < \mu < 1. \tag{5.2}$$

The constant ε is assumed to be small and the constant μ is assumed to be close to 1.

Let the variables δ and δ' be given by the formulas

$$\delta = s - \varepsilon \xi^{1/\gamma}, \quad \delta' = \xi^{1/\gamma}(\mu - \varepsilon).$$

Taking into account that $\xi \in (0, t - s^\gamma]$ we see that the variables δ, δ' are positive and

$$s - \delta > 0, \quad s - \delta + \delta' < \xi^{1/\gamma}, \quad \delta < (t - \xi)^{1/\gamma}. \tag{5.3}$$

The inequality in the middle implies that

$$u(\xi) \in \overline{B}_{s-\delta+\delta'}(R) \tag{5.4}$$

and thus the term X is estimated as follows

$$\begin{aligned} X &\le C \int_0^{t-s^\gamma} \|f(\xi, u(\xi))\|_{s-\delta} \, d\xi \le C^2 \int_0^{t-s^\gamma} \frac{1}{\delta'^\alpha} \, d\xi \\ &\le \frac{C^2}{(\mu-\varepsilon)^\alpha} \int_0^{t-s^\gamma} \frac{d\xi}{\xi^\chi} = \frac{C^2}{(1-\chi)(\mu-\varepsilon)^\alpha} (t - s^\gamma)^{1-\chi}. \end{aligned} \tag{5.5}$$

We shall estimate the term Y.

Introduce a function ψ by the formula

$$\psi(y) = y^{1/\gamma} + (1-y)^{1/\gamma} - 1.$$

The function ψ is positive on the interval $(0, 1)$. Define a constant I as follows

$$I = \int_0^1 \frac{dy}{(\psi(y))^\alpha}.$$

Let the constant μ be as above. We redefine the variables δ, δ' by the formulas

$$\delta = \mu(t - \xi)^{1/\gamma}, \quad \delta' = \mu \xi^{1/\gamma} + \delta - s.$$

Now the variable ξ belongs to the interval $[t - s^\gamma, t]$ and thus the variables δ, δ' are positive and satisfy inequalities (5.3).

It is only not trivial to show that the variable δ' is positive. Let us prove this. Indeed,

$$\delta' = \mu\xi^{1/\gamma} + \mu(t-\xi)^{1/\gamma} - s = t^{1/\gamma}\Big(\mu y^{1/\gamma} + \mu(1-y)^{1/\gamma} - \frac{s}{t^{1/\gamma}}\Big), \tag{5.6}$$

here and below we use the notation $y = \xi/t$. Form (5.6) it follows that

$$\delta' > t^{1/\gamma}\mu\psi(y). \tag{5.7}$$

By the same argument as above, inclusion (5.4) is fulfilled with the new δ and δ'.

We are ready to estimate the term Y. By (5.7) it follows that

$$\begin{aligned} Y &\le C\int_{t-s^\gamma}^{t} \|f(\xi,u(\xi))\|_{s-\delta}\,d\xi \le C^2\int_{t-s^\gamma}^{t}\frac{d\xi}{\delta'^\alpha} \\ &\le \frac{C^2t^{1-\chi}}{\mu^\alpha}\int_{1-s^\gamma/t}^{1}\frac{dy}{(\psi(y))^\alpha} \le \frac{C^2I}{\mu^\alpha}t^{1-\chi}. \end{aligned} \tag{5.8}$$

Now the assertion the of lemma follows from formulas (5.1), (5.5) and (5.8). □

Corollary 1. *Formulas (5.5), (5.8) imply that if* $0 < s^\gamma < t \le T_*$ *and* $v(t) = F(u), \quad u \in W(T_*)$ *then*

$$\|v(t)\|_s \le c_2 t^{1-\chi},$$

here c_2 *is a positive constant independent on* u, t, s.

Lemma 6. *The set* $F(W(T_*))$ *is precompact in* $E(T_*)$.

Proof. By proposition 2 it is sufficient to prove that the set $F(W(T_*))$ is uniformly continuous.

Take a function $u \in W(T_*)$ and let $v(t) = F(u)$. We must show that if $t', t'' \ge \tau, \quad \tau \in (0, T_*)$ then for any $\mu \in (0,1)$ one has

$$\sup_{u\in W(T_*)} \|v(t') - v(t'')\|_{\mu\tau^{1/\gamma}} \to 0, \quad \text{as} \quad |t' - t''| \to 0.$$

Indeed, for definiteness assume that $t'' > t'$ then

$$\begin{aligned} v(t'') - v(t') &= \int_{t'}^{t''} e^{A(t''-\xi)}f(\xi,u)\,d\xi \\ &\quad + \Big(e^{A(t''-t')} - \mathrm{id}_{E_s}\Big)\int_0^{t'} e^{A(t'-\xi)}f(\xi,u)\,d\xi, \quad s^\gamma < \tau. \end{aligned} \tag{5.9}$$

Choose a positive constant δ such that $(s+\delta)^\gamma < \tau$ and using formula (2.2) estimate the first term from the right side of this formula

$$\begin{aligned} \Big\|\int_{t'}^{t''} e^{A(t''-\xi)}f(\xi,u)\,d\xi\Big\|_s &\le C\int_{t'}^{t''}\|f(\xi,u)\|_s\,d\xi \le C^2\int_{t'}^{t''}\frac{d\xi}{\delta^\alpha} \\ &= \frac{C^2}{\delta^\alpha}(t''-t'). \end{aligned}$$

So that the first term in the right side of (5.9) is vanished uniformly.

Consider a set

$$U=\bigcup_{\tau\le t'\le T_*}\Big\{\int_0^{t'}e^{A(t'-\xi)}f(\xi,u)\,d\xi\mid u\in W(T_*)\Big\}.$$

By lemma 5 the set U is bounded in any space $E_{\mu'\tau^{1/\gamma}}$ with $1>\mu'>\mu$ thus it is compact in $E_{\mu\tau^{1/\gamma}}$. By lemma 4 we get

$$\sup_{w\in U}\|e^{A(t''-t')}w-w\|_{\mu\tau^{1/\gamma}}\to 0,\quad \text{as}\quad t''-t'\to 0.$$

This shows that the second term in the right side of formula (5.9) is vanished uniformly. ☐

Corollary 2. *The set $F(W(T_*))$ is uniformly continuous with respect to the variable t.*

Lemma 7. *The mapping $F:W(T_*)\to W(T_*)$ is continuous with respect to the topology of the space $E(T_*)$.*

Proof. Suppose a sequence $\{v_l\}\subset W(T_*)$ to be convergent to the element $v\in W(T_*)$ as $l\to\infty$. We need to show that for any $s^\gamma<\tau<T_*$ the sequence

$$\sup_{\tau\le t\le T_*}\Big\|\int_0^t e^{A(t-\xi)}f(\xi,v_l(\xi))\,d\xi-\int_0^t e^{A(t-\xi)}f(\xi,v(\xi))\,d\xi\Big\|_s$$

vanishes as $l\to\infty$.

By corollary 2 the sequence

$$\Big\{\int_0^t e^{A(t-\xi)}f(\xi,v_l(\xi))\,d\xi\Big\} \tag{5.10}$$

is uniformly continuous on the interval $[\tau,T_*]$. The uniform convergence of such a sequence is equivalent to its pointwise convergence [8]. Thus it is sufficient to prove that sequence (5.10) is convergent in E_s for each $t\in[\tau,T_*]$.

Fix $t\in[\tau,T_*]$ and let constants ε,μ satisfy inequality (5.2). Then using the argument of lemma 5 write

$$\begin{aligned}\Big\|\int_0^t e^{A(t-\xi)}&(f(\xi,v_l(\xi))-f(\xi,v(\xi)))\,d\xi\Big\|_s\\ &\le\int_0^{t-s^\gamma}\|f(\xi,v_l(\xi))-f(\xi,v(\xi))\|_{\varepsilon\xi^{1/\gamma}}\,d\xi\\ &+\int_{t-s^\gamma}^t\|f(\xi,v_l(\xi))-f(\xi,v(\xi))\|_{s-\mu(t-\xi)^{1/\gamma}}\,d\xi.\end{aligned} \tag{5.11}$$

Since the function f is continuous, for a fixed ξ we have:

$$\|f(\xi,v_l(\xi))-f(\xi,v(\xi))\|_{\varepsilon\xi^{1/\gamma}}\to 0,\quad \xi\in[0,t-s^\gamma],$$
$$\|f(\xi,v_l(\xi))-f(\xi,v(\xi))\|_{s-\mu(t-\xi)^{1/\gamma}}\to 0,\quad \xi\in[t-s^\gamma,t],\quad \text{as}\quad l\to\infty.$$

Moreover by formulas (5.5), (5.8) both of these expressions are majorized with the L^1-integrable function:

$$\begin{aligned}\|f(\xi,v_l(\xi))-f(\xi,v(\xi))\|_{\varepsilon\xi^{1/\gamma}} &\le \|f(\xi,v_l(\xi))\|_{\varepsilon\xi^{1/\gamma}}+\|f(\xi,v(\xi))\|_{\varepsilon\xi^{1/\gamma}}\\ &\le \frac{2C^2}{(\mu-\varepsilon)^\alpha\xi^\chi},\end{aligned}$$

and

$$\|f(\xi,v_l(\xi))-f(\xi,v(\xi))\|_{s-\mu(t-\xi)^{1/\gamma}}\le \frac{2C^2}{t^\chi\mu^\alpha(\psi(\xi/t))^\alpha}.$$

Therefore by the Dominated convergence theorem the integrals in the right side of (5.11) are vanished as $l\to\infty$. □

So by theorem 3 and lemmas 5, 6, 7 we obtain a fixed point of the mapping F, say u:

$$F(u)=u\in W(T_*).$$

Let us show that this fixed point is the desired solution to problem (2.5). Suppose that $t, t+h>s^\gamma$. First consider the case $h>0$. Differentiate the function $u(t)$ explicitly:

$$\begin{aligned}u_t(t)&=\lim_{h\to0}h^{-1}\Big(\int_0^{t+h}e^{A(t+h-\xi)}f(\xi,u(\xi))\,d\xi-\int_0^t e^{A(t-\xi)}f(\xi,u(\xi))\,d\xi\Big)\\ &=\lim_{h\to0}h^{-1}\int_t^{t+h}e^{A(t+h-\xi)}f(\xi,u(\xi))\,d\xi\\ &+\lim_{h\to0}h^{-1}(e^{Ah}-\mathrm{id}_{E_s})\int_0^t e^{A(t-\xi)}f(\xi,u(\xi))\,d\xi.\end{aligned}\tag{5.12}$$

Lemma 5 implies that $\int_0^t e^{A(t-\xi)}f(\xi,u(\xi))\,d\xi\in E_{s'}$ with $s^\gamma<s'^\gamma<t,t+h$ hence formula (2.2) gives

$$h^{-1}(e^{Ah}-\mathrm{id}_{E_s})\int_0^t e^{A(t-\xi)}f(\xi,u(\xi))\,d\xi\to A\int_0^t e^{A(t-\xi)}f(\xi,u(\xi))\,d\xi\tag{5.13}$$

in E_s as $h\to0$.

Let us prove that

$$h^{-1}\int_t^{t+h}e^{A(t+h-\xi)}f(\xi,u(\xi))\,d\xi\to f(t,u(t))\tag{5.14}$$

in E_s as $h\to0$.

Indeed, observe that

$$\begin{aligned}&h^{-1}\int_t^{t+h}e^{A(t+h-\xi)}f(\xi,u(\xi))\,d\xi-f(t,u(t))\\ &=h^{-1}\Big(\int_t^{t+h}e^{A(t+h-\xi)}(f(\xi,u(\xi))-f(t,u(t)))\,d\xi\\ &+\int_t^{t+h}(e^{A(t+h-\xi)}-\mathrm{id}_{E_s})f(t,u(t))\,d\xi\Big).\end{aligned}$$

The first integral in the right side of this formula is estimated as follows:

$$\Big\|\int_t^{t+h} e^{A(t+h-\xi)}(f(\xi,u(\xi))-f(t,u(t)))\,d\xi\Big\|_s$$
$$\le Ch\max_{t\le\xi\le t+h}\|f(\xi,u(\xi))-f(t,u(t))\|_s=o(h).$$

Since the semigroup e^{At} is strongly continuous for the second integral we get

$$\Big\|\int_t^{t+h}(e^{A(t+h-\xi)}-\mathrm{id}_{E_s})f(t,u(t))\,d\xi\Big\|_s$$
$$\le h\max_{t\le\xi\le t+h}\|(e^{A(t+h-\xi)}-\mathrm{id}_{E_s})f(t,u(t))\|_s=o(h).$$

If $h<0$ then instead of formula (5.12) one must use the following expression

$$u_t(t)=\lim_{h\to 0}h^{-1}\Big((\mathrm{id}_{E_s}-e^{-Ah})\int_0^{t+h}e^{A(t+h-\xi)}f(\xi,u(\xi))\,d\xi$$
$$-\int_{t+h}^{t}e^{A(t-\xi)}f(\xi,u(\xi))\,d\xi\Big).$$

In this case only the proof of the formula

$$\lim_{h\to 0}h^{-1}(\mathrm{id}_{E_s}-e^{-Ah})\int_0^{t+h}e^{A(t+h-\xi)}f(\xi,u(\xi))\,d\xi$$
$$=A\int_0^t e^{A(t-\xi)}f(\xi,u(\xi))\,d\xi$$

differs from the previous argument.

Let us prove this formula. Obviously we have

$$(\mathrm{id}_{E_s}-e^{-Ah})\int_0^{t+h}e^{A(t+h-\xi)}f(\xi,u(\xi))\,d\xi$$
$$=(\mathrm{id}_{E_s}-e^{-Ah})u(t)+(\mathrm{id}_{E_s}-e^{-Ah})(u(t+h)-u(t)). \tag{5.15}$$

The set

$$V=\Big\{\frac{u(t+h)-u(t)}{\|u(t+h)-u(t)\|_{s'}}\Big|h\in(h',0)\Big\}$$

with $h'<0$ close to zero is bounded in $E_{s'}$, $\quad s^\gamma<s'^\gamma<t+h'$. Consequently V is a compact set in E_s. By lemma 4 the set

$$(A_{-h}-A)V,\quad A_{-h}=\frac{1}{h}\Big(\mathrm{id}_{E_s}-e^{-Ah}\Big)$$

is bounded in E_s and thus the set $A_{-h}V$ is also bounded.

Thus taking into account that the function $u(t)$ is continuous we yield

$$\Big\|\frac{1}{h}\Big(\mathrm{id}_{E_s}-e^{-Ah}\Big)(u(t+h)-u(t))\Big\|_s$$
$$=\|u(t+h)-u(t)\|_{s'}\cdot\Big\|A_{-h}\frac{u(t+h)-u(t)}{\|u(t+h)-u(t)\|_{s'}}\Big\|_s=o(1).$$

For the second term of the right side of (5.15) this implies

$$\|(\mathrm{id}_{E_s}-e^{-Ah})(u(t+h)-u(t))\|_s=o(h).$$

The first term of the right side of formula (5.15) is estimated as follows

$$\|(\mathrm{id}_{E_s}-e^{-Ah})u(t)-hAu(t)\|_s=o(h).$$

Substituting formulas (5.13) and (5.14) to (5.12) we see that the function u is a solution to equation (2.5).

Formula (2.8) follows from corollary 1.

Theorem 1 is proved.

References

[1] J. M. Arrieta A. N. Carvalho Abstract parabolic problems with critical nonlinearities and applications to Navier-Stokes and Heat equations. Transl. of the Amer. Math. Soc. V. 352, 1, pp. 285-310.
[2] F. E. Browder A new generalization of the Schauder fixed point theorem, Math. Ann. 174, (1967), 285-290.
[3] A. N. Carvalho Abstract parabolic problems in odered Banach spaces. Cadernos de Mathematica 02, 141-146, March (2001) Artigo Numero SMA No.104.
[4] El-Maati Ouhabaz Analysis of Heat Equations on Domains. Princeton University Press, 2004.
[5] H. Fujita, T. Kato On the Navier-Stokes initial value problem I. Arch. Rat. Mech. Anal. 16 (1964), 269-315. MR 29:3774.
[6] T. Kato, H. Fujita On the nonstationary Navier-Stokes system. Rend. Sem. Math. Univ. Padova 32, (1962), 243-260. MR 26:495.
[7] K. Ohkitani and H. Okamoto, Blow-up problems modeled from the strain-vorticity dynamics, Proceedings of "Tosio Kato's Method and Principles for Evolution Equations in Mathematical Physics" (Eds. H. Fujita, S. T. Kuroda and H. Okamoto), RIMS Kokyuroku 1234 (2001), pp. 240–250.
[8] L. Schwartz Analyse Mathématique, Hermann, 1967.
[9] M. E. Taylor Partial Differential Equations, Springer, New York, 1996.
[10] O. Zubelevich On Some Topological View on the Abstract Cauchy-Kowalewski Problem. Complex Variables, August 15, 2004, vol. 49, no. 10, pp. 703-709(7).

E-mail address: ozubel@yandex.ru
Current address: 2-nd Krestovskii Pereulok 12-179, 129110, Moscow, Russia